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Relativistic Quantum Mechanics: exercises

ABSTRACT: This document contains exercises on symmetries in quantum mechanics, the Klein-Gordon equation and the Dirac equation. Electromagnetic interactions are also studied in the context of the minimal coupling prescription.

Symmetries

1. Prove Wigner's theorem on the representation of symmetries on Hilbert spaces. (See Appendix A, chapter 2 of Weinberg vol I, or the document accompanying this problem set.

2. Show explicitly that the generator of the translation is the momentum operator. (Hint: Taylor expand the Hamiltonian calculated at a displaced point, compare the expression you find with the general commutation rules of $\left[\hat{p}, \hat{O}(x)\right]$, and finally check that imposing the invariance of the Hamiltonian is equivalent to $\left[\hat{p}, \hat{H}\right] = 0$). Generalize it to the case of a system of two point particles.

3. Verify that the Schroedinger equation is invariant under Galilean transformations:
$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi'(x',t')}{\partial x'^2} + V(x')\,\psi'(x',t') = i\hbar\frac{\partial\psi'(x',t')}{\partial t'} \tag{1}$$
if the potential V is invariant and the probability amplitude transforms as
$$\psi'(x',t') = \psi(x,t)\exp\left[\frac{i}{\hbar}\left(-mvx + \frac{1}{2}mv^2t\right)\right]. \tag{2}$$
How do you interpret the change in the probability amplitude? (Hint: Find first the rules of transformation for the operators $\partial/\partial x$ and $\partial/\partial t$).

4. Starting from the expression of the infinitesimal boost in the z direction, $\Lambda = I_4 - i\omega K_x$, where I_4 is the identity matrix and K_x the $(2,1)$ and $(1,2)$ elements equal to $-i$ and all the rest zero, find the expression for the finite boost, i.e. show that
$$\Lambda(\omega) = \mathrm{e}^{-i\omega K_x}. \tag{3}$$
(Hint: use the usual Taylor expansion of the exponential function).

5. Using the explicit form of the generators of the boosts and the rotations, check that the following commutation relations are obeyed:
$$[J_x, J_y] = iJ_z, \quad [K_x, K_y] = -iJ_z, \quad [J_x, K_y] = iK_z.$$

The Klein Gordon equation

1. Consider the Klein-Gordon equation in the presence of an EM field $A^\mu = (A^0, \mathbf{A})$. Introduce the interaction by minimal coupling. Take the non-relativistic limit and show that the equation reduces to the Schroedinger equation with minimal coupling.

2. Discuss the charge and current densities in the presence of an EM field. Consider, as an example, the Coulomb potential. What happens to the charge density close to the source of the potential?

3. (***) Solve exactly the eigenvalue problem in the presence of a Coulomb potential. Comment on the conditions to be met to have a meaningful solution.

The Dirac Equation

1 Show that an alternative choice for the matrices $\vec{\alpha}$ and β in the Dirac equation is

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$
$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Writing the wavefunction as

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \mathrm{e}^{i\vec{p}.\vec{x} - iEt} \quad (\hbar = c = 1)\,,$$

obtain the equations satisfied by ϕ and χ. Verify that for consistency one requires $E^2 = \vec{p}^2 + m^2$. Show that the ϕ, χ equations decouple if $m = 0$.
Interpret the ϕ and χ equations in this case *i.e.* what are the properties of the particles described by them? In particular consider the *helicities* of the particles, where helicity is the component of spin along the direction of motion). Find explicit forms for ϕ and χ in the case $\vec{p} = p\,(\sin\theta, 0, \cos\theta)$ and $m = 0$, satisfying $\phi^\dagger\phi = \chi^\dagger\chi = 1$.

2 Find the normalisation constant N (dependent on the energy E) such that the free particle positive energy Dirac solutions

$$\psi = N \begin{pmatrix} \phi \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\phi \end{pmatrix} \mathrm{e}^{-ip.x}, \quad E = \sqrt{\vec{p}^2 + m^2} \quad (\hbar = c = 1)\,,$$

are normalised to $\psi^\dagger\psi = E/m$. Why should one want $\psi^\dagger\psi$ to behave under Lorentz transformations as an energy?
Show that for the negative energy solution it is also possible to choose the normalisation so that $\psi^\dagger\psi$ is positive.

3 A positive energy electron is incident on a *one dimensional* potential step of height V. What boundary condition would you impose on the wavefunction at the step? [Hint: think about ρ and $\vec{j}$]. Write down the appropriate Dirac equation and solve it using your boundary condition. Show that if V is *large* enough the reflected current is larger than the incident current. Suggest a physical interpretation for this result.

4 Why can the time-dependent Schroedinger equation not correctly predict the behavior of relativistic particles? Why is the Klein-Gordon equation not a satisfactory alternative?
The Dirac equation for a free spin-$\frac{1}{2}$ particle of mass m, energy E and momentum $\vec{p}$ may be written

$$(\vec{\alpha}.\vec{p} + \beta m)\, u = E\, u.$$

In the Pauli representation the matrices $\vec{\alpha}$ and β are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{4}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

where the components of $\vec{\sigma}$ are the 2×2 Pauli spin matrices and I is the 2×2 unit matrix. Find four normalized solutions of the Dirac equation that are eigenstates of the helicity operator

$$\hat{s} \equiv \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|} \text{ where } \vec{\Sigma} = \frac{1}{2}\hbar \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

