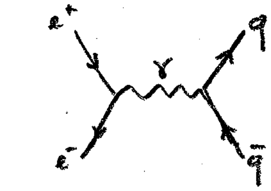
QCD in e^+e^-

* **Lowest order:** $e^+e^- \to \gamma \to q\bar{q}$ We neglect the masses.



electron: $p_1 \equiv \left(0, 0, \frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}\right)$

positron: $p_2 \equiv \left(0, 0, -\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}\right)$

- phase space

$$\int d\Phi_2 = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2)\,(2\pi)\delta(k_1^2)\,(2\pi)\delta(k_2^2)$$
$$= \frac{1}{(2\pi)^2}\int \frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}\,\delta^{(4)}(p_1+p_2-k_1-k_2)$$
$$= \frac{1}{16\pi^2}\int_0^\infty \frac{E_1^2\, dE_1}{E_1^2}\, d\cos(\theta)\, d\phi\; \delta(\sqrt{s}-2E_1)$$
$$= \frac{1}{16\pi}\int_{-1}^{1} d\cos(\theta) \qquad \Rightarrow \begin{cases} k_1 \equiv \frac{\sqrt{s}}{2}\left(\sin(\theta), 0, \cos(\theta), 1\right) \\ k_2 \equiv \frac{\sqrt{s}}{2}\left(-\sin(\theta), 0, -\cos(\theta), 1\right)\end{cases}$$

- Mandelstam variables:

$$s = (p_1+p_2)^2 = (k_1+k_2)^2 = 2p_1\cdot p_2 = 2k_1\cdot k_2$$
$$t = (p_1-k_1)^2 = (p_2-k_2)^2 = -2p_1\cdot k_1 = -2p_2\cdot k_2 = -\frac{s}{2}\left[1-\cos(\theta)\right]$$
$$u = (p_1-k_2)^2 = (p_2-k_1)^2 = -2p_1\cdot k_2 = -2p_2\cdot k_1 = -\frac{s}{2}\left[1+\cos(\theta)\right]$$

- Matrix element

$$\overline{\sum}|\mathcal{M}|^2 = \frac{1}{4}N_c\, e^4 e_q^2\; \mathrm{tr}(\not{p}_1\gamma_\mu \not{p}_2\gamma_\rho)\,\frac{g^{\mu\nu}g^{\rho\sigma}}{s^2}\;\mathrm{tr}(\not{k}_1\gamma_\nu\not{k}_2\gamma_\sigma)$$
$$= 64\pi^2\alpha_e^2 e_q^2 N_c \frac{1}{s^2}\left(p_1^\mu p_2^\rho + p_1^\rho p_2^\mu - p_1\cdot p_2\, g^{\mu\rho}\right)\left(k_{1\mu}k_{2\rho} + k_{1\rho}k_{2\mu} - k_1\cdot k_2\, g_{\mu\rho}\right)$$
$$= 64\pi^2\alpha_e^2 e_q^2 N_c \frac{1}{s^2}\left(2p_1\cdot k_1\, p_2\cdot k_2 + 2p_1\cdot k_2\, p_2\cdot k_1 + 4p_1\cdot p_2\, k_1\cdot k_2 - 4p_1\cdot p_2\, k_1\cdot k_2\right)$$
$$= 32\pi^2\alpha_e^2 e_q^2 N_c \frac{t^2+u^2}{s^2} \qquad t^2+u^2 = +\frac{s^2}{2}\left[1+\cos^2(\theta)\right]$$
$$= 16\pi^2\alpha_e^2 e_q^2 N_c \left[1+\cos^2(\theta)\right]$$

- Cross-section

$$d\sigma = \frac{1}{2s}|\mathcal{M}|^2 d\Phi_2 \Rightarrow \frac{d\sigma}{d\cos(\theta)} = \frac{1}{2s}\frac{1}{16\pi}\,16\pi^2\alpha_e^2 e_q^2 N_c\left[1+\cos^2(\theta)\right]$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\cos(\theta)} = e_q^2 N_c \frac{\pi\alpha_e^2}{2s}\left[1+\cos^2(\theta)\right] = e_q^2 N_c\, \pi\alpha_e^2\, \frac{t^2+u^2}{s^2}}$$

Integrating over $\cos(\theta)$, we get

$$\boxed{\sigma(e^+e^-\to q\bar{q}) = N_c\left(\sum_q e_q^2\right)\sigma_0} \qquad \boxed{\sigma_0 = \frac{4\pi\alpha_e^2}{3s}}$$

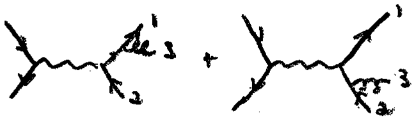
Thus, computing $R = \frac{\sigma(e^+e^-\to q\bar{q})}{\sigma(e^+e^-\to\mu^+\mu^-)}$ allows to test QCD for

- the N_c colours
- the $\neq$ quark flavours.

Note: There is also a $e^+e^-\to Z\to$ hadrons contribution. We do not discuss it for simplicity

* First order α_s corrections : $e^+e^- \to q\bar{q}g$

- **diagrams**

2 real emissions: 

3 virtual corrections: 

- 3-particle phase space:

$$\int d\Phi_3 = \prod_{i=1}^{3} \frac{d^3k_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2-k_3)$$

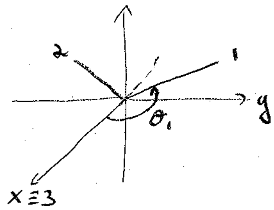
Introducing $\boxed{x_i = \frac{2E_i}{\sqrt{s}}}$ we have $\frac{d^3k_i}{(2\pi)^3 2E_i} = \frac{s}{8}\frac{1}{(2\pi)^3} x_i\, dx_i\, d\Omega_i$

The 6 angles can be set as follows : 3 Euler angles $\alpha \in [0,2\pi]$ (zxz convention)
$\beta \in [0,\pi]$
$\gamma \in [0,2\pi]$

such that

- k_3 is along the x axis
- k_1 is in the xy plane (with y>0)

+ $\theta_1 \equiv \theta_{13}$ in $[0,\pi]$, $\theta_2 \equiv \theta_{23}$ in $[0,\pi]$, ϕ_2 in $[0,2\pi]$



$$\Rightarrow \int d\Phi_3 = \frac{1}{(2\pi)^5}\left(\frac{s}{8}\right)^3 \frac{16}{s^2} \int d\alpha\, d\cos(\beta)\, d\gamma\, dx_1\, dx_2\, dx_3\, d\cos(\theta_1)\, d\cos(\theta_2)\, d\phi_2 \;\; \delta(2-x_1-x_2-x_3)\, x_1 x_2 x_3$$
$$\delta(x_2\sin(\theta_2)\sin(\phi_2))\,\delta(x_2\sin(\theta_2)\cos(\phi_2)+x_1\sin(\theta_1))\,\delta(x_3+x_2\cos(\theta_2)+x_1\cos(\theta_1))$$

We can remove θ_1, θ_2 & ϕ_2 using the $3\,\delta(\ldots)$. It gives

$$\boxed{\int d\Phi_3 = \frac{s}{32(2\pi)^5}\int dx_1\, dx_2\, d\alpha\, d\cos(\beta)\, d\gamma}$$

with

$$\cos(\theta_{13}) = -\frac{x_1^2+x_3^2-x_2^2}{2x_1x_3} \quad , \quad \cos(\theta_{23}) = -\frac{x_2^2+x_3^2-x_1^2}{2x_2x_3} \quad , \quad x_1+x_2+x_3 = 2$$

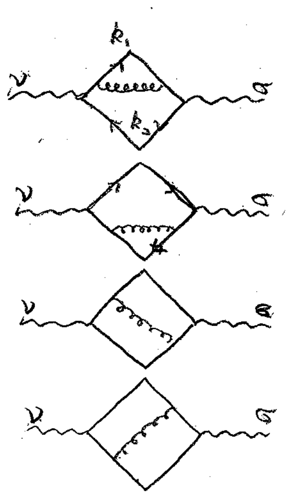
With our convention for the Euler angles, we can show that

$$p_{\substack{1\\2}}\cdot k_1 = \frac{s}{4}\cdot x_1\left(1 \mp \sin(\beta)\sin(\gamma+\theta_1)\right)$$
$$p_{\substack{1\\2}}\cdot k_2 = \frac{s}{4}\, x_2\left(1 \mp \sin(\beta)\sin(\gamma-\theta_2)\right)$$
$$p_{\substack{1\\2}}\cdot k_3 = \frac{s}{4}\, x_3\left(1 \mp \sin(\beta)\sin(\gamma)\right)$$

• Matrix element

$$\bar{\sum}|\mathcal{M}|^2 = \frac{1}{4} e^4 e_q^2 g^2 \sum_{ABa}\left(t^a_{BA}\, t^a_{AB}\right)\cdot(-)\,\mathrm{tr}\left(\not p_1 \gamma_\mu \not p_2 \gamma_\rho\right) \frac{g^{\mu\nu} g^{\rho\sigma}}{s^2}$$

$$\left[\frac{(-)^2}{(k_1+k_3)^4}\,\mathrm{tr}\left(\not k_1 \gamma_\alpha (\not k_1+\not k_3)\gamma_\nu \not k_2 \gamma_\sigma (\not k_1+\not k_3)\gamma^\alpha\right)\right.$$
$$+\frac{(-)^2}{(k_2+k_3)^4}\,\mathrm{tr}\left(\not k_2 \gamma_\alpha (\not k_2+\not k_3)\gamma_\sigma \not k_1 \gamma_\nu (\not k_2+\not k_3)\gamma^\alpha\right)$$
$$+\frac{(-)^3}{(k_1+k_3)^2(k_2+k_3)^2}\,\mathrm{tr}\left(\not k_1 \gamma^\alpha (\not k_1+\not k_3)\gamma_\nu \not k_2 \gamma_\alpha (\not k_2+\not k_3)\gamma_\sigma\right)$$
$$\left.+\frac{(-)^3}{(k_1+k_3)^2(k_2+k_3)^2}\,\mathrm{tr}\left(\not k_1 \gamma_\nu (\not k_2+\not k_3)\gamma_\alpha \not k_2 \gamma_\sigma (\not k_1+\not k_3)\gamma^\alpha\right)\right]$$



in the Feynman gauge.

This expression can be written as $\left(\sum_{ABa} t^a_{BA} t^a_{AB} = C_F N_c = N_c \cdot \frac{N_c^2-1}{2N_c}\right)$

$$\bar{\sum}|\mathcal{M}|^2 = \frac{(4\pi)^3}{s^2}\,\alpha_s \alpha_e^2\, C_F N_c \left(A_{11} + A_{22} - A_{12}\right)$$

with A_{11}, A_{22} and A_{12} corresponding, in the $[\ldots]$ in the above expression, to the contrib from the gluon emitted from the quark (A_{11}), the antiquark (A_{22}) and the interference term (A_{12})

Simplifying the traces, we find

$$A_{11} = \frac{-1}{(k_1+k_3)^4}\left[p_1^\nu p_2^\sigma + p_1^\sigma p_2^\nu - g^{\nu\sigma} p_1\cdot p_2\right] \mathrm{tr}\left(\not k_1 \gamma_\alpha (\not k_1+\not k_3)\gamma_\nu \not k_2 \gamma_\sigma (\not k_1+\not k_3)\gamma^\alpha\right)$$
$$= \frac{8}{(k_1\cdot k_3)}\left(p_1\cdot k_2\; p_2\cdot k_3 + p_1\cdot k_3\; p_2\cdot k_2\right)$$

$$A_{22} = \frac{-1}{(k_2+k_3)^4}\left[p_1^\nu p_2^\sigma + p_1^\sigma p_2^\nu - g^{\nu\sigma} p_1\cdot p_2\right] \mathrm{tr}\left(\not k_2 \gamma_\alpha (\not k_2+\not k_3)\gamma_\sigma \not k_1 \gamma_\nu (\not k_2+\not k_3)\gamma^\alpha\right)$$
$$= \frac{8}{(k_2\cdot k_3)}\left(p_1\cdot k_1\; p_2\cdot k_3 + p_1\cdot k_3\; p_2\cdot k_1\right)$$

$$A_{12} = \frac{-1}{(k_1+k_3)^2(k_2+k_3)^2}\left[p_1^\nu p_2^\sigma + p_1^\sigma p_2^\nu - g^{\nu\sigma} p_1\cdot p_2\right]\left[\mathrm{tr}\left(\not k_1 \gamma^\alpha (\not k_1+\not k_3)\gamma_\nu \not k_2 \gamma_\alpha (\not k_2+\not k_3)\gamma_\sigma\right)\right.$$
$$\left.+\,\mathrm{tr}\left(\not k_1 \gamma_\nu (\not k_2+\not k_3)\gamma_\alpha \not k_2 \gamma_\sigma (\not k_1+\not k_3)\gamma^\alpha\right)\right]$$

$$= \frac{8}{(k_1\cdot k_3)(k_2\cdot k_3)}\Big[2\, p_1\cdot k_1\; p_2\cdot k_1\; k_2\cdot k_3 + 2\, p_1\cdot k_2\; p_2\cdot k_2\; k_1\cdot k_3 - \left(\tfrac{s}{2} + k_1\cdot k_2\right)\left(p_1\cdot k_1\; p_2\cdot k_2 + p_1\cdot k_2\; p_2\cdot k_1\right)$$
$$- p_1\cdot k_1\; k_1\cdot k_2\; p_2\cdot k_3 - p_2\cdot k_2\; k_1\cdot k_2\; p_1\cdot k_3 - p_1\cdot k_2\; k_1\cdot k_2\; p_2\cdot k_3 - p_2\cdot k_1\; k_1\cdot k_2\; p_1\cdot k_3\Big]$$

Summing the 3 contributions, we get after straightforward algebra (*)

$$\boxed{\bar{\sum}|\mathcal{M}|^2 = 4(4\pi)^3 \alpha_e^2 \alpha_s\, C_F N_c\, \frac{(p_1\cdot k_1)^2 + (p_1\cdot k_2)^2 + (p_2\cdot k_1)^2 + (p_2\cdot k_2)^2}{s\,(k_1\cdot k_3)\,(k_2\cdot k_3)}}$$

(*) We have used i) $k_3 = p_1 + p_2 - k_1 - k_2$

ii) $k_3^2 = 0 \Rightarrow p_1\cdot p_2 + k_1\cdot k_2 = p_1\cdot k_1 + p_1\cdot k_2 + p_2\cdot k_1 + p_2\cdot k_2$

This result has several interesting consequences.
First, let us calculate the differential cross-section

$$d\sigma = \frac{1}{2s}\, d\Phi_3\, \bar{\Sigma}|\mathcal{M}|^2$$

We will integrate out the Euler angle and keep the x_1, x_2 dependence.

$$\frac{d^2\sigma}{dx_1\,dx_2} = \frac{1}{2s}\,\frac{s}{32(2\pi)^5}\,4(4\pi)^3\alpha_e^2\alpha_s C_F N_c \int d\alpha\, d\cos(\beta)\, d\gamma\, \frac{(p_1\cdot k_1)^2+(p_1\cdot k_2)^2+(p_2\cdot k_1)^2+(p_2\cdot k_2)^2}{s\,(k_1\cdot k_3)(k_2\cdot k_3)}$$

Using the abovementionned expressions of the scalar products in terms of the Euler angles, we have

$$\int d\alpha\, d\cos(\beta)\, d\gamma\, (p_1\cdot k_1)^2+(p_2\cdot k_1)^2 = \left(\frac{s}{4}\right)^2\cdot x_1^2\cdot \int d\alpha\, d\cos(\beta)\, d\gamma\; 2\left[1+\sin^2(\beta)\sin^2(\gamma+\theta_1)\right]$$

$$= \left(\frac{s}{4}\right)^2\cdot x_1^2\cdot (2\pi)^2 \int d\cos(\beta)\; \underbrace{2+\sin^2(\beta)}_{3-\cos^2(\beta)}$$

$$= \left(\frac{s}{4}\right)^2 x_1^2\,(2\pi)^2\, 2\left(3-\frac{1}{3}\right)$$

$$= \frac{1}{3}\, s^2 x_1^2\,(2\pi)^2$$

And similarly for $(p_2\cdot k_2)^2+(p_1\cdot k_2)^2 = \frac{1}{3}s^2(2\pi)^2x_2^2$. Hence,

$$\frac{d^2\sigma}{dx_1\,dx_2} = \frac{1}{2s}\,\frac{1}{(2\pi)^2}\,\alpha_e^2\alpha_s C_F N_c\,\frac{1}{3}\, s^2\, \frac{x_1^2+x_2^2}{(k_1\cdot k_3)(k_2\cdot k_3)}\,(2\pi)^2$$

Using

$$(k_1\cdot k_3)(k_2\cdot k_3) = \frac{s^2}{16}\cdot x_1x_2x_3^2\cdot\left[1-\cos(\theta_{13})\right]\left[1-\cos(\theta_{23})\right]$$

with

$$1-\cos(\theta_{13}) = 1+\frac{x_1^2+x_3^2-x_2^2}{2x_1x_3} = \frac{(x_1+x_3)^2-x_2^2}{2x_1x_3} = \frac{(x_1+x_2+x_3)(x_1+x_3-x_2)}{2x_1x_3} = \frac{2(1-x_2)}{x_1x_3}$$

$$\Rightarrow (k_1\cdot k_3)(k_2\cdot k_3) = \frac{s^2}{4}(1-x_1)(1-x_2)$$

We finally have

$$\frac{d^2\sigma}{dx_1\,dx_2} = \frac{2}{3s}\, e_q^2 N_c\,\alpha_s C_F\cdot \alpha_e^2\cdot \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

Factorising out the Born-level $e^+e^-\to\mu^+\mu^-$ cross-section $\sigma_0 = \frac{4\pi\alpha_e^2}{3s}$, we obtain

$$\boxed{\frac{d^2\sigma}{dx_1\,dx_2} = e_q^2 N_c\,\sigma_0\,\frac{\alpha_s C_F}{2\pi}\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}} \qquad \text{with} \quad 0\le x_1, x_2, x_3\le 1$$

From which most of the forthcoming physical discussion is based.

* Divergences

$\frac{d^2\sigma}{dx_1 dx_2}$ has 2 divergences when x_1 and/or x_2 go to 1.

Since we have

$$1-x_2 = \frac{1}{2} x_1 x_3 \left[1-\cos(\theta_{13})\right]$$
$$1-x_1 = \frac{1}{2} x_2 x_3 \left[1-\cos(\theta_{23})\right]$$

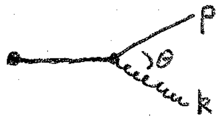
we see that we have 2 types of divergence (both infrared):

(i) $x_3 \to 0$: this means that the energy of the emitted gluon is small ($E_g \to 0$)
It is the SOFT DIVERGENCE

(ii) $\theta_{13} \to 0$ (or $\theta_{23} \to 0$): In this limit the gluon becomes collinear with the quark that emits it
It is the COLLINEAR DIVERGENCE.

(Note that even though $x_1 \to 1$ when $x_2 \to 0$, there is no divergence when $x_2 \to 0$ as, in that case, the phase-space for x_1 ($1-x_2 \le x_1 \le 1$) shrinks to 0 so that the integral is finite).

These soft and collinear divergences are fundamental situations in QCD. They are related to the fact that, when an emitted gluon becomes soft or collinear (to the emitter), one propagator goes on shell:



if $p^2 = 0$, $\frac{1}{(p+k)^2} \to 0$ when $\begin{cases} \text{(i) } E_k \to 0 \quad \text{(soft limit)} \\ \text{(ii) } \theta \to 0 \text{ (or } k \propto p) \quad \text{(collinear limit)} \end{cases}$

This is a very general property of QCD that the emission of gluons, from a quark, an antiquark or a gluon, will have a soft and a collinear divergence (both logarithmic).(*)

In practice, these divergences are cancelled by virtual corrections (here proportional to $\delta(1-x_1)\delta(1-x_2)$) so that the total cross-section $\sigma(q\bar{q} \to \gamma \to \text{hadrons})$ is finite at order α_s.

To briefly summarize how that works, we use dimensional regularisation and go to $d = 4-2\varepsilon$ dimensions. The real emission cross-section becomes

$$\frac{d^2\sigma}{dx_1 dx_2} = e_q^2 N_c \sigma_0 \frac{\alpha_s C_F}{2\pi} T(\varepsilon) \frac{x_1^2 + x_2^2 - \varepsilon(2-x_1-x_2)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1-\varepsilon}}$$

with

$$T(\varepsilon) = \frac{3(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} = 1 + \left(\frac{2}{3} - 2\gamma_E\right)\varepsilon + \mathcal{O}(\varepsilon^2)$$

(*) As we shall explicitly prove later, the probability to radiate collinear & soft gluons goes as

$\alpha_s \frac{d\theta}{\theta}$ for collinear gluons (with θ the splitting angle)

$\alpha_s \frac{dz}{z}$ for soft gluons (with z the fraction of the parent momentum carried by the gluon)

Integrating over x_1 & x_2, we have, for the real contribution

$$\sigma^{q\bar{q}g}_{real} = e_q^2 N_c \sigma_0 \frac{\alpha_s C_F}{2\pi} T(\varepsilon)\left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + O(\varepsilon)\right]$$

The virtual correction can also be computed in $d = 4-2\varepsilon$ dimensions and is

$$\sigma^{q\bar{q}g}_{virt.} = e_q^2 N_c \sigma_0 \frac{\alpha_s C_F}{2\pi} T(\varepsilon)\left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + O(\varepsilon)\right]$$

Hence, it is obvious that the divergences (the $1/\varepsilon^2$ & $1/\varepsilon$ terms) cancel between real and virtual contributions, to give

$$\sigma^{q\bar{q}g}_{O(\alpha_s)} = e_q^2 N_c \sigma_0 \frac{3\alpha_s C_F}{4\pi} = e_q^2 N_c \sigma_0 \frac{\alpha_s}{\pi}$$

Which gives (using $C_F = 4/3$)

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c\left(\sum_q e_q^2\right)\left[1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right]$$

At higher orders, we also have a dependence on the renormalisation scheme and hence on the choice of a renormalisation scale. At first order in α_s, one usually take $\alpha_s(\mu) \equiv \alpha_s(\sqrt{s})$.

This cancellation between real and virtual parts of the amplitude is a fundamental property of the perturbation theory. This cancellation actually happens to all orders in the perturbative expansion. (see the Bloch-Nordsieck and Kinoshita-Lee-Nauenberg theorems).

For more exclusive quantities, we can give a criterium for this cancellation between real and virtual gluons to occur. If we write the expansion of an observable $\mathcal{O}$ as

$$\mathcal{O} = \sum_{n=2}^{\infty} \int d\Phi_n(k_1, \dots, k_n) \frac{d\sigma}{d\Phi_n}(k_1, \dots, k_n) \cdot \mathcal{O}_n(k_1^\mu, \dots, k_n^\mu)$$

the condition are

(i) infrared safety:

$$\lim_{k_n^\mu \to 0} \mathcal{O}_n(k_1^\mu, \dots, k_n^\mu) = \mathcal{O}_{n-1}(k_1^\mu, \dots, k_{n-1}^\mu)$$

(ii) collinear safety

$$\mathcal{O}_n(k_1^\mu, \dots, \lambda k_{n-1}^\mu, (1-\lambda) k_{n-1}^\mu) = \mathcal{O}_{n-1}(k_1^\mu, \dots, k_{n-1}^\mu) \qquad (0 \leq \lambda \leq 1)$$

When those conditions are satisfied the observable is said INFRARED AND COLLINEAR SAFE

We shall explicitly discuss examples on IRC safe quantities later (jets & event shapes) but before, let us show different interesting reformulations of the $e^+e^- \to q\bar{q}g$ matrix element.

• collinear branching

instead of expressing the differential x-sect° in terms of x_1 and x_2, let us use

$$z \equiv x_3 \qquad \text{and} \qquad \theta \equiv \theta_{13}.$$

If $t = 1-\cos(\theta_{13}) = 1-\cos(\theta)$, we have

$$2(1-x_2) = x_1 x_3 (1-\cos(\theta_{13})) \Rightarrow 2-2x_2 = zt(2-x_2-z) \Rightarrow 2-z(2-z)t = (2-zt)x_2$$

$$\Rightarrow x_2 = \frac{2-z(2-z)t}{2-zt}, \qquad 1-x_2 = \frac{z(1-z)t}{2-zt}$$

$$x_1 = 2-x_2-x_3 = \frac{(2-z)(2-zt)-2+z(2-z)t}{2-zt} \Rightarrow x_1 = \frac{2(1-z)}{2-zt}, \qquad 1-x_1 = \frac{z(2-t)}{2-zt}$$

We thus have

$$dx_1\,dx_2 = \begin{vmatrix} \frac{2z(1-z)}{(2-zt)^2} & \frac{-2}{2-zt} + \frac{2(1-z)t}{(2-zt)^2} \\ \frac{-z(2-z)}{2-zt} + \frac{2z-z^2(2-z)t}{(2-zt)^2} & \frac{-2(1-z)t}{2-zt} + \frac{2t-z(2-z)t^2}{(2-zt)^2} \end{vmatrix} dz\,dt$$

$$= \frac{1}{(2-zt)^4} \begin{vmatrix} 2z(1-z) & 2t-4 \\ -2z(1-z) & -z^2t^2-2(1-2z)t \end{vmatrix} dz\,dt$$

$$= \frac{2z(1-z)}{(2-zt)^4}\left(4-2t+z^2t^2+2t-4zt\right) dz\,dt$$

$$= \frac{2z(1-z)}{(2-zt)^2}\, dz\,dt$$

and

$$\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} = \frac{4[1+(1-z)^2]-4z(2-z)t+z^2(2-z)^2t^2}{z^2(1-z)\,t(2-t)}$$

Thus

$$d\sigma = (e_q^2 N_c \sigma_0)\frac{\alpha_s C_F}{2\pi}\,\frac{4[1+(1-z)^2]-4z(2-z)t+z^2(2-z)^2t^2}{z\,t(2-t)(2-zt)^2}\,dz\,dt$$

It is interesting to consider the collinear limit, $\theta \ll 1$, where $t \simeq \frac{\theta^2}{2}$ and

$$d\sigma = (e_q^2 N_c \sigma_0)\frac{\alpha_s C_F}{2\pi}\,\frac{1+(1-z)^2}{z}\,dz\,\frac{d\theta^2}{\theta^2}.$$

In this expression, on top of the $(e_q^2 N_c \sigma_0)$ prefactor corresponding to the 0th order result, we have

(i) the logarithmic collinear divergence $\frac{d\theta^2}{\theta^2}$

(ii) a factor $\frac{\alpha_s}{2\pi} C_F \frac{1+(1-z)^2}{z} dz$. This correspond to the branching probability for one quark to split in another quark and a collinear gluon, with the quark and gluon carrying fractions $(1-z)$ and z of the parent quark's momentum.

This is known as the P_{gq} splitting function that we will meet again in DIS

Note also that this contains the $\frac{dz}{z}$ soft divergence.

• soft gluon emission

Let us show how the one-gluon emission simplifies if we assume that the emitted gluon is soft.
We start from the $q\bar{q}g$ matrix element

$$\overline{\sum}|\mathcal{M}|^2_{q\bar{q}g} = 32(2\pi)^3 \alpha_s \alpha_e^2 N_c C_F e_q^2 \cdot \frac{(p_1\cdot k_1)^2+(p_1\cdot k_2)^2+(p_2\cdot k_1)^2+(p_2\cdot k_2)^2}{s(k_1\cdot k_3)(k_2\cdot k_3)}$$

When k_3 is soft, we can factorise out the matrix element for $e^+e^- \to q\bar{q}$ at LO

$$\overline{\sum}|\mathcal{M}_0|^2_{q\bar{q}} = 4(2\pi)^2 \alpha_e^2 N_c e_q^2 \frac{(p_1\cdot k_1)^2+(p_1\cdot k_2)^2+(p_2\cdot k_1)^2+(p_2\cdot k_2)^2}{(k_1\cdot k_2)^2}$$

to obtain ($s \simeq 2k_1\cdot k_2$)

$$\overline{\sum}|\mathcal{M}_0|^2_{q\bar{q}g} = \left(\overline{\sum}|\mathcal{M}_0|^2_{q\bar{q}}\right) \cdot 8\pi\alpha_s C_F \cdot \frac{k_1\cdot k_2}{(k_1\cdot k_3)(k_2\cdot k_3)}$$

Since the 3-particle phase space can be written

$$d\Phi_3 = d\Phi_2 \frac{d^4k_3}{(2\pi)^4}(2\pi)\delta^+(k_3^2)$$

where we have used the fact that, in the soft limit, $\delta^{(4)}(p_1+p_2-k_1-k_2-k_3) \simeq \delta^{(4)}(p_1+p_2-k_1-k_2)$, we have

$$d\sigma_{q\bar{q}g} = (d\sigma_{q\bar{q}}) \cdot \frac{\alpha_s C_F}{\pi^2} \frac{k_1\cdot k_2}{(k_1\cdot k_3)(k_2\cdot k_3)} \delta(k_3^2)\, d^4k_3$$

The important point here is that the emission of the soft gluon k_3 can be factorised out of the "hard" process ($d\sigma_{q\bar{q}}$).

In general, to the emission of a soft gluon from a hard "quark dipole" k_1, k_2, corresponds a factor (gluon $\equiv k_3$)

$$\boxed{\frac{\alpha_s C_F}{\pi^2} \frac{(k_1\cdot k_3)}{(k_1\cdot k_3)(k_2\cdot k_3)} d^4k_3\, \delta(k_3)}$$

This is known as the antenna formula

Note: in the large-N_c approximation,

(i) $C_F = \frac{N_c^2-1}{2N_c} \to \frac{N_c}{2}$

(ii) the emission of a soft gluon from a gluon dipole k_1, k_2 is the same with the replacement

$$C_F \to C_A = N_c$$

Corresponding to the fact that, at large N_c, a gluon is "made" of 2 quarks

This expression is useful when it comes to resumming multiple soft-gluon radiation.

* jet shapes : an example of IRC safe observable

In e^+e^-, a series of variables (the EVENT SHAPE VARIABLE) have been introduce to characterise the "shape" of the event, i.e. to quantify if the output particles are distributed uniformly, in a plane or along axes. Those quantities are computable order by order in perturbation theory and the comparison of these predictions with experimental measurements provide great tests of QCD as a fundamental theory for strong interactions. They also allow for precision measurements like the determination of α_s as we shall illustrate below.

Let us introduce the THRUST as

$$T(k_1,\dots,k_n) = \max_{|\vec{u}|=1} \frac{\sum_{i=1}^{n} |\vec{k}_i\cdot\vec{u}|}{\sum_{i=1}^{n} |\vec{k}_i|}$$

The vector $\vec{u}$ that maximises T is called the THRUST AXIS. $(\vec{t})$

For a uniformly distributed event (i.e. spherically isotropic), we get $T\simeq 1/2$, while for a "pencil-like" event, we rather obtain $T\simeq 1$.

Other event shape variables can be introduced, like

the sphericity : $$S = \left(\frac{4}{\pi}\right)^2 \min_{|\vec{u}|=1}\left(\frac{\sum_i |\vec{k}_i\times\vec{u}\,|}{\sum_i |\vec{k}_i|}\right)^2$$

the major : $$M = \max_{|\vec{u}|=1,\,\vec{u}\cdot\vec{t}=0} \frac{\sum_i |\vec{k}_i\cdot\vec{u}|}{\sum_i |\vec{k}_i|}$$

but their treatment goes roughly as the thrust one (at least for what we are concerned about), so we shall concentrate on the thrust from now on.

The first property is that the thrust is IRC safe. This is straightforward to check as

(i) adding a soft particle, $k_{n+1}\to 0$, will not change $\sum|\vec{k}_i\cdot\vec{u}|$ and $\sum|\vec{k}_i|$, hence will not change T

(ii) for $0\le\lambda\le 1$, $|\lambda\vec{k}\cdot\vec{u}|+|(1-\lambda)\vec{k}\cdot\vec{u}| = |\vec{k}\cdot\vec{u}|$ and $|\lambda\vec{k}|+|(1-\lambda)\vec{k}|=|\vec{k}|$, hence a collinear splitting does not change T.

Then, in perturbative QCD, the thrust distribution $\frac{1}{\sigma}\frac{d\sigma}{dT}$. E.g. for the $O(\alpha_s)$ contribution, we can use $\frac{d\sigma^{q\bar{q}g}}{dx_1dx_2}$ computed earlier, together with the fact that, in the $q\bar{q}g$ case, $T=\max\{x_i\}$, to get

$$\frac{1}{\sigma}\frac{d\sigma}{T} = \frac{\alpha_s C_F}{2\pi}\left[\frac{2(2-3T+3T^2)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right) - \frac{3(2-T)(3T-2)}{1-T}\right]$$

Notes : (i) the log divergence when $T\to 1$ is directly coming from the soft & collinear divergences of the one-gluon emission

(ii) This allows for tests of QCD. It is a function of T, so it is more informative than the total cross section (which is just a number). For example, comparison with the LEP data allow to show that the gluon is a vector particle, not a scalar.

* jets

Because of the collinear singularity, one expects e^+e^- events to be jet-like, i.e., to have the form of a few bunches of collimated particles (coming from collinear splitting of a parent parton). We would thus like to introduce quantities (computable in perturbative QCD) that reflect this behaviour and are IRC safe.

Since at first order, we are thus likely to have 2 "jets", and situations with more than 2 are expected when we increase the order of the perturbation theory.

Since the limit between a collinear and "non-collinear" splitting is somehow arbitrary (especially if we go to higher orders in α_s, where even the notion of a parton becomes ambiguous), there is no unique definition of a "jet".

· Cone: The first attempt is due to Sterman & Weinberg (1977). They classified an event as being a 2-jet event, if at least a fraction $(1-\varepsilon)$ of the total energy is contained in 2 cones of opening angle $2R$. Each of the 2 cones then corresponds to a jet. Similarily, one can define n-jet events as events where at least a fraction $(1-\varepsilon)$ of the total energy is contained in n (disjoint) cones of opening angle $2R$, and not in $(n-1)$. (ε & R are fixed)

This definition is IRC safe as

(i) the parameter R ensures that, after a collinear splitting, the resulting particle will belong to the same cone, hence the event will be classified as if there were no splitting

(ii) the parameter ε ensures that particles with infinitely small energies can be radiated as they will not modify the fraction of energy in the "hard" particles.

As a consequence, we expect behaviours containing $\alpha_s \log(\varepsilon)$ and $\alpha_s \log(R)$ (both finite!) for small ε and R, due to the soft and collinear radiation probability $\alpha_s \frac{d\theta}{\theta} \frac{dz}{z}$.

Given ε and R, one can compute the probability to have 2 or 3 jets at first order in α_s (obviously higher multiplicities only start at higher orders). Note that, as usual for IRC-safe quantities, the computation can be done in 4 dimensions as it is free of singularities. For this precise example, one would compute f_3^{cone}, the fraction of 3-jet events which can be obtained from $\frac{d\sigma^{q\bar{q}g}}{dx_1\,dx_2}$ and is free of divergences. One then gets $f_2^{\text{cone}} = 1 - f_3^{\text{cone}}$ (the fraction of 2-jet events) directly (rather than computing the $q\bar{q}$ and $q\bar{q}g$ contributions that are separately infinite).

For this jet definition, the resulting f_2 & f_3 are rather complicated expressions. This is mostly due to the fact that the boundary between 2 and 3 jets is complicated:

$$\begin{cases} x_1, x_2, x_3 > \varepsilon \\ \theta_{12}, \theta_{13}, \theta_{23} > R \end{cases} \Leftrightarrow \begin{cases} x_i > \varepsilon \\ (2-\rho x_i)(2-\rho x_j) > 4-2\rho \end{cases} \quad \text{with } \rho = 1-\cos(R).$$

Hence, this "cone algorithm" is not well-suited for perturbative computations.

Note also that at higher orders/multiplicities, we additionally face the problem of cones having the tendency to overlap. We shall address that issue when considering jets at hadronic colliders.

• JADE

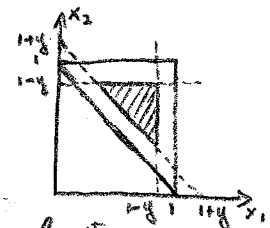
- Because of the complicated perturbative behaviour of the cone algorithm, other jet definitions have been introduced. The first important one we shall discuss here is the JADE algorithm. It is based on the observation that if one parton branches in 2 child partons, their invariant mass will be small in the collinear limit. The idea is therefore to search for pairs of particles having a small invariant mass and to recombine them as they are likely to come from a collinear splitting. More precisely, the algorithm follows this recipe:

1) Among all possible pairs, find the one which minimises
$$m_{ij}^2 = (k_i + k_j)^2 = 2E_i E_j (1-\cos(\theta_{ij}))$$
2) Recombine those 2 particles in a single one
3) Repeat while the minimal m_{ij}^2 is smaller than $y_{cut} \cdot s$.

which has y_{cut} as a parameter.

- Since a soft and collinear branching will correspond to $m_{ij}^2 \to 0$, leading to immediate recombination, the JADE algorithm is IRC safe.
- Again, at 1st order in α_s, the 3-jet probability is given by a finite integration of $\frac{d\sigma^{q\bar{q}g}}{dx_1 dx_2}$.

Since
$$m_{ij}^2 = \frac{s}{2} x_i x_j (1-\cos(\theta_{ij})) = \frac{s}{4}\left[(x_i + x_j)^2 - x_k^2\right] = s(1-x_k)$$
The event will be classified as 3-jet when
$$1 - x_i > y \quad \text{or} \quad \begin{array}{l} x_1 < 1-y \\ x_2 < 1-y \\ x_1 + x_2 > 1+y \end{array}$$



NB $\left.\begin{array}{l} x_2 = 1-y \\ x_1 + x_2 = 1+y \end{array}\right\} \Rightarrow x_1 = 2y$

i.e. in a region free of soft and collinear singularities.
We thus get
$$f_3^{JADE} = \frac{1}{\sigma}\int_{2y}^{1-y} dx_1 \int_{1+y-x_1}^{1-y} dx_2 \, \frac{d\sigma^{q\bar{q}g}}{dx_1 dx_2}$$
$$= \frac{\alpha_s C_F}{2\pi}\int_{2y}^{1-y} dx_1 \int_{1+y-x_1}^{1-y} dx_2 \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$
$$= \frac{\alpha_s C_F}{2\pi}\left[3(1-2y)\log\left(\frac{y}{1-2y}\right) + 2\log^2\left(\frac{y}{1-y}\right) + 4 Li_2\left(\frac{y}{1-y}\right) - \frac{\pi^2}{3} + \frac{5-12y-9y^2}{2}\right]$$

Note the $\alpha_s \log^2(y)$ behaviour for small values of y reminiscent of the soft & collinear behaviour. At order α_s^n, we would have terms $\propto \alpha_s^n \log^{2n}(y)$, so the perturbative series breaks down. Hence, unless a specific resummation of those terms is carried, this QCD prediction has to be taken with care at small y.

- This expression allows for QCD tests, the most obvious being a direct comparison of f_n^{JADE} at fixed s, as a function of y_{cut} (see figure for a comparison with LEP OPAL data at $O(\alpha_s^2)$).
Then, studying the s dependence (of f_n at fixed y_{cut}, or of $\langle f_n \rangle_{y_{cut}}$), one can directly study the running of α_s and determine $\alpha_s(M_Z)$. Not that at fixed y, and $O(\alpha_s)$, f_3 is directly proportional to α_s! (see figure for an $O(\alpha_s^2)$ extraction of $\alpha_s(M_Z)$ from $f_3(0.08)$)

- k_t algorithm (Durham algorithm)
  - Even if the JADE algorithm gives solid results, its behaviour (starting at $O(\alpha_s^2)$) presents some unwanted features. The best illustration is the following situation with 2 soft gluons (at small y)

can be classified as a 3-jet event after the merging of the 2 soft gluons, while, naively, this should be a 2-jet event. Because of this type of events, the resummation of the $\alpha_s^n \log^{2n}(y)$ at small y cannot be performed.

  - Hence, new algorithm have been introduced. They are based on the same "successive recombination" procedure as JADE but with a different distance measure. The most well-known is the k_t algorithm for which

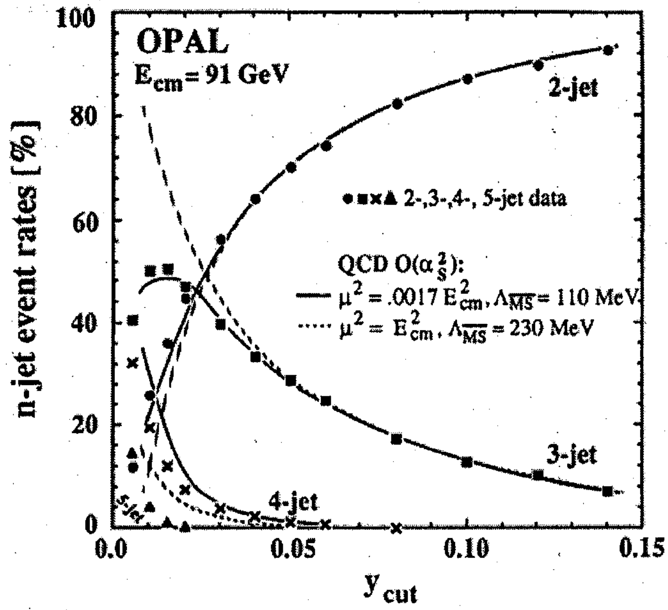
$$d_{ij}^{k_t} = 2\min(E_i^2, E_j^2)\,[1-\cos(\theta_{ij})]$$

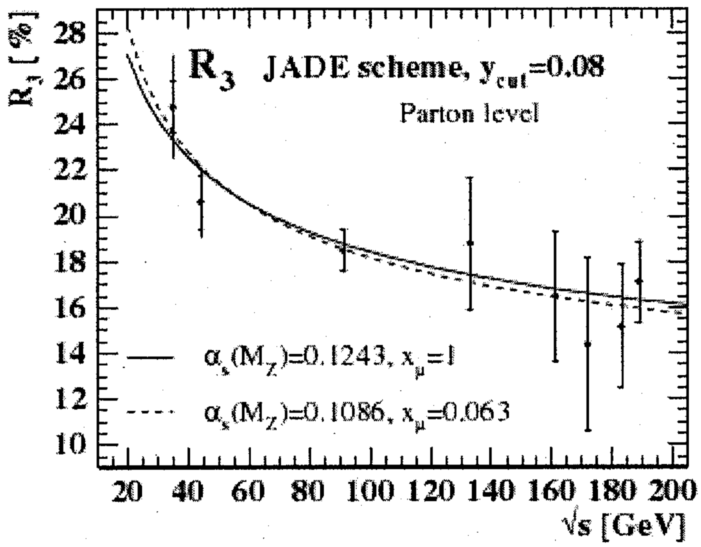
  - With this new measure, the above event is classified as 2 jets.
As for the previous case, the algorithm is IRC-safe and one can compute the jet rates f_n in perturbation theory. On top of that, the ressummation of the dominant log terms at small y can be performed, which allows for better comparisons with experiments
  - Another often-used algorithm is the Cambridge/Aachen algorithm. The distance measure is

$$d_{ij}^{C/A} = 2\,[1-\cos(\theta_{ij})]$$

i.e. simply the geometric distance. Note that, in this case, the k_t distance is used as a stopping criterion (i.e. the 2 particles with the smallest $d_{ij}^{C/A}$ are merged, one stops when $d_{ij}^{k_t} > y_{cut}\, s$).
  - An extraction of $\alpha_s(M_Z)$ at $O(\alpha_s^2)$ has been performed using the k_t and C/A algorithms (see figure).

Comparison of the jet rates (as a function of y_{cut}) with QCD predictions at order α_s^2.
(obtained with the JADE algorithm)





Variation with energy of the fraction of 3-jet events (at fixed y_{cut} = 0.08)
Compared with 2nd order QCD prediction.
(JADE algorithm)

Running of the coupling as extracted from QCD fits to the dijet rates. $Q = \sqrt{s}$
(k_t & Cambridge/Aachen algorithms).