QCD REDUX

PART I

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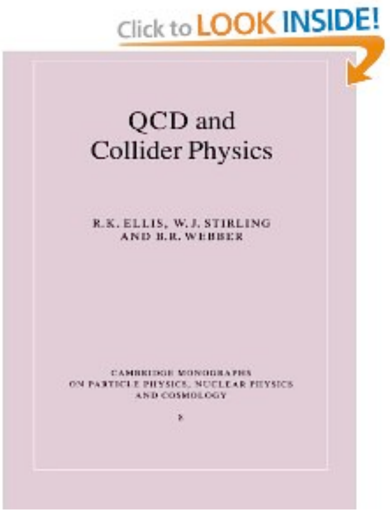
Minimal References

- Ellis, Stirling and Webber: The Pink Book



- Excellent lectures on the archive
by M. Mangano, P. Nason,
and more recently by G. Salam, P. Skands.

QCD : THE FUNDAMENTALS

1. QCD is a good theory for strong interactions: facts
2. From QED to QCD: the importance of color
3. Renormalization group and asymptotic freedom

Strong interactions

Strong interactions are characterized at moderate energies by a single* dimensionful scale, Λ_S , of few hundreds of MeV:

$$\sigma_h \cong 1/\Lambda_s^2 \cong 10 \text{ mb}$$
$$\Gamma_h \cong \Lambda_s$$
$$R \cong 1/\Lambda_s \cong 1 \text{ fm}$$

No hint to the presence of a small parameter! Very hard to understand and many attempts...

*neglecting quark masses..!!!

Strong interactions

Nowadays we have a satisfactory model of strong interactions based on a non-abelian gauge theory, i.e.. Quantum Chromo Dynamics.

Why is QCD a good theory?

1. Hadron spectrum
2. Scaling
3. QCD: a consistent QFT
4. Low energy symmetries
5. MUCH more....

Hadron spectrum

- Hadrons are made up of spin 1/2 quarks, of different flavors (d,u,s,c,b,[t])
- Each flavor comes in three colors, thus quarks carry a flavor and and color index

$$\psi_i^{(f)}$$

- The global SU(3) symmetry acting on color is exact:

$$\psi_i \to \sum_k U_{ik} \psi_k$$

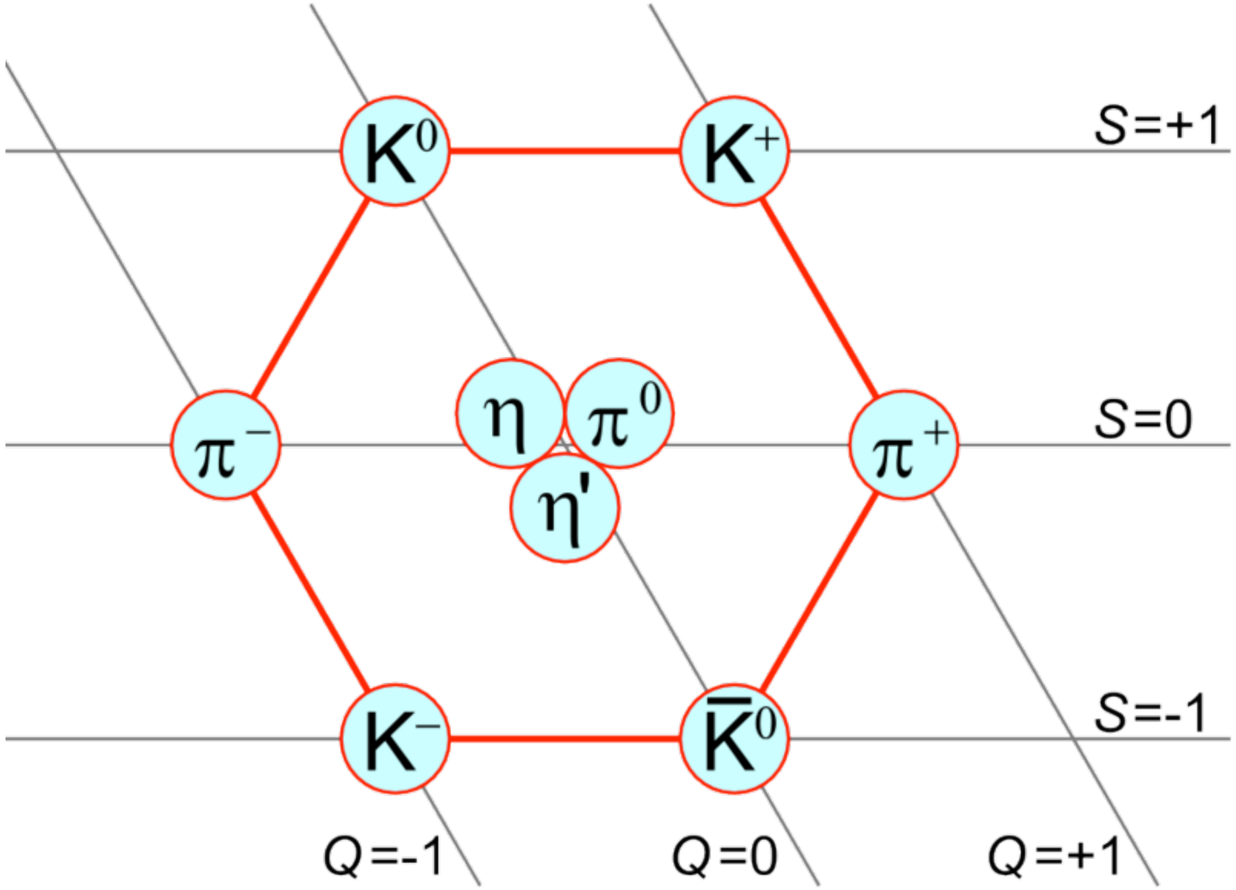
$$\sum_k \psi_k^* \psi_k \longleftarrow \textbf{Mesons}$$

$$\sum_{ijk} \epsilon^{ijk} \psi_i \psi_j \psi_k \longleftarrow \textbf{Baryons}$$

Hadron spectrum

Note that physical states are classified in multiplets of the FLAVOR SU(3)$_f$ group!

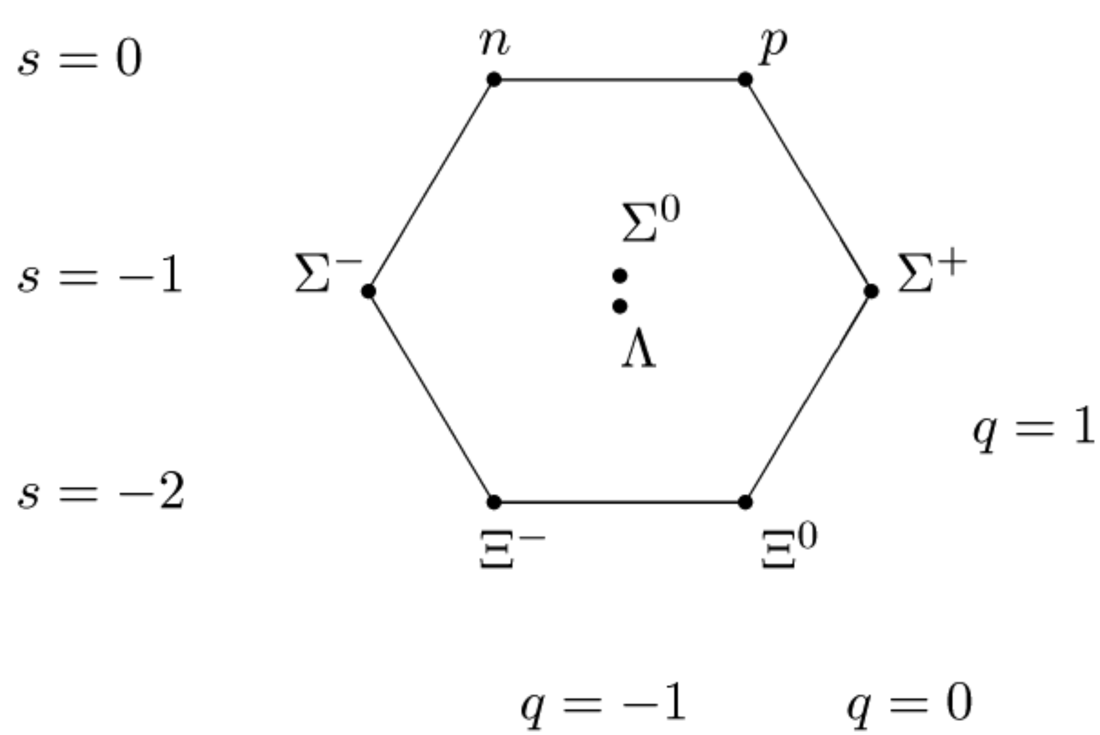
$$3_f \otimes \bar{3}_f = 8_f \oplus 1_f$$

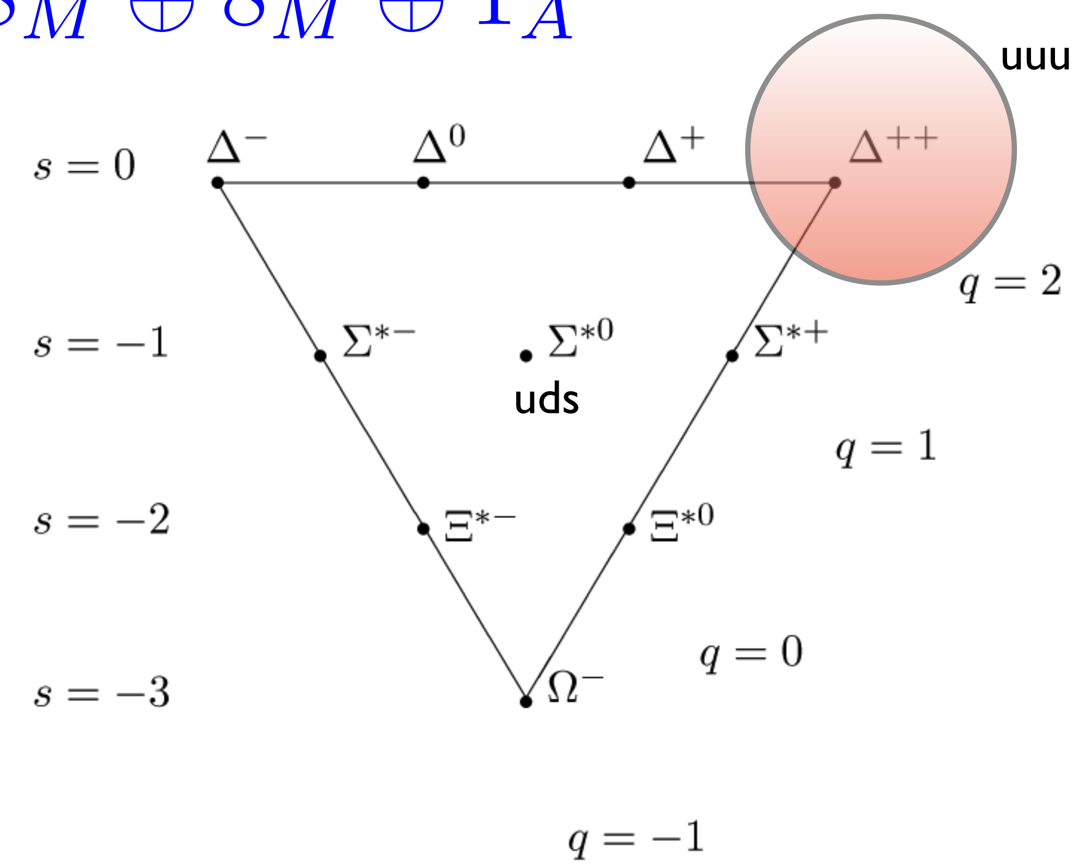


Hadron spectrum

Note that physical states are classified in multiplets of the FLAVOR SU(3)$_f$ group!

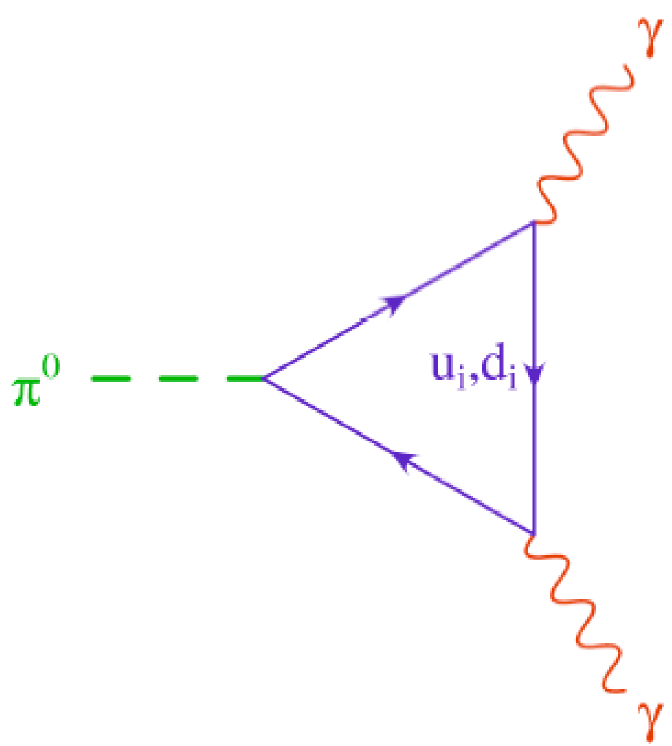
$$3_f \otimes 3_f \otimes 3_f = 10_S \oplus 8_M \oplus 8_M \oplus 1_A$$

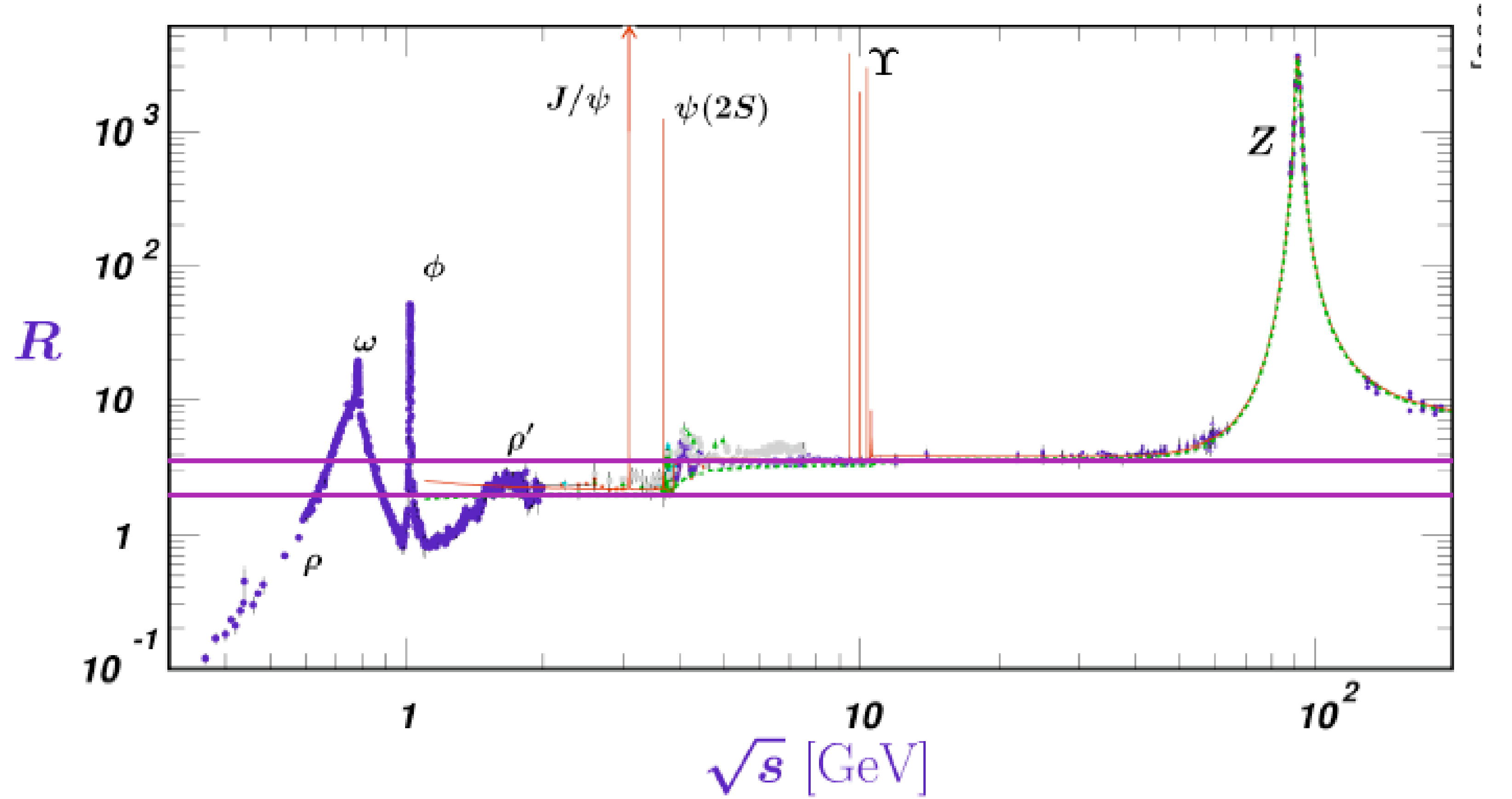




We need an extra quantum number (color) to have the Δ^{++} with similar properties to the Σ^{*0}. All particles in the multiplet have symmetric spin, flavour and spatial wave-function. Check that n_q - n_{qbar} = $n \times N_c$, with n integer.

How many colors?





$$\Gamma \sim N_c^2 \left[Q_u^2 - Q_d^2\right]^2 \frac{m_\pi^3}{f_\pi^2}$$

$$\Gamma_{TH} = \left(\frac{N_c}{3}\right)^2 7.6\,\text{eV}$$

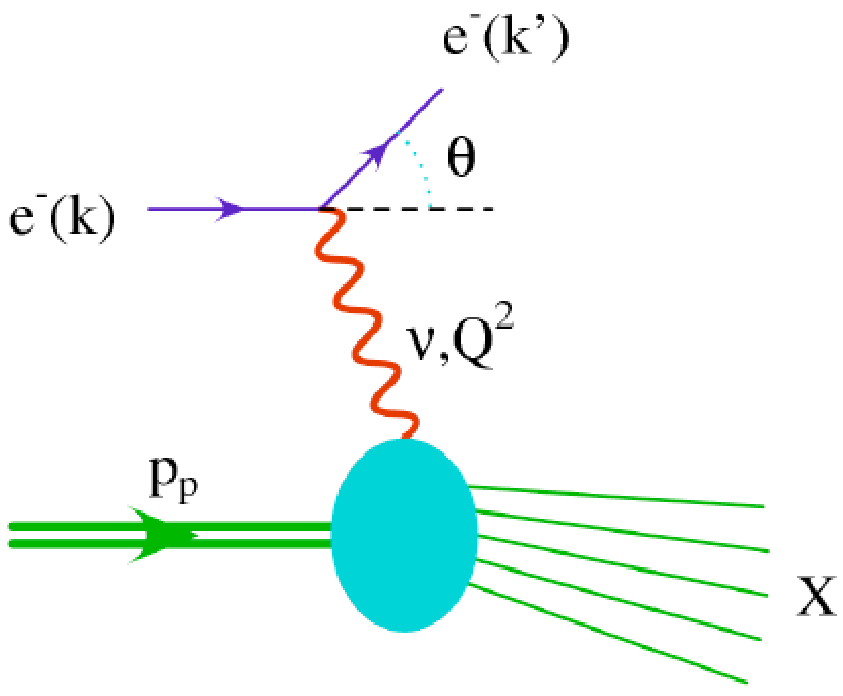
$$\Gamma_{EXP} = 7.7 \pm 0.6 \quad \text{eV}$$

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \sim N_c \sum_q e_q^2$$

$$= 2(N_c/3) \quad q = u, d, s$$

$$= 3.7(N_c/3) \quad q = u, d, s, c, b$$

Scaling



$s = (P+k)^2$ cms energy²

$Q^2 = -(k-k')^2$ momentum transfer²

$x = Q^2/2(P\cdot q)$ scaling variable

$\nu = (P\cdot q)/M = E - E'$ energy loss

$y = (P\cdot q)/(P\cdot k) = 1 - E'/E$ rel. energy loss

$W^2 = (P+q)^2 = M^2 + \frac{1-x}{x}Q^2$ recoil mass

$$\frac{d\sigma_{\text{elastic}}}{dq^2} = \left(\frac{d\sigma}{dq^2}\right)_{\text{point}} \cdot F^2_{\text{elastic}}(q^2)\,\delta(1-x)\,dx$$

$$\frac{d\sigma_{\text{inelastic}}}{dq^2} = \left(\frac{d\sigma}{dq^2}\right)_{\text{point}} \cdot F^2_{\text{inelastic}}(q^2,x)\,dx$$

What should we expect for $F(q^2,x)$?

Scaling

Two plausible and one crazy scenarios for the $|q^2| \to \infty$ (Bjorken) limit:

1. Smooth electric charge distribution: (classical picture)

$$F^2_{elastic}(q^2) \sim F^2_{inelastic}(q^2) << 1$$

i.e., external probe penetrates the proton as knife through the butter!

2. Tightly bound point charges inside the proton: (bound quarks)

$$F^2_{elastic}(q^2) \sim 1 \text{ and } F^2_{inelastic}(q^2) << 1$$

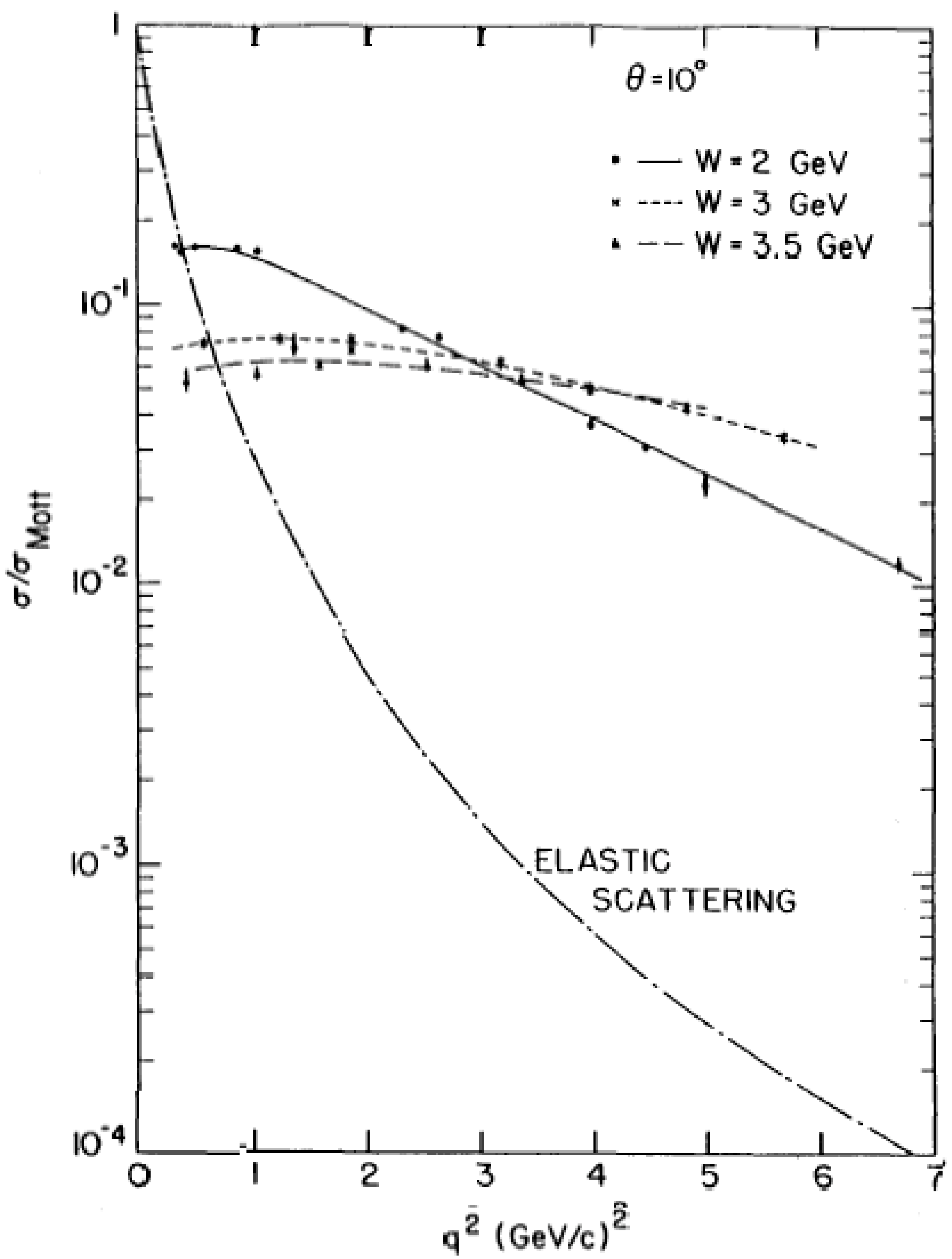
i.e., quarks get hit as single particles, but momentum is immediately redistributed as they are tightly bound together (confinement) and cannot fly away.

3. And now the crazy one: (free quarks)

$$F^2_{elastic}(q^2) << 1 \text{ and } F^2_{inelastic}(q^2) \sim 1$$

i.e., there are points (quarks!) inside the protons, however the hit quark behaves as a free particle that flies away without feeling or caring about confinement!!!

Scaling



$$\frac{d^2\sigma^{\mathrm{EXP}}}{dx\,dy} \sim \frac{1}{Q^2}$$

Remarkable!!! Pure dimensional analysis!

The right hand side does not depend on Λ_S !

This is the same behaviour one may find in a renormalizable theory like in QED.

Other stunning example is again $e^+e^- \to$ hadrons.

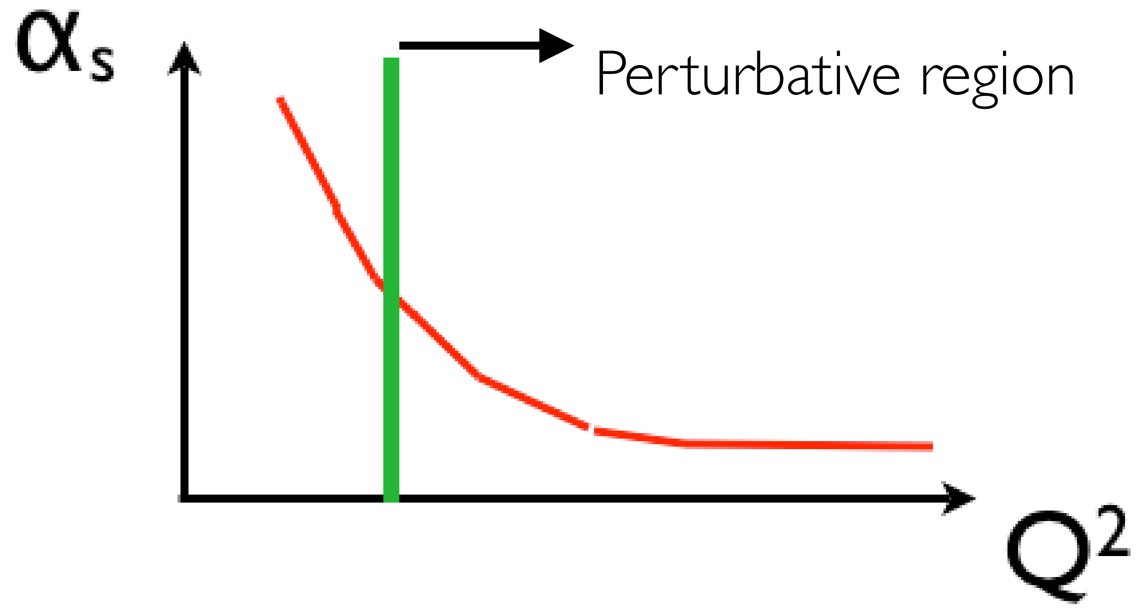
This motivated the search for a weakly-coupled theory at high energy!

Asymptotic freedom

Among QFT theories in 4 dimension only the non-Abelian gauge theories are "asymptotically free".

It becomes then natural to promote the global color SU(3) symmetry into a local symmetry where color is a charge.

This also hints to the possibility that the color neutrality of the hadrons could have a dynamical origin



In renormalizable QFT's scale invariance is broken by the renormalization procedure and couplings depend logarithmically on scales.

The QCD Lagrangian

$$\mathcal{L} = \boxed{-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}} + \sum_f \boxed{\bar{\psi}_i^{(f)}(i\not{\partial} - m_f)\psi_i^{(f)}} - \boxed{\bar{\psi}_i^{(f)}(g_s t^a_{ij} \not{A}_a)\psi_j^{(f)}}$$

Gauge Fields | Matter | Interaction

$$[t^a, t^b] = i f^{abc} t^c$$ → Algebra of SU(N)

$$\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$ → Normalization

Very similar to the QED Lagrangian.. we'll see in a moment where the differences come from!

The symmetries of the QCD Lagrangian

Now we know that strong interacting physical states have very good symmetry properties like the isospin symmetry: particles in the same multiplets (n,p) or (π^+,π^-,π^0) have nearly the same mass. Are these symmetries accounted for?

$$\mathcal{L}_F = \sum_f \bar{\psi}_i^{(f)} \left[(i\partial\!\!\!/ - m_f)\delta_{ij} - g_s t^a_{ij} A\!\!\!/_a \right] \psi_j^{(f)}$$

$$\psi^{(f)} \to \sum_{f'} U^{ff'} \psi^{(f')}$$

Isospin transformation acts only f=u,d.

It is a simple EXERCISE to show that the lagrangian is invariant if $m_u=m_d$ or m_u, $m_d \to 0$. It is the second case that is more appealing. If the masses are close to zero the QCD lagrangian is MORE symmetric:

CHIRAL SYMMETRY

The symmetries of the QCD Lagrangian

$$\mathcal{L}_F = \sum_f \left\{ \bar{\psi}_L^{(f)} \left(i\partial\!\!\!/ - g_s t^a A\!\!\!/_a \right) \psi_L^{(f)} + \bar{\psi}_R^{(f)} \left(i\partial\!\!\!/ - g_s t^a A\!\!\!/_a \right) \psi_R^{(f)} \right\}$$

$$- \sum_f m_f \left(\left\{ \bar{\psi}_R^{(f)} \psi_L^{(f)} + \bar{\psi}_L^{(f)} \psi_R^{(f)} \right) \right\}$$

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi$$

$$\psi_R = \frac{1}{2}(1 + \gamma_5)\psi$$

$$\psi_L^{(f)} \to e^{i\phi_L} \sum_{f'} U_L^{ff'} \psi_L^{(f')}$$

$$\psi_R^{(f)} \to e^{i\phi_R} \sum_{f'} U_R^{ff'} \psi_R^{(f')}$$

Do these symmetries have counterpart in the real world?

-The vector subgroup is realized in nature as the isospin
-The corresponding U(1) is the baryon number conservation
-The axial $U_A(1)$ is not there due the axial anomaly
-The remaining axial transformations are spontaneously broken and the goldstone bosons are the pions.

$$SU_L(N) \times SU_R(N) \times U_L(1) \times U_R(1)$$

This is amazing! Without knowing anything about the dynamics of confinement we correctly describe isospin, the small mass of the pions, the scattering properties of pions, and many other features.

Why do we believe QCD is a good theory of strong interactions?

- QCD is a non-abelian gauge theory, is renormalizable, is asymptotically free, is a one-parameter theory [Once you measure α_S (and the quark masses) you know everything fundamental about (perturbative) QCD].
- It explains the low energy properties of the hadrons, justifies the observed spectrum and catch the most important dynamical properties.
- It explains scaling (and BTW anything else we have seen up to now!!) at high energies.
- It leaves EW interaction in place since the SU(3) commutes with SU(2) x U(1). There is no mixing and there are no enhancements of parity violating effect or flavor changing currents.

ok, then. Are we done?

Motivations for QCD predictions

- Accurate and experimental friendly predictions for collider physics range from being very useful to strictly necessary.
- Confidence on possible excesses, evidences and eventually discoveries builds upon an intense (and often non-linear) process of description/prediction of data via MC's.
- Measurements and exclusions always rely on accurate predictions.
- Predictions for both SM and BSM on the same ground.

no QCD $\Rightarrow$ no PARTY !

QCD : THE FUNDAMENTALS

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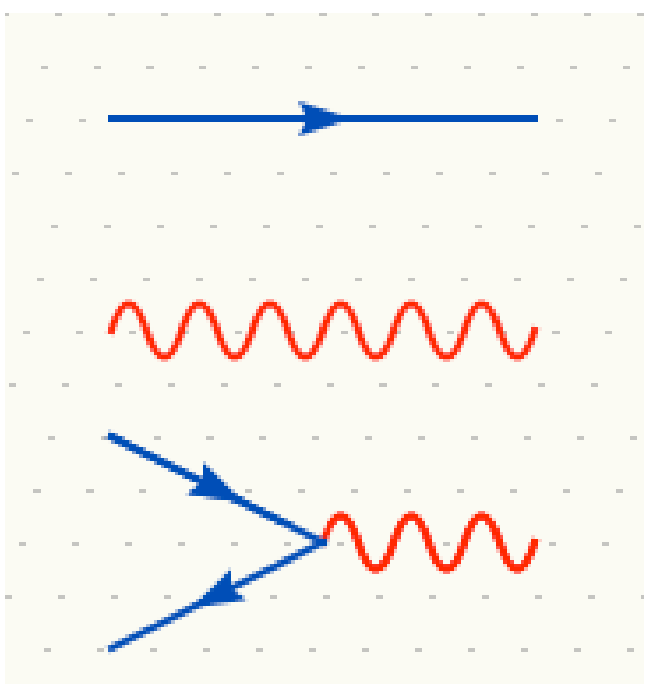
2. From QED to QCD: the importance of color
3. Renormalization group and asymptotic freedom

From QED to QCD

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - eQ\bar{\psi}\not{A}\psi$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$



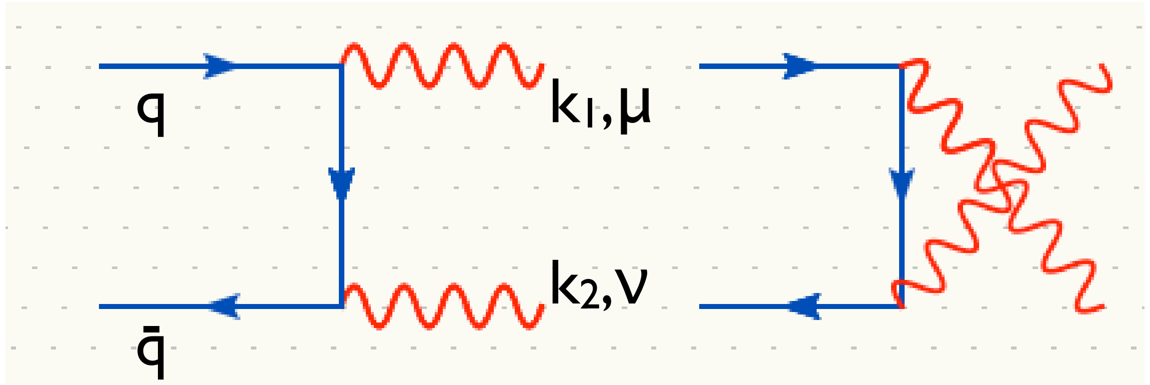
$$= \frac{i}{\not{p} - m + i\epsilon}$$

$$= \frac{-i g_{\mu\nu}}{p^2 + i\epsilon}$$

$$= -ie\gamma_\mu Q$$

From QED to QCD

We want to focus on how gauge invariance is realized in practice.
Let's start with the computation of a simple process $e^+e^- \to \gamma\gamma$. There are two diagrams:



$= D_1 + D_2$

$$i\mathcal{M} = \mathcal{M}_{\mu\nu}\epsilon_1^{*\mu}\epsilon_2^{*\nu} = D_1 + D_2 = e^2\left(\bar{v}(\bar{q})\not{\epsilon}_2\frac{1}{\not{q}-\not{k}_1}\not{\epsilon}_1 u(q) + \bar{v}(\bar{q})\not{\epsilon}_1\frac{1}{\not{q}-\not{k}_2}\not{\epsilon}_2 u(q)\right)$$

Gauge invariance requires that:

$$\epsilon_1^{*\mu}k_2^{\nu}\mathcal{M}_{\mu\nu} = \epsilon_2^{*\nu}k_1^{\mu}\mathcal{M}_{\mu\nu} = 0$$

From QED to QCD

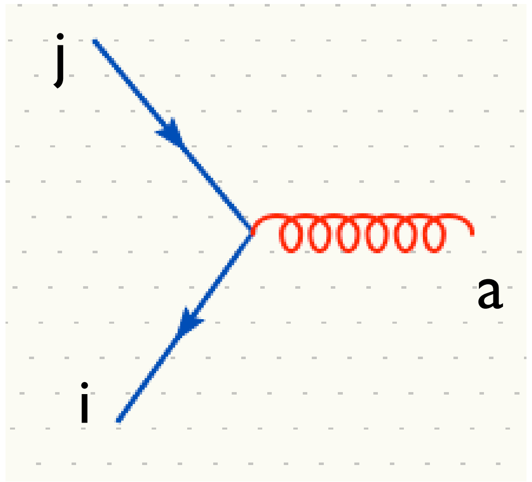
$$\mathcal{M}_{\mu\nu} k_1^{*\mu} \epsilon_2^{*\nu} = D_1 + D_2 = e^2 \left(\bar{v}(\bar{q}) \not{\epsilon}_2 \frac{1}{\not{q} - \not{k}_1} (\not{k}_1 - \not{q}) u(q) + \bar{v}(\bar{q}) (\not{k}_1 - \not{\bar{q}}) \frac{1}{\not{k}_1 - \not{\bar{q}}} \not{\epsilon}_2 u(q) \right)$$
$$= -\bar{v}(\bar{q}) \not{\epsilon}_2 u(q) + \bar{v}(\bar{q}) \not{\epsilon}_2 u(q) = 0$$

Only the sum of the two diagrams is gauge invariant. For the amplitude to be gauge invariant it is enough that one of the polarizations is longitudinal. The state of the other gauge boson is irrelevant.

Let's try now to generalize what we have done for SU(3). In this case we take the (anti-)quarks to be in the (anti-)fundamental representation of SU(3), 3 and 3*. Then the current is in a 3 ⊗ 3* = 1 ⊕ 8. The singlet is like a photon, so we identify the gluon with the octet and generalize the QED vertex to :

with $[t^a, t^b] = i f^{abc} t^c$

$$-i g_s t^a_{ij} \gamma^\mu$$



So now let's calculate qq → gg and we obtain

$$\frac{i}{g_s^2} M_g \equiv (t^b t^a)_{ij} D_1 + (t^a t^b)_{ij} D_2$$

$$M_g = (t^a t^b)_{ij} M_\gamma - g^2 f^{abc} t^c_{ij} D_1$$

From QED to QCD

To satisfy gauge invariance we still need:

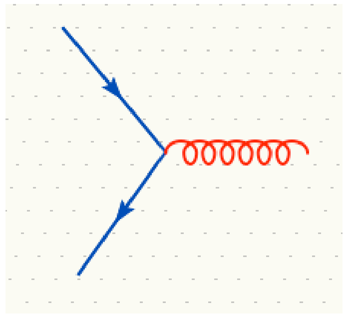
$$k_1^\mu {\epsilon_2}^\nu M_g^{\mu,\nu} = k_2^\nu \epsilon_1^\mu M_g^{\mu,\nu} = 0.$$

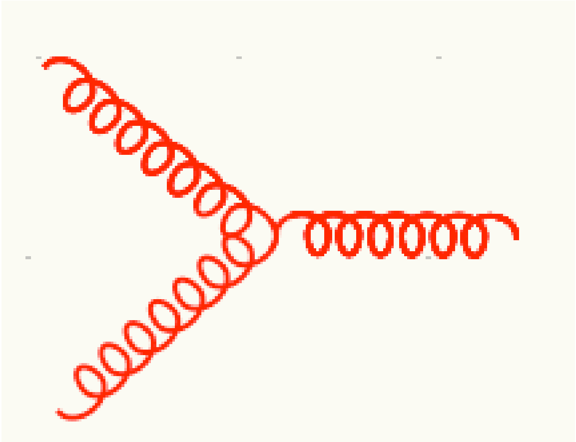
But in this case one piece is left out

$$k_{1\mu} M_g^\mu = -g_s^2 f^{abc} t_{ij}^c \bar{v}_i(\bar{q}) \not{\epsilon}_2 u_i(q)$$

$$k_{1\mu} M_g^\mu = i(-g_s f^{abc} \epsilon_2^\mu)(-i g_s t_{ij}^c \bar{v}_i(\bar{q}) \gamma_\mu u_i(q))$$

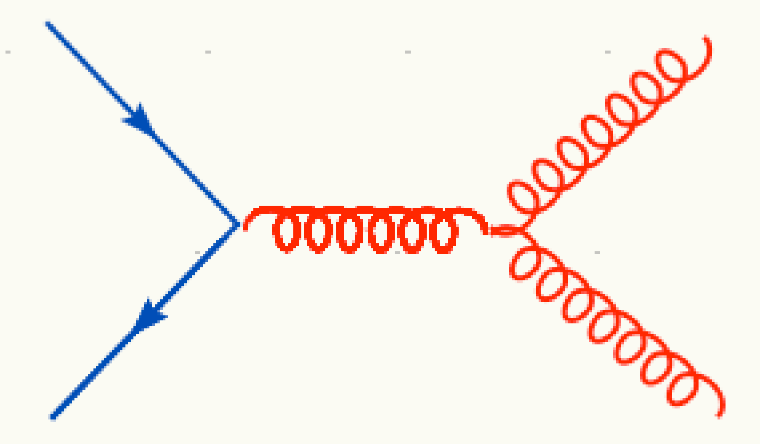
We indeed see that we interpret as the normal vertex times a new 3 gluon vertex:





$$-g_s f^{abc} V_{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

From QED to QCD



$$-ig_s^2 D_3 = \left(-ig_s t^a_{ij}\bar{v}_i(\bar{q})\gamma^\mu u_j(q)\right) \times \left(\frac{-i}{p^2}\right) \times$$

$$\left(-gf^{abc}V_{\mu\nu\rho}(-p,k_1,k_2)\epsilon_1^\nu(k_1)\epsilon_2^\rho(k_2)\right)$$

How do we write down the Lorentz part for this new interaction? We can impose

1. Lorentz invariance : only structure of the type $g_{\mu\nu}\, p_\rho$ are allowed
2. fully anti-symmetry : only structure of the type remain $g_{\mu 1 \mu 2}\ (k_1)_{\mu 3}$ are allowed...
3. dimensional analysis : only one power of the momentum.

that uniquely constrain the form of the vertex:

$$V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = V_0\left[(p_1-p_2)_{\mu_3}g_{\mu_1\mu_2} + (p_2-p_3)_{\mu_1}g_{\mu_2\mu_3} + (p_3-p_1)_{\mu_2}g_{\mu_3\mu_1}\right]$$

With the above expression we obtain a contribution to the gauge variation:

$$k_1\cdot D_3 = g^2 f^{abc}t^c V_0\left[\bar{v}(\bar{q})\not{\epsilon}_2 u(q) - \frac{k_2\cdot\epsilon_2}{2k_1\cdot k_2}\bar{v}(\bar{q})\not{k}_1 u(q)\right]$$

The first term cancels the gauge variation of $D_1 + D_2$ if $V_0=1$, the second term is zero IFF the other gluon is physical!!

One can derive the form of the four-gluon vertex using the same heuristic method.

The QCD Lagrangian

By direct inspection and by using the form non-abelian covariant derivation, we can check that indeed non-abelian gauge symmetry implies self-interactions. This is not surprising since the gluon itself is charged (In QED the photon is not!)

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \sum_f \bar{\psi}_i^{(f)} (i\partial\!\!\!/ - m_f) \psi_i^{(f)} - \bar{\psi}_i^{(f)} (g_s t^a_{ij} A\!\!\!/_a) \psi_j^{(f)}$$

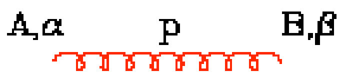
Gauge Fields and their interact.

Matter

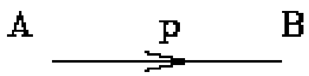
Interaction

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

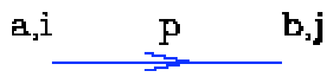
The Feynman Rules of QCD



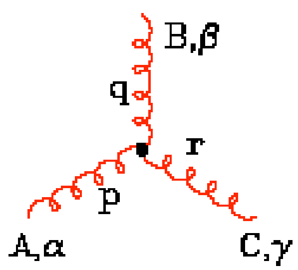
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\frac{i}{(p^{2}+i\epsilon)}$$

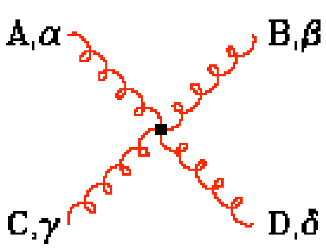


$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

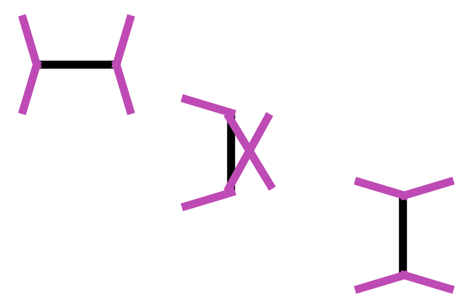


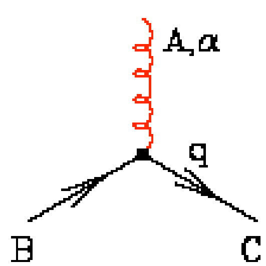
$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

(all momenta incoming)



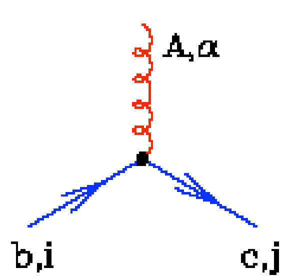
$$-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$





$$g\ f^{ABC}q^{\alpha}$$

← what is this?



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

From QED to QCD: physical states

In QED, due to abelian gauge invariance, one can sum over the polarization of the external photons using:

$$\sum_{pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu}$$

I In fact the longitudinal and time-like component cancel each other, no matter what the choice for $\boldsymbol{\varepsilon}_2$ is. The production of any number of unphysical photons vanishes.

In QCD this would give a wrong result!!

We can write the sum over the polarization in a convenient form using the vector $k=(k_0, 0,0,-k_0)$.

$$\sum_{phys\,pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu} + \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k \cdot \bar{k}}$$

For gluons the situation is different, since $k_1 \cdot M \sim \boldsymbol{\varepsilon}_2 \cdot k_2$. So the production of two unphysical gluons is not zero!!

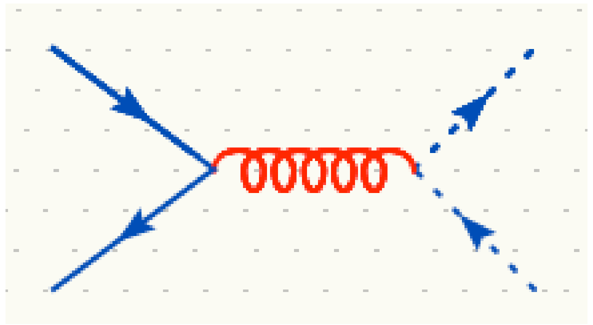
From QED to QCD: physical states

In the case of non-Abelian theories it is therefore important to restrict the sum over polarizations (and the off-shell propagators) to the physical degrees of freedom.

Alternatively, one has to undertake a formal study of the implications of gauge-fixing in non-physical gauges. The outcome of this approach is the appearance of two color-octet scalar degrees of freedom that have the peculiar property that behave like fermions.

Ghost couple only to gluons and appear in internal loops and as external states (in place of two gluons). Since they break the spin-statistics theorem their contribution can be negative, which is what is require to cancel the the non-physical dof in the general case.

Adding the ghost contribution gives

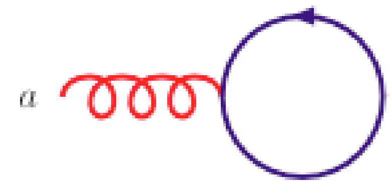


$$\Rightarrow -\left| i g_s^2 f^{abc} t^a \frac{1}{2k_1 \cdot k_2} \bar{v}(\bar{q}) \not{k}_1 u(q) \right|^2$$

which exactly cancels the non-physical polarization in a covariant gauge.

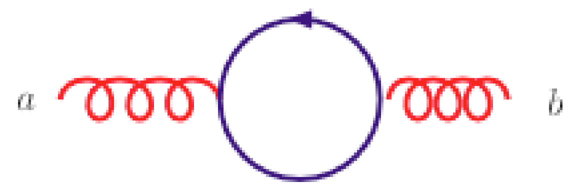
The color algebra

$$\mathrm{Tr}(t^a) = 0$$



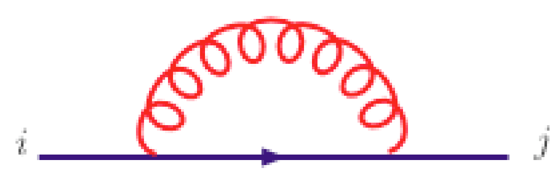
= 0

$$\mathrm{Tr}(t^a t^b) = T_R \delta^{ab}$$



= T_R *

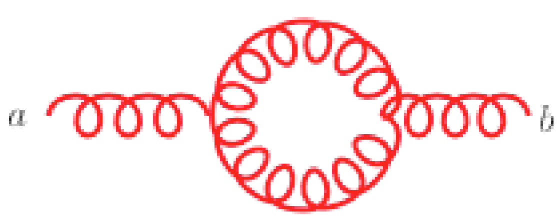
$$(t^a t^a)_{ij} = C_F \delta_{ij}$$



= C_F *

$$\sum_{cd} f^{acd} f^{bcd}$$

$$= (F^c F^c)_{ab} = C_A \delta_{ab}$$

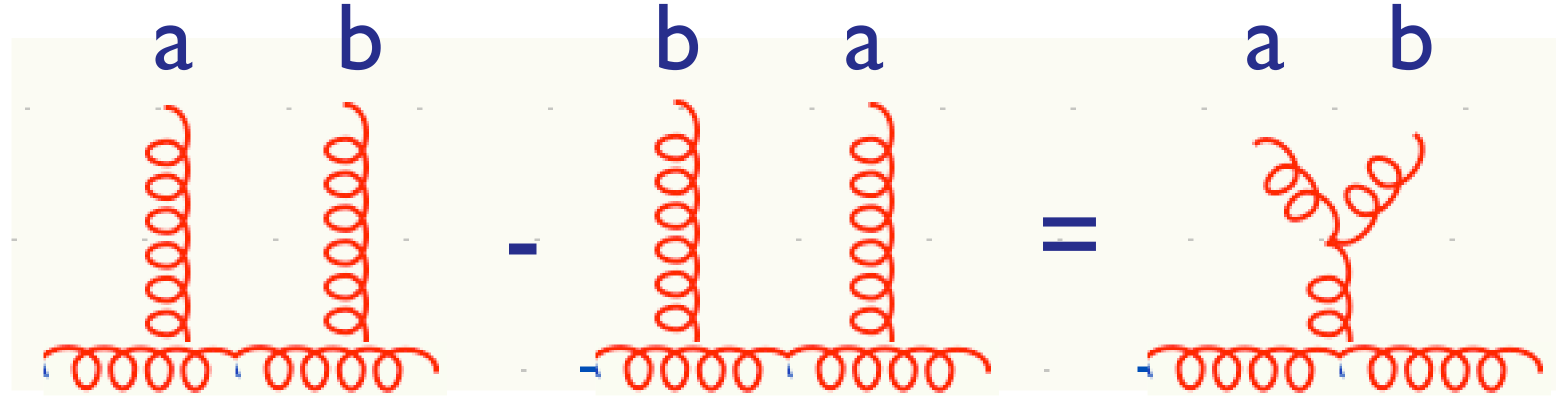


= C_A*

The color algebra

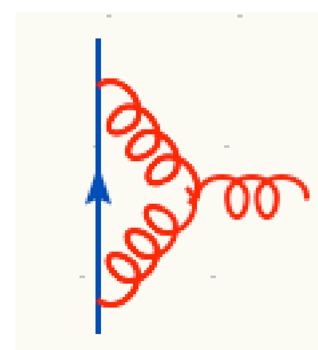
$$[t^a, t^b] = i f^{abc} t^c$$

$$[F^a, F^b] = i f^{abc} F^c$$

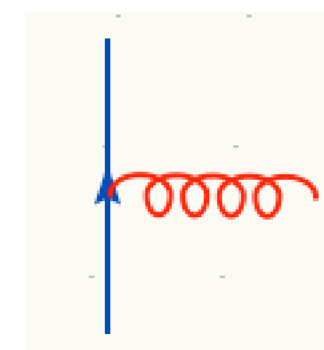


1-loop vertices

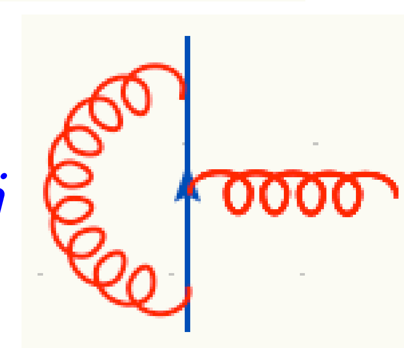
$$i f^{abc} (t^b t^c)_{ij} = \frac{C_A}{2} t^a_{ij}$$



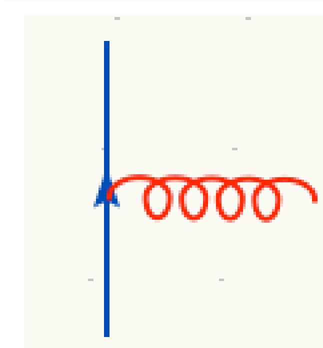
= C_A/2 *



$$(t^b t^a t^b)_{ij} = (C_F - \frac{C_A}{2}) t^a_{ij}$$

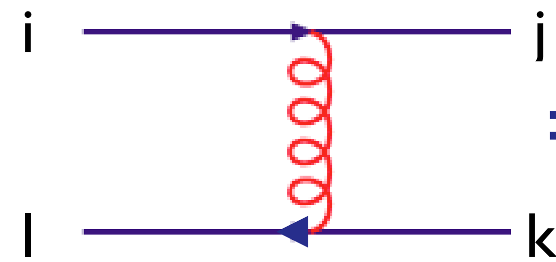


= -1/2/Nc *

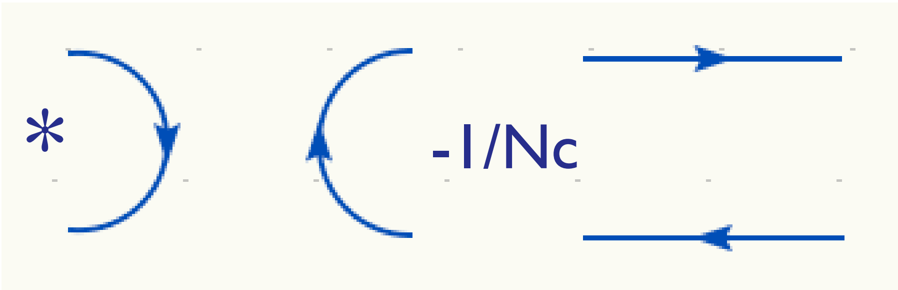


The color algebra

$$t^a_{ij}t^a_{kl} = \frac{1}{2}(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl})$$

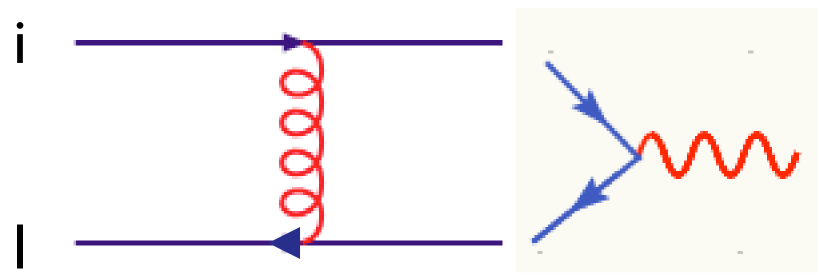


= 1/2 *



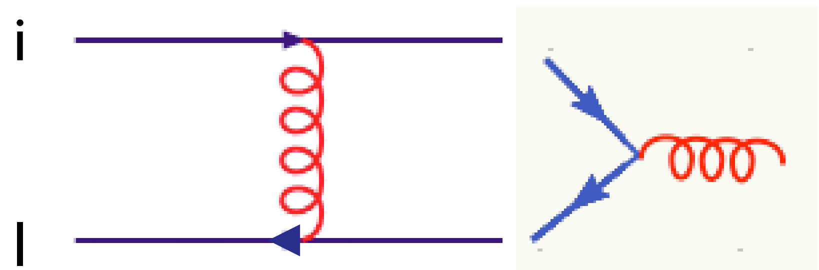
Problem: Show that the one-gluon exchange between quark-antiquark pair can be attractive or repulsive. Calculate the relative strength.

Solution: a q qb pair can be in a singlet state (photon) or in octet (gluon) : $3 \otimes \bar{3} = 1 \oplus 8$



$$\frac{1}{2}(\delta_{ik}\delta_{lj} - \frac{1}{N_c}\delta_{ij}\delta_{lk})\delta_{ki} = \frac{1}{2}\delta_{lj}(N_c - \frac{1}{N_c}) = C_F\delta_{lj}$$

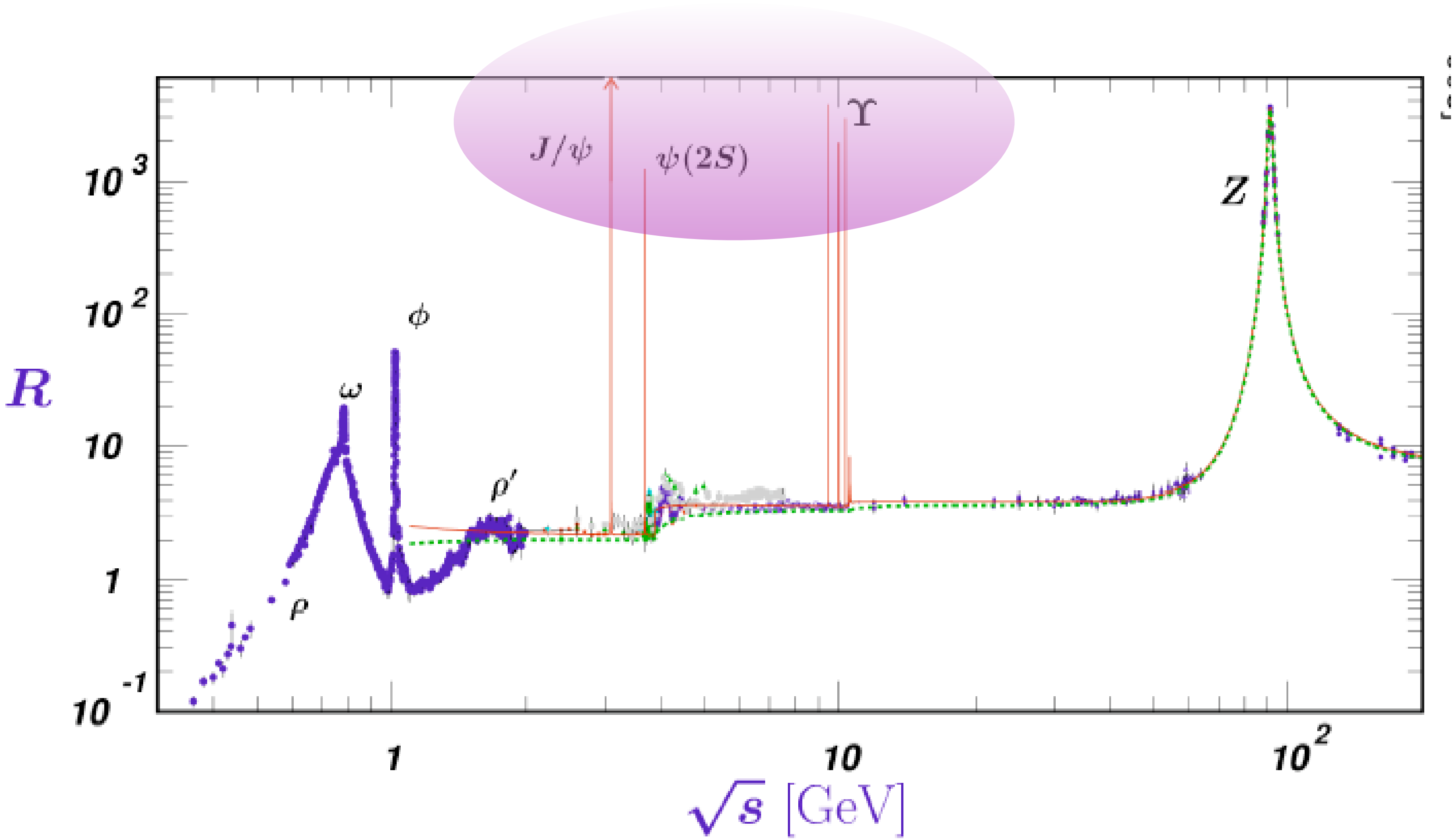
>0, attractive

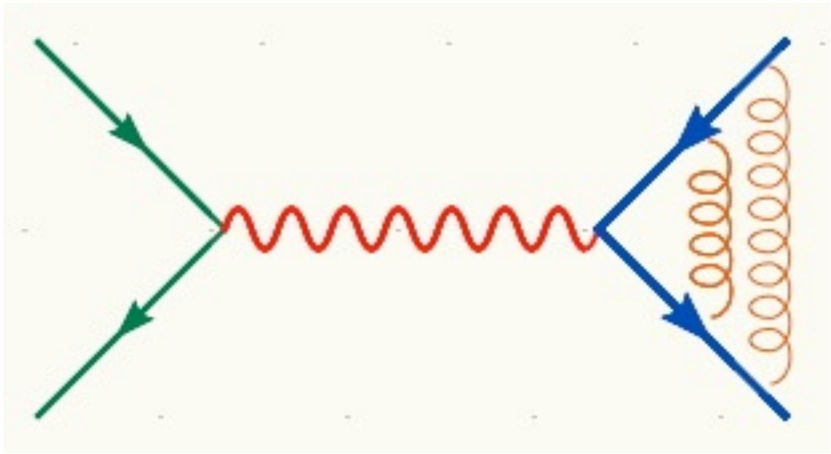


$$\frac{1}{2}(\delta_{ik}\delta_{lj} - \frac{1}{N_c}\delta_{ij}\delta_{lk})t^a_{ki} = -\frac{1}{2N_c}t^a_{lj}$$

<0, repulsive

Quarkonium states

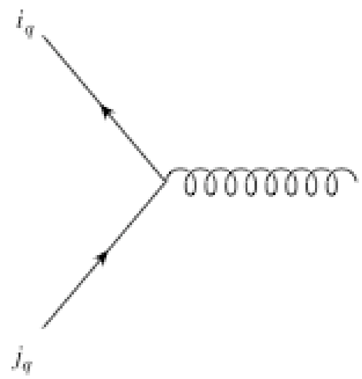


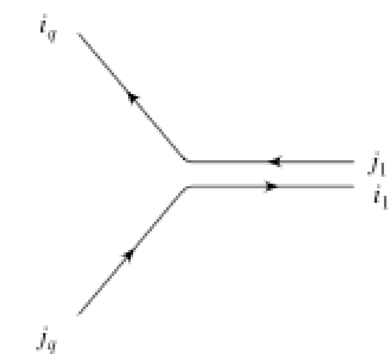


Very sharp peaks => small widths (~ 100 KeV) compared to hadronic resonances (100 MeV) => very long lived states. QCD is "weak" at scales >> ΛQCD (asymptotic freedom), non-relativistic bound states are formed like positronium!

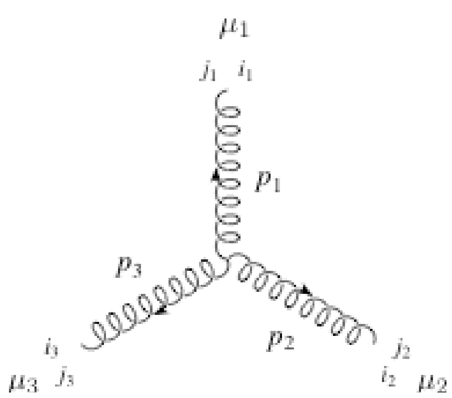
The QCD-Coulomb attractive potential is like: $V(r) \simeq -C_F \frac{\alpha_S(1/r)}{r}$

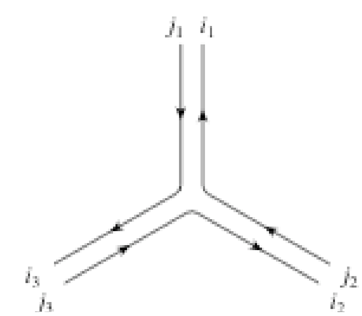
Color algebra: 't Hooft double line





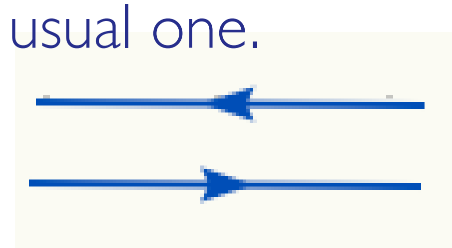
$$i\frac{g}{\sqrt{2}}\gamma_1^{\mu}\delta_{j_1}^{i_q}\delta_{j_q}^{i_1}$$





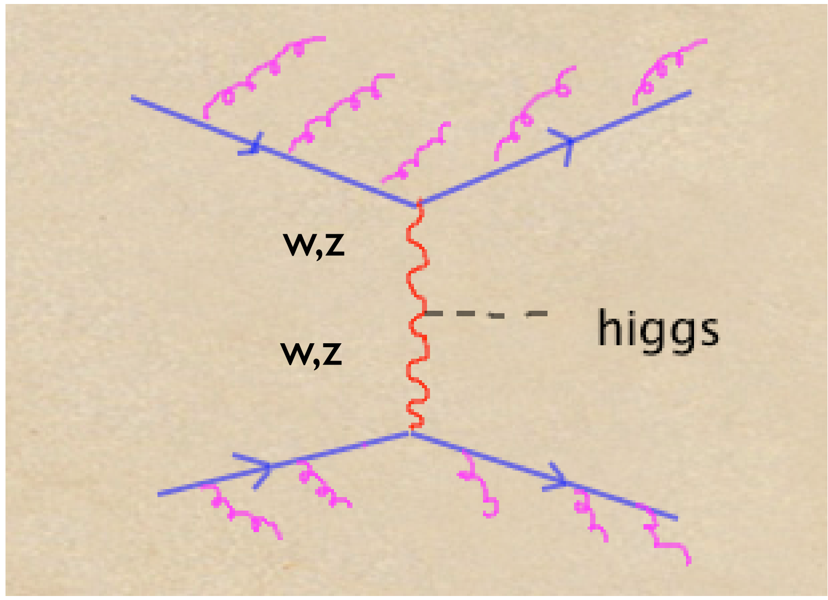
$$i\frac{g}{\sqrt{2}}\sum K^{\mu_1\mu_2\mu_3}\delta_{j_1}^{i_3}\delta_{j_2}^{i_1}\delta_{j_3}^{i_2}$$

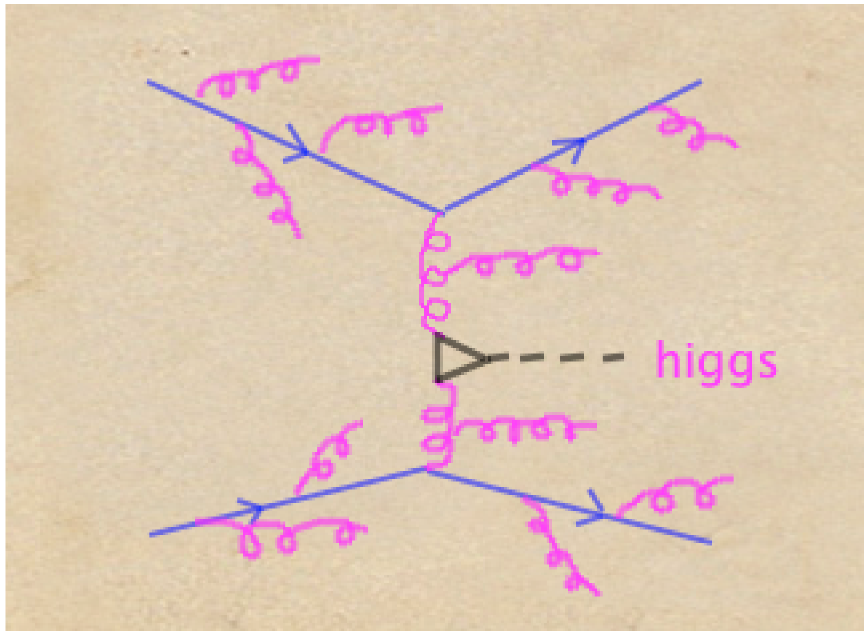
This formulation leads to a graphical representation of the simplifications occuring in the large Nc limit, even though it is exactly equivalent to the usual one.

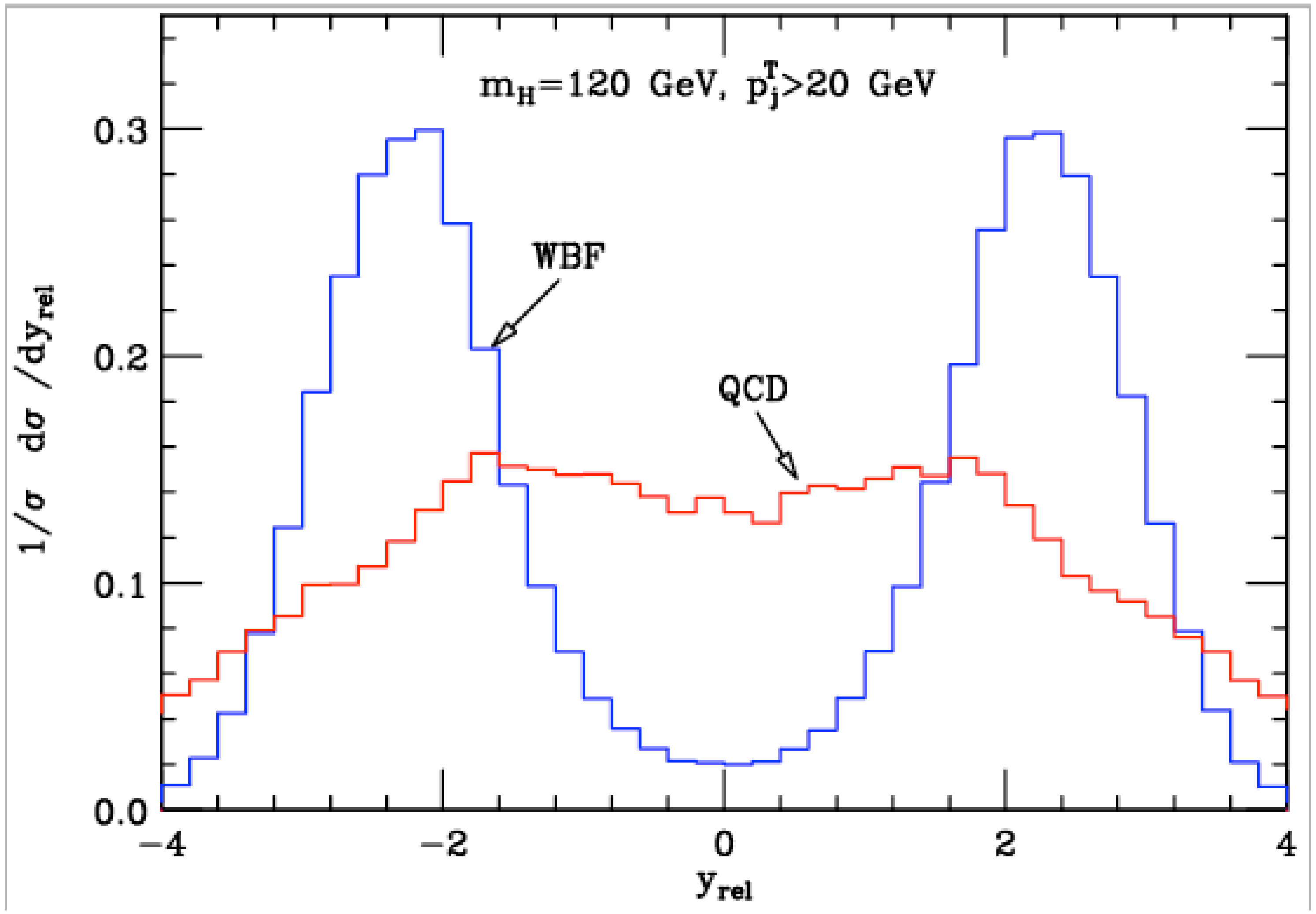
 ≈ 1/2 

In the large Nc limit, a gluon behaves as a quark-antiquark pair. In addition it behaves classically, in the sense that quantum interference, which are effects of order $1/Nc^2$ are neglected. Many QCD algorithms and codes (such a the parton showers) are based on this picture.

Example: VBF fusion



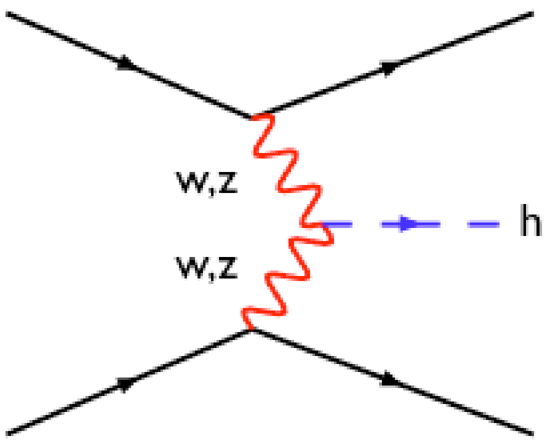




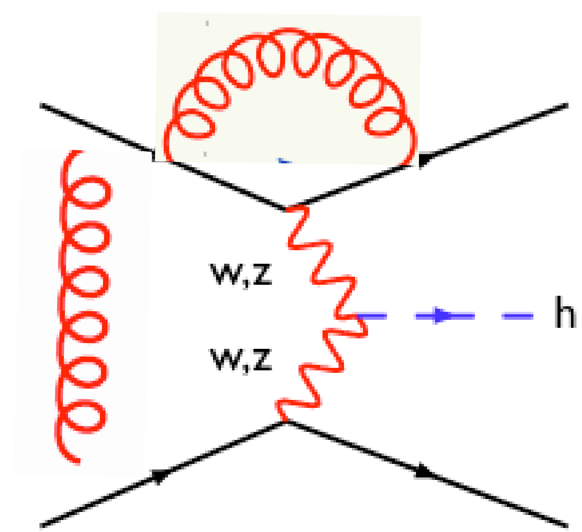
Third jet distribution

Example: VBF fusion

Consider VBF: at LO there is no exchange of color between the quark lines:



$$\delta_{ij}\delta_{kl}$$



$$C_F\delta_{ij}\delta_{kl} \Rightarrow$$

$$M_{\rm tree}M^*_{1-{\rm loop}} = C_F N_c^2 \simeq N_c^3$$

$$\frac{1}{2}(\delta_{ik}\delta_{lj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}) \Rightarrow$$

$$M_{\rm tree}M^*_{1-{\rm loop}} = 0$$

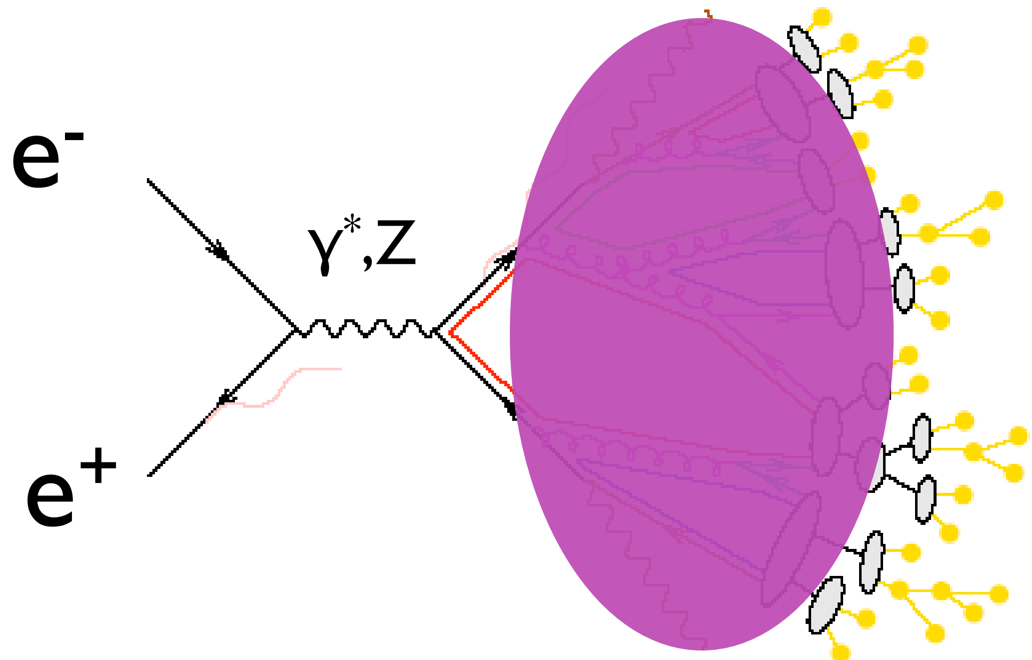
Also at NLO there is no color exchange! With one little exception....
At NNLO exchange is possible but it suppressed by $1/Nc^2$

QCD : THE FUNDAMENTALS

1. QCD is a good theory for strong interactions: facts
2. From QED to QCD: the importance of color
3. Renormalization group and asymptotic freedom

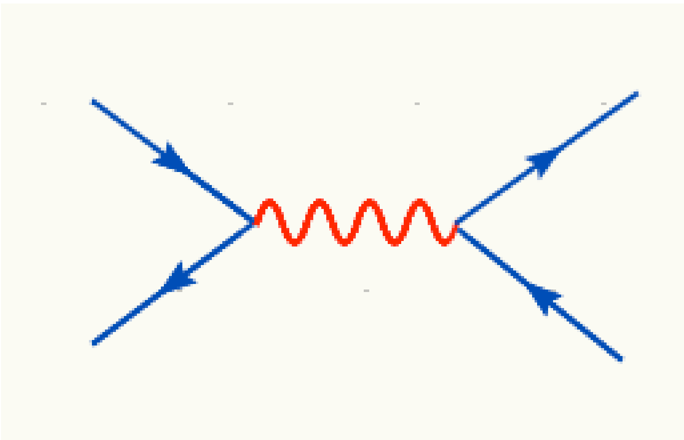
Ren. group and asymptotic freedom

Let us consider the process:
$e^-e^+ \to$ hadrons and for a $Q^2 >> \Lambda_S$.
At this pont (though we will!) we don't have an idea how to calculate the details of such a process.
So let's take the most inclusive approach ever: we just want to count how many events with hadrons in the final state there are wrt to a pair of muons.



Zeroth Level: e+ e- → qq

$$R_0 = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_f Q_f^2$$



Very simple exercise. The calculation is exactly the same as for the μ+μ-.

Ren. group and asymptotic freedom

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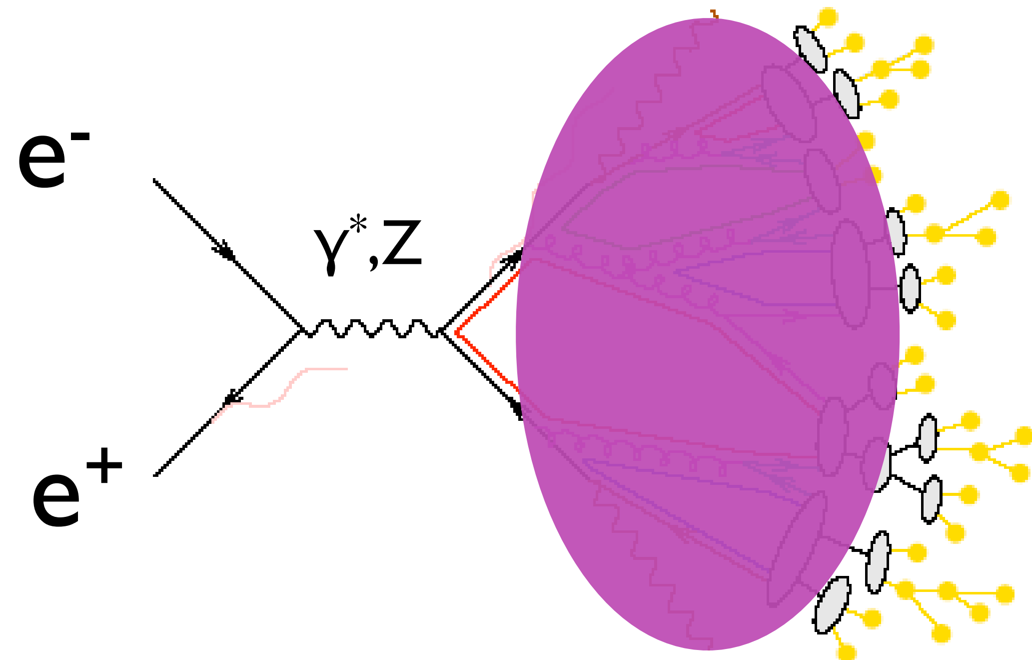
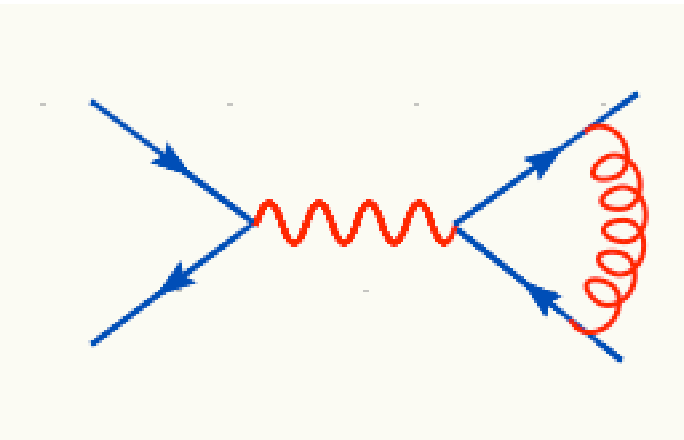


First improvement: e+ e- → qq at NLO
Already a much more difficult calculation!
There are real and virtual contributions. There are:
* UV divergences coming from loops
* IR divergences coming from loops and real diagrams. Ignore the IR for the moment (they cancel anyway) We need some kind of trick to regulate the divergences. Like dimensional regularization or a cutoff M. At the end the result is VERY SIMPLE:

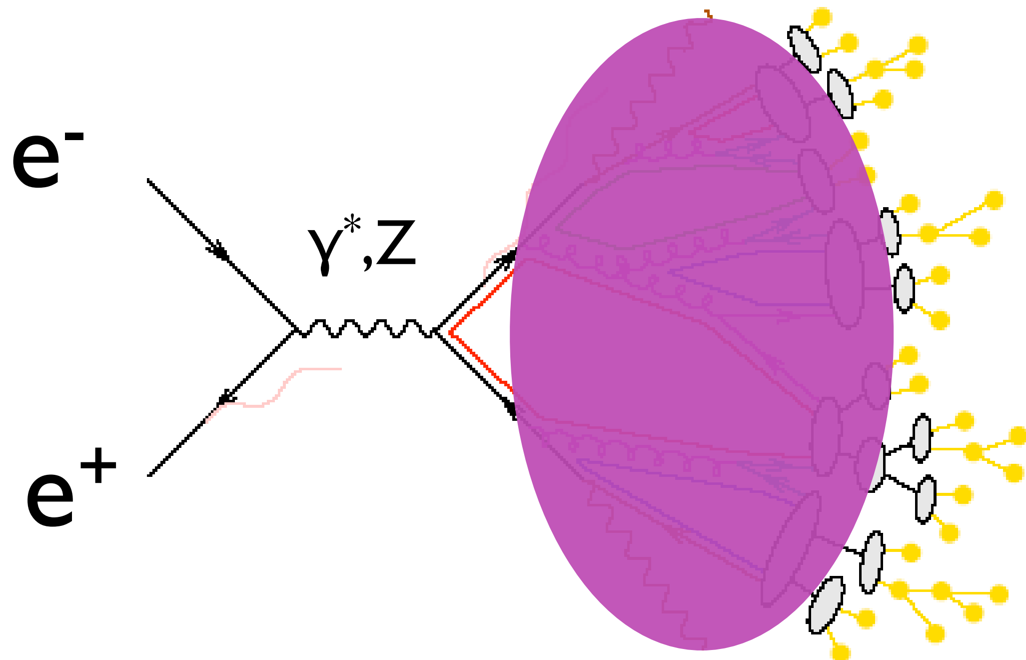


$$R_1 = R_0 \left(1 + \frac{\alpha_S}{\pi}\right)$$

No renormalization is needed! Electric charge is left untouched by strong interactions!

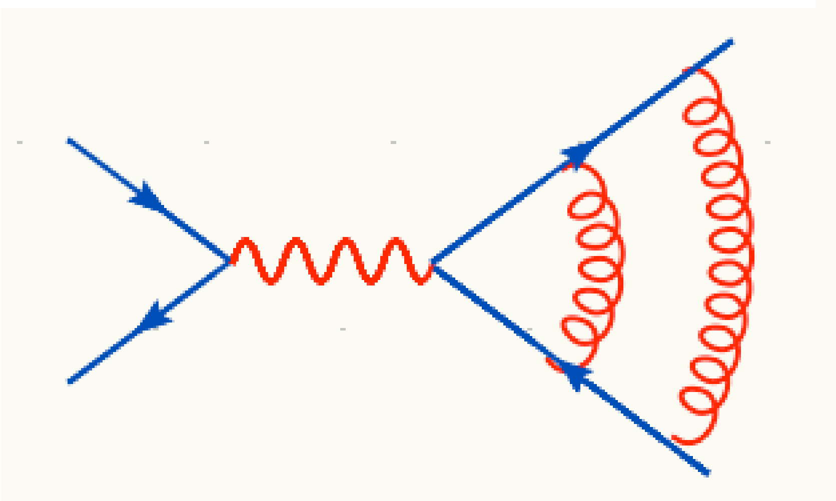
Ren. group and asymptotic freedom

Let us consider the process:
$e^-e^+ \rightarrow$ hadrons and for a $Q^2 >> \Lambda_S$.
At this pont (though we will!) we don't have an idea how to calculate the details of such a process.
So let's take the most inclusive approach ever: we just want to count how many events with hadrons in the final state there are wrt to a pair of muons.



Second improvement: e+ e- → qq at NNLO
Extremely difficult calculation!
Something new happens:

$$R_2 = R_0 \left(1 + \frac{\alpha_S}{\pi} + \left[c + \pi b_0 \log \frac{M^2}{Q^2}\right] \left(\frac{\alpha_S}{\pi}\right)^2\right)$$



The result is explicitly dependent on the arbitrary cutoff scale. We need to perform normalization of the coupling and since QCD is renormalizable we are guaranteed that this fixes all the UV problems at this order.

$$\alpha_S(\mu) = \alpha_S + b_0 \log \frac{M^2}{\mu^2} \alpha_S^2$$

Ren. group and asymptotic freedom

(1) $$R_2^{\rm ren}(\alpha_S(\mu), \frac{\mu^2}{Q^2}) = R_0 \left(1 + \frac{\alpha_S(\mu)}{\pi} + \left[c + \pi b_0 \log\frac{\mu^2}{Q^2}\right]\left(\frac{\alpha_S(\mu)}{\pi}\right)^2\right)$$

(2) $$\alpha_S(\mu) = \alpha_S + b_0 \log\frac{M^2}{\mu^2}\alpha_S^2 \qquad b_0 = \frac{11 N_c - 2 n_f}{12\pi} > 0$$

Comments:

1. Now R_2 is finite but depends on an arbitrary scale **μ**, directly and through **α_s**. We had to introduce **μ** because of the presence of M.

2. Renormalizability guarantees than any physical quantity can be made finite with the SAME substitution. If a quantity at LO is $A\alpha_s^N$ then the UV divergence will be N A b_0 log M^2 α_s^{N+1}.

3. R is a physical quantity and therefore cannot depend on the arbitrary scale **μ**!! One can show that at order by order:

$$\mu^2 \frac{d}{d\mu^2} R^{\rm ren} = 0 \Rightarrow R^{\rm ren}(\alpha_S(\mu), \frac{\mu^2}{Q^2}) = R^{\rm ren}(\alpha_S(Q), 1)$$

which is obviously verified by Eq. (1). Choosing **μ** ≈ Q the logs ...are resummed!

Ren. group and asymptotic freedom

(2) $\alpha_S(\mu) = \alpha_S + b_0 \log \frac{M^2}{\mu^2} \alpha_S^2$ $\qquad b_0 = \frac{11 N_c - 2 n_f}{12\pi}$ >0

4. From (2) one finds that:

$$\beta(\alpha_S) \equiv \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} = -b_0 \alpha_S^2 \qquad \Rightarrow \qquad \alpha_S(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}}$$

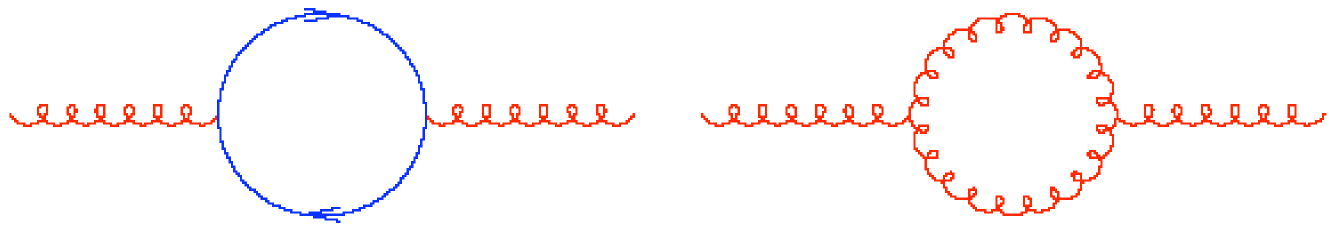
This gives the running of α_S. Since $b_0 > 0$, this expression make sense for all scale $\mu > \Lambda$.
In general one has:

$$\frac{d\alpha_S(\mu)}{d \log \mu^2} = -b_0 \alpha_S^2(\mu) - b_1 \alpha_S^3(\mu) - b_2 \alpha_S^4(\mu) + \ldots$$

where all b_i are finite (renormalization!). At present we know the b_i up to b_3 (4 loop calculation!!). b_1 and b_2 are renormalization scheme independent. Note that the expression for $\alpha_S(\mu)$ changes accordingly to the loop order. At two loops we have:

$$\alpha_S(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}} \left[1 - \frac{b_1}{b_0^2} \frac{\log \log \mu^2/\Lambda^2}{\log \mu^2/\Lambda^2} \right]$$

Why is the beta function negative in QCD?



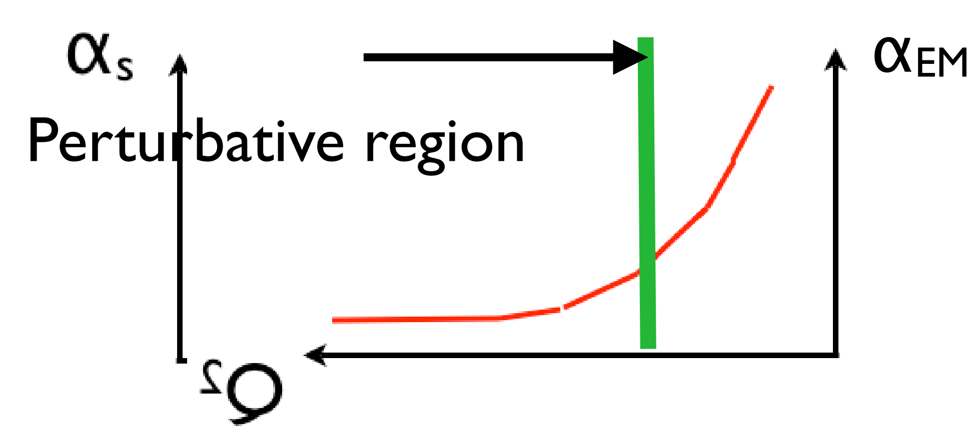
(a) (b)

Roughly speaking, quark loop diagram (a) contributes a negative N_f term in b_0, while the gluon loop, diagram (b) gives a positive contribution proportional to the number of colors N_c, which is dominant and make the overall beta function negative.

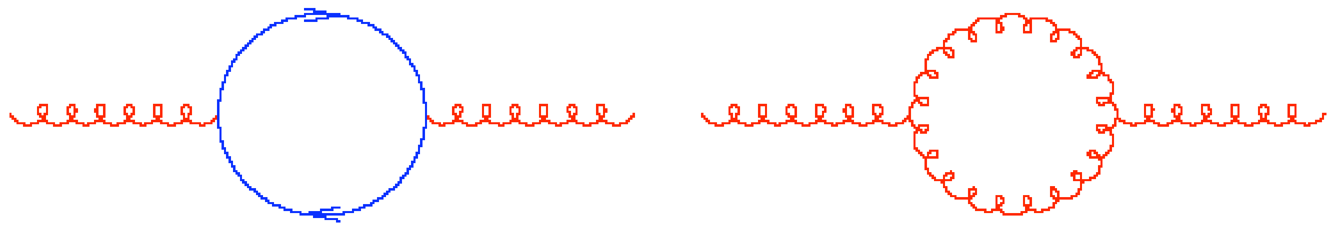
$$b_0 = \frac{11N_c - 2n_f}{12\pi} \quad >0 \quad \Rightarrow \quad \beta(\alpha_S)<0 \text{ in QCD}$$

$$b_0 = -\frac{n_f}{3\pi} \quad <0 \quad \Rightarrow \quad \beta(\alpha_S)>0 \text{ in QED}$$

$$\alpha_{EM}(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2_{\mathrm{QED}}}}$$



Why is the beta function negative in QCD?

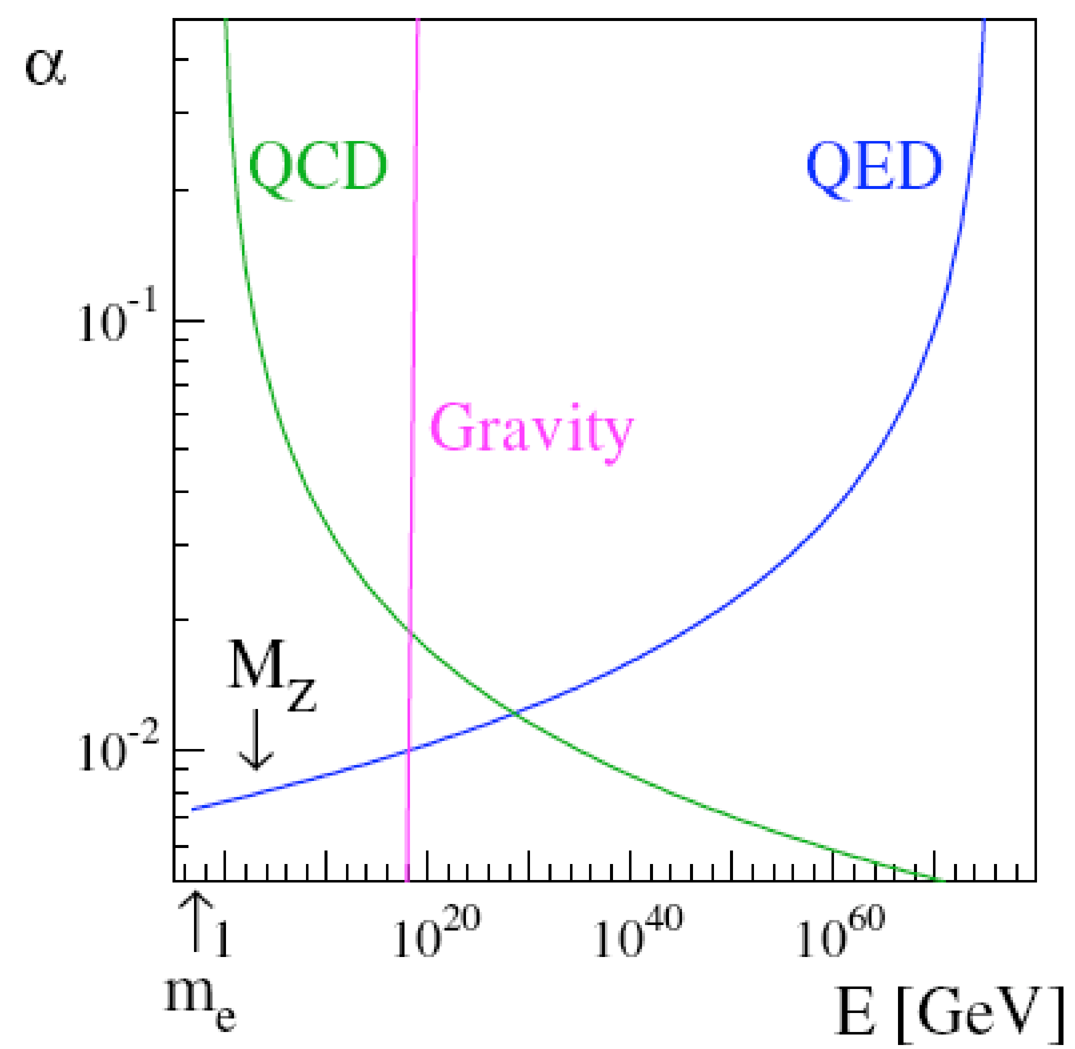

(a)

Roughly speaking, quark loop diagram (a) contributes a loop, diagram (b) gives a positive contribution proport is dominant and make the overall beta function negativ

$$b_0 = \frac{11N_c - 2n_f}{12\pi} \quad >0 \quad \Rightarrow$$

$$b_0 = -\frac{n_f}{3\pi} \quad <0 \quad \Rightarrow$$

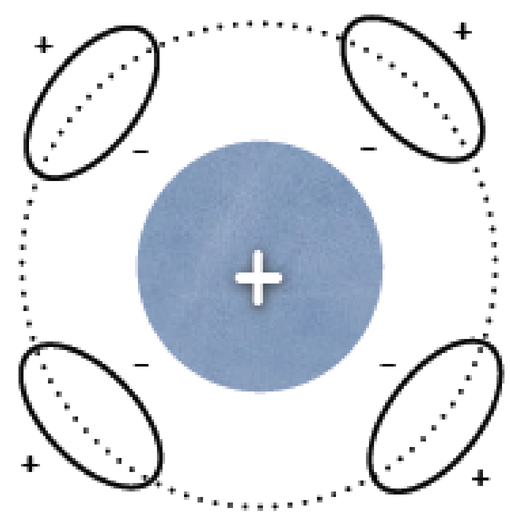
$$\alpha_{EM}(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2_{\rm QED}}}$$



Why is the beta function negative in QCD?

QED

charge screening



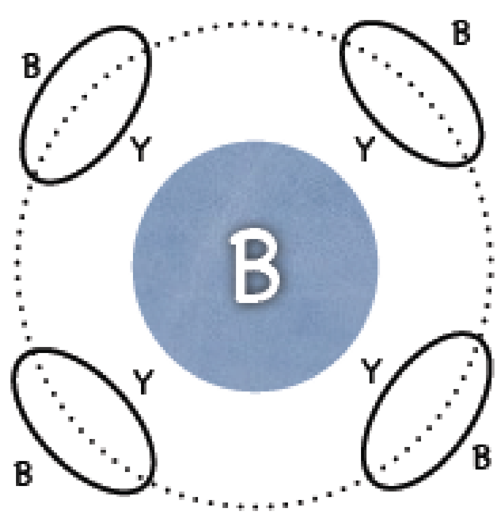
as a result the charge increases as you get closer to the center

DIELECTRIC $\varepsilon > 1$

Why is the beta function negative in QCD?

QCD

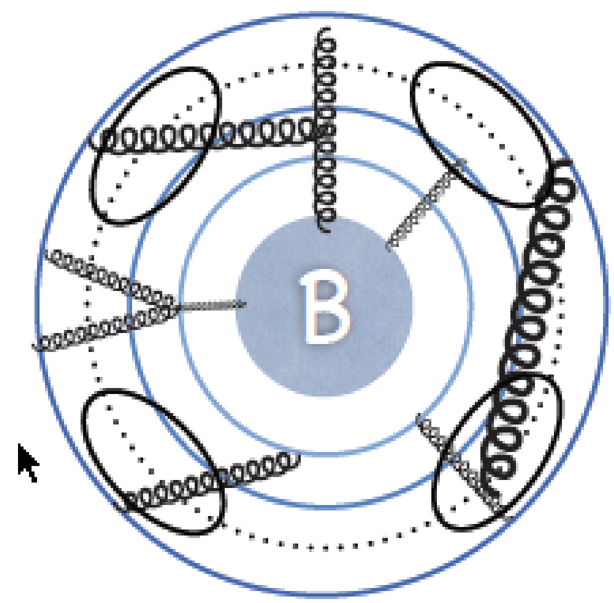
charge screening
from quarks



DIAMAGNETIC **μ**<1
(=DIELECTRIC **ε**>1, SINCE **με**=1)

$$\delta\mu = -(-1/3 + (2 \times \frac{1}{2})^2)q^2 = -\frac{2}{3}q^2$$

charge anti-screening
from gluons



gluons align as little magnets along the color lines and make the field increase at larger distances.

PARAMAGNETIC **μ**>1

$$\delta\mu = (-1/3 + 2^2)q^2 = \frac{11}{3}q^2$$

Ren. group and asymptotic freedom

Given

$$\alpha_S(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}} \qquad b_0 = \frac{11N_c - 2n_f}{12\pi}$$

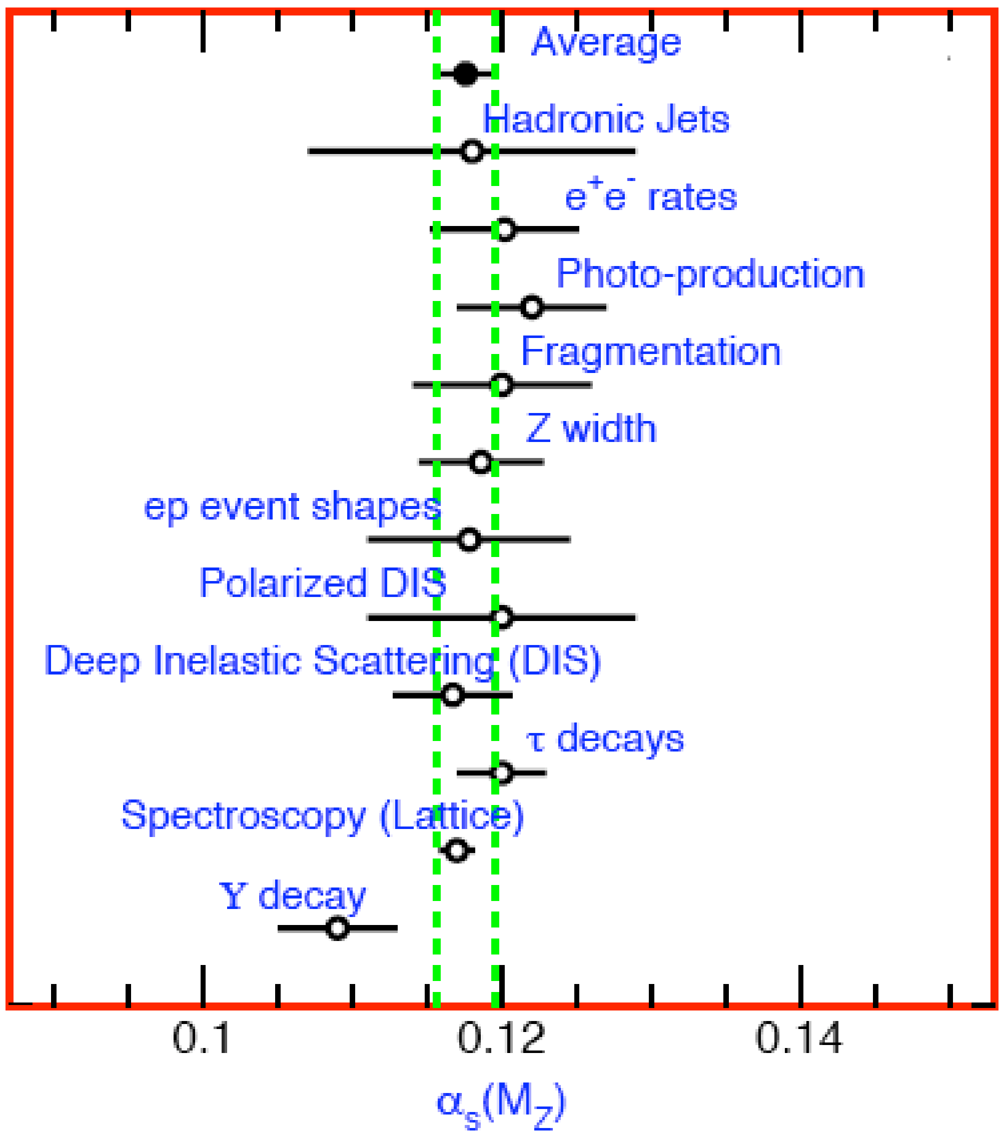
It is tempting to use identify Λ with Λ_S=300 MeV and see what we get for LEP I

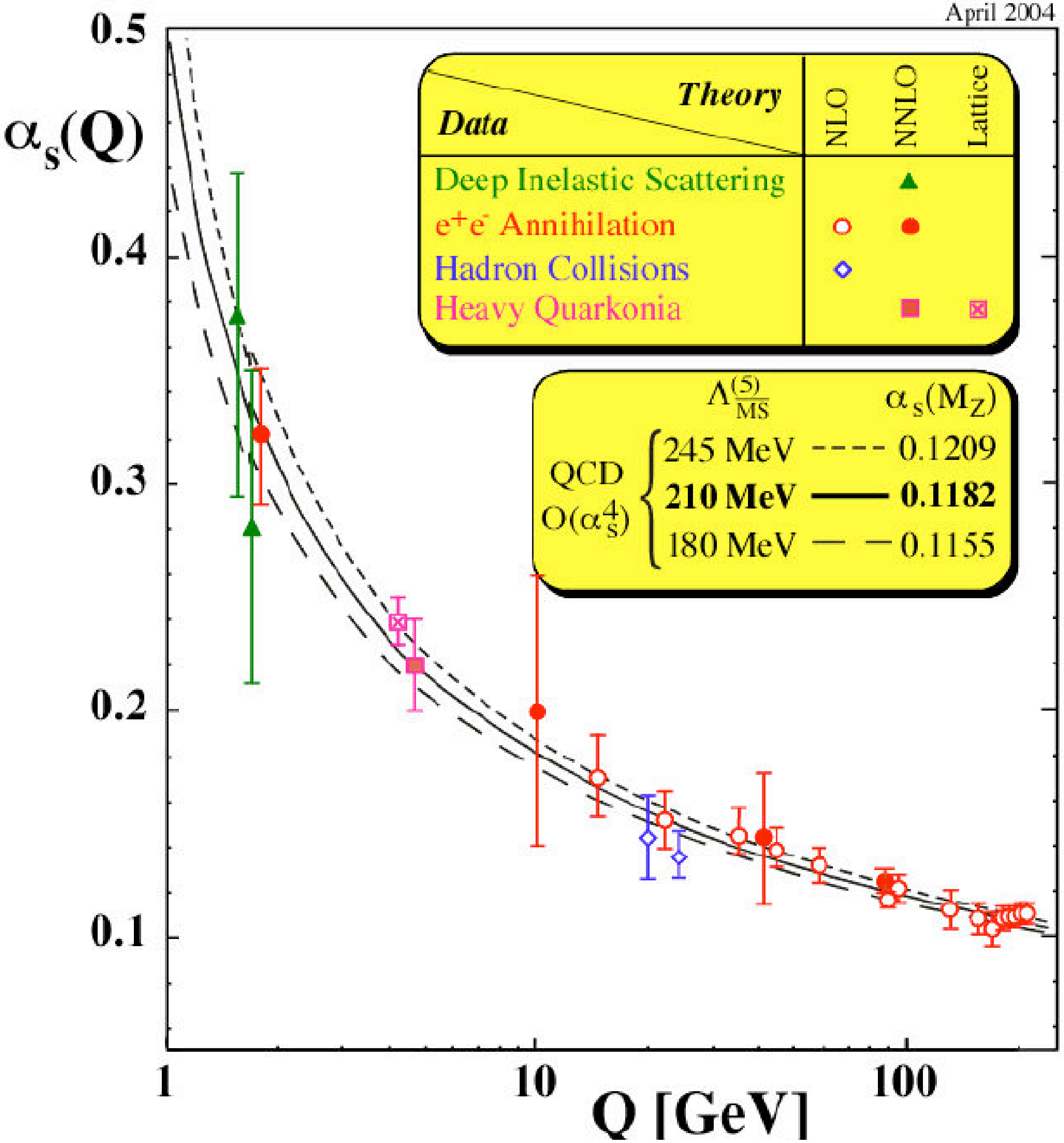
$$R(M_Z) = R_0 \left(1 + \frac{\alpha_S(M_Z)}{\pi}\right) = R_0(1 + 0.046)$$

which is in very reasonable agreement with LEP.

This example is very sloppy since it does not take into account heavy flavor thresholds, higher order effects, and so on. However it is important to stress that had we measured 8% effect at LEP I we would have extracted Λ= 5 GeV, a totally unacceptable value...

α_S: Experimental results





Many measurements at different scales all leading to very consistent results once evolved to the same reference scale, M_Z.

Scale dependence

$$R_2^{\rm ren}(\alpha_S(\mu), \frac{\mu^2}{Q^2}) = R_0 \left(1 + \frac{\alpha_S(\mu)}{\pi} + \left[c + \pi b_0 \log\frac{\mu^2}{Q^2}\right]\left(\frac{\alpha_S(\mu)}{\pi}\right)^2\right)$$

As we said, at all orders physical quantities do not depend on the choice of the renormalization scale. At fixed order, however, there is a residual dependence due to the non-cancellation of the higher order logs:

$$\frac{d}{d\log\mu}\sum_{n=1}^{N} c_n(\mu)\alpha_S^n(\mu) \sim \mathcal{O}\left(\alpha_S^n(\mu)^{N+1}(\mu)\right)$$

So possible (related) questions are:

* Is there a systematic procedure to estimate the residual uncertainty in the theoretical prediction?

* Is it possible to identify a scale corresponding to our best guess for the theoretical prediction?

BTW: The above argument proves that the more we work the better a prediction becomes!

CHOOSING THE SCALE IN E+E- → HADRONS

Cross section for e+e- → hadrons:

$$\sigma_{tot} = \frac{12\pi\alpha^2}{s}\left(\sum_q q_f^2\right)(1+\Delta)$$

Let's take our best TH prediction

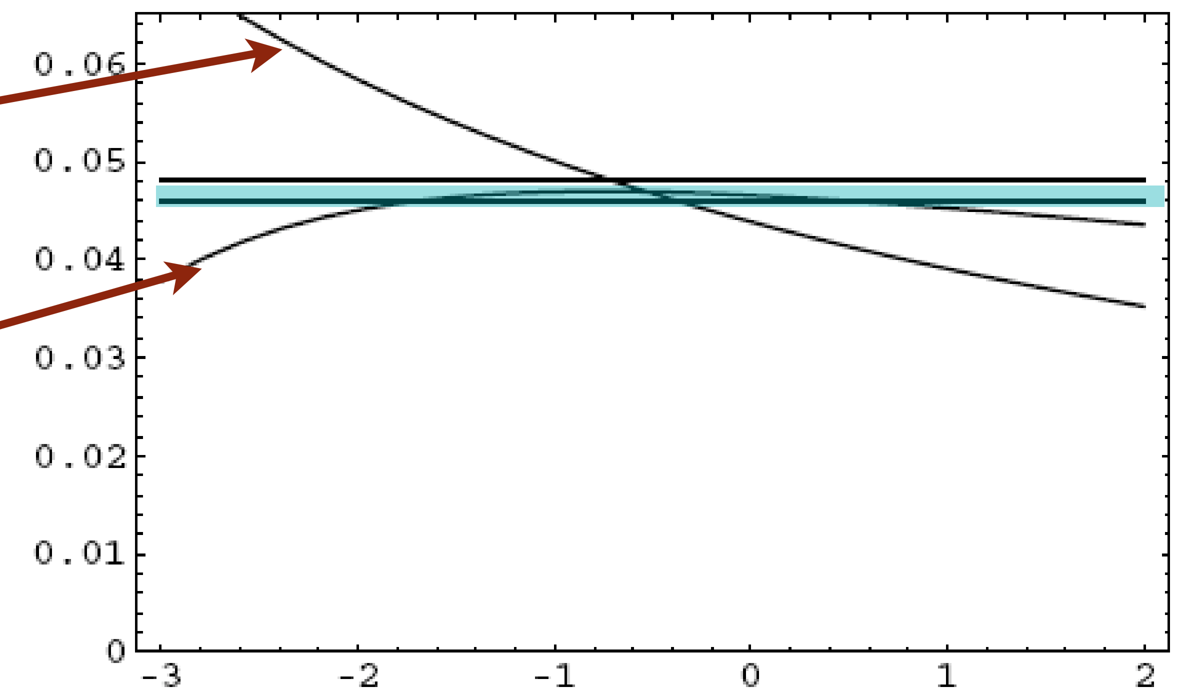
$$\Delta(\mu) = \frac{\alpha_S(\mu)}{\pi} + [1.41 + 1.92\log(\mu^2/s)]\left(\frac{\alpha_S(\mu)}{\pi}\right)^2$$
$$= [-12.8 + 7.82\log(\mu^2/s) + 3.67\log^2(\mu^2/s)]\left(\frac{\alpha_S(\mu)}{\pi}\right)^3$$

CHOOSING THE SCALE IN E+E- → HADRONS

Take $\alpha_s(M_z) = 0.117$, $\sqrt{s} = 34$ GeV, 5 flavors and let's plot $\Delta(\mu)$ as function of p where $\mu=2^p \sqrt{s}$.

First curve Δ_1

Second curve Δ_2



Possible choice:

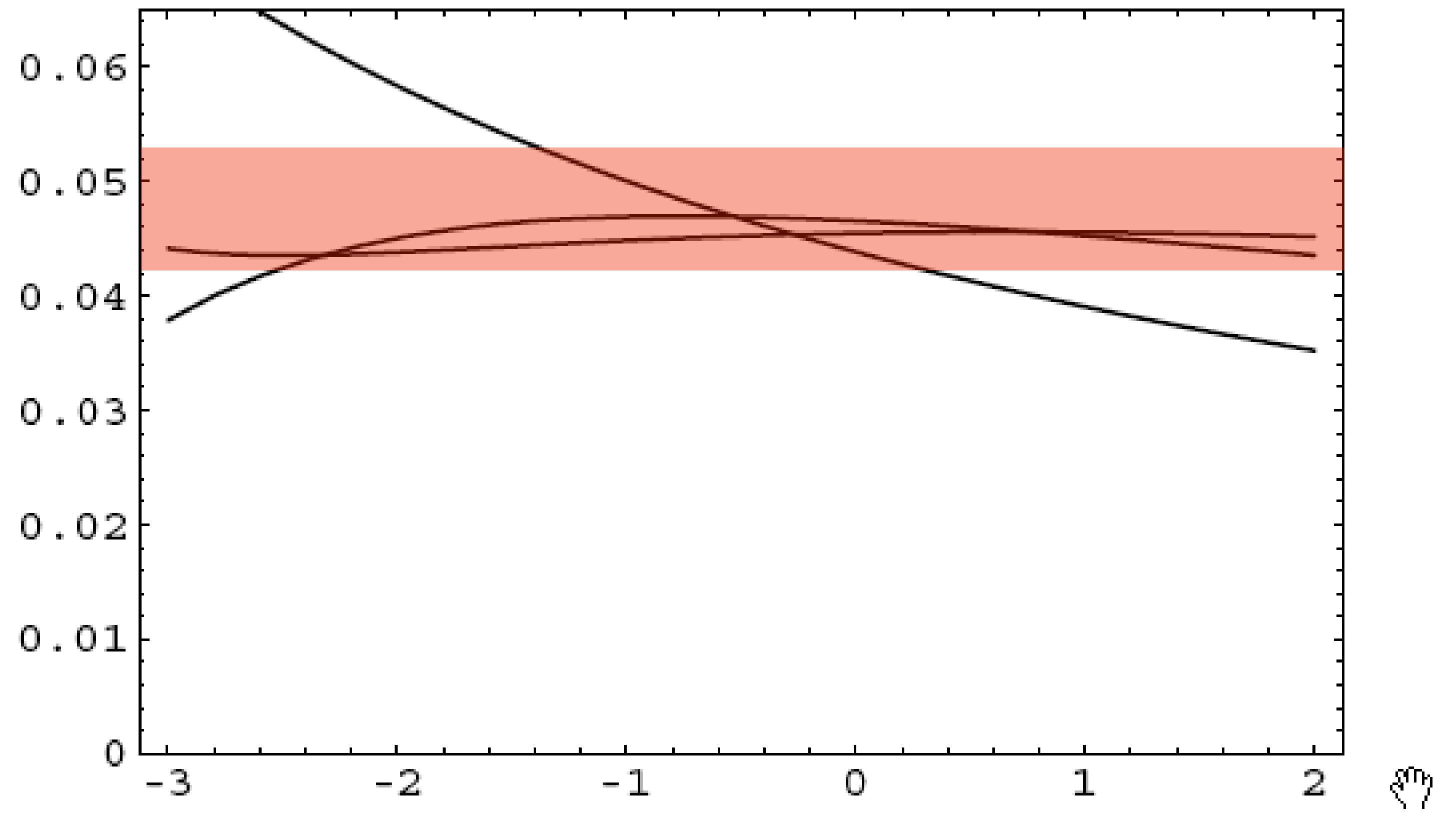
$\Delta_{PMS} = \Delta(\mu_0)$ where at μ_0 $d\Delta/d\mu=0$
and error band $p\in[1/2,2]$

Principle of mimimal sensitivity!

Improvement of a factor of two from LO to NLO!
How good is our error estimate?

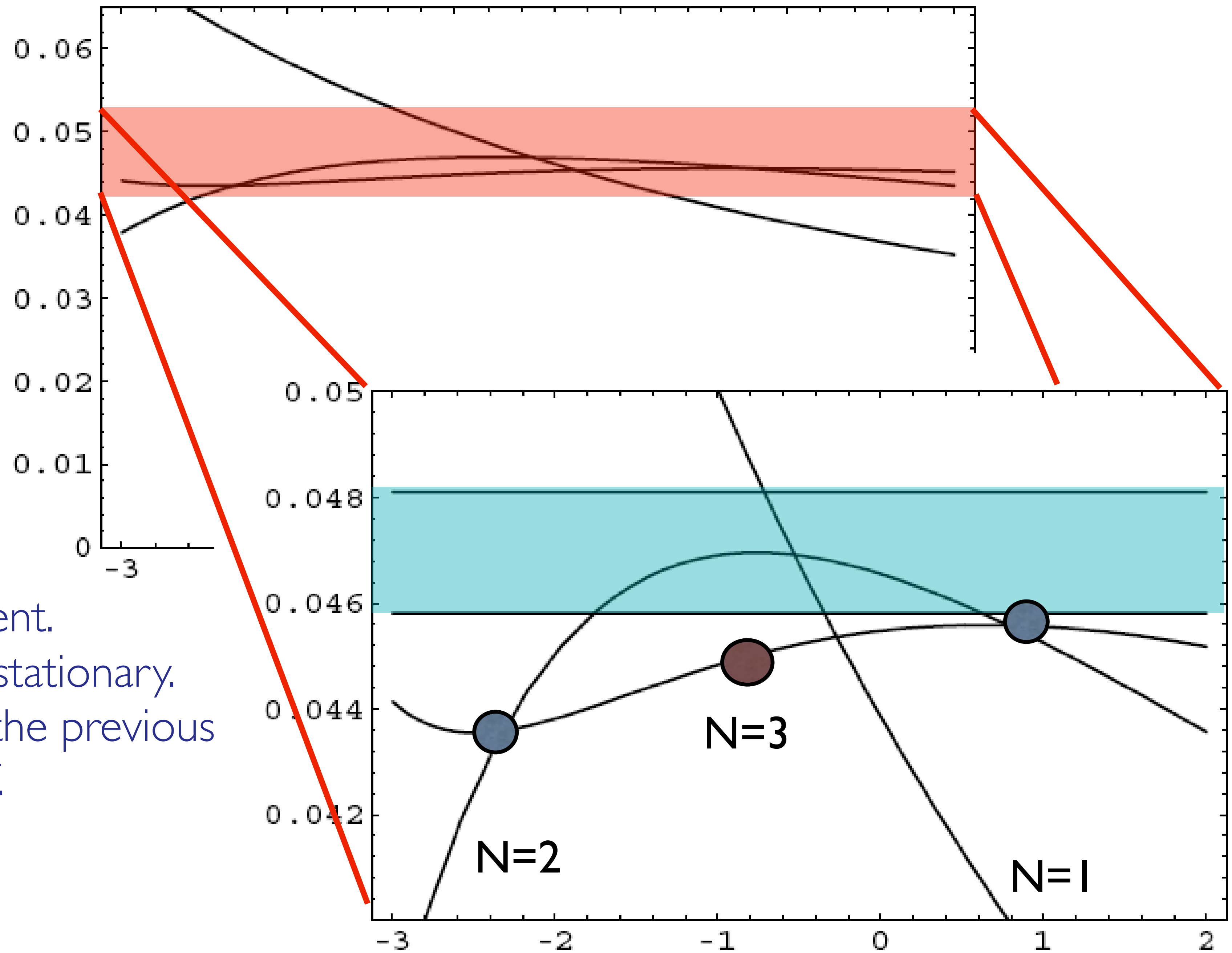
CHOOSING THE SCALE IN E+E- → HADRONS

What happens at α_s^3?



CHOOSING THE SCALE IN E+E- → HADRONS

What happens at α_s^3?



N=3 less scale dependent.
Two places where μ is stationary.
Take the average, then the previous estimate was sligthly off.

CHOOSING THE SCALE IN E+E- → HADRONS

Bottom line

There is no theorem that states the right 95% confidence interval for the uncertainty associated to the scale dependence of a theoretical predictions.

There are however many recipes available, where educated guesses (meaning physical). For example the so-called BLM choice.

In hadron-hadron collisions things are even more complicated due to the presence of another scale, the factorization scale, and in general also on a multi-scale processes...

Summary

1. We have given evidence of why we think QCD is a good theory: hadron spectrum, scaling, QCD is a renormalizable and asymptotically free QFT, low energy (broken) symmetries.

2. We have seen how gauge invariance is realized in QCD starting from QED.

3. We have illustrated with an example the use of the renormalization group and the appearance of asymptotic freedom.

QCD redux

part II

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New set of questions

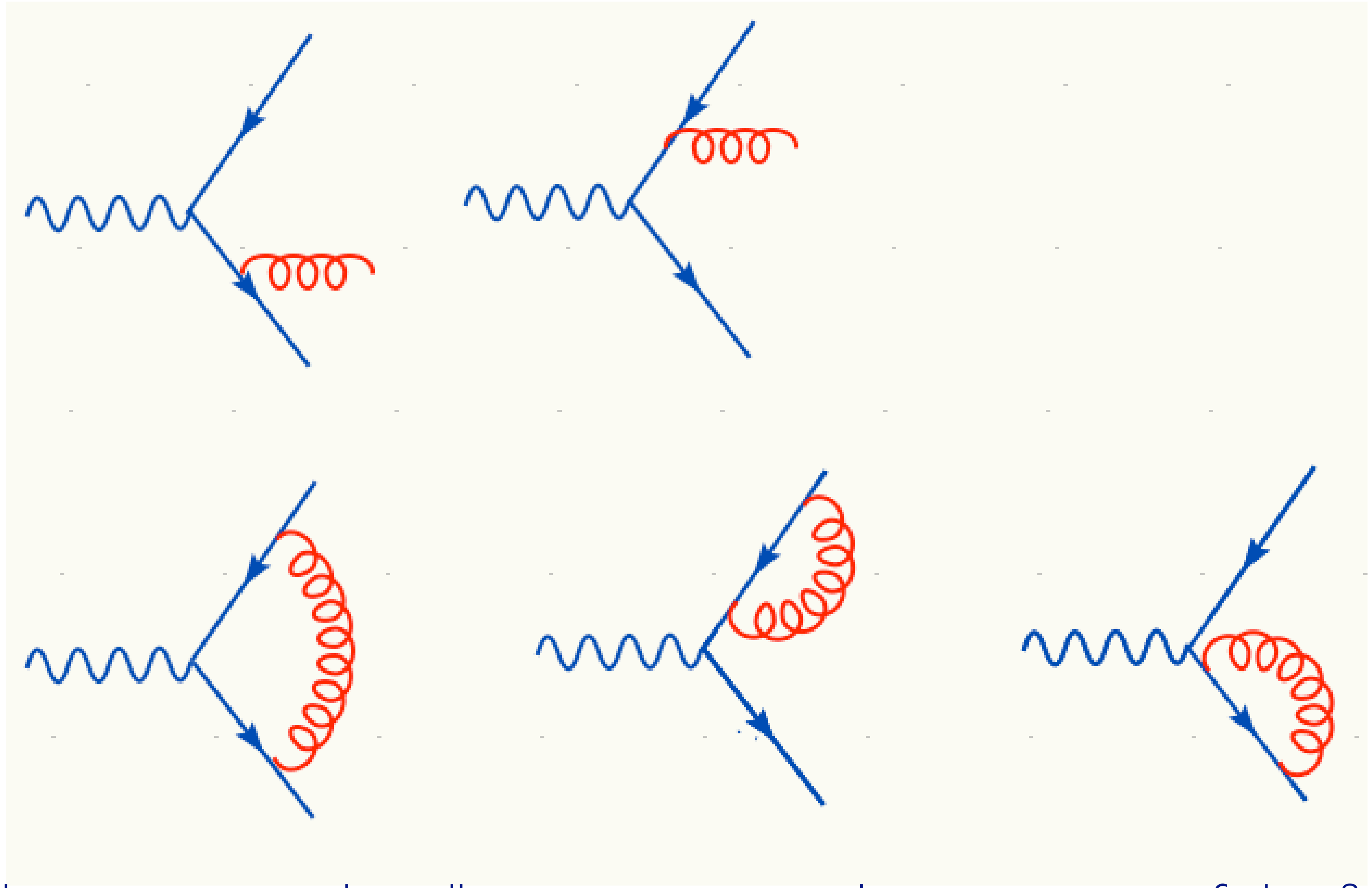
The "infrared" behaviour of QCD

1. How can we identify a cross sections for producing quarks and gluons with a cross section for producing hadrons?

2. Given the fact that free quarks are not observed, why is the computed Born cross section so good?

3. Are there other calculable, i.e., that do not depend on the non-perturbative dynamics (like hadronization), quantities besides the total cross section?

Anatomy of a NLO calculation



Real

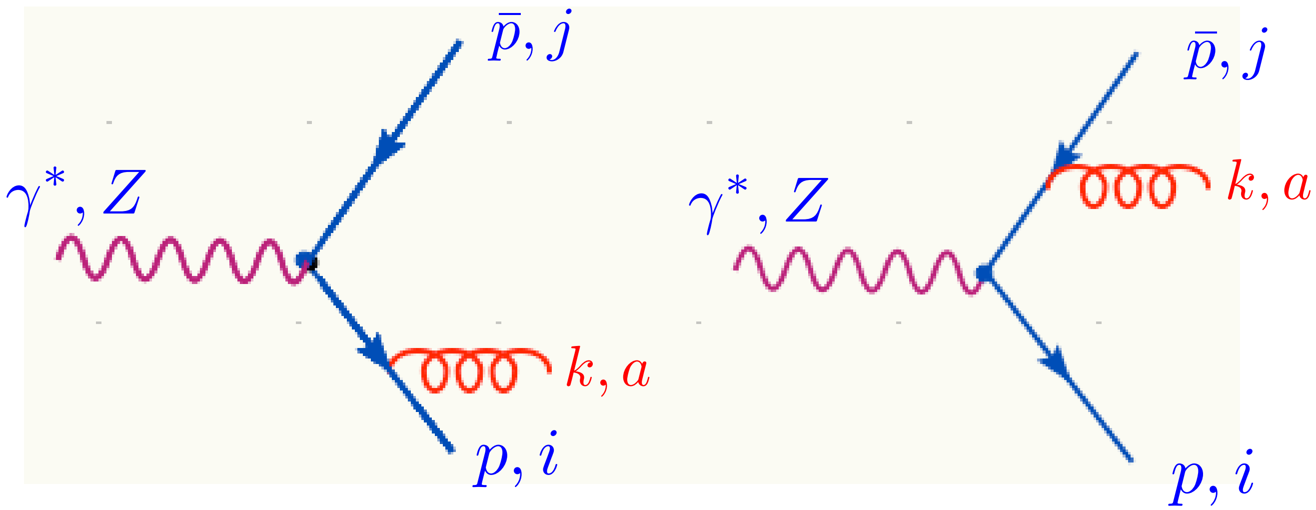
Virtual

The KLN theorem states that divergences appear because some of the final state are physically degenerate but we treated them as different. A final state with a soft gluon is nearly degenerate with a final state with no gluon at all (virtual).

$$\int \frac{d^d k}{(2\pi)^d} \cdots$$

$$\sigma^{\rm NLO} = \int_R |M_{real}|^2 d\Phi_3 + \int_V 2Re\left(M_0 M^*_{virt}\right) d\Phi_2 = \text{finite!}$$

Anatomy of a NLO calculation



Let's consider the real gluon emission corrections to the process e+e- →qq. The full calculation is a little bit tedious, but since we in any case interested in the issues arising in the infra-red, we already start in that approximation.

$$A = \bar{u}(p)\not{\epsilon}(-ig_s)\frac{-i}{\not{p}+\not{k}}\Gamma^\mu v(\bar{p})t^a + \bar{u}(p)\Gamma^\mu\frac{i}{\not{\bar{p}}+\not{k}}(-ig_s)\not{\epsilon}v(\bar{p})t^a$$

$$= -g_s\left[\frac{\bar{u}(p)\not{\epsilon}(\not{p}+\not{k})\Gamma^\mu v(\bar{p})}{2p\cdot k} - \frac{\bar{u}(p)\Gamma^\mu(\not{\bar{p}}+\not{k})\not{\epsilon}v(\bar{p})}{2\bar{p}\cdot k}\right]t^a$$

The denominators $2p\cdot k = p_0k_0(1-\cos\theta)$ give singularities for collinear ($\cos\theta \to 1$) or soft ($k_0 \to 0$) emission. By neglecting k in the numerators and using the Dirac equation, the amplitude simplifies and factorizes over the Born amplitude:

$$A_{soft} = -g_s t^a\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{\bar{p}\cdot\epsilon}{\bar{p}\cdot k}\right)A_{Born} \qquad A_{Born} = \bar{u}(p)\Gamma^\mu v(\bar{p})$$

Factorization: Independence of long-wavelength (soft) emission form the hard (short-distance) process. Soft emission is universal!!

Anatomy of a NLO calculation

By squaring the amplitude we obtain:

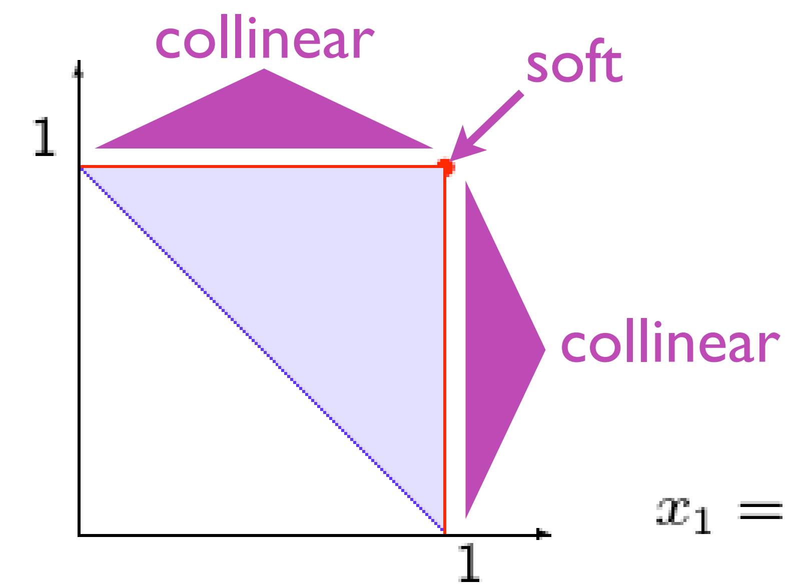
$$\sigma_{q\bar{q}g}^{\rm REAL} = C_F g_s^2 \sigma_{q\bar{q}}^{\rm Born} \int \frac{d^3k}{2k^0(2\pi)^3} 2 \frac{p\cdot\bar{p}}{(p\cdot k)(\bar{p}\cdot k)}$$

$$= C_F \frac{\alpha_S}{2\pi} \sigma_{q\bar{q}}^{\rm Born} \int d\cos\theta \frac{dk^0}{k^0} \frac{4}{(1-\cos\theta)(1+\cos\theta)}$$

Two collinear divergences and a soft one. Very often you find the integration over phase space expressed in terms of x_1 and x_2, the fraction of energies of the quark and anti-quark:

$$x_1 = 1 - x_2 x_3 (1-\cos\theta_{23})/2$$
$$x_2 = 1 - x_1 x_3 (1-\cos\theta_{13})/2$$
$$x_1 + x_2 + x_3 = 2$$
$$0 \le x_1, x_2 \le 1, \quad \text{and} \quad x_1 + x_2 \ge 1$$

$$x_2 = \frac{2E_{\bar{q}}}{\sqrt{s}}$$



$$x_1 = \frac{2E_q}{\sqrt{s}}$$

So we can now predict the divergent part of the virtual contribution, while for the finite part an explicit calculation is necessary:

$$\sigma_{q\bar{q}}^{\rm VIRT} = -\sigma_{q\bar{q}}^{\rm Born} C_F \frac{\alpha_S}{2\pi} \int d\cos\theta' \frac{dk'_0}{k'_0} \frac{1}{1-\cos^2\theta} 2\delta(k'_0)[\delta(1-\cos\theta')+\delta(1+\cos\theta')] + \ldots$$

Anatomy of a NLO calculation

Summary:

$$\sigma^{\text{REAL}} + \sigma^{\text{VIRT}} = \infty - \infty = ?$$

Solution: regularize the "intermediate" divergences, by giving a gluon a mass (see later) or going to d=4-2**ε** dimensions.

$$\int^1 \frac{1}{1-x} dx = -\log 0 \overset{\text{regularization}}{\rightarrow} \int^1 \frac{(1-x)^{-2\epsilon}}{1-x} dx = -\frac{1}{2\epsilon}$$

This gives:

$$\sigma^{\text{REAL}} = \sigma^{\text{Born}} C_F \frac{\alpha_S}{2\pi} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 \right)$$

$$\sigma^{\text{VIRT}} = \sigma^{\text{Born}} C_F \frac{\alpha_S}{2\pi} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 \right)$$

$$\lim_{\epsilon \to 0} (\sigma^{\text{REAL}} + \sigma^{\text{VIRT}}) = C_F \frac{3}{4} \frac{\alpha_S}{\pi} \sigma^{\text{Born}}$$

$$R_1 = R_0 \left(1 + \frac{\alpha_S}{\pi} \right)$$ as presented before

New set of questions

1. How can we identify a cross sections for producing (few) quarks and gluons with a cross section for producing (many) hadrons?

2. Given the fact that free quarks are not observed, why is the computed Born cross section so good?

Answers:

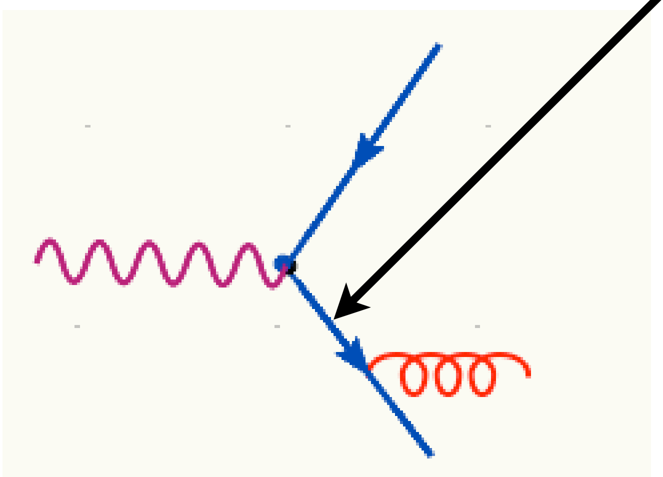
The Born cross section IS NOT the cross section for producing q qbar, since the coefficients of the perturbative expansion are infinite! But this is not a problem since we don't observe q qbar and nothing else. So there is no contradiction here.

On the other hand the cross section for producing hadrons is finite order by order and its lowest order approximation IS the Born.

A further insight can be gained by thinking of what happens in QED and what is different there. For instance soft and collinear divergence are also there. In QED one can prove that cross section for producing "only two muons" is zero...

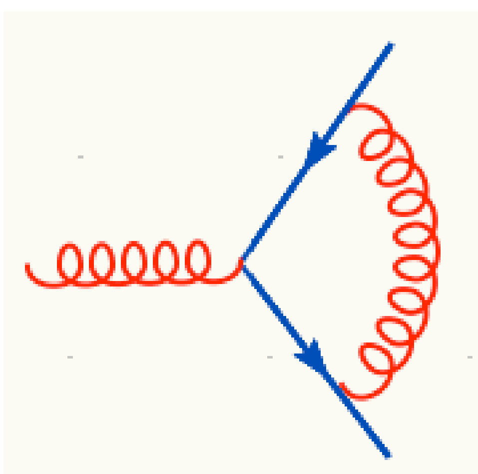
Infrared divergences

$$A_{soft} = -g_s t^a \left(\frac{p \cdot \epsilon}{p \cdot k} - \frac{\bar{p} \cdot \epsilon}{\bar{p} \cdot k} \right) A_{Born}$$



Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored.

This is because there are configurations in phase space for gluons and quarks, i.e. when gluons are soft and/or when are pairs of partons are collinear.

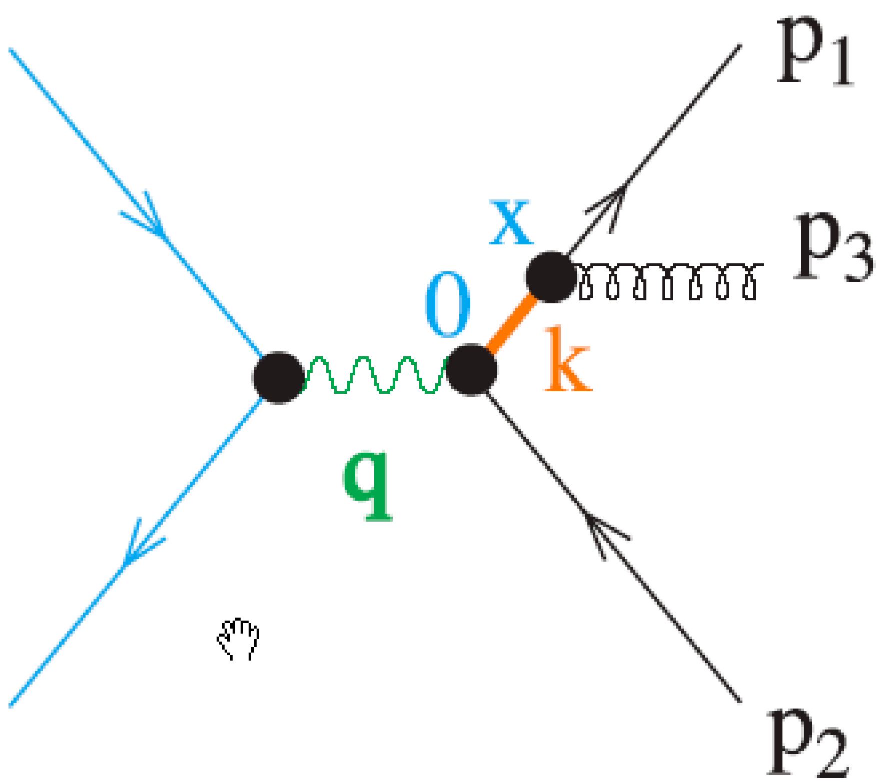