Electromagnetic interactions and the Dirac equation

1 The stationary-state Dirac equation for a spin-$\frac{1}{2}$ particle of mass m and energy E is (with $\hbar = c = 1$)

$$(-i\vec{\alpha}.\nabla + \beta m)\,\psi = E\psi. \tag{6}$$

Explain why the quantities $\vec{\alpha}$, β have to satisfy the relations

$$\begin{aligned} \alpha_i^2 = \beta^2 &= 1 \\ \alpha_i\alpha_j + \alpha_j\alpha_i &= 0 \ \forall \ i \neq j \\ \alpha_i\beta + \beta\alpha_i &= 0. \end{aligned}$$

Why are $\vec{\alpha}$ and β Hermitian?
These requirements are satisfied by the choices shown in equations 4 and 5. The three Pauli matrices σ_i satisfy $\sigma_i\sigma_j = \delta_{ij} + \sum_k i\epsilon_{ijk}\sigma_k$. The stationary-state Dirac equation for a spin-$\frac{1}{2}$ particle of mass m and charge e in a static electromagnetic field with vector potential $\vec{A}$ is obtained from equation 6 by replacing $-i\nabla$ by $\vec{D}$ where $\vec{D} = -i\nabla - e\vec{A}$. Verify that

$$\left(\vec{\alpha}.\vec{D}\right)^2 = \vec{D}^2 I_4 - e\vec{\Sigma}.\vec{B},$$

where I_4 is the 4×4 unit matrix, $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, and $\vec{B} = \nabla \times \vec{A}$.

For the particular case $\vec{A} = (0, xB, 0)$, show, by considering solutions of the form $\psi = \exp\left[i\left(p_y y + p_z z\right)\right] u\left(x\right)$, or otherwise, that the energy eigenvalues E of a relativistic electron in a constant magnetic induction $\vec{B}$ are given by

$$E^2 = m^2 + p_z^2 + (2n+1)\,|eB| \pm eB,$$

for $n = 0, 1, 2,...$ [Hint: you may assume that the eigenvalues of $-\left(\mathrm{d}^2/\mathrm{d}x^2\right) + \omega^2 x^2$ are $(2n+1)\,|\omega|$].

2 Verify explicitly that the matrices $\vec{\alpha}$ and β given in equations 4 and 5 satisfy the Dirac anti-commutation relations. Defining the four γ matrices $\gamma^\mu = \left(\gamma^0, \vec{\gamma}\right)$ where $\gamma^0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$ show that the Dirac equation can be written in the covariant form:

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0.$$

Find the anti-commutation relations between the γ matrices.
Define the *conjugate spinor* $\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0$ and use the result above to find the equation satisfied by $\bar{\psi}$ in the γ matrix notation.
The Dirac probability current can be written $j^\mu = \bar{\psi}\gamma^\mu\psi$, hence show that it satisfies the conservation law $\partial_\mu j^\mu = 0$.

3 A simple treatment of the Dirac equation predicts the gyromagnetic ratio of the electron, g, to be 2.

(i) The Dirac equation can be written as $(\vec{\alpha}.\vec{p} + \beta m)\,u = Eu$, where u is a four-component spinor which can be decomposed into a pair of two-component spinors u_A and u_B,

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}.$$

Taking an explicit matrix representation of $\vec{\alpha}$ and β:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

gives a pair of equations

$$\vec{\sigma}.\vec{p}u_B = (E - m)\,u_A \tag{7}$$

$$\vec{\sigma}.\vec{p}u_A = (E + m)\,u_B. \tag{8}$$

Use these to show that for a non-relativistic electron, travelling with a velocity v, u_B is smaller than u_A by a factor of approximately v/c.

(ii) To introduce an interaction between an electron and the electromagnetic field substitute $p^\mu \rightarrow p^\mu + eA^\mu$ into equations 7 and 8. Then act on equation 7 with the operator $\left(E + eA^0 + m\right)$, taking $eA^0 \ll m$. Finally, eliminate u_B using equation 8. Show that the resulting expression for u_A can be written as

$$\vec{\sigma}.\left(\vec{p} + e\vec{A}\right)\sigma\left(\vec{p} + e\vec{A}\right)u_A = 2m\left(E_{nr} + eA^0\right)u_A,$$

where $E_{nr} = E - m$.

(iii) The operators in the above equation do not commute, show that for the general operators $\vec{a}$ and $\vec{b}$

$$(\vec{\sigma}.\vec{a})\,(\vec{\sigma}.b) = \vec{a}.\vec{b} + i\vec{\sigma}.\left(\vec{a}\times\vec{b}\right),$$

given that

$$\{\sigma_i, \sigma_j\} = \delta_{ij} \text{ and } [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

Hence, using

$$\vec{p}\times\vec{A} + \vec{A}\times\vec{p} = -i\nabla\times\vec{A}$$

show that the wavefunction $\psi_A = u_A \exp\left(-ip^\mu x_\mu\right)$ satisfies the Schroedinger-Pauli equation,

$$\left(\frac{1}{2m}\left(\vec{p}+e\vec{A}\right)^2 + \frac{e}{2m}\vec{\sigma}.\vec{B} - eA^0\right)\psi_A = E_{nr}\psi_A,$$

where $\vec{\sigma}.\vec{B}$ is the magnetic field.

(iv) From this last equation show that the predicted gyromagnetic ratio of the electron, g, is 2.

(v) Experimentally g is slightly larger than 2, the actual value can be predicted very accurately using *Quantum Electrodynamics* (QED). What is the essential difference between the approach here and QED?