Next-to-Leading Order

Rikkert Frederix
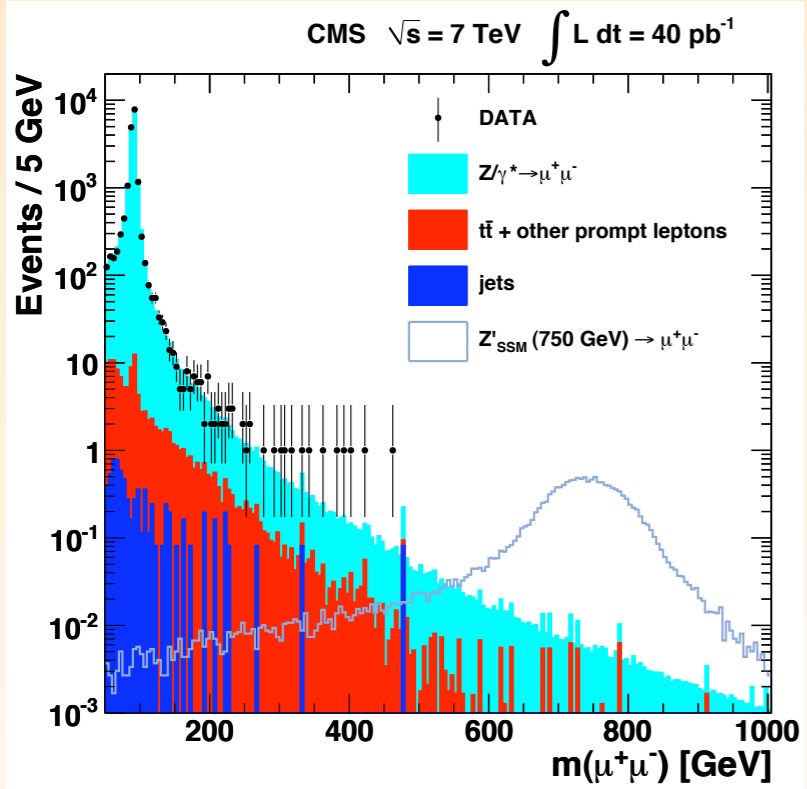### University of Zurich

SEARCHING FOR NEW PHYSICS

[MLM 2008]

Peak

$pp \rightarrow Z' \rightarrow \mu^+\mu^-$

Shape

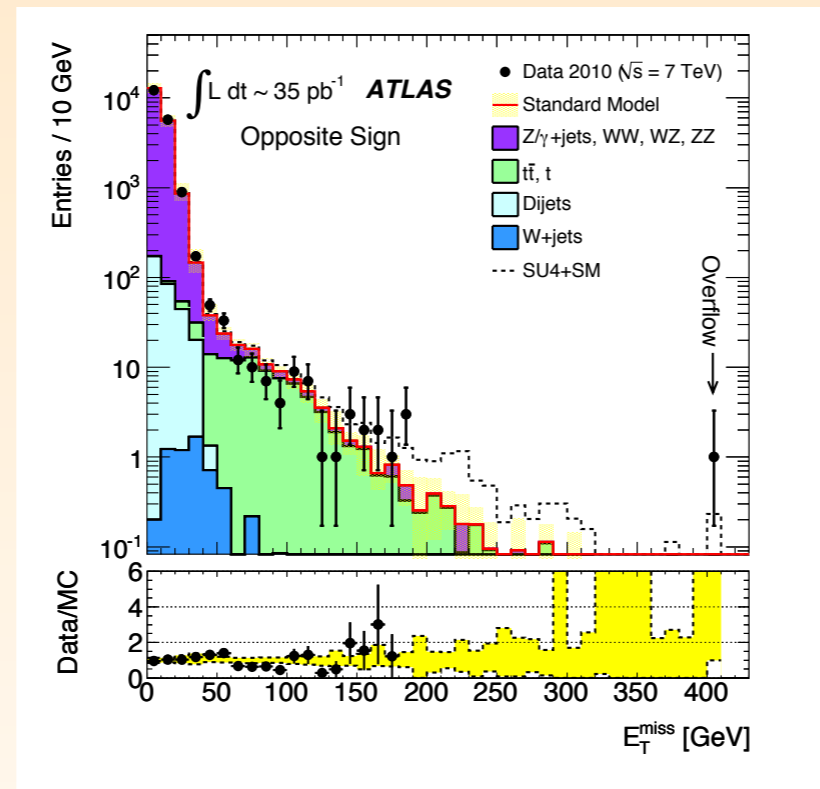Supersymmetry

Rate

$pp \rightarrow H \rightarrow W^+W^-$

"Easy"

Background can be measured from data and interpolated. Theory needed for parameter extraction (normalization, acceptance, ...)
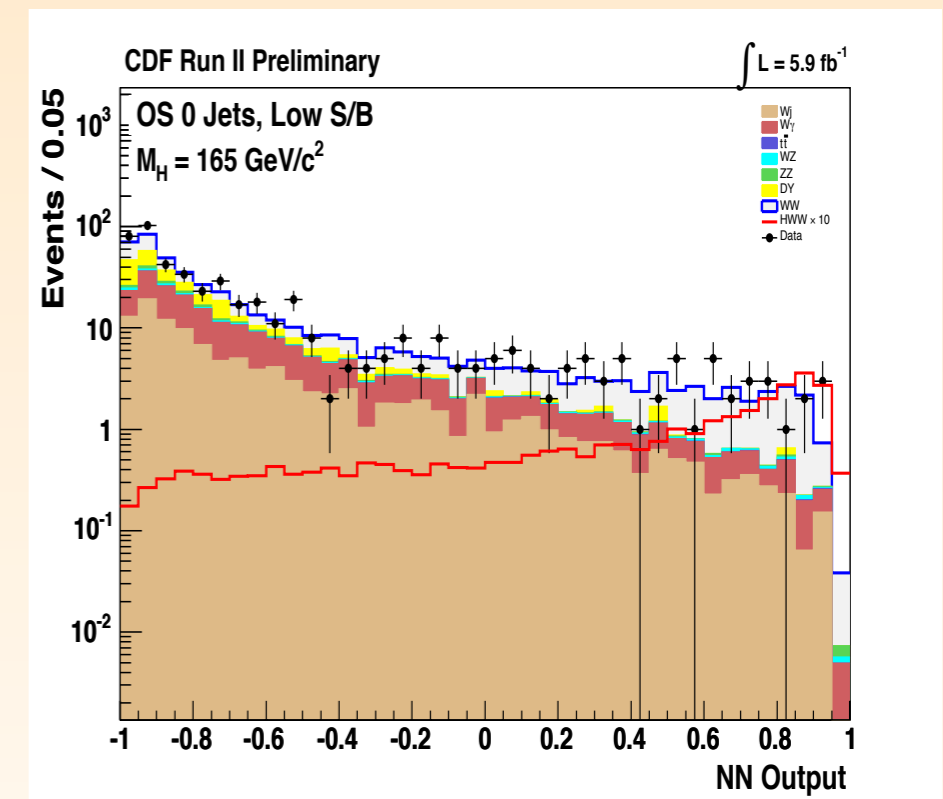
Hard

Background shapes needed from theory. Flexible MC for both signal and background tuned and validated against data

Very Hard

Background normalization and shapes need to be known very well. Interplay between the best theory prediction (via MC) and data

Leading order

- For many of the theory predictions needed in the searches for new physics as well as measuring properties of the SM, leading order predictions are used

- The reasons for this are clear:

  - In many regions of phase-space they do a decent job, in particular for shapes of distributions

  - Parton showers and hadronizations models are tuned to data

  - **Many flexible lowest order (LO) tools are readily available**

- Unfortunately LO predictions describe total rates rather poorly

Need for NLO

* If we would have the same **flexible** tools available at NLO, the experimental analyses will benefit a various ways:

  * NLO predictions predict rates much more precisely

  * Reduced theoretical uncertainties due to meaningful scale dependence

  * Shapes are better described

  * Correct estimates for PDF uncertainties

  * Even data-driven analyses might benefit: smaller uncertainty due to interpolation from control region to signal region

* These accurate theoretical predictions are particularly needed for

  * searches of signal events in large backgrounds samples and

  * precise extraction of parameters (couplings etc.) when new physics signals have been found

B PAIR PRODUCTION AT THE LHC
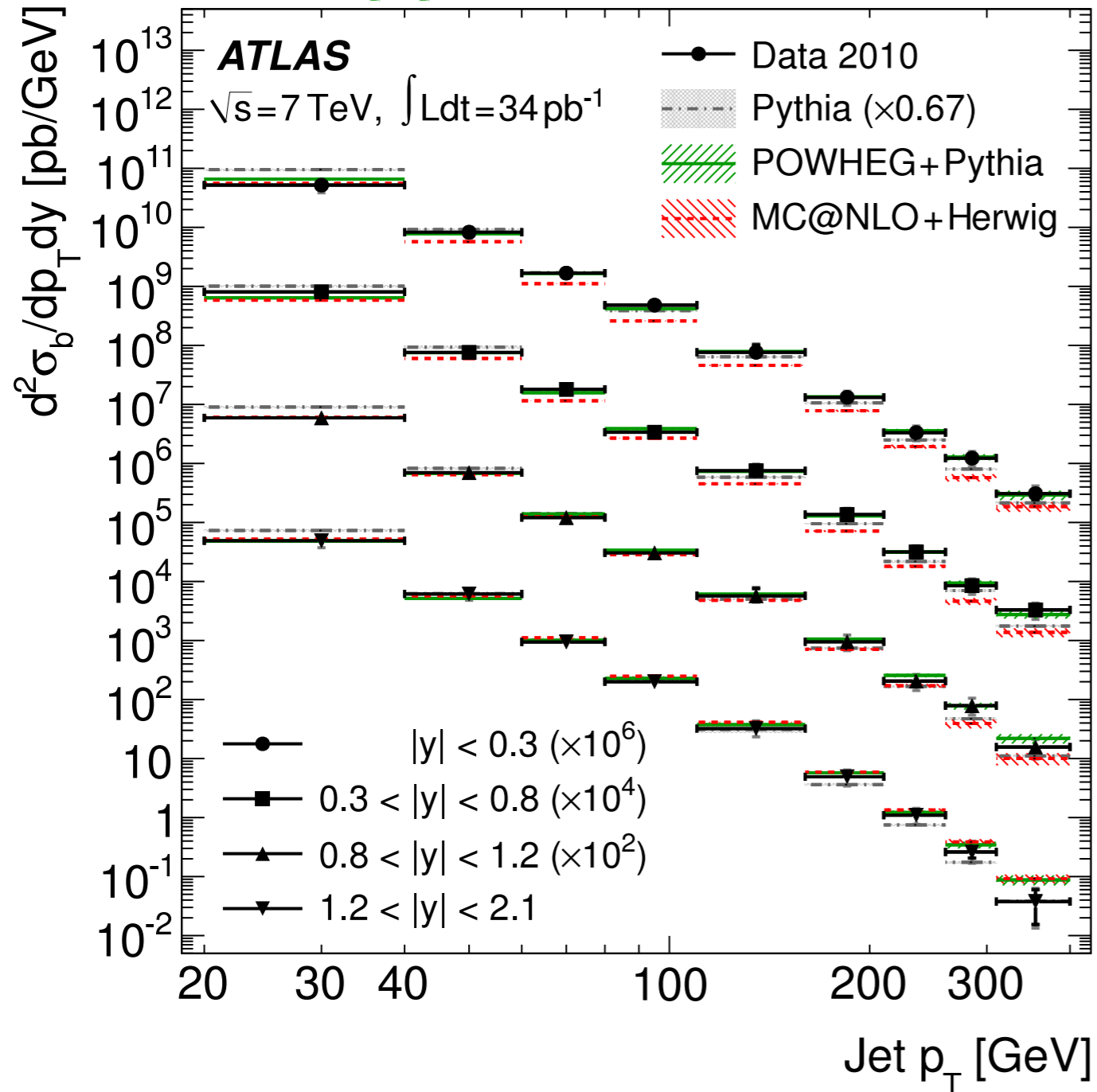
- b-jet transverse momentum in b pair production

- LO: Pythia,
  NLO: POWHEG + Pythia
  NLO: MC@NLO + Herwig

- Shapes are well described by LO and by NLO

- Normalization is well predicted at NLO

- At NLO, theory uncertainties can be studied systematically (not shown in this plot)

WHY AN AUTOMATIC TOOL?

※ To save time

Less human time spending on computing matrix elements means more time available on physics and phenomenology.

※ Robustness

Modular code structure means that elements can be checked systematically and extensively once and for all. Trust can easily be build.

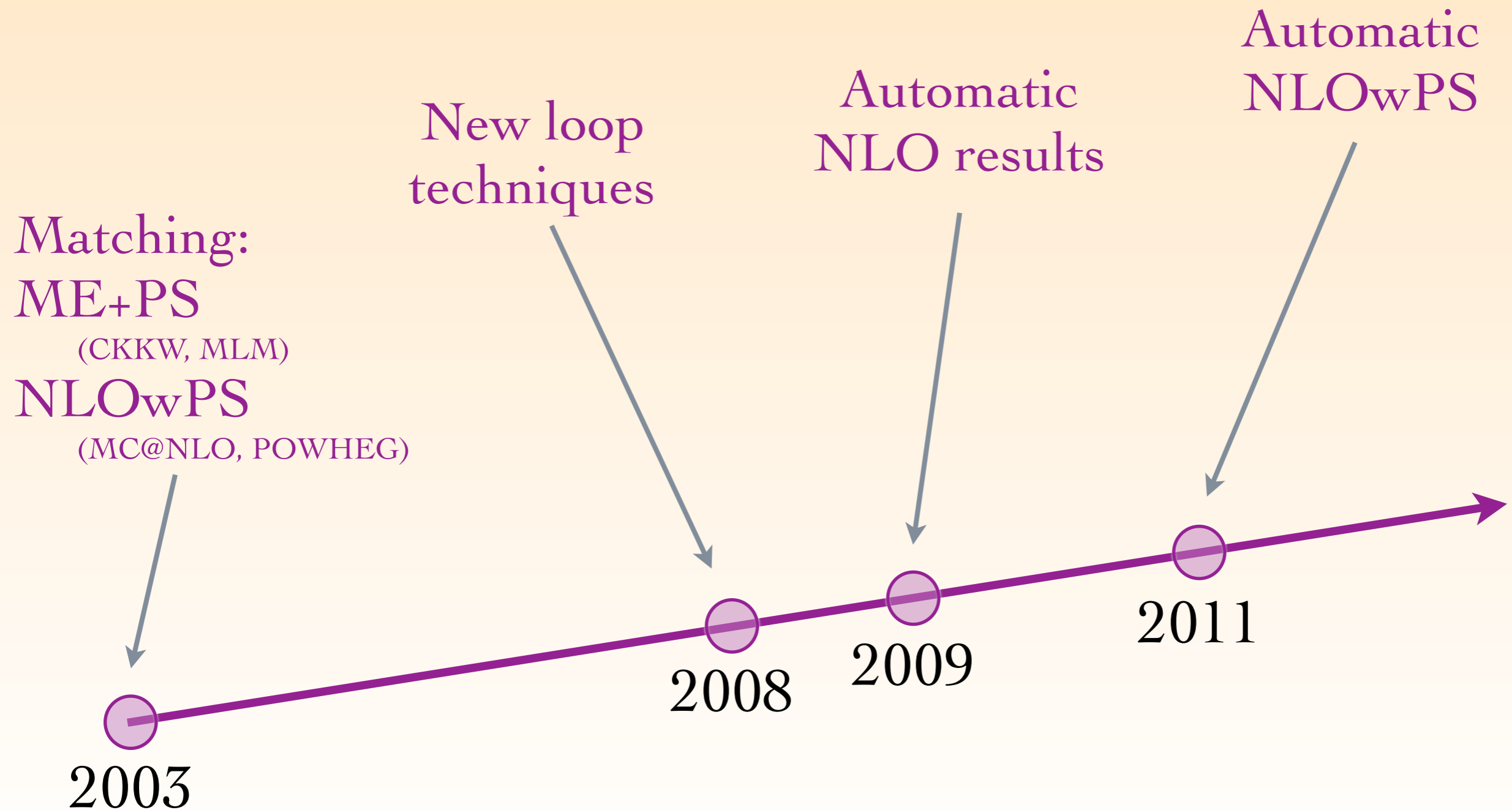※ Wide accessibility

One framework for all. Available to everybody for an unlimited set of applications. Suitable for Experimental collaborations.
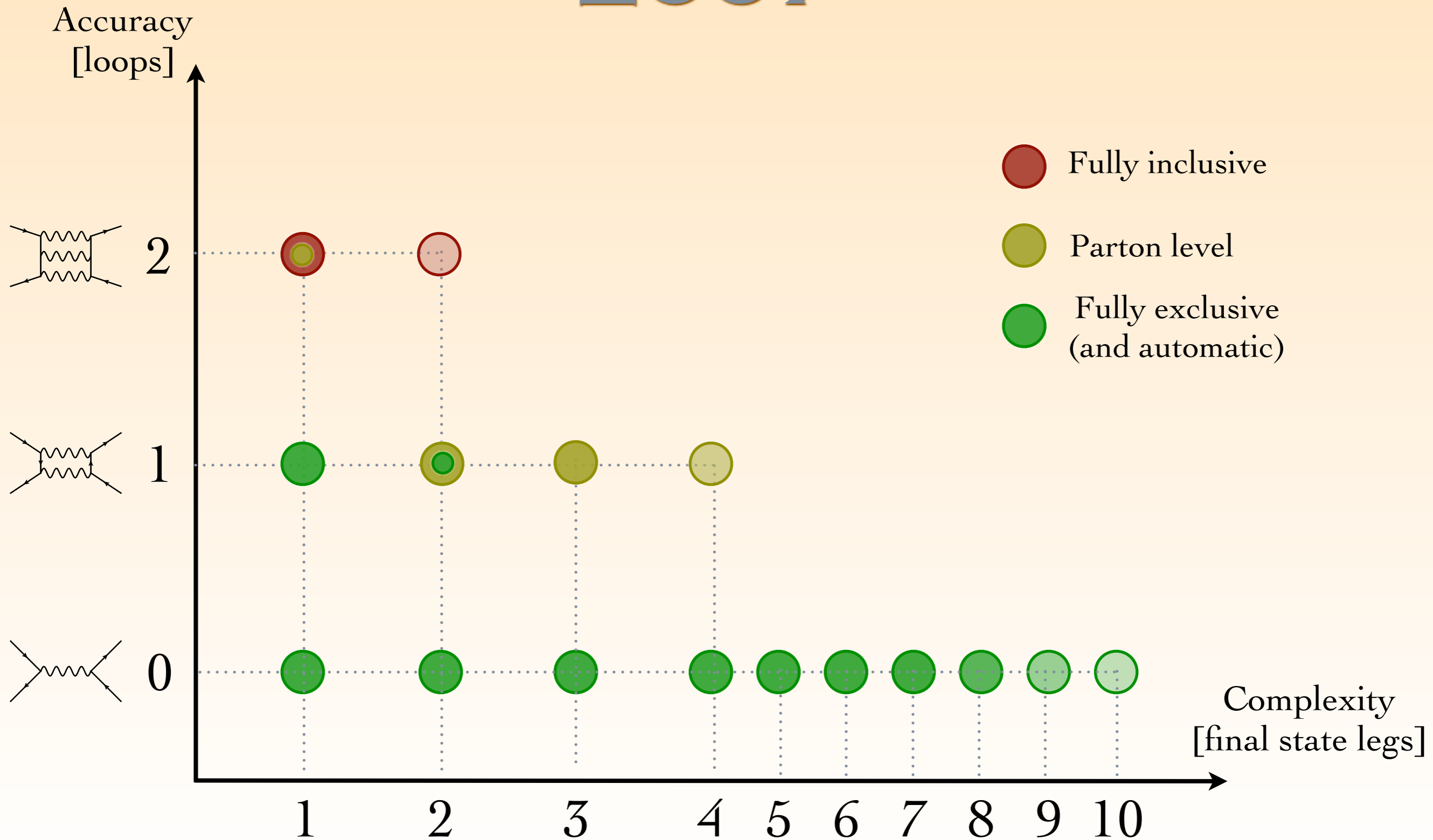
QCD AND MC PROGRESS (SIMPLIFIED)
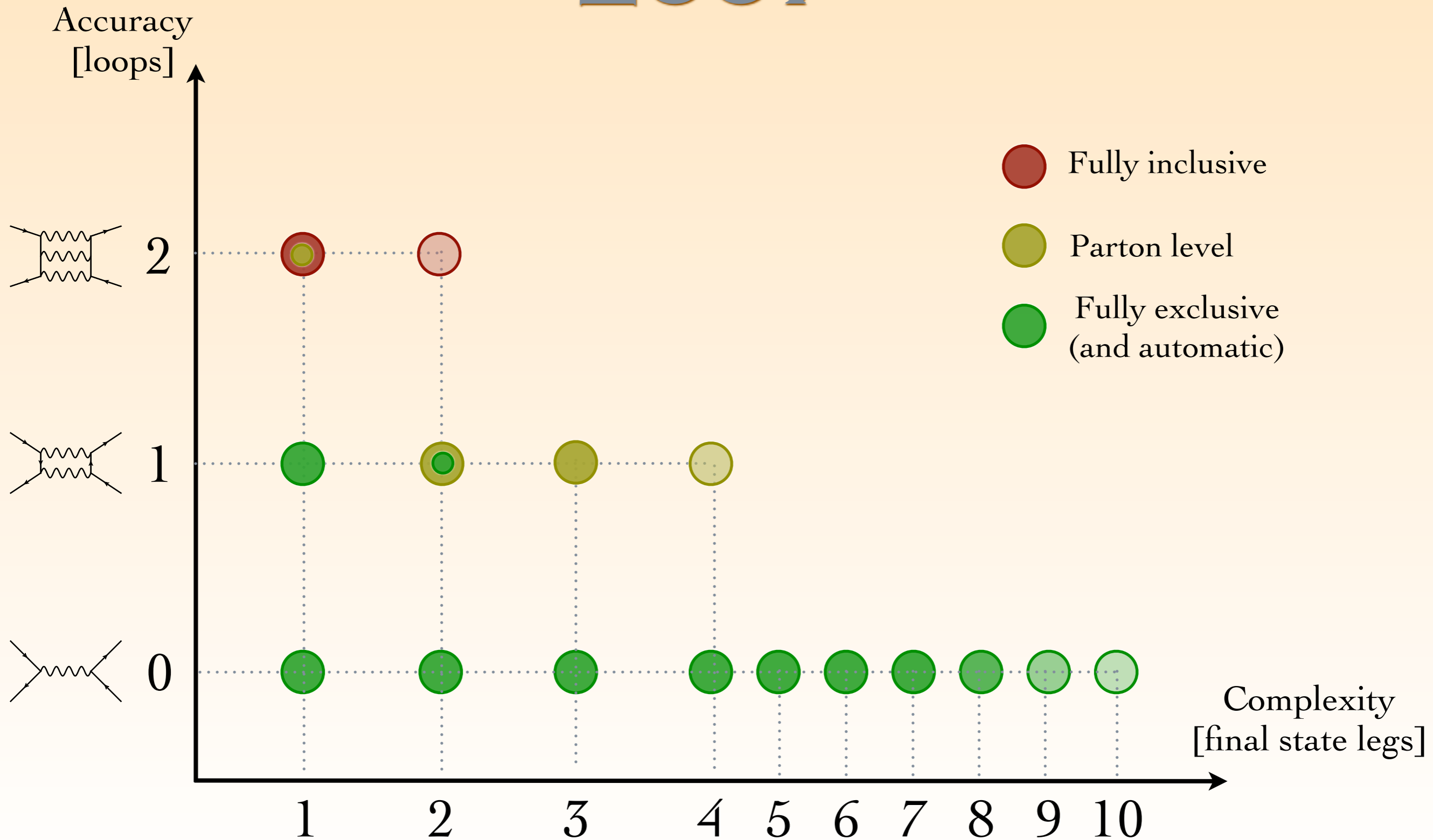
SM STATUS: SINCE A FEW MONTHS

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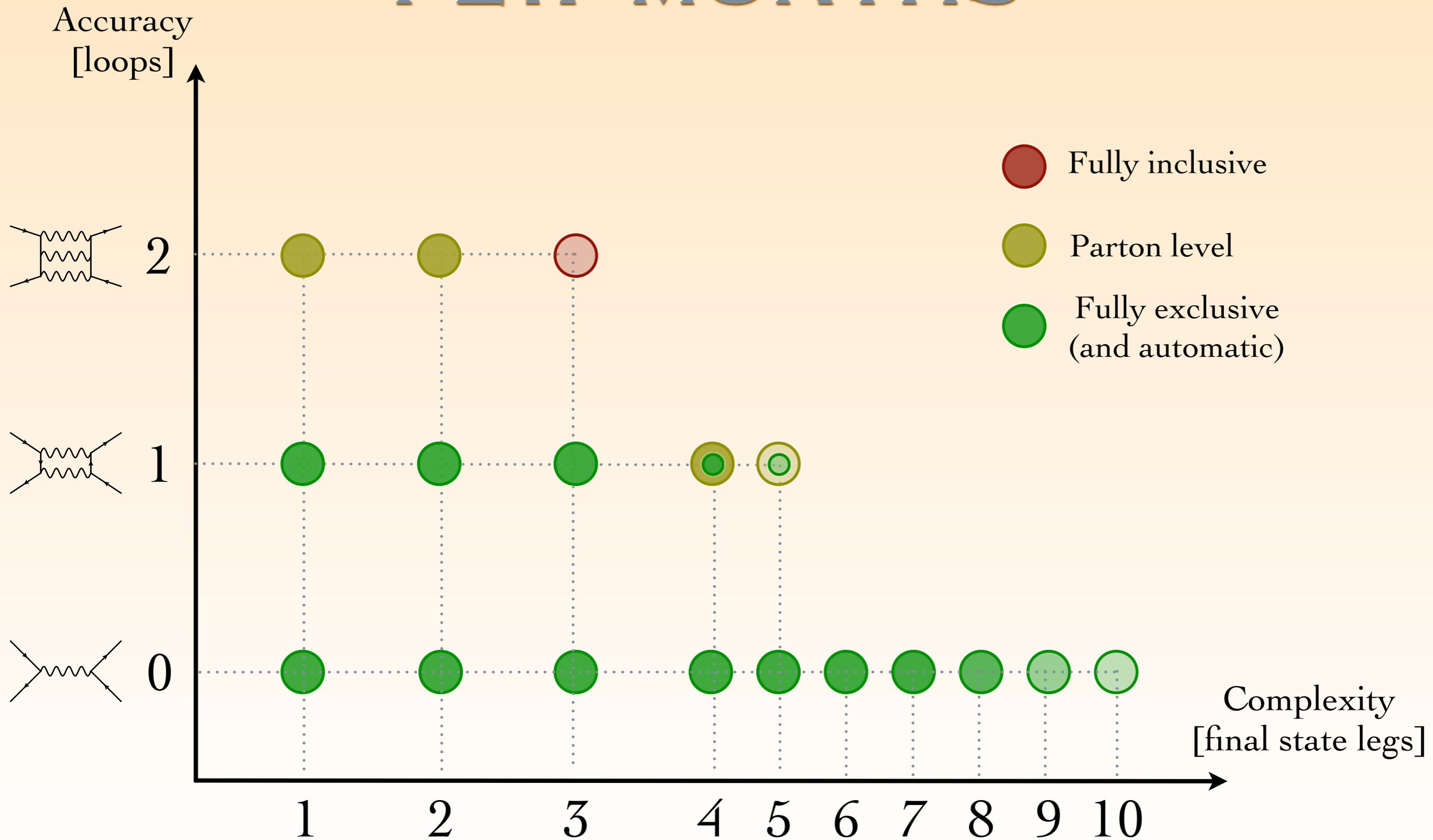
Contents of these lectures

- These lectures will be about the concepts behind the computation of NLO corrections (as implemented in MadGraph)

  - Computing Loop corrections without doing integrals and using only tree-level matrix elements

  - Cancellation of infrared singularities (FKS subtraction)

  - Matching NLO to the parton shower: MC@NLO formalism

  - Tutorial

- Please, interrupt me at any time if something is not clear!

NOT YET PUBLICLY AVAILABLE

❋ Disclaimer:
Although NLO corrections are included in MadGraph (version 4) they are not yet publicly available. The reason is that the code is

❋ still a bit slow (not possible to run on a single desktop for anything beyond a $2 \rightarrow 2$ process)

❋ not yet completely general (some processes cannot be computed; and no warning is given if one tries)

❋ The code is being rewritten in MG5. When this is done, it will be made publicly available

❋ The tutorial will be based on MC@NLO
[Frixione, Webber; + collaborators]

NLO computations

Master equation for hadron colliders

$$\sum_{a,b} \int dx_1 dx_2 d\Phi_{\text{FS}} \, f_a(x_1, \mu_F) f_b(x_2, \mu_F) \, \hat{\sigma}_{ab \to X}(\hat{s}, \mu_F, \mu_R)$$

Phase-space integral   Parton density functions   Parton-level cross section

✸ Parton-level cross section from matrix elements: model and process dependent

✸ Parton density (or distribution) functions: process independent

✸ Differences between colliders given by parton luminosities

Perturbative expansion

$$\hat{\sigma}_{ab \to X}(\hat{s}, \mu_F, \mu_R) \quad \text{Parton-level cross section}$$

✳ The parton-level cross section can be computed as a series in perturbation theory, using the coupling constant as an expansion parameter

$$\hat{\sigma} = \sigma^{\text{Born}}\left(1 + \frac{\alpha_s}{2\pi}\sigma^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \sigma^{(2)} + \left(\frac{\alpha_s}{2\pi}\right)^3 \sigma^{(3)} + \dots\right)$$
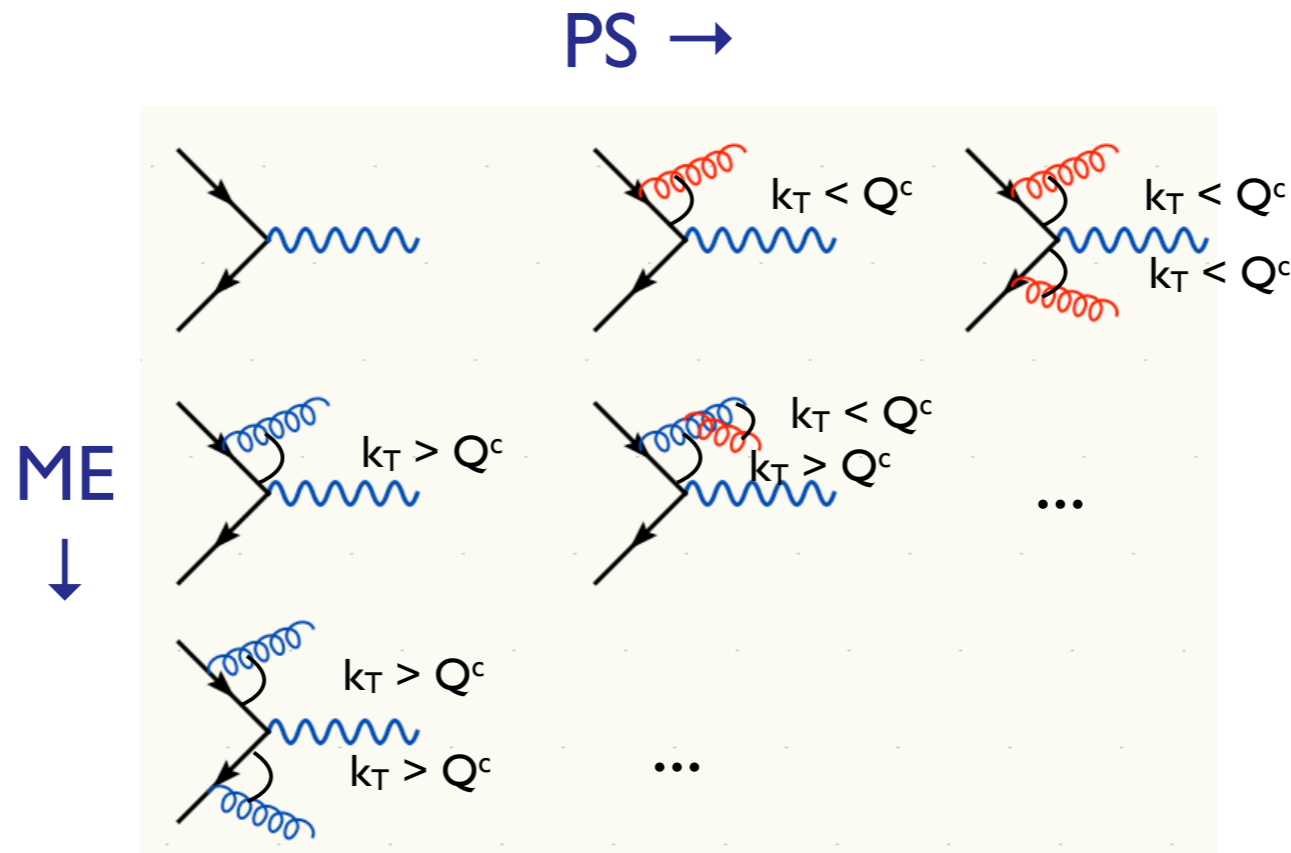
| LO predictions | NLO corrections | NNLO corrections | NNNLO corrections |

✳ Including higher corrections improves predictions and reduces theoretical uncertainties

Merging ME with PS

[Mangano]
[Catani, Krauss, Kuhn, Webber]

PS →

ME ↓

$k_T < Q^c$

$k_T < Q^c$
$k_T < Q^c$

$k_T > Q^c$

$k_T < Q^c$
$k_T > Q^c$

...

$k_T > Q^c$
$k_T > Q^c$

...

⭐ **Using matching is not doing a (N...)NLO computation**

⭐ However, part of the NLO contributions are included, which is the reason that shapes are so well described, but normalization is off

⭐ Uncertainties are still large in a matched sample (although reduced from a prediction without matching

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16

NLO CORRECTIONS

※ There are two types of contributions to the NLO corrections:

  ※ Virtual (or Loop) corrections: formed by an amplitude with a closed loop of particles interfered with the Born amplitudes

  ※ Real emission corrections: formed by amplitudes with one extra parton compared to the Born process

※ Both have one power of α_s extra compared to the Born process

$$\sigma^{\mathrm{NLO}} = \int_{m+1} d^{(d)}\sigma^R + \int_m d^{(d)}\sigma^V + \int_m d^{(4)}\sigma^B$$

NLO PREDICTIONS

☀ As an example, consider Drell-Yan production

$$\hat{\sigma} = \sigma^{\text{Born}}\left(1 + \frac{\alpha_s}{2\pi}\sigma^{(1)} + \dots\right)$$
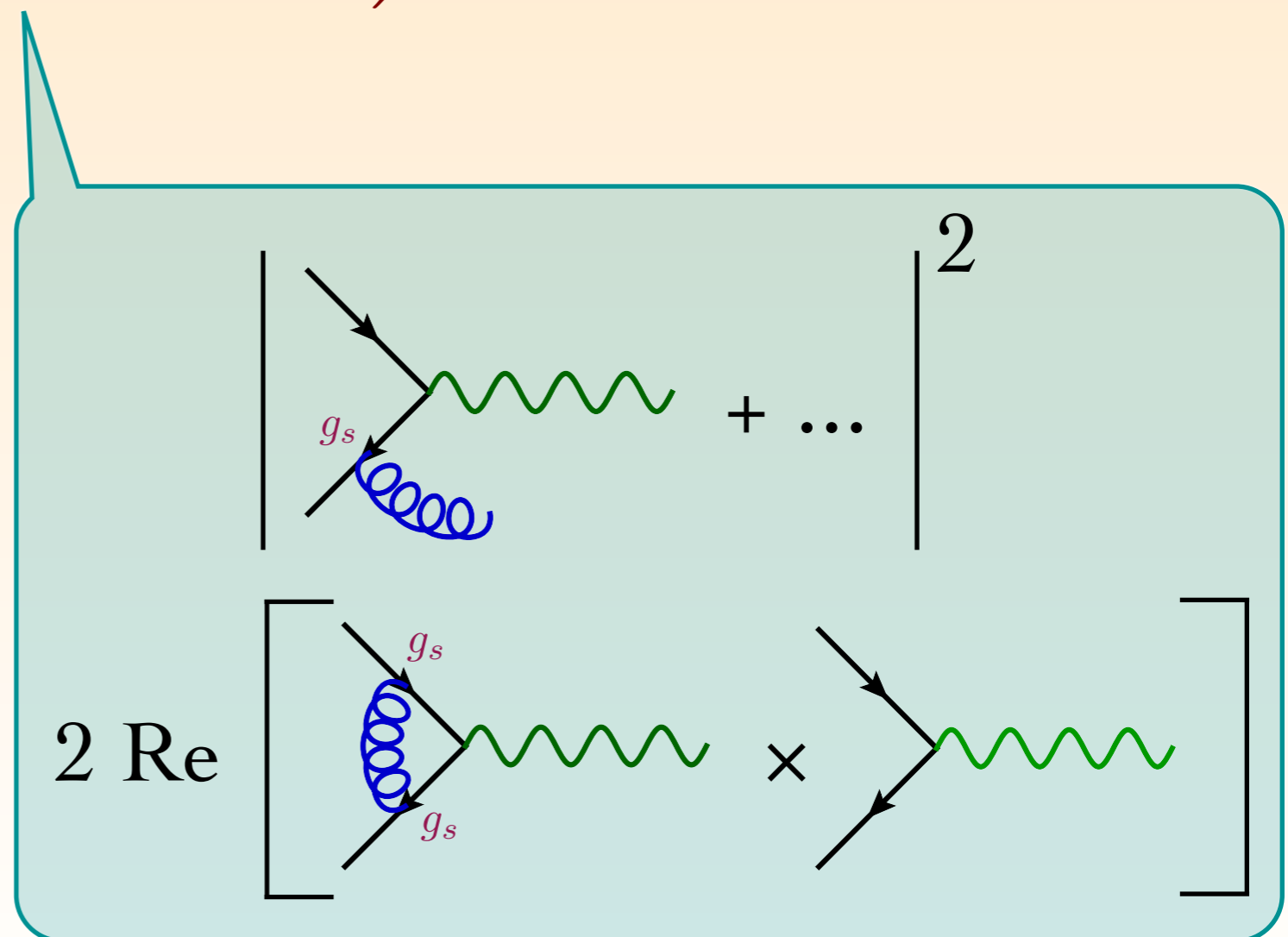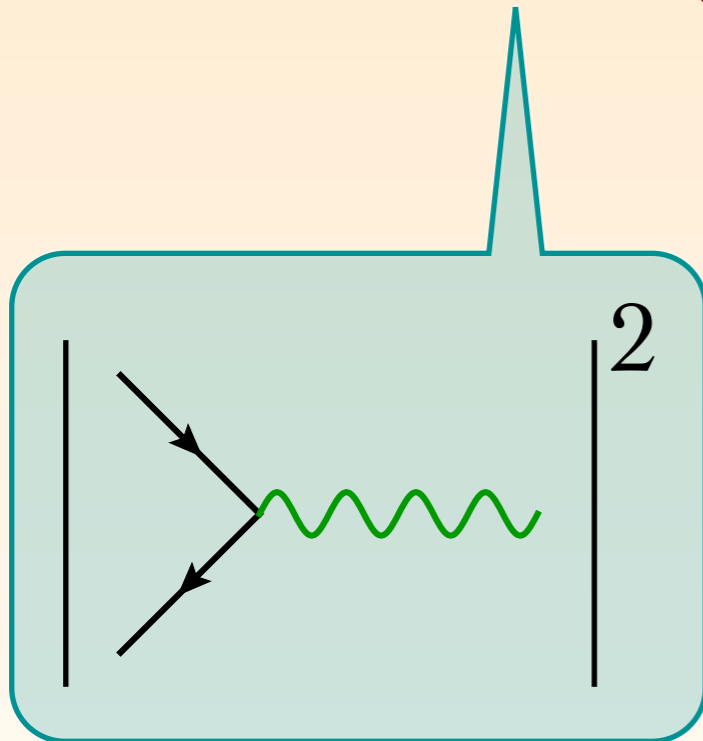
Bottlenecks
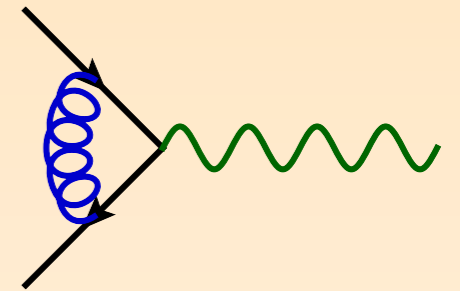
* Virtual amplitudes: how to compute the loops automatically in a reasonable amount of time

* How to deal with infra-red divergences: virtual corrections and real-emission corrections are separately divergent and only their sum is finite (for IR-safe observables) according to the KLN theorem

* How to match these processes to a parton shower without double counting

Loops

ONE-LOOP INTEGRAL

k_1

k_2

k_n

l

$l + k_1 = l + p_1$

D_0 D_1

k_3

D_{m-1} D_2 $l + k_1 + k_2 + k_3 = l + p_2$

k_4

D_3

$l + k_1 + \ldots + k_6 = l + p_3$

k_5

k_6
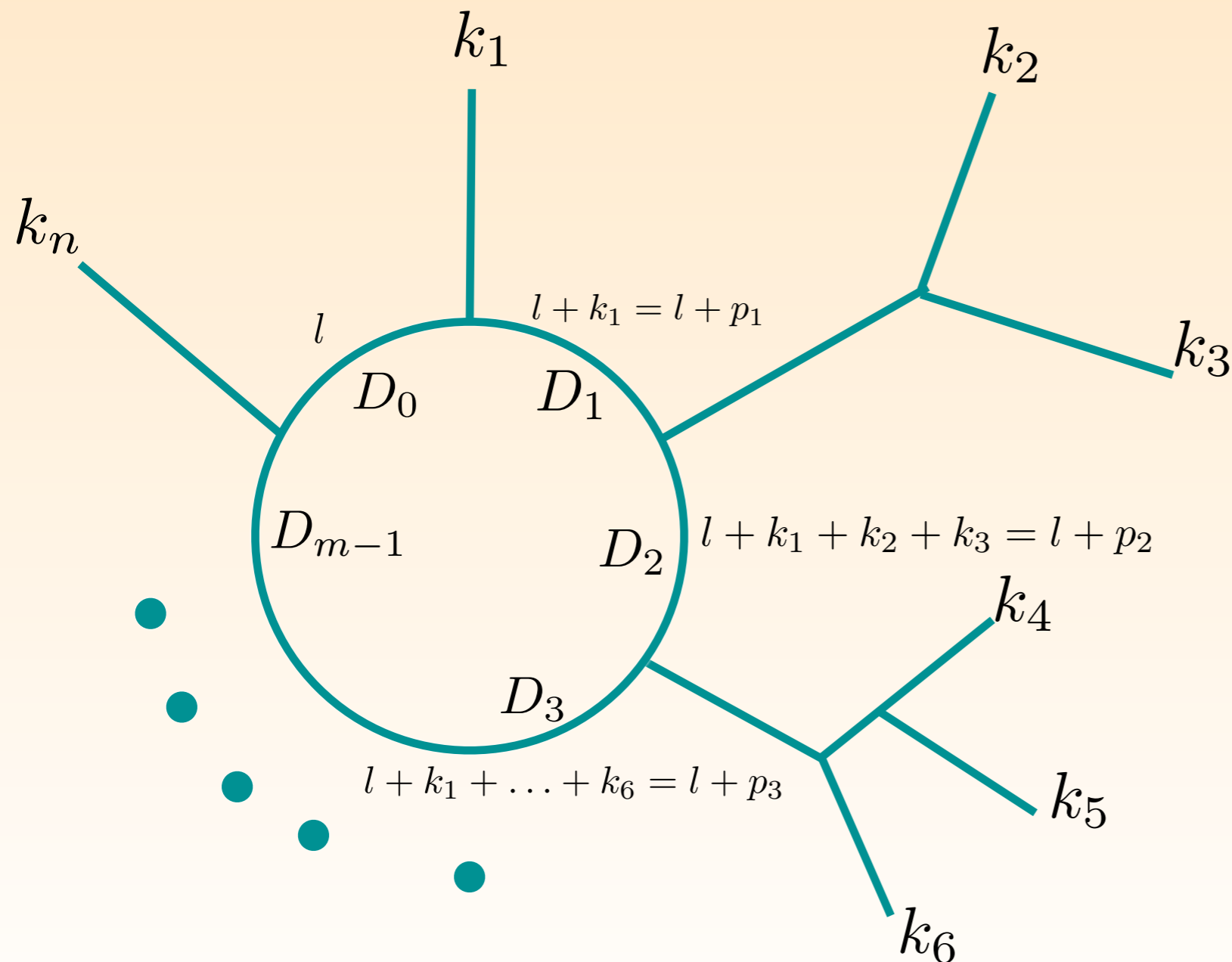
☀ Consider this *m*-point loop diagram with *n* external momenta

☀ The integral to compute is

$$\int d^d l \, \frac{N(l)}{D_0 D_1 D_2 \cdots D_{m-1}}$$

$$D_i = (l + p_i)^2 - m_i^2$$

Standard Approach

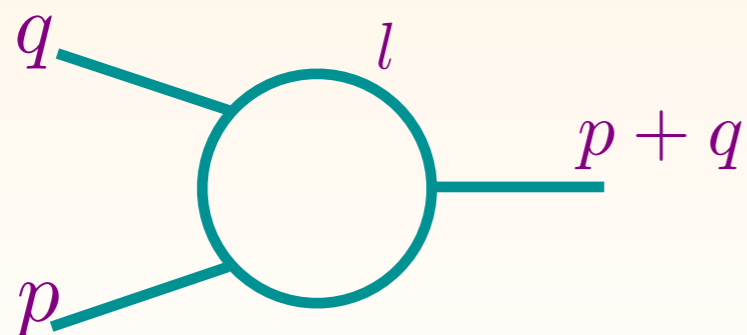※ Passarino-Veltman reduction:

$$\int d^d l \, \frac{N(l)}{D_0 D_1 D_2 \cdots D_{m-1}} \to \sum_i \text{coeff}_i \int d^d l \, \frac{1}{D_0 D_1 \cdots}$$

※ Reduce a general integral to "scalar integrals" by "completing the square"

※ Let's do an example:
Suppose we want to calculate this triangle integral

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)}$$

* ✺ The only independent four vectors are p^μ and q^μ. Therefore, the integral must be proportional to those. We can set-up a system of linear equations and try to solve for C_1 and C_2

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)} = \left(\begin{array}{cc} p^\mu & q^\mu \end{array} \right) \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right)$$

* ✺ We can solve for C_1 and C_2 by contracting with p and q

$$\left(\begin{array}{c} R_1 \\ R_2 \end{array} \right) = \left(\begin{array}{c} [2l \cdot p] \\ [2l \cdot q] \end{array} \right) = G \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right) \equiv \left(\begin{array}{cc} 2p \cdot p & 2p \cdot q \\ 2p \cdot q & 2q \cdot q \end{array} \right) \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right)$$

where $[2l \cdot p] = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot p}{l^2 (l+p)^2 (l+q)^2}$ (For simplicity, the masses are neglected here)

* ✺ By expressing $2l.p$ and $2l.q$ as a sum of denominators we can express R_1 and R_2 as a sum of simpler integrals, e.g.

$$R_1 = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot p}{l^2 (l+p)^2 (l+q)^2} = \int \frac{d^n l}{(2\pi)^n} \frac{(l+p)^2 - l^2 - p^2}{l^2 (l+p)^2 (l+q)^2}$$

$$= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+q)^2} - \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p)^2 (l+q)^2} - p^2 \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2 (l+q)^2}$$

⁂ And similarly for R_2

$$R_2 = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot q}{l^2 (l+p)^2 (l+q)^2} = \int \frac{d^n l}{(2\pi)^n} \frac{(l+q)^2 - l^2 - q^2}{l^2 (l+p)^2 (l+q)^2}$$

$$= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2} - \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p)^2 (l+q)^2} - q^2 \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2 (l+q)^2}$$

⁂ Now we can solve the equation

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l \cdot p] \\ [2l \cdot q] \end{pmatrix} = G \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p \cdot p & 2p \cdot q \\ 2p \cdot q & 2q \cdot q \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

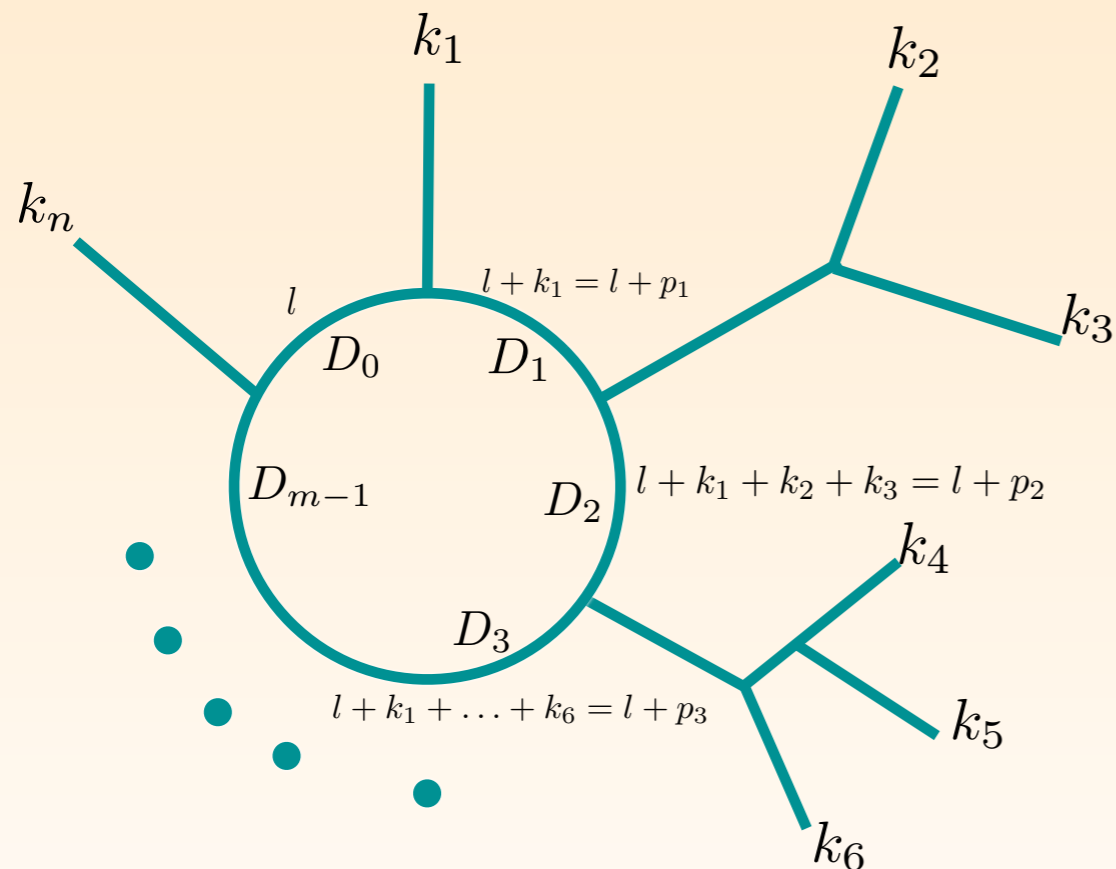by inverting the "Gram" matrix G

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = G^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

and we have expressed our original integral

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

in terms of known, simpler integrals and we are done!

Higher point integrals

- For loop integrals with many legs, the reduction to scalar integrals can still be performed

- Only up to 4-point scalar integrals are needed (in 4 dimensions)!

- The proof is beyond the scope of these lectures (it is straight forward by using the Van Neerven-Vermaseren basis for the loop momentum); it is related to the fact that in 4 dimensions only four 4-vectors can be linearly independent

Basis of scalar integrals

$$\mathcal{M}^{\text{1-loop}} = \sum_{i_0 < i_1 < i_2 < i_3} d_{i_0 i_1 i_2 i_3} \text{Box}_{i_0 i_1 i_2 i_3}$$

$$+ \sum_{i_0 < i_1 < i_2} c_{i_0 i_1 i_2} \text{Triangle}_{i_0 i_1 i_2}$$

$$+ \sum_{i_0 < i_1} b_{i_0 i_1} \text{Bubble}_{i_0 i_1}$$

$$+ \sum_{i_0} a_{i_0} \text{Tadpole}_{i_0}$$

$$+ R + \mathcal{O}(\epsilon)$$

✳ The a, b, c, d and R coefficients depend only on external parameters and momenta

$$D_i = (l + p_i)^2 - m_i^2$$

$$\text{Tadpole}_{i_0} = \int d^d l \frac{1}{D_{i_0}}$$

$$\text{Bubble}_{i_0 i_1} = \int d^d l \frac{1}{D_{i_0} D_{i_1}}$$

$$\text{Triangle}_{i_0 i_1 i_2} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2}}$$

$$\text{Box}_{i_0 i_1 i_2 i_3} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}}$$

✳ All these scalar integrals are known and available in computer libraries (FF [v. Oldenborgh], QCDLoop [Ellis, Zanderighi], OneLOop [v. Hameren])

About the R term

- In our example the decomposition to scalar integrals was "exact", i.e. there were no left-over terms.

- This is true for most integrals. Only if the rank of the integral is

$$r \geq \max\Big\{(N-1), 2)\Big\}$$

  there are some extra contributions which are called "Rational terms" that are not proportional to a scalar integral

- They are of UV origin and come from the ϵ (dimensional regulator) dependence of the integral times a scalar integral that is UV divergent

$$\text{Rational terms } \sim \epsilon B_0(p, m_1, m_2)$$

  (The Bubble integrals are the only UV divergent integrals)

- When taking the limit $\epsilon \to 0$, only the leading contribution remains, which are independent from the scalar integral itself

Automation

* Advantage:

  * The method above can be straight-forwardly generalized to any one-loop integral (appearing in a renormalizable theory)

* Disadvantage:

  * For relatively simple processes, the number of terms already explodes (several 100 MB of code is no exception for the matrix elements of a $2 \rightarrow 3$ process); simplifications require hard work and are difficult to do in a general way

  * Does only work when the integrals are known analytically

One indicator of NLO progress

$pp \rightarrow W + 0$ jet	1978	Altarelli, Ellis, Martinelli
$pp \rightarrow W + 1$ jet	1989	Arnold, Ellis, Reno
$pp \rightarrow W + 2$ jets	2002	Campbell, Ellis
$pp \rightarrow W + 3$ jets	2009	BH+Sherpa
		Ellis, Melnikov, Zanderighi
$pp \rightarrow W + 4$ jets	2010	BH+Sherpa

Slide from Lance Dixon

Slide from L. Dixon

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