Lectures on Parton Showers

Stefan Prestel

SLAC NATIONAL ACCELERATOR LABORATORY

MadGraph school 2015

Shanghai, November 22-27, 2015

1) **Introduction**
2a) **Parton showering**
2b) **Soft physics (multiparton interactions)**

Many previous lectures can be found at
`http://users.phys.psu.edu/∼cteq` and `montecarlonet.org`.
Further references at the end of the slides.

Part 1: Introduction

a) Why do we need Event Generators?
b) Event generation at hadron colliders.

How will we find what is out there?

Know what we want to look for...

Know what we're facing...

Assess if there is a realistic chance with our current experiments
...and check before building a new experiment.

How will we find what is out there?

Know what we want to look for...

Missing E_T and jets (a.k.a. classical SUSY)?

Compressed masses?

Dark sectors?

New bound states?

Know what we're facing...

Assess if there is a realistic chance with our current experiments
...and check before building a new experiment.

How will we find what is out there?

Know what we want to look for...

Missing E_T and jets (a.k.a. classical SUSY)?

Compressed masses?

Dark sectors?

New bound states?

Know what we're facing...

QCD,

QCD,

QCD.

Assess if there is a realistic chance with our current experiments
...and check before building a new experiment.

Know what we want to look for...

Missing E_T and jets (a.k.a. classical SUSY)?

Compressed masses?

Dark sectors?

New bound states?

Know what we're facing...

QCD,

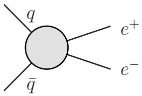
QCD,

QCD.

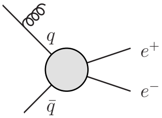
Assess if there is a realistic chance with our current experiments
...and check before building a new experiment.

We need an accurate representation of "known" and "unknown" physics that comes as close to data as possible!
$\Longrightarrow$ **Event generators**

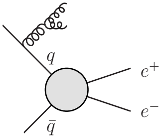
High-energy scattering $ab \to ABC\ldots$ of fundamental particles at the "core" of the collision



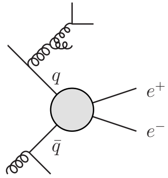
Highly accelerated particles decellerate by radiating (especialy QCD emissions) arbitrarily often,



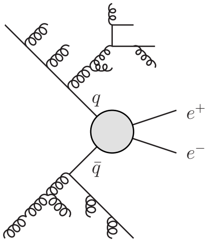
Highly accelerated particles decellerate by radiating (especialy QCD emissions) arbitrarily often,



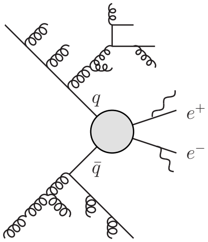
Highly accelerated particles decellerate by radiating (especialy QCD emissions) arbitrarily often,



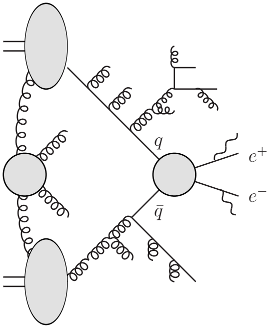
Highly accelerated particles decellerate by radiating (especialy QCD emissions) arbitrarily often,



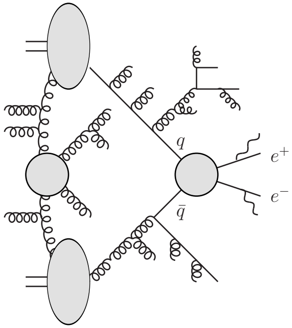
...but even massive W- or Z-bosons can be radiated at very high energies.



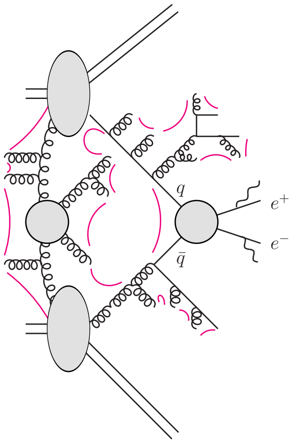
Colliding composite protons means there can be many interactions between the proton constituents



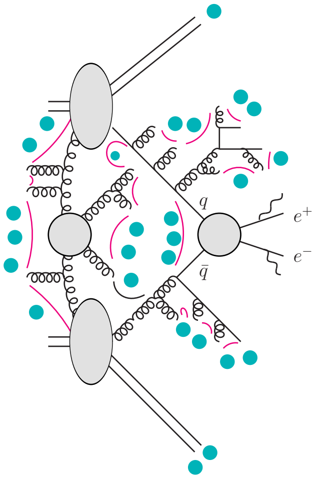
...which all produce yet more radiation.



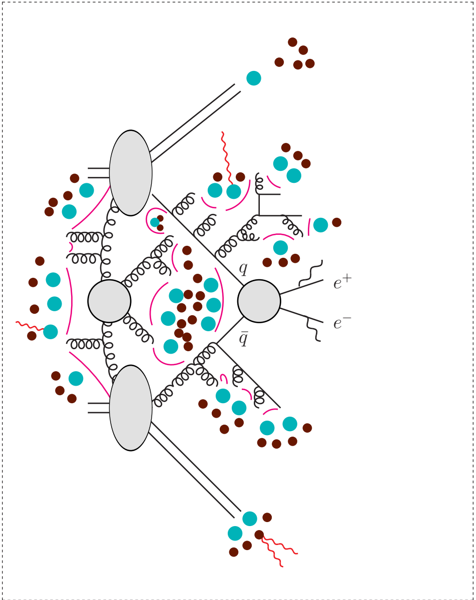
If all energies are small, we have a phase transition to a colour-neutral state (by transitioning to "proto-hadron" colour strings)



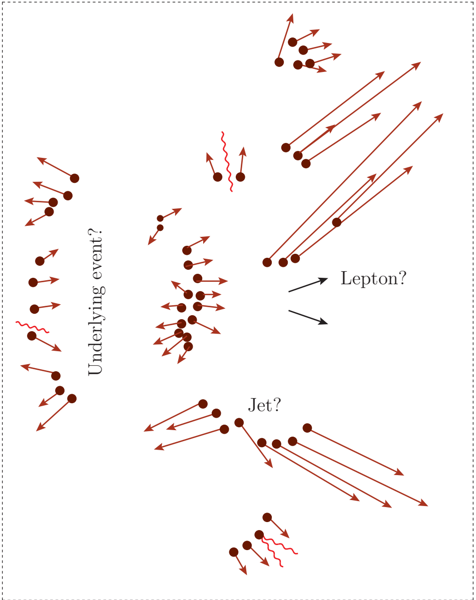
The colour-neutral strings then break up into tiny pieces forming (highly excited) hadrons,



and the excited hadrons decay into the particles (protons, pions, photons, electrons ...) we see in the detector.



After all these steps, the simulation should match as closely as possible what detectors record.



The three commonly used General Purpose Event Generators are

HERWIG	**PYTHIA**	**SHERPA**
Basic ME generator	Basic ME generator	Mature ME generator
Angular ordered $\tilde{q}$ shower and $p_\perp$-ordered CS dipole shower	$p_\perp$-ordered dipoles with ME-corrections, VINCIA antenna shower, DIRE dipole parton shower.	$p_\perp$-ordered CS dipole shower, DIRE dipole parton shower
YFS for soft photons MPI afterburner	Photons from shower Interleaved MPI	YFS for soft photons MPI afterburner
Cluster hadronisation	String hadronisation	Cluster hadronisation

(Warning: No purists in this game. Every theorist has to learn and compromise)

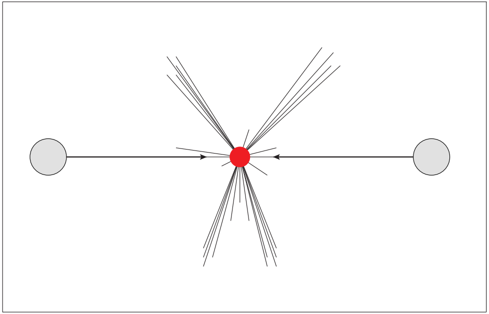
Part 2: Parton showering

a) Factorisation
b) From probabilities to parton showers
c) Parton shower details

Jets

Hard scattering + Radiation cascade + Hadronisation + Hadron decays

$\longrightarrow$ Leads to collimated sprays of particles called **Jets**

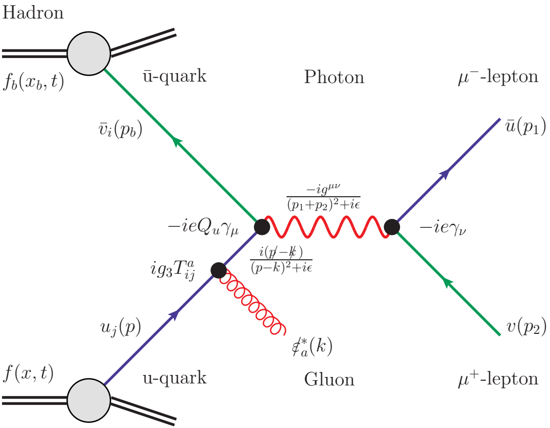


Jets are complicated objects to model in detail. But measuring only the "macroscopic" properties of jets reduces the complexity considerably (1000s of partices $\rightarrow$ to 10s of jets)

Jets are indispensable to be able to calculate scattering properties.

To model jets, we need to model the radiation cascade!

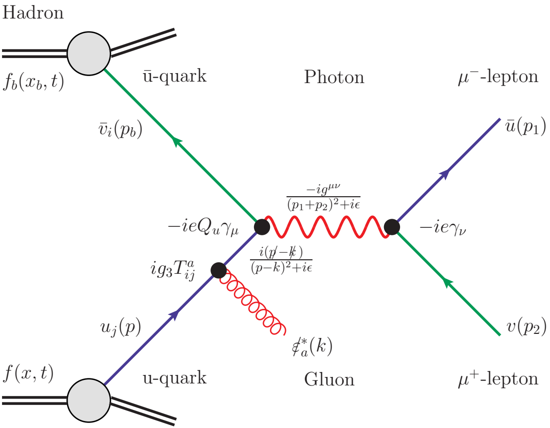
Factorisation: Divide and conquer



The hadronic cross section is

$$d\sigma(\mathrm{pp} \to \mu^+\mu^-\mathrm{g} + X) = dx dx_b f(x,t) f_b(x_b,t) d\hat{\sigma} \quad , \quad d\hat{\sigma} = \frac{\left|\mathcal{M}(\mathrm{u\bar{u}} \to \mu^+\mu^-\mathrm{g})\right|^2 d\Phi_{n+1}}{4\sqrt{(pp_b)^2}}$$

Factorisation: Divide and conquer

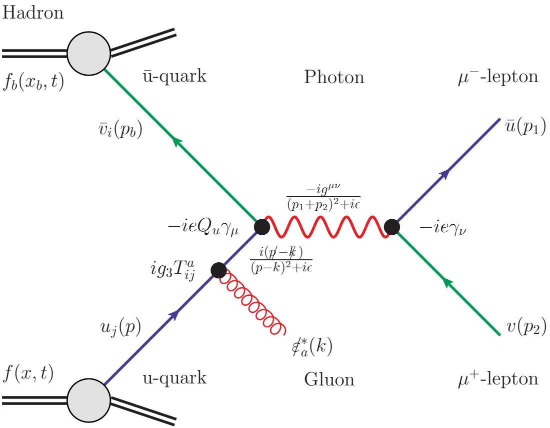


$E_{(p-k)} \approx zE_p$ and small gluon $p_\perp \Rightarrow$ Interal quark almost on-shell. Then:

$$\frac{i(\not{p}-\not{k})}{(p-k)^2} \approx \frac{u(p_a)\bar{u}(p_a)}{p_a^2} \quad , \quad d\Phi_{n+1} \approx d\Phi_n \frac{d\phi dz dp_\perp^2}{4(2\pi)^3(1-z)} \quad , \quad \frac{1}{4\sqrt{(pp_b)^2}} \approx \frac{z}{4\sqrt{(p_a p_b)^2}}$$

$\Longrightarrow$ Matrix element, phase space integration and flux factors factorise!

Factorisation: Divide and conquer



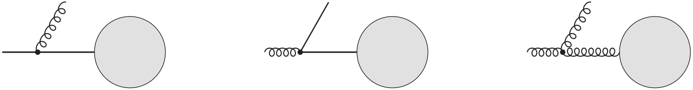
Matrix element, phase space integration and flux factors factorise:

$$d\sigma(\mathrm{pp} \to \mu^+\mu^-\mathrm{g} + X) = d\sigma(\mathrm{pp} \to \mu^+\mu^- + X) \int \frac{dp_\perp^2}{p_\perp^2}\frac{dz}{z}\frac{\alpha_s}{2\pi}C_F\frac{f(\frac{x_a}{z},t)}{f_a(x_a,t)}\frac{1+z^2}{1-z}$$

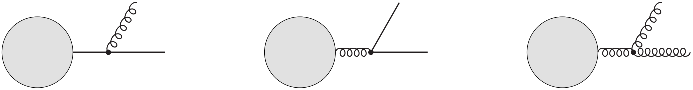
Every cross section containing an additional collinear parton can be factorised as

$$d\sigma(\mathrm{pp} \to Y + \mathrm{g} + X) = d\sigma(\mathrm{pp} \to Y + X) \int \frac{dp_\perp^2}{p_\perp^2} \frac{dz}{z} \frac{\alpha_s}{2\pi} \frac{f(\frac{x_a}{z}, t)}{f_a(x_a, t)} P(z)$$

with the splitting kernels $P(z)$



$$P_{qq} = C_F \frac{1+z^2}{1-z} \qquad P_{qg} = T_R[z^2 + (1-z)^2] \qquad P_{gg} = C_A \frac{(1-z(1-z))^2}{z(1-z)}$$



This is independent of the process $\mathrm{pp} \to Y + X$!

The splitting kernels

... are independent of the "hard" scattering;

... have a probabilistic interpretation:

$$\int_{p^2_{\perp min}}^{p^2_{\perp max}} \frac{dp^2_\perp}{p^2_\perp} \int_{z_{min}}^{z_{max}} dz \frac{\alpha_s}{2\pi} P(z) \equiv$$

aa

Probability of emitting a parton with momentum fraction $1 - z \in [z_{min}, z_{max}]$ and transverse momentum $p_\perp \in [p_{\perp min}, p_{\perp max}]$.

Also, note

$$\frac{dp^2_\perp}{p^2_\perp} = \frac{dQ^2}{Q^2} = \frac{d\Theta^2}{\Theta^2} = \frac{d\rho}{\rho} \quad (\text{for} \quad \rho = f(z) p^2_\perp)$$

$\Longrightarrow$ Many variables can be used to characterise the collinear limit!

...and note that we've put the z-range $[z_{min}, z_{max}]$. The lower limit z_{min} comes from the constraint $\frac{x_a}{z} < 1$, the upper limit when conserving 4-momentum.

Emission probabilities

Integrating the splitting probability, we get

$$\int_{p^2_{\perp min}}^{p^2_{\perp max}} \frac{dp^2_\perp}{p^2_\perp} \int_{z_{min}}^{z_{max}} dz \frac{\alpha_s}{2\pi} P(z) \approx \int_{p^2_{\perp min}}^{p^2_{\perp max}} \frac{dp^2_\perp}{p^2_\perp} \int_{z_{min}}^{z_{max}} dz \frac{\alpha_s}{2\pi} \frac{2C_{F/A}}{(1-z)}$$

$$\approx \alpha_s \ln\left(\frac{p^2_{\perp max}}{p^2_{\perp min}}\right) \ln\left(\frac{z_{max}}{z_{min}}\right)$$

More generally, we can write

$$d\sigma(\mathrm{pp} \to Y + \mathrm{g} + X) = d\sigma(\mathrm{pp} \to Y + X) \otimes \left(\alpha_s c_2 L^2 + \alpha_s c_1 L + \alpha_s c_0\right)$$

with $L = \ln\left(Q^2/p^2_{\perp min}\right)$, $Q^2 = \mathcal{O}(p^2_{\perp max})$, $p^2_{\perp min} = \mathcal{O}(\Lambda_{\mathrm{QCD}})$.

Even more generally

$$d\sigma(\mathrm{pp} \to Y + n\mathrm{g}) = d\sigma(\mathrm{pp} \to Y) \otimes \alpha_s^n \left(c_{2n} L^{2n} + c_{2n-1} L^{2n-1} + \cdots + c_0\right)$$

$$d\sigma(\mathrm{pp} \to Y + n\mathrm{g}) \approx d\sigma(\mathrm{pp} \to Y) \alpha_s^n c_{2n} L^{2n}$$

⇒ Multi-parton cross sections approximated by multiplying splitting kernels.
⇒ Largest terms comes from "dressing" states with many collinear partons!

Comments on iterating the collinear approximation

- (Multiple) gluon emission give the largest contribution to this multi-parton cross section...but other QCD splittings (e.g. $g \to q\bar{q}$) give very important sub-leading contributions, and are included!
- A careful analysis shows: The dominant contributions to the cross section are produced by ordered emissions
  ...but any ordering

  $$\rho_0 > \rho_1 > \rho_2 > \ldots$$

  is allowed if $\frac{d\rho}{\rho} = \frac{dp_\perp^2}{p_\perp^2}$. The many ways of choosing an ordering are one major difference between parton shower Monte Carlo's (see later).
- Sensible predictions respects energy-momentum conservation!

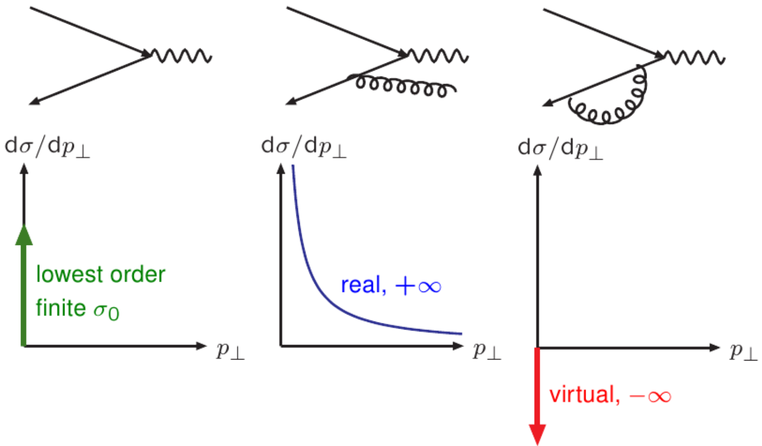
Most important, just multiplying splitting kernels gives $d\sigma(\mathrm{pp} \to Y)\alpha_s^n c_{2n} L^{2n}$, which is divergent as $p_{\perp min} \to 0$.

$\Longrightarrow$ Largest contribution for sure, but to give a sensible approximation of the multi-parton cross section, we need to do more!

Real-virtual cancellations and the Kinoshita-Lee-Nauenberg theorem

Pen-and-paper: Add Born + Real + Virtual

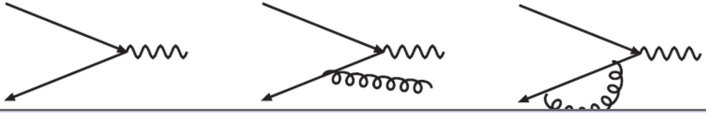
$$\langle \mathcal{O} \rangle^{\text{NLO}} = \int \mathrm{B}_n \mathcal{O}(\Phi_n) d\Phi_n + \int \mathrm{B}_{n+1} \mathcal{O}(\Phi_n) d\Phi_{n+1} + \int \mathrm{V}_n \mathcal{O}_n(\Phi_n) d\Phi_n$$



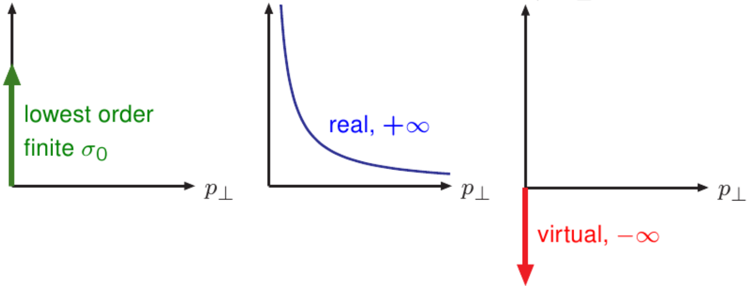
Real-virtual cancellations and the Kinoshita-Lee-Nauenberg theorem

Pen-and-paper: Add Born + Real + Virtual

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int \mathrm{B}_n \mathcal{O}(\Phi_n) d\Phi_n + \int \mathrm{B}_{n+1} \mathcal{O}(\Phi_n) d\Phi_{n+1} + \int \mathrm{V}_n \mathcal{O}_n(\Phi_n) d\Phi_n$$



Sensible predictions include *all* contributions at the necessary order.



To give a sensible collinear approximation of the multi-parton cross section, we also need approximate virtual corrections!

By looking at multi-loop integrals in the collinear limit, we find that that approximate virtual corrections form an all-order Sudakov form factor

$$\begin{aligned}\Pi(\rho_0, \rho_{min}) &= \exp\left(-\int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int dz \frac{\alpha_s}{2\pi} P(z)\right) \\ &= 1 - \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int_{z_{min}}^{z_0} dz_1 \frac{\alpha_s}{2\pi} P_1(z_1) \\ &\quad + \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int dz_1 \frac{\alpha_s}{2\pi} P_1(z_1) \int_{\rho_{min}}^{\rho_1} \frac{d\rho_1}{\rho_1} \int dz_2 \frac{\alpha_s}{2\pi} P_2(z_2) \ + \ldots\end{aligned}$$

But how do we calculate this?

$\Longrightarrow$ Taking probabilities seriously!

To give a sensible collinear approximation of the multi-parton cross section, we also need approximate virtual corrections!

By looking at multi-loop integrals in the collinear limit, we find that that approximate virtual corrections form an all-order Sudakov form factor

$$\Pi(\rho_0, \rho_{min}) = \exp\left(-\int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int dz \frac{\alpha_s}{2\pi} P(z)\right)$$

$$= 1 - \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int_{z_{min}}^{z_0} dz_1 \frac{\alpha_s}{2\pi} P_1(z_1)$$

1st-order virtual correction.

$$+ \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int dz_1 \frac{\alpha_s}{2\pi} P_1(z_1) \int_{\rho_{min}}^{\rho_1} \frac{d\rho_1}{\rho_1} \int dz_2 \frac{\alpha_s}{2\pi} P_2(z_2) \; + \ldots$$

But how do we calculate this?

$\Longrightarrow$ Taking probabilities seriously!

To give a sensible collinear approximation of the multi-parton cross section, we also need approximate virtual corrections!

By looking at multi-loop integrals in the collinear limit, we find that that approximate virtual corrections form an all-order Sudakov form factor

$$\begin{aligned}
\Pi(\rho_0, \rho_{min}) &= \exp\left(-\int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int dz \frac{\alpha_s}{2\pi} P(z) \right) \\
&= 1 - \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int_{z_{min}}^{z_0} dz_1 \frac{\alpha_s}{2\pi} P_1(z_1) \\
&\quad + \int_{\rho_{min}}^{\rho_0} \frac{d\rho_1}{\rho_1} \int dz_1 \frac{\alpha_s}{2\pi} P_1(z_1) \int_{\rho_{min}}^{\rho_1} \frac{d\rho_1}{\rho_1} \int dz_2 \frac{\alpha_s}{2\pi} P_2(z_2) \;+ \ldots
\end{aligned}$$

2nd-order virtual correction.

But how do we calculate this?

$\Longrightarrow$ Taking probabilities seriously!

Taking probabilities seriously

We have already found:

$$\frac{\delta p_\perp^2}{p_\perp^2}\int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z) \equiv$$

Probability of an emission with $1-z \in [z_1, z_0]$ and $p_\perp^2$ in the range $[p_{\perp min}^2, p_{\perp min}^2 + \delta p_\perp^2]$.

Then the probability of no emission is

$$1 - \frac{\delta p_\perp^2}{p_\perp^2}\int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)$$

or, if $\delta p_\perp^2$ is divided into n parts, and the no-emission probabilities are independent

$$\left[1 - \frac{\delta p_\perp^2/n}{p_\perp^2}\int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)\right]^n \underset{n\to\infty}{\rightarrow} \exp\left(-\int_{p_{\perp min}^2}^{p_{\perp min}^2+\delta p_\perp^2} \frac{dp_\perp^2}{p_\perp^2}\int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)\right)$$

The Sudakov factor is the probability of no resolvable emission in the range $[p_{\perp min}^2, p_{\perp min}^2 + \delta p_\perp^2]$, where resolvable means $1-z \in [z_1, z_0]$.

Thus:

$$\int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z) \quad \equiv$$

Probability of a resolvable emission with $p_\perp^2$ in the range $[\rho_{min}, \rho_0^2]$.

$$\exp\left(- \int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z) \right) \equiv$$

Probability of no resolvable emission with $p_\perp^2$ in the range $[\rho_{min}, \rho_0]$.

We can construct an all-legs and all-loops result based on probabilities!
$\Longrightarrow$ Ideal for numerical iteration with random numbers.
$\Longrightarrow$ Monte Carlo parton showers.

An algorithm to produce multiple emissions

0. Construct a state with no emissions.
1. Begin algorithm at a "largest $p_\perp$" ρ_{max} (evolution parameter).
2. Propose a state change by emitting at $\rho < \rho_{max}$.
3. Decide if a new state should be constructed according to the splitting function probability. If yes, construct the new state.
4. Set $\rho_{max} = \rho$. Start from 1. (possibly with a new input state - need momentum conservation in step 3 to be able to iterate!).

An algorithm to produce multiple emissions

0. Construct a state with no emissions.

1. Begin algorithm at a "largest $p_\perp$" ρ_{max} (evolution parameter).

2. Propose a state change by emitting at $\rho < \rho_{max}$.

3. Decide if a new state should be constructed according to the splitting function probability. If yes, construct the new state.

4. Set $\rho_{max} = \rho$. Start from 1. (possibly with a new input state - need momentum conservation in step 3 to be able to iterate!).

When the "$p_\perp$" is decreased by $\delta\rho$, there are two possibilities:

- The algorithm produced an emission at scale ρ.
- The algorithm did not produce an emission.

P(No emission above ρ_{min}) $+$ P(No emission above ρ)$\times$ P(One emission at ρ)

$$= d\sigma \otimes \Pi_0(\rho_0, \rho_{min})\ \mathcal{O}_0 \qquad + \ d\sigma \otimes \int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)\ \Pi_0(\rho_0, \rho)\ \mathcal{O}_1$$

Each of parton shower cross section is a finite result including all orders in QCD, because of Sudakov suppression:

$$d\sigma_B(pp \to X) \otimes \int_{\rho_{min}}^{\rho_0} \frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)\ \Pi_0(\rho_0, \rho)\ \mathcal{O}_1 \underset{\rho_{min} \to 0}{\longrightarrow} \text{finite}$$

Now remember that we derived the no-emission probability Π from

$$P_{\text{emission}} + P_{\text{no emission}} = 1$$

$\Longrightarrow$ Summing over all emissions, the PS never changes the cross section, it only changes shapes:

The PS takes a small, finite (!) part of the 0-parton cross section and reinterprets it as 1-parton cross section. This is called parton shower unitarity.

Remember:

$$\Pi(\rho_0, \rho_1) = \exp\left(-\int_{\rho_1}^{\rho_0} \frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z)\right) \equiv$$

Probability of no resolvable emission with evolution scale in the range $[\rho_1, \rho_0]$.

$$\frac{d\rho}{\rho} \int_{z_1}^{z_0} dz \frac{\alpha_s}{2\pi} P(z) \Pi(\rho_0, \rho) \quad \equiv$$

Probability of a exactly one resolvable emission, with evolution scale ρ.

Note: We've cheated by quietly dropping PDFs before! Keeping the PDFs, we would have arrived at

No-emission probability:

$$\Pi(\rho_0, \rho_1) = \exp\left(-\int_{\rho_1}^{\rho_0}\frac{d\rho}{\rho}\int_{z_1}^{z_0}\frac{dz}{z}\frac{\alpha_s}{2\pi}\frac{f_1(\frac{x}{z}, \rho)}{f_0(x, \rho)}P(z)\right)$$

Probability of an emission with $x_{new} = \frac{x}{z}$ at evolution scale ρ:

$$\frac{d\rho}{\rho}\int_{z_1}^{z_0}\frac{dz}{z}\frac{\alpha_s}{2\pi}\frac{f_1(\frac{x}{z}, \rho)}{f_0(x, \rho)}P(z)\Pi(\rho_0, \rho)$$

Note

$$\frac{d\ln\Pi}{d\ln\rho} = \int_{z_1}^{z_0}\frac{dz}{z}\frac{\alpha_s}{2\pi}\frac{f_1(\frac{x}{z}, \rho)}{f_0(x, \rho)}P(z)$$

$\Rightarrow$ A shower of incoming partons reproduces the DGLAP evolution of the sructure functions, but PDFS are crucial for that!

Backward evolution

Remember: PDFs evolve according to the DGLAP equation, from small virtuality Q^2 to larger virtuality Q_0^2. PDFs are small at large Q_0^2.

Should parton showers do the same?

No, since it would be **very** unlikely to "hit" a resonance (i.e. a Higgs or Z-boson propagator) in a narrow virtuality window at large Q_0^2.
$\Longrightarrow$ Simulating high-scale physics would be nearly impossible!

$\Longrightarrow$ Instead, reformulate DGLAP to evolve from large Q_0^2 and small x to smaller Q^2 and larger x/z.
$\Longrightarrow$ Backwards evolution.

DGLAP	: Sums up all emissions by evolving from Q^2 to Q_0^2
Backward evolution:	Performs all emissions (that had previously been summed up) by evolving from Q_0^2 to Q^2. It "unintegrates" the PDFs!

Review

Achievements so far:

- Found a way to approximate (one of) the largest contributions to a n-parton cross section: the collinear approximation
  ...and devised a probabilistic algorithm to produce this result.
- The parton shower produces finite results by introducing all-order (resummed) virtual corrections.
- This extends the validity of perturbation theory even to small $p_\perp$.
- We know how to treat emissions off final and initial state partons.

To get there, we needed

- To derive emission and no-emission probabilities.
- Find a prescription for momentum conservation - otherwise, we cannot iterate the procedure.
- We had to order in an evolution scale ρ to reproduce the dominant terms.

But...

Review

Achievements so far:

- Found a way to approximate (one of) the largest contributions to a n-parton cross section: the collinear approximation
  ...and devised a probabilistic algorithm to produce this result.
- The parton shower produces finite results by introducing all-order (resummed) virtual corrections.
- This extends the validity of perturbation theory even to small $p_\perp$.
- We know how to treat emissions off final and initial state partons.

To get there, we needed

- To derive emission and no-emission probabilities.
- Find a prescription for momentum conservation - otherwise, we cannot iterate the procedure.
- We had to order in an evolution scale ρ to reproduce the dominant terms.

But...

- The evolution scale ρ can be defined freely, as long as $d\rho/\rho = dp_\perp^2/p_\perp^2$. This e.g. allows (relative) angle, virtuality, $p_\perp^2$ ...

Achievements so far:

- Found a way to approximate (one of) the largest contributions to a n-parton cross section: the collinear approximation
  ...and devised a probabilistic algorithm to produce this result.
- The parton shower produces finite results by introducing all-order (resummed) virtual corrections.
- This extends the validity of perturbation theory even to small $p_\perp$.
- We know how to treat emissions off final and initial state partons.

To get there, we needed

- To derive emission and no-emission probabilities.
- Find a prescription for momentum conservation - otherwise, we cannot iterate the procedure.
- We had to order in an evolution scale ρ to reproduce the dominant terms.

But...

- The evolution scale ρ can be defined freely, as long as $d\rho/\rho = dp_\perp^2/p_\perp^2$. This e.g. allows (relative) angle, virtuality, $p_\perp^2$ ...
- Momentum conservation can be implemented in many different ways.

Backward evolution in the initial state means evolving from a “hard process” at large momentum transfer to smaller momentum transfers.

The hard process is the “starting point” of the radiation cascade.

We want to start from an “exact” result, i.e. a good description of the inclusive cross section with n partons, and produce approximate higher order corrections.

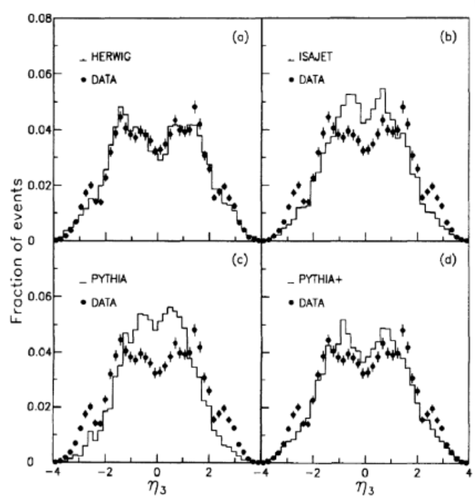
If the evolution scale is defined such that after some emissions, a “harder” process is generated, then the exact starting point is obscured, and we cannot do backward evolution.

$\Longrightarrow$ Initial state showers suggest to use a “hardness” ordering, i.e. where large momentum transfers happen early in the cascade (e.g. Q^2 or $p_\perp^2$).

Choosing an ordering variable: Is virtuality ordering safe?

Ordering:
PYTHIA: Virtuality,
HERWIG: Something else.

⟹ Something is missing.



⟹ Virtuality ordering did not capture the physics!
⟹ Missing another important ingredient!

The soft limit and QM interference

When trying to find an approximation of additional gluon emissions, we found that the largest contribution to $Q_i(p_i + k) \to Q_i'(p_i) + g(k)$ arose from an on-shell propagator

$$u(p_i)\not{\varepsilon}\frac{(\not{p}_i + \not{k})}{(p_i + k)^2} = u(p_i)\frac{p_i\varepsilon}{2p_ik} = u(p_i)\frac{p_i\varepsilon}{(1-z)E_{Q_i}^2(1-\cos\Theta_{Q_ig})}$$

Apart from collinear divergence $\Theta_{Q_ig} \to 0$, there is also a soft divergence $z \to 1$.

$\Longrightarrow$ We were missing the soft piece before!

For $z \to 1$, already the amplitudes universally factorise. Thus, upon squaring

$$d\sigma_{n+1} = d\sigma_n \int \frac{dw}{w}\frac{d\Omega}{2\pi}\frac{\alpha_s}{2\pi}\sum_{ij} C_{ij}W_{ij}$$

$$\text{with} \quad W_{ij} = \frac{1-\cos\Theta_{Q_iQ_j}}{(1-\cos\Theta_{Q_ig})(1-\cos\Theta_{Q_jg})}$$

$\Longrightarrow$ QM interference between gluon emission off partons Q_i and Q_j!

How can soft emissions be independent?

How can soft emissions be independent?
Let us write

$$W_{ij} = W_{ij}^1 + W_{ij}^2 \quad \text{with} \quad W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{(1-\cos\Theta_{Q_ig})} - \frac{1}{(1-\cos\Theta_{Q_jg})}\right)$$

Then, after integrating over the azimuthal angle, we get

$$\int \frac{d\phi_{Q_ig}}{2\pi} W_{ij}^i = \begin{cases} \frac{1}{(1-\cos\Theta_{Q_ig})} & \text{for} \quad \Theta_{Q_ig} < \Theta_{Q_iQ_j} \\ 0 & \text{else} \end{cases}$$

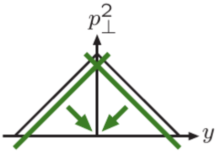
Soft emissions are independent if ordered in emission angle!
Another (opening cone) argument shows: $p_\perp$-ordered final state emissions are okay as well.

HERWIG had angular ordering in the CDF plot. Color coherence necessary to describe data! But angle does not define hardness!

Choosing an ordering variable: Hardness vs. angle

We found: Hardness ordering (Q^2, $p_\perp^2$) motivated by ISR, Θ ordering by soft limit. Both mutually exclusive!

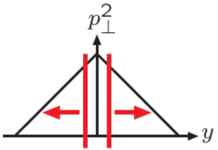
Q^2



Virtuality

- Defines hardness, as necessary in ISR.
- No coherence. Additional vetoes necessary.

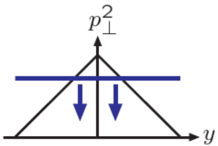
$E^2\Theta^2$



Angle

- Does not define hardness. Additional vetoes necessary.
- Coherence by construction.

$p_\perp^2$



$p_\perp$

- Defines hardness, as necessary in ISR.
- Coherence in FSR. ISR not clear.

Is it hopeless? No! $\Longrightarrow$ Dipole/antenna showers.

Dipoles / antennae

In the soft limit, we found

$$d\sigma_{n+1} = d\sigma_n \int \frac{dw}{w} \frac{d\Omega}{2\pi} \frac{\alpha_s}{2\pi} \sum_{ij} C_{ij} W_{ij}$$

and after writing

$$W_{ij} = W_{ij}^1 + W_{ij}^2 \quad \text{with} \quad W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{(1-\cos\Theta_{Q_i g})} - \frac{1}{(1-\cos\Theta_{Q_j g})} \right)$$

derived angular ordering.

But we could have directly used W_{ij} as splitting probability ($\equiv$ QCD antenna), or partitioned cleverly ($\equiv$ QCD dipole).

Both antennae and dipoles can be inferred from NLO subtraction methods. This means they come with a well-defined phase space mapping.

Energy-momentum conservation

We have stressed the importance of energy-momentum conservation, but not given a prescription.

NLO subtraction formalisms give a one-to-one correspondence

$$d\Phi_{n+1} = d\Phi_n d\hat{\Phi} = d\Phi_n J(\rho, z, \phi) d\rho dz \frac{d\phi}{2\pi}$$

which maps an on-shell n-particle phase space point unto an on-shell $n+1$-particle configuration. The $n+1$-particle is completely covered.

This momentum conservation can be achieved by

aborbing the "recoil" of a $1 \to 2$ splitting with a spectator (dipoles).

performing $2 \to 3$ splittings (antennae).

$\Rightarrow$ All modern showers are designed in this way!

Momentum conservation in each intermediate step is the major advantage compared to analytical tools. It also makes systematic step-by-step improvements possible ($\Longrightarrow$ Andreas' lecture).

Freedom in the recoil scheme is an uncertainty of the very exclusive parton shower predictions.

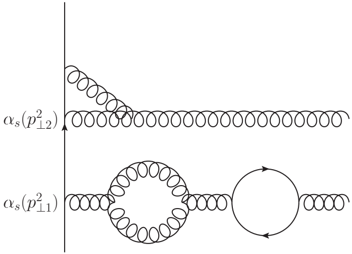
Until now, we have found:

- Parton showers generate the leading collinear logarithms. Angular ordering (or modern showers) include the soft limit as well.
- Local momentum conservation (formally beyond LL) is included.
- Initial state radiation requires PDF evaluations at dynamical scales (e.g. Q^2, $p_\perp^2$ of the branching).

Another important improvement is evaluation of α_s at dynamical scales $\alpha_s = \alpha_s(p_\perp^2)$.

This is known as Modified Leading Log Approximation. This resums dominant universal propagator corrections to all orders.

After this improvement, many more soft emissions are produced. The PS must ensure to avoid the Landau pole (e.g. $p_{\perp min} > \Lambda_{QCD}$).



Effects (almost) beyond the reach of parton showers

Parton showers already include many sub-dominant contributions. But remember that

1. Any parton shower is only sensible in the collinear regime:
   Only very collimated parton cascades are reliably modelled.
2. Showers are always leading order correct for (very) inclusive observables.
3. Non-relativistic threshold effects are not included.
4. High-energy (i.e. low-x) enhancements $\ln(\hat{s}/\hat{t})$ are hardly included.
5. Traditionally, showers only include QCD.
   Photon emissions and final-state $\gamma \to f\bar{f}$ usually included.
   Recent efforts to include W/Z emissions.

More developments still necessary!

Major industry of improvements for points 1. and 2. (Matrix element corrections, Merging, NLO Matching, NLO Merging, NNLO Matching).
$\Longrightarrow$ Andreas' lecture

Parton showers are usually part of event generator frameworks.
Commonly used event generators for LHC physics are

HERWIG++: Improved angular ordered $\tilde{q}$ shower
$p_\perp$-ordered Catani-Seymour dipole shower

PYTHIA 8 : $p_\perp$-ordered dipole shower with DGLAP+ME- corrections
DIRE dipole shower
VINCIA antenna shower (FSR)

SHERPA : $p_\perp$-ordered Catani-Seymour dipole shower
DIRE dipole shower

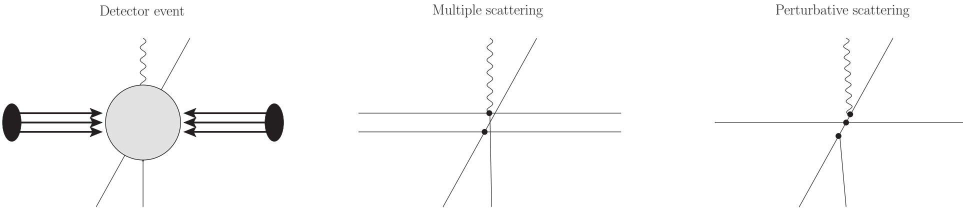
All three include QED radiation, EW effects, underlying event, diffractive modelling, hadronisation, higher-order improvements, hadron decays...

Summary of Part 2: Parton showering

- QCD scattering cross sections factorise in the soft / collinear limits.
- The factorisation is universal, and can be viewed as probabilistic.
- The existence of emission and no-emission probabilities makes all-order (all-legs) numerical implementations possible.
- Parton showers require an ordering criterion. Hardness and angle are well-motivated, but not without pitfalls.
- Almost all modern showers are based on antennae or dipoles instead.
- With the inclusion of soft effects, momentum conservation and running scales, many (all-order) refinements are already added.
- Some effects are beyond the parton shower approximation. But systematic enhancements are possible.

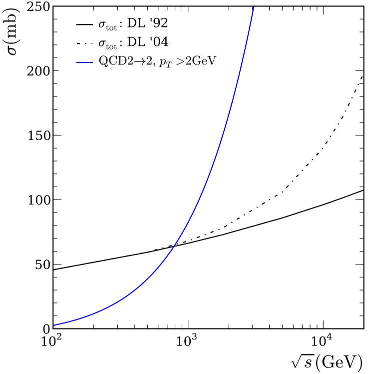
...now we can describe collimated jets of partons in isolation, but
a) How do we convert to a colour-neutral final state?
b) What about all the other activity that proton scatterings produce in-between jets?

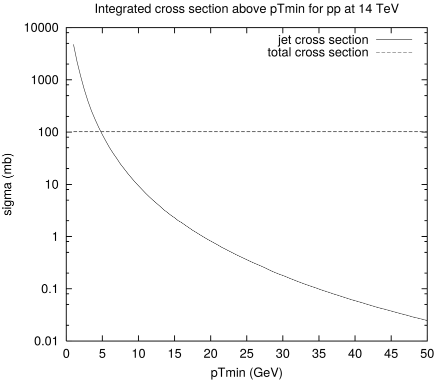
No description of a) given in this lectures. See other MCnet/CTEQ lectures for that.



When colliding composite objects, many constituent scatterings "compete" for the collision energy. **For example, two simulaneous scatterings can look like single complicated scatterings! Which effect is more important?**

The issue with the dijet process





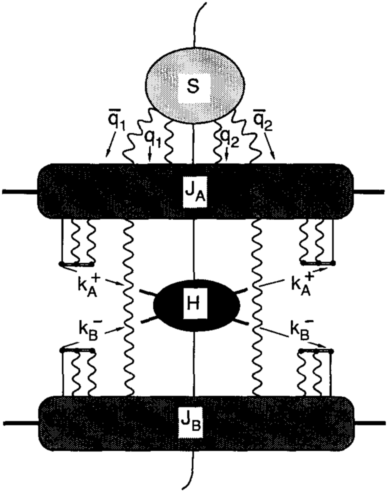
Remember that the perturbative cross section

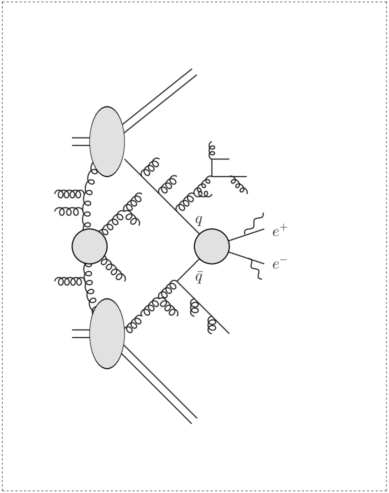
$$\sigma(pp \to jj + X) = \int_{p_{\perp min}}^{\frac{E_{cm}}{2}} dx_1 dx_2 f_1(x_1) f_2(x_2) \frac{d\hat{\sigma}}{dp_\perp} dp_\perp > \sigma(pp \to anything) \text{ for } \frac{p_{\perp min}}{E_{cm}} \to 0$$

as $f(x)$ not small (enough) for low $x \approx \frac{p_{\perp min}}{E_{cm}}$ to suppress $\frac{p_{\perp min}}{E_{cm}} \to 0$ divergence!

Hints from factorisation

Still consistent with perturbative QCD: PDFs are the *inclusive* probability to find parton at x, with all other interactions above $x \approx \frac{p_{\perp min}}{E_{cm}}$ integrated out!





Multiple interactions (indirect evidence)

Collider observables are not "inclusive" enough and can "see" these additional interactions.

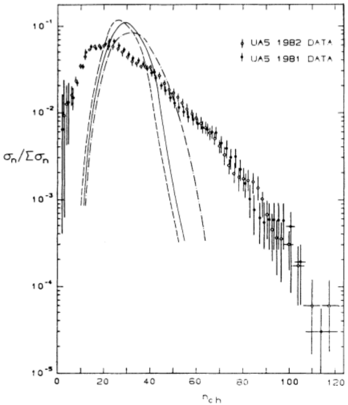


FIG. 3. Charged-multiplicity distribution at 540 GeV, UA5 results (Ref. 32) vs simple models: dashed low p_T only, full including hard scatterings, dash-dotted also including initial- and final-state radiation.

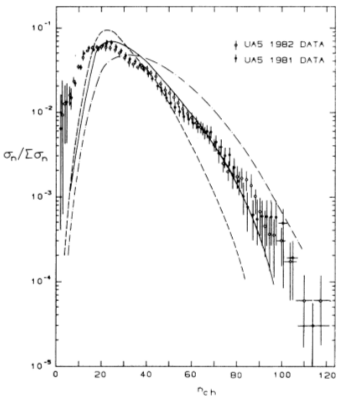
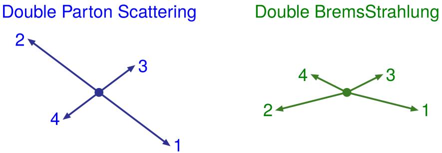


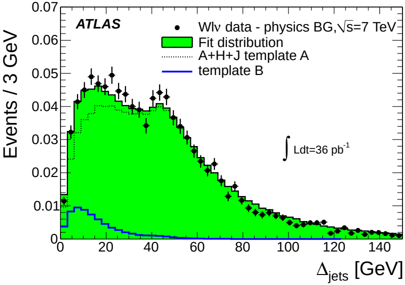
FIG. 5. Charged-multiplicity distribution at 540 GeV, UA5 results (Ref. 32) vs impact-parameter-independent multiple-interaction model: dashed line, $p_{T\min}=2.0$ GeV; solid line, $p_{T\min}=1.6$ GeV; dashed-dotted line, $p_{T\min}=1.2$ GeV.

Double parton scattering has typical kinematics...



Double Parton Scattering: $|\mathbf{p}_{\perp 1} + \mathbf{p}_{\perp 2}| \approx 0$, $|\mathbf{p}_{\perp 3} + \mathbf{p}_{\perp 4}| \approx 0$

Double BremsStrahlung: $|\mathbf{p}_{\perp 1} + \mathbf{p}_{\perp 2}| \gg 0$, $|\mathbf{p}_{\perp 3} + \mathbf{p}_{\perp 4}| \gg 0$



...that's seen in the data (ATLAS New J.Phys. 15 (2013) 033038)

A scattering above $p_{\perp min}$ is accompanied by an average $\langle n(p_{\perp min})\rangle$ interactions, so that

$$d\sigma^{inc}(p_{\perp min}, E_{cm}) = \langle n(p_{\perp min})\rangle \cdot \sigma^{ND}(p_{\perp min}, E_{cm})$$

where

$$\sigma^{ND} < \sigma(pp \to anything)$$

is the inelastic scattering cross-section[1].

Now the average number of scatterings is unconstrained, but the (single scattering) cross section remains small.

[1] more precisely, the inelastic non-diffractive cross-section defined implicitly by total, elastic cross section and diffractive parametrisation:
$\sigma^{ND} \approx \left(\sigma^{tot} - \sigma^{el}\right) - \int (d\sigma_{SD} - d\sigma_{DD} - d\sigma_{CD})$

Question: So can we just overlay many scatterings to approximate the result?
Answer: **No!**

Consistency condition: For large $p_\perp$, model must preserve the perturbative hard scattering cross section, otherwise factorisation of inclusive cross section violated!

Multiparton interactions from a “consistency condition”

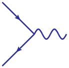
Question: So can we just overlay many scatterings to approximate the result?
Answer: **No!**

Consistency condition: For large $p_\perp$, model must preserve the perturbative hard scattering cross section, otherwise factorisation of inclusive cross section violated!

Solution: Subtract what you add!
For every additional scattering, we need ”virtual corrections”

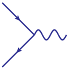
(This should sound familiar from parton shower derivation (and KLN)!)

Start from a single scattering...



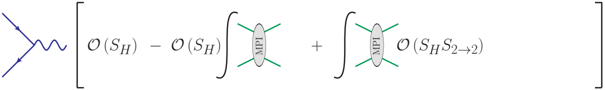
$\mathcal{O}(S_H)$

...and add another scattering

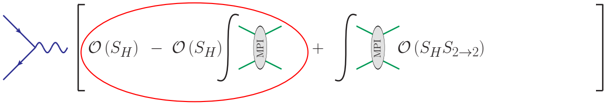


$\Big[\ \mathcal{O}(S_H) \quad + \int$

$\mathcal{O}(S_H S_{2\to 2})\ \Big]$

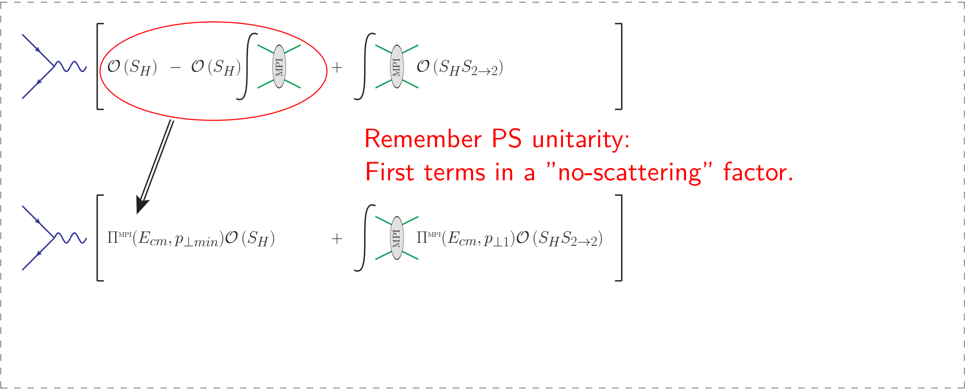

$\mathcal{O}(S_H)$
$-$
$\mathcal{O}(S_H)$
MPI
$+$
MPI
$\mathcal{O}(S_H S_{2\to2})$

...then include "virtual corrections" to preserve the cross section

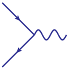


Remember PS unitarity:
First terms in a "no-scattering" factor.

This is like a first parton shower step!



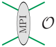
As in the parton shower, we can continue with this...



$$\Bigg[\mathcal{O}(S_H) - \mathcal{O}(S_H) \int$$

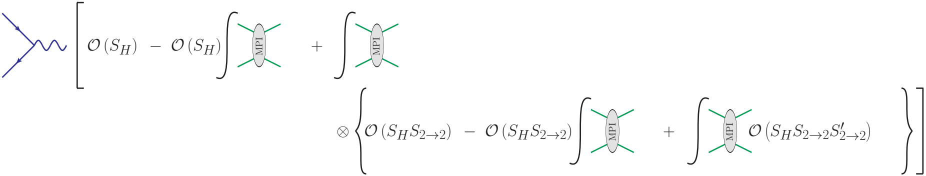
$$+ \int$$

$$\otimes \Bigg\{ \mathcal{O}(S_H S_{2\to 2}) \quad + \int$$

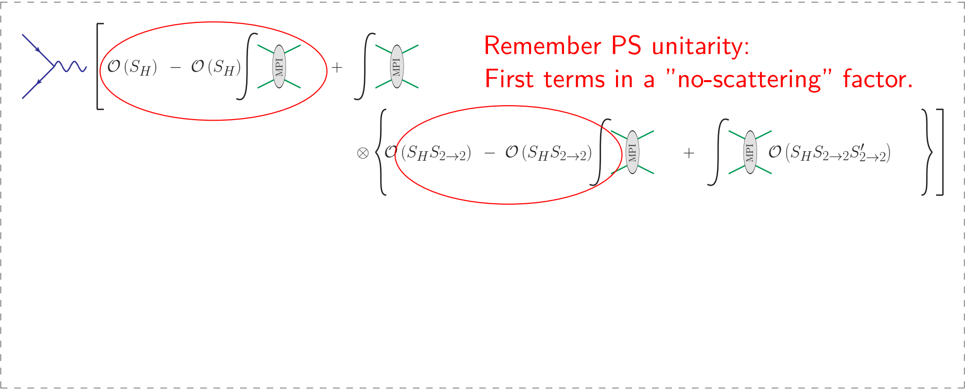


$$\mathcal{O}(S_H S_{2\to 2} S'_{2\to 2}) \Bigg\} \Bigg]$$

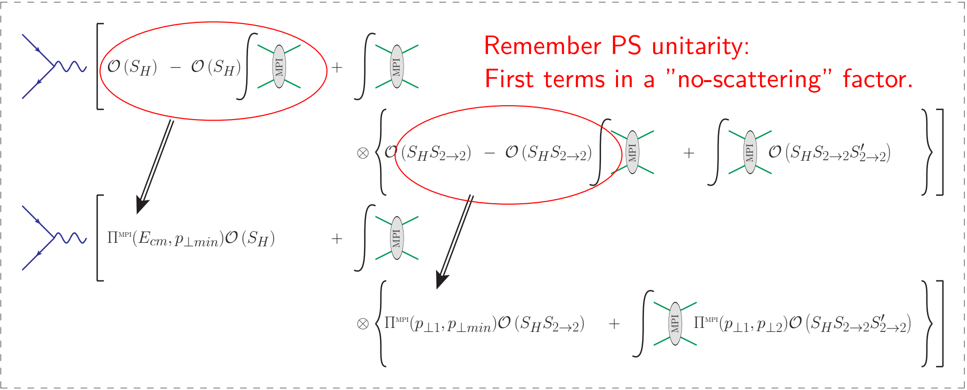
As in the parton shower, we can continue with this...



As in the parton shower, we can continue with this...



...and we have a multiple scattering model!



If you have a hammer...

...everything looks like a parton shower. Assume

$\delta p_\perp \langle n(p_\perp)\rangle \equiv$ Probability for scattering with $p_\perp \in [p_{\perp min}, p_{\perp min} + \delta p_\perp]$.

Then the probability of no scattering is

$$1 - \delta p_\perp \langle n(p_\perp)\rangle$$

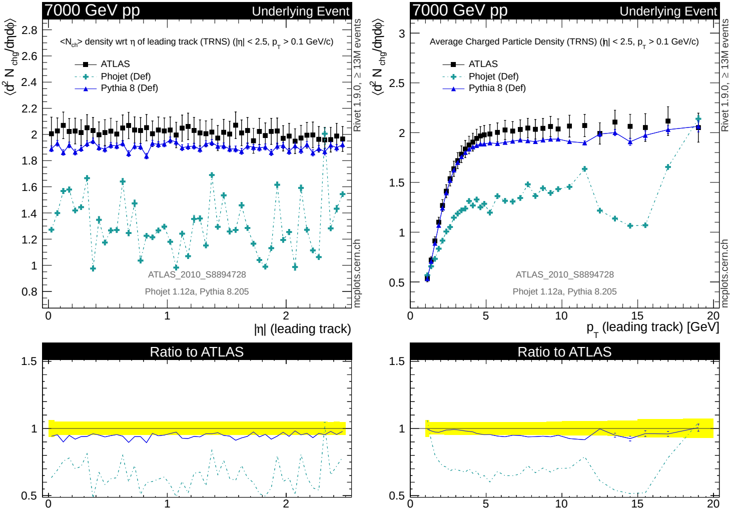
or, if $\delta p_\perp$ is divided into m parts, and the scattering probabilities are independent

$$[1 - \delta p_\perp/m\ \langle n(p_\perp)\rangle]^m \underset{m\to\infty}{\rightarrow} \exp\left(-\int_{p_{\perp min}}^{p_{\perp min}+\delta p_\perp} dp_\perp \langle n(p_\perp)\rangle\right) \equiv \Pi^{\mathsf{MPI}}(p_{\perp min}+\delta p_\perp, p_{\perp min})$$

We can define a no-additional scattering probability which contains "all-order virtual corrections" – just like a parton shower Sudakov factor.

$\Longrightarrow$ Can recycle the PS algorithm to produce additional scatterings.

The pedestal, or *how entangled are MPI?*



Activity uniform in rapidity. More additional particles for harder core scatterings (trigger bias!) $\Rightarrow$ Not captured by model where all scatterings have same probability. $\Rightarrow$ Model matter overlap, i.e. introduce impact parameter dependence.

Redefine the naive scattering as

$$\langle n(p_\perp)\rangle \rightarrow \langle n(p_\perp)\rangle \;\cdot\; \mathcal{A}(b)/\langle\mathcal{A}\rangle$$

The probability for any partonic scattering is still σ^{inc}, but the no-additional-scattering probability becomes

$$\Pi^{\mathsf{MPI}}(b, p_{\perp 0}, p_\perp)\;\exp\left(-\frac{\mathcal{A}(b)}{\langle\mathcal{A}\rangle}\int_{p_\perp}^{p_{\perp 0}} d\Phi_{2\to 2}\;\frac{d\hat{\sigma}_{2\to 2}(\Phi_{2\to 2})}{\sigma^{ND}}\right)$$

$\Rightarrow$ Perturbative MPI *model* keeping the inc. cross section. Unknowns:

- σ^{ND} depends on σ^{tot}, σ^{elast} and diffractive model. The parameters of these thus influence MPI.
- Most MPI very soft, but σ^{inc} is still divergent for $p_{\perp min} \to 0$, i.e. needs extra regulator parameter $p_{\perp 0}$. Regulator should be larger if E_{cm} becomes larger (to not violate the total cross section), i.e. get parameters for energy scaling of $p_{\perp 0}$
- Cut $p_{\perp c}$ deciding when to convert to hadrons.
- Parameters for matter profile $\mathcal{A}(b)$.

MPI models in general-purpose event generators:
HERWIG: Pick interactions prior to running according to Poissonian,
SHERPA: MPI after hard process evolution in a $p_\perp$-ordered sequence.
PYTHIA: MPI + ISR + FSR combined in one single $p_\perp$ sequence.

Summary of Part 2b: Multiparton interactions

- Collider events are always accompanied with low-energy activity.
- The factorisation of QCD into long- and short-distance physics gives hints how to describe this.
- Multiparton interactions are built upon a consistency condition: Integration over all MPI degrees of freedom will recover the perturbative result, while each scattering has a finite probability.
- Multiparton interactions are generated almost identical to parton shower emissions.
- Lacking a rigorous foundation, MPI models come with parameters. The most important input is the proton matter profile.

- Collider events are always accompanied with low-energy activity.
- The factorisation of QCD into long- and short-distance physics gives hints how to describe this.
- Multiparton interactions are built upon a consistency condition: Integration over all MPI degrees of freedom will recover the perturbative result, while each scattering has a finite probability.
- Multiparton interactions are generated almost identical to parton shower emissions.
- Lacking a rigorous foundation, MPI models come with parameters. The most important input is the proton matter profile.

Thanks for your time! Questions?

Exercise: Compare the no-emission probabilities with Sudakov factors

Remember:

$$\Pi(\rho_0,\rho_1) = \exp\left(-\int_{q_\perp^2}^{Q^2}\frac{dp_\perp^2}{p_\perp^2}\int_{z_1}^{z_0}dz\frac{\alpha_s}{2\pi}P(z)\right) \equiv$$

Probability of no resolvable emission with $p_\perp^2 \in [Q^2, q_\perp^2]$.

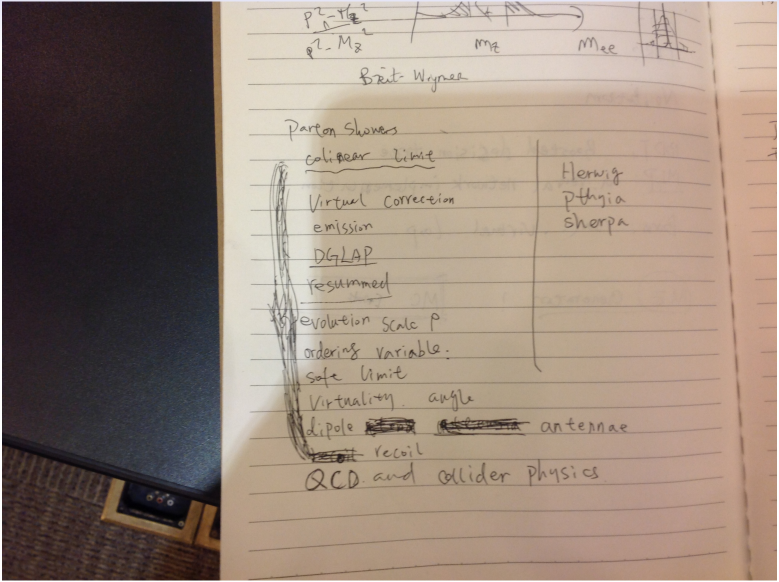
We will often call this a Sudakov (form) factor. The quark Sudakov form factor for a massless quark can be calculated analytically in QCD. For $q_\perp \to 0$, it reads

$$\Delta(\rho_0,\rho_1) = \exp\left(-\int_{q_\perp^2}^{Q^2}\frac{dp_\perp^2}{p_\perp^2}\frac{\alpha_s}{2\pi}C_F\left[\ln\left(\frac{Q^2}{p_\perp^2}\right)-\frac{3}{2}+\mathcal{O}\left(\frac{p_\perp^2}{Q^2}\right)\right]\right) \qquad (1)$$

a) Assume that the parton shower splitting kernel is $P(z) = C_F\frac{1+z^2}{1-z}$, and that $z_1 = a_1\frac{p_\perp}{Q} + a_2\frac{p_\perp^2}{Q^2}$, $0 < z_1 < 1$ and $z_0 = 1 - z_1$. Write the no-emission probability, for $\frac{p_\perp}{Q} \to 0$, in the form of eq. (1). (Hint: Rewrite $P(z)$ so that you can clearly identify which term gives the logarithm and which term gives the constant) What are the phase space boundary z_0, z_1 necessary to match eq. (1)?

b) Now assume the splitting kernel $P(z, p_\perp^2) = C_F\frac{2(1-z)}{(1-z)^2+p_\perp^2/Q^2} - (1+z)$. What is the form of z_0, z_1 now?

Which phase space is larger?



Yesterday, you rightfully asked to explain some technical terms that appeared in the lecture (see picture).

Explanation of some technical terms

Yesterday, you rightfully asked to explain some technical terms that appeared in the lecture. I'll try to do so here. For details, consider the books *"QCD and collider physics", by Ellis, Stirling and Webber, or "Quantum Chromodynamics" by Dissertori, Knowles and Schmelling*.

Collinear limit: The limit in which the three-momenta of two or more particles are perfectly aligned, and all of these particles move in the same direction. This limit can be characterised in many different ways, e.g. by using that the relative angle between the particles vanishes.

Virtual corrections: Quantum fluctuations that get reabsorbed directly so that the momenta of external particles do not get changed. Rather, the total probability of an interactions gets re-adjusted.

Emission: Any higher-order correction to a scattering that adds new particles to the final state. These corrections change the momentum distribution of external particles w.r.t. the state without the correction.

DGLAP: Short for Dokshitzer-Gribov-Lipatov-Altarelli-Parisi, the authors of an important result for massless gauge theories: The DGLAP evolution equation, which tells us how the functions encoding the structure of hadrons at low-energies can be translated to the structure functions at large momentum transfers that we need for high-energy collider physics. This is done by "evolving" the structure functions extracted from fits of experimental measurements at low energies to the necessary high energies through an all-order perturbative approximation of the massless theory.

Resummed: By that, we mean that we find closed analytical expressions that, when expanded, contain any order in the coupling parameter. An example of this is $\exp(-\alpha_s F) = \sum_{n=0}^{\infty} (-\alpha_s F)^n / n!$.

Evolution scale: We imagine that a set of high-energy particles evolves to a larger set of low-energy partons by successive decays and emissions. The state changes (by e.g. additing particles to the final state) can be characterised by the momentum transfers – or energy scales – at which they occur. It thus convenient to think that the set of particles expands by following a decreasing sequence of "evolution scales".

Ordering variable: A function of the particle momenta that is used in parton shower calculations to classify how close to the singular limits a phase space point is. Common examples are the relative angle between particles, the relative transverse momentum of particles, or the squared sum of their four-momenta.

Soft limit: The limit in which the energy of one (or more) massless external particle vanishes.

Virtuality: The squared sum of four-momenta of two external particles. If these particles were connected by a single vertex, then the virtuality $(p_1 + p_2)^2$ is the invariant mass of an internal particle, i.e. the off-shellness of an internal propagator.

Angle: The relative angle of two particles. This function of particle momenta can be used to define the collinear limit.

Dipoles and anntenas: In the soft limit, the pattern of radiation from a system of two color charges looks, in the simplest approximation, like the electric field (of photons) of an electric dipole in electrical engineering. Radio antennas produce such an electric field. Because of this analogy, we call QCD radiation cascade programs that produce radiation by referring to two color charges *dipole or antenna* showers. These programs have two advantages over simpler parton showers: a) The soft limit of QCD is described in a more natural way, and b) Momentum conservation is apparent in a much simpler way. Dipole and antenna showers are slight variations on how to implement the soft limit in detail. Unfortunately, what is called "dipole shower" and what is called "antenna shower" depends on the physics publication, depending on which tradition the authors want to be associated with.

References (Parton Showers)

Introduction
Good references for event generators in general are:
MCnet report (Phys. Rept. 504 (2011) 145-233)
Many older lectures of MCNet (montecarlonet.org) and CTEQ schools.
Peter Skands' TASI lectures (arXiv:1207.2389)
Stefan Höche's TASI lectures (http://slac.stanford.edu/ shoeche/tasi14/ws/tasi.pdf)
Factorisation: Divide and conquer
The book: Collins, Perturbative Quantum Chromodynamics
Collins, Soper, Sterman (Nucl.Phys.B250(1985)199)
Backward evolution
The ISR paper: PLB 175 (1985) 321
Choosing an ordering variable: Is virtuality ordering safe?
Plot taken from CDF (PRD 50 (1994) 5562)
Dipoles / antennae
Ariadne (CPC 71 (1992) 15)
Catani, Seymour (Nucl.Phys.B485(1997)291)
Kosower antennae (Phys.Rev. D57 (1998) 5410)
Nagy, Soper (JHEP 0709 (2007) 114)
Vincia (Phys.Rev. D78 (2008) 014026)
Dinsdale, Ternick, Weinzierl (Phys.Rev. D76 (2007) 094003)
Sherpa CS (JHEP 0803 (2008) 038)
Sherpa ANTS (JHEP 0807 (2008) 040)
Herwig++ CS (JHEP 1101 (2011) 024)
Running scales
Amati et al. (Nucl.Phys. B173 (1980) 429)
Common event generator frameworks
Herwig++ (JHEP 0312 (2003) 045,JHEP 1101 (2011) 024)
Pythia 8 (Comput.Phys.Commun. 178 (2008) 852-867)
Vincia (Phys.Rev. D78 (2008) 014026, Phys.Rev. D84 (2011) 054003, Phys.Lett. B718 (2013) 1345-1350)
Sherpa (JHEP 0803 (2008) 038, JHEP 0807 (2008) 040)
Other showers:
HERWIRI (Phys.Lett.B685(2010)283, Phys.Rev.D81(2010)076008, Phys.Lett.B719(2013)367, arXiv:1305.0023)
DEDUCTOR (JHEP 1406 (2014) 097, JHEP 1406 (2014) 178, JHEP 1406 (2014) 179)

References (MPI)

Factorisation and soft gluons:
Collins, Soper, Sterman (Nucl. Phys. B 308 (1988) 833)
The book: Collins, Perturbative Quantum Chromodynamics

Multiparton interactions:
The original article: Sjostrand, van Zijl (PRD D36 (1987) 2019)
A good introduction to the HERWIG model: M. Bähr, Underlying Event Simulation in the Herwig++ Event Generator, Dissertation, ITP Karlsruhe (see `https://www.itp.kit.edu/prep/phd/PSFiles/Diss_Baehr.pdf`)