Next-to-Leading Order

lecture III

Rikkert Frederix
University of Zurich

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Summary of lecture II

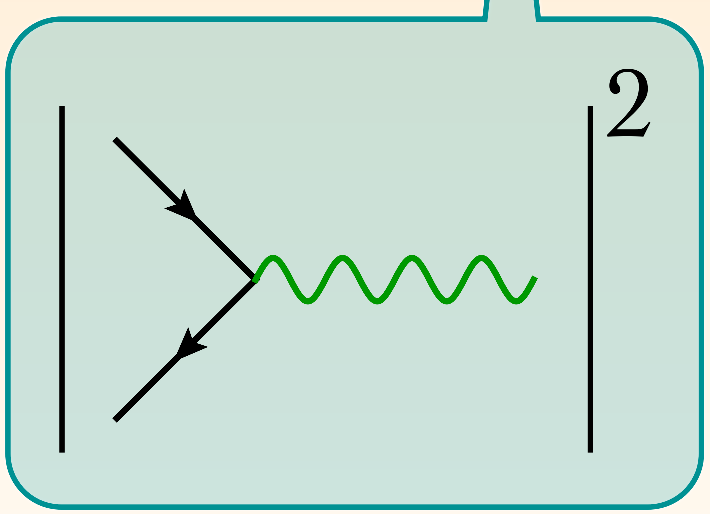
- One-loop integrals can be written as coefficients a, b, c and d times scalar functions and a rational part R
- The traditional approach for computing one-loop diagrams (Passarino-Veltman reduction) becomes more and more complicated and difficult to automate when the number of external particles increases
- The OPP reduction works at the integrand level: choosing specific values of the loop momentum results in a linear system of equations, which can be solved numerically
- MadGraph has been extended to compute loops by using the OPP reduction as implemented in the CutTools computer code
- MadLoop generates loop diagrams by cutting them open, which results in tree-level diagrams with two extra external particles
- CutTools provides the values for which the numerator should be computed numerically and solves the resulting system of equations numerically

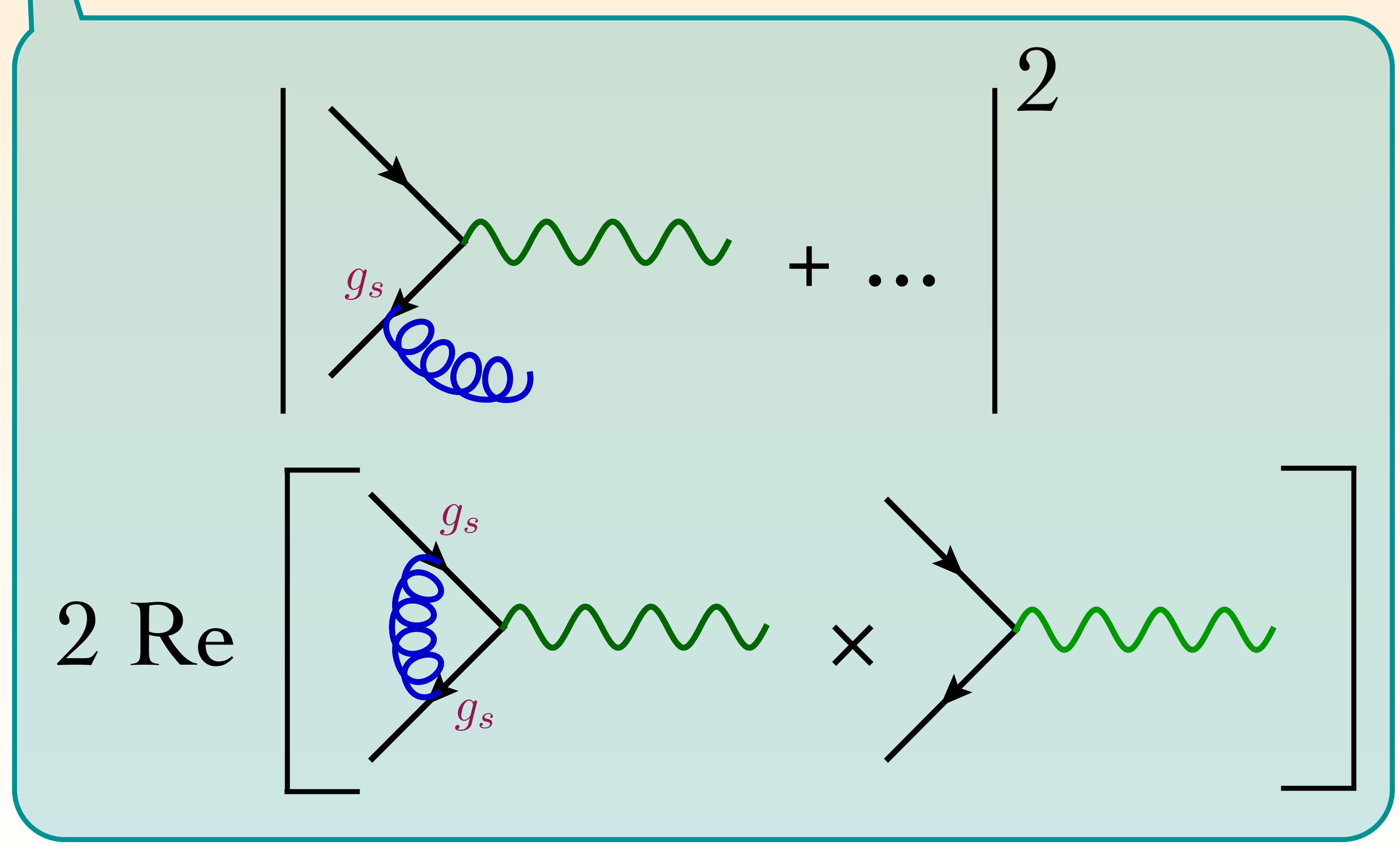
Canceling infrared divergences: FKS subtraction

NLO predictions

- As an example, consider Drell-Yan production

$$\hat{\sigma} = \sigma^{\text{Born}} \left(1 + \frac{\alpha_s}{2\pi} \sigma^{(1)} + \ldots \right)$$





Divergences

$$\mathcal{M}^{\text{1-loop}} = \sum_{i_0<i_1<i_2<i_3} d_{i_0 i_1 i_2 i_3} \text{Box}_{i_0 i_1 i_2 i_3} + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \text{Triangle}_{i_0 i_1 i_2} + \sum_{i_0<i_1} b_{i_0 i_1} \text{Bubble}_{i_0 i_1} + \sum_{i_0} a_{i_0} \text{Tadpole}_{i_0} + R + \mathcal{O}(\epsilon)$$

$$D_i = (l+p_i)^2 - m_i^2$$

$$\text{Tadpole}_{i_0} = \int d^d l \frac{1}{D_{i_0}}$$

$$\text{Bubble}_{i_0 i_1} = \int d^d l \frac{1}{D_{i_0} D_{i_1}}$$

$$\text{Triangle}_{i_0 i_1 i_2} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2}}$$

$$\text{Box}_{i_0 i_1 i_2 i_3} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}}$$

- The coefficients d, c, b and a are finite and do not contain poles in $1/\epsilon$ and are finite
- The $1/\epsilon$ dependence is in the scalar integrals (and the UV renormalization)
- When we have solved this system (and included the UV renormalization) we have the full dependence on the soft/collinear divergences in terms of coefficients in front of the poles. These divergences should cancel against divergences in the real emission corrections (according to KLN theorem)

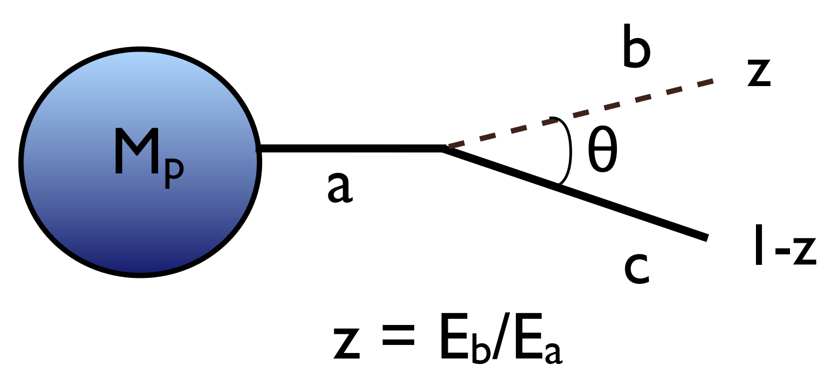
$$\text{Virtual} \sim v_0 + \frac{v_1}{\epsilon} + \frac{v_2}{\epsilon^2}$$

Origin of IR divergences

- In the virtual corrections: if a particles in the loop is soft, or collinear to an external particle we get soft and collinear divergences
- For the real emission it is clear where the IR divergences are coming from. From Johan's lecture:

Matrix elements involving q →q g (or g → gg) are strongly enhanced when the final state particles are close in the phase space:

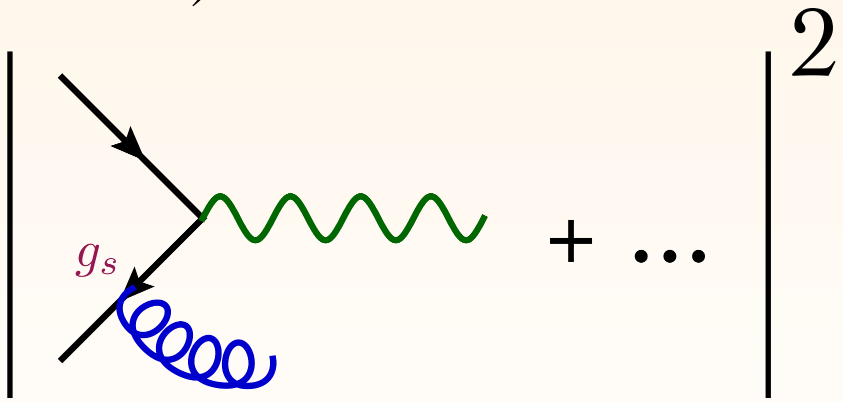
$$\frac{1}{(p_b+p_c)^2} \simeq \frac{1}{2E_bE_c(1-\cos\theta)} = \frac{1}{t}$$



soft and collinear divergencies

Infrared cancellation

- The KLN theorem tells us that divergences from virtual and real-emission corrections cancel in the sum (for observables insensitive to soft and collinear radiation)
- When doing an analytic calculation in dimensional regularization this can be explicitly seen in the cancellation of the $1/\epsilon$ and $1/\epsilon^2$ terms (with ϵ the regulator, $\epsilon \to 0$)
- In the real emission corrections, the explicit poles enter after the phase-space integration (in d dimensions)



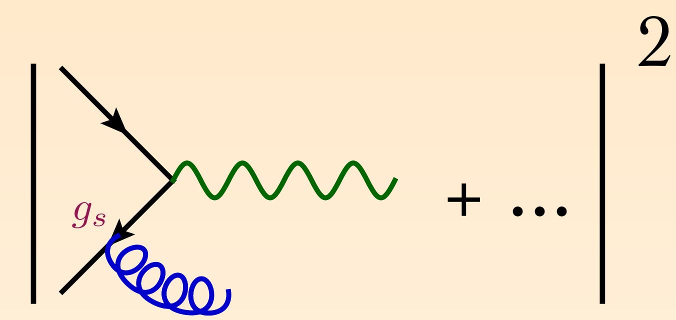
PHASE-SPACE INTEGRATION

- For complicated processes we have to result to numerical phase-space integration techniques ("Monte Carlo integration"), which can only be performed in an integer number of dimensions
    - Cannot use a finite value for the dimensional regulator and take the limit to zero in a numerical code
- But we still have to cancels the divergences explicitly
- Two solutions exists
    - Phase-space slicing
    - Subtraction method

Example

- Suppose we want to compute the integral ("real emission radiation", where the 1-particle phase-space is referred to as the 1-dimensional x)

$$\int_0^1 dx\, f(x)$$



where $f(x) = \dfrac{g(x)}{x}$ and $g(x)$ is finite everywhere

- Let's introduce a regulator

$$\lim_{\epsilon \longrightarrow 0} \int_0^1 dx\, \frac{f(x)}{x^{1+\epsilon}} = \lim_{\epsilon \longrightarrow 0} \int_0^1 dx\, x^{-\epsilon} f(x)$$

for any non-integer non-zero value for ϵ this integral is finite

- We would like to factor out the explicit poles in ϵ so that they can be canceled explicitly against the "virtual corrections"

Phase-space slicing

$$\lim_{\epsilon\to 0}\int_0^1 dx\, x^{-\epsilon} f(x) \qquad\qquad f(x)=\frac{g(x)}{x}$$

- Introduce a small parameter δ

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_0^1 dx\, x^{-\epsilon} f(x) &= \lim_{\epsilon\to 0}\left[\int_0^\delta dx\, x^{-\epsilon} f(x) + \int_\delta^1 dx\, x^{-\epsilon} f(x)\right]\\
&= \lim_{\epsilon\to 0}\left[\int_0^\delta dx\, x^{-\epsilon} \frac{g(0)}{x} + \int_\delta^1 dx\, x^{-\epsilon} \frac{g(x)}{x^{1+\epsilon}}\right]\\
&= \lim_{\epsilon\to 0}\frac{\delta^{-\epsilon}}{-\epsilon} g(0) + \int_\delta^1 dx\,\frac{g(x)}{x}\\
&= \lim_{\epsilon\to 0}\left[\frac{-1}{\epsilon}+\log\delta\right] g(0) + \int_\delta^1 dx\,\frac{g(x)}{x}
\end{aligned}$$

- We get the explicit pole in ϵ and a finite integral that can be computed numerically

Subtraction method

$$\lim_{\epsilon\to 0}\int_0^1 dx\, x^{-\epsilon} f(x) \qquad\qquad f(x) = \frac{g(x)}{x}$$

- Add and subtract the same term

$$\lim_{\epsilon\to 0}\int_0^1 dx\, x^{-\epsilon} f(x) = \lim_{\epsilon\to 0}\int_0^1 dx\, x^{-\epsilon}\left[\frac{g(0)}{x} + f(x) - \frac{g(0)}{x}\right]$$

$$= \lim_{\epsilon\to 0}\int_0^1 dx\left[g(0)\frac{x^{-\epsilon}}{x} + \frac{g(x)-g(0)}{x^{1+\epsilon}}\right]$$

$$= \lim_{\epsilon\to 0}\frac{-1}{\epsilon}g(0) + \int_0^1 dx\,\frac{g(x)-g(0)}{x}$$

- Like before, we have factored out the explicit divergence. The coefficient in front of the $1/\epsilon$ pole is the same in both methods (as it should be!)

- According to the KLN theorem the divergence cancels against the virtual corrections

Slicing vs Subtraction

Slicing: $$\int_\delta^1 dx\,\frac{g(x)}{x} + g(0)\log\delta$$

Subtraction: $$\int_0^1 dx\,\frac{g(x)-g(0)}{x}$$ ← "Plus distribution"

- Terms of order δ are neglected in the slicing method; the subtraction method is exact
  - One has to proof that any observable is independent of δ when $\delta \to 0$
- Both methods feature cancellations between large numbers: if for an observable O, if $\lim_{x\to 0} O(x) \neq O(0)$ or we choose the bin-size too small, instabilities render the computation useless
  - We already knew that! KLN is sufficient; one must have infra-red safe observables and cannot ask for infinite resolution
- Subtraction method is more flexible -> method of choice in automation

NLO with Subtraction

$$\sigma^{\mathrm{NLO}} \sim \int d^4\Phi_m\, B(\Phi_m) + \int d^4\Phi_m \int_{\mathrm{loop}} d^d l\, V(\Phi_m) + \int d^d\Phi_{m+1}\, R(\Phi_{m+1})$$

- With the subtraction method this is replace by

$$\sigma^{\mathrm{NLO}} \sim \int d^4\Phi_m\, B(\Phi_m)$$
$$+ \int d^4\Phi_m \left[\int_{\mathrm{loop}} d^d l\, V(\Phi_m) + \int d^d\Phi_1 G(\overline{\Phi}_{m+1})\right]_{\epsilon\to 0}$$
$$+ \int d^4\Phi_{m+1} \left[R(\Phi_{m+1}) - G(\overline{\Phi}_{m+1})\right]$$

- Terms between the brackets are finite. Can integrate them numerically and independent from one another in 4 dimensions

Subtraction methods

- $G(\overline{\Phi}_{m+1})$ should be defined such that

  1) it exactly matches the singular behavior of $R(\Phi_{m+1})$

  2) its form is convenient for MC integration techniques

  3) it is exactly integrable in d dimensions over the one-particle subspace $\int d^d\Phi_1 G(\overline{\Phi}_{m+1})$, leading to soft and/or collinear divergences as explicit poles in the dimensional regulator

  4) it is universal, i.e. "process independent"
  → "overall factor" times the Born process

Two methods

- Catani-Seymour dipole subtraction
  - Most used method
  - Clear written paper on how to use this method in practice
  - Method evolved from cancellation of the soft divergence
  - Proven to work for simple as well as complicated processes
  - Automation in publicly available packages: MadDipole, AutoDipole, Helac-Dipoles, Sherpa
- FKS subtraction
  - Not so well-known
  - (Probably) more efficient, because less subtraction terms are needed
  - Collinear divergences as a starting point
  - Proven to work for simple as well as complicated processes
  - Preferred method when interfacing NLO to a parton shower
  - Implemented in MadFKS and (a)MC@NLO

FKS subtraction

- FKS subtraction: Frixione, Kunszt & Signer 1996. Standard subtraction method in MC@NLO and POWHEG, but can also be used for 'normal' NLO computations
- Also known as "residue subtraction"
- Based on using plus-distributions to regulate the infrared divergences of the real emission matrix elements

Phase-space partitions

- Easiest to understand by starting from real emission:

$$d\sigma^R = |M^{n+1}|^2 d\phi_{n+1}$$

- $|M^{n+1}|^2$ blows up like $\dfrac{1}{\xi_i^2}\dfrac{1}{1-y_{ij}}$ with $\xi_i = E_i/\sqrt{\hat{s}}$, $y_{ij} = \cos\theta_{ij}$

- Partition the phase space in such a way that each partition has at most one soft and one collinear singularity

$$d\sigma^R = \sum_{ij} S_{ij} |M^{n+1}|^2 d\phi_{n+1} \qquad\qquad \sum_{ij} S_{ij} = 1$$

- Use plus distributions to regulate the singularities

$$d\tilde{\sigma}^R = \sum_{ij} \left(\frac{1}{\xi_i}\right)_+ \left(\frac{1}{1-y_{ij}}\right)_+ \xi_i(1-y_{ij}) S_{ij} |M^{n+1}|^2 d\phi_{n+1}$$

Regularized by plus-prescription

$$d\tilde{\sigma}^R = \sum_{ij} \left(\frac{1}{\xi_i}\right)_+ \left(\frac{1}{1-y_{ij}}\right)_+ \xi_i (1-y_{ij}) S_{ij} |M^{n+1}|^2 d\phi_{n+1}$$

- Definition plus distribution

$$\int d\xi \left(\frac{1}{\xi}\right)_+ g(\xi) = \int d\xi \, \frac{g(\xi)-g(0)}{\xi}$$

- One event has maximally three counter events:
  - Soft: $\xi_i \longrightarrow 0$
  - Collinear: $y_{ij} \longrightarrow 1$
  - Soft-collinear: $\xi_i \longrightarrow 0 \qquad y_{ij} \longrightarrow 1$

Regularized by plus-prescription

$$d\tilde{\sigma}^R = \sum_{ij} \left(\frac{1}{\xi_i}\right)_{\xi_{cut}} \left(\frac{1}{1-y_{ij}}\right)_{\delta_O} \xi_i(1-y_{ij}) S_{ij} |M^{n+1}|^2 d\phi_{n+1}$$

- Modified definition plus distribution (include counter terms only when event is close to being singular)

$$\int d\xi \left(\frac{1}{\xi}\right)_{\xi_{cut}} g(\xi) = \int d\xi \, \frac{g(\xi) - g(0)\Theta(\xi_{cut}-\xi)}{\xi}$$

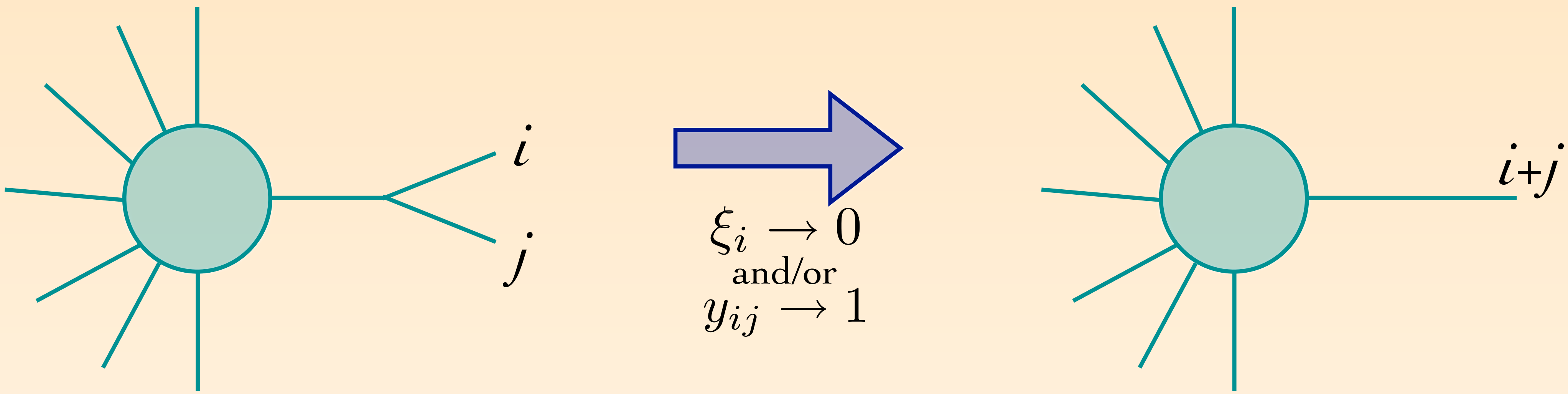
- One event has maximally three counter events:
  - Soft: $\xi_i \longrightarrow 0$
  - Collinear: $y_{ij} \longrightarrow 1$
  - Soft-collinear: $\xi_i \longrightarrow 0 \qquad y_{ij} \longrightarrow 1$

Subtraction terms

$$\sigma^{\mathrm{NLO}} \sim \int d^4\Phi_m \, B(\Phi_m)$$
$$+ \int d^4\Phi_m \left[\int_{\mathrm{loop}} d^d l \, V(\Phi_m) + \int d^d\Phi_1 G(\overline{\Phi}_{m+1}) \right]_{\epsilon\to 0}$$
$$+ \int d^4\Phi_{m+1} \left[R(\Phi_{m+1}) - G(\overline{\Phi}_{m+1}) \right]$$

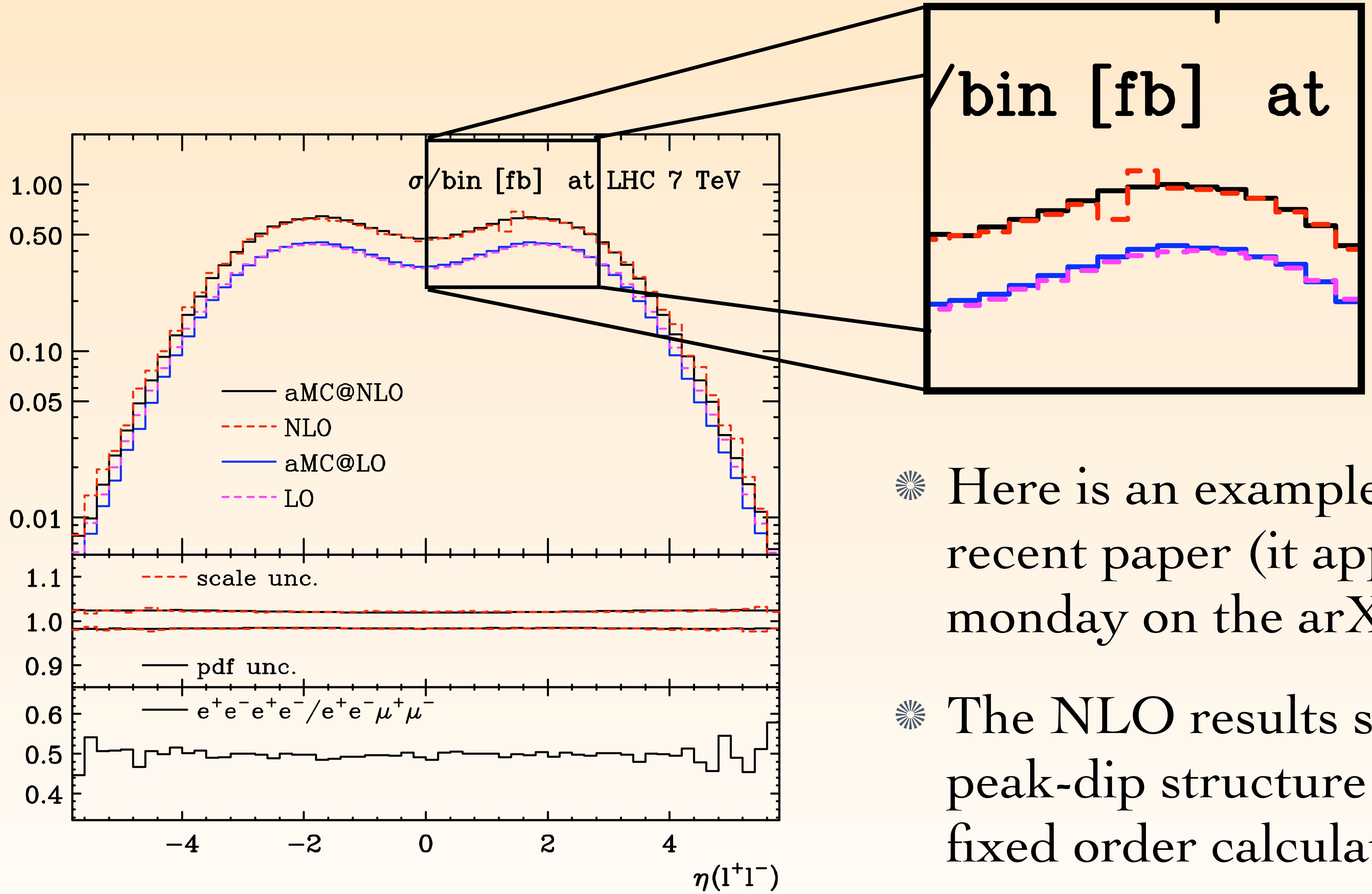
- This defines the subtraction terms for the reals
- They need to be integrated over the one-parton phase space (analytically) to get the explicit poles $1/\epsilon$ and added to the virtual corrections so that these poles cancel
    - these are process-independent terms proportional to the (color-linked) Borns
- All formulae can be found in the MadFKS paper, arXiv:0908.4247

Kinematics of counter events



- If *i* and *j* are two on-shell particles that are present in a splitting that leads to an singularity, for the counter events we need to combine their momenta to a new on-shell parton that's the sum of *i+j*
- This is not possible without changing any of the other momenta in the process
- When applying cuts or making plots, events and counter events might end-up in different bins
  - Use IR-safe observables and don't ask for infinite resolution! (KLN theorem)

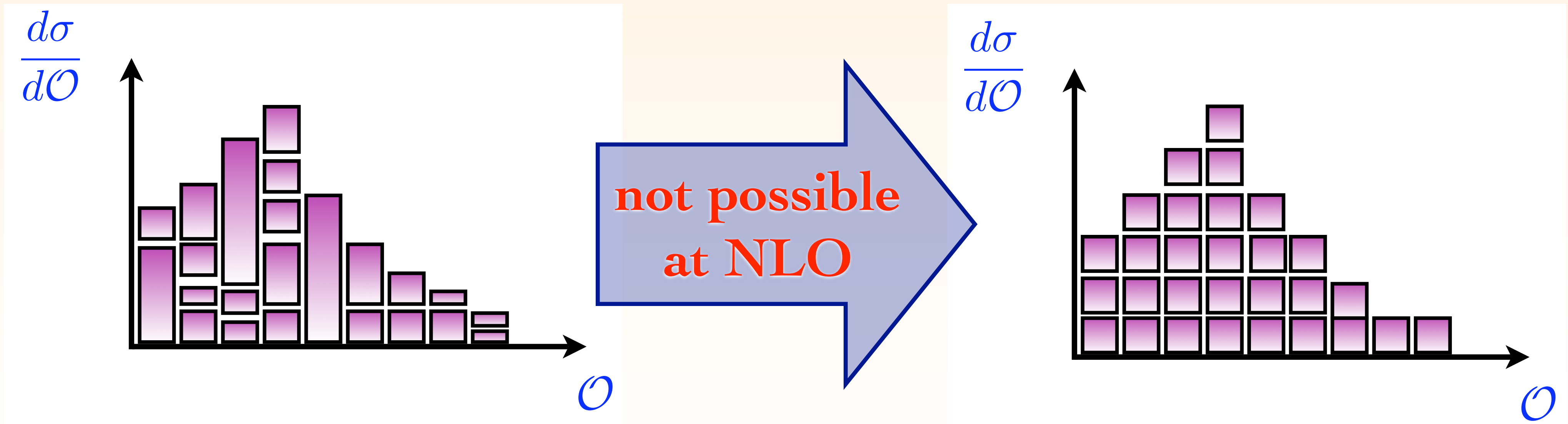
Example in 4 charged lepton production



- Here is an example of a very recent paper (it appeared last monday on the arXiv!)
- The NLO results shows a typical peak-dip structure that hampers fixed order calculations

Event unweighting?

- It is not possible to generate unweighted events in this set-up
  - Even though the integrals are finite, they are not bounded (compare with $\int_0^1 dx \, \frac{1}{\sqrt{x}}$), so there is no maximum to unweight against: a single event can have an arbitrarily large weight
  - Furthermore, event and counter event have different kinematics: which one to use for the unweighted event?



Summary

- Both the virtual and real-emission corrections are IR divergent, but their sum is finite
- We can use the slicing or subtraction methods to cancel the poles explicitly
  - Preferred method is the subtraction method (no approximations needed and proven to work very well for complicated processes)
- This generates events and counter events with slightly different kinematics
- When making plots or applying cuts, use only IR safe observables with finite resolution
- Phase-space integrals are finite, but not bounded: cannot unweight the events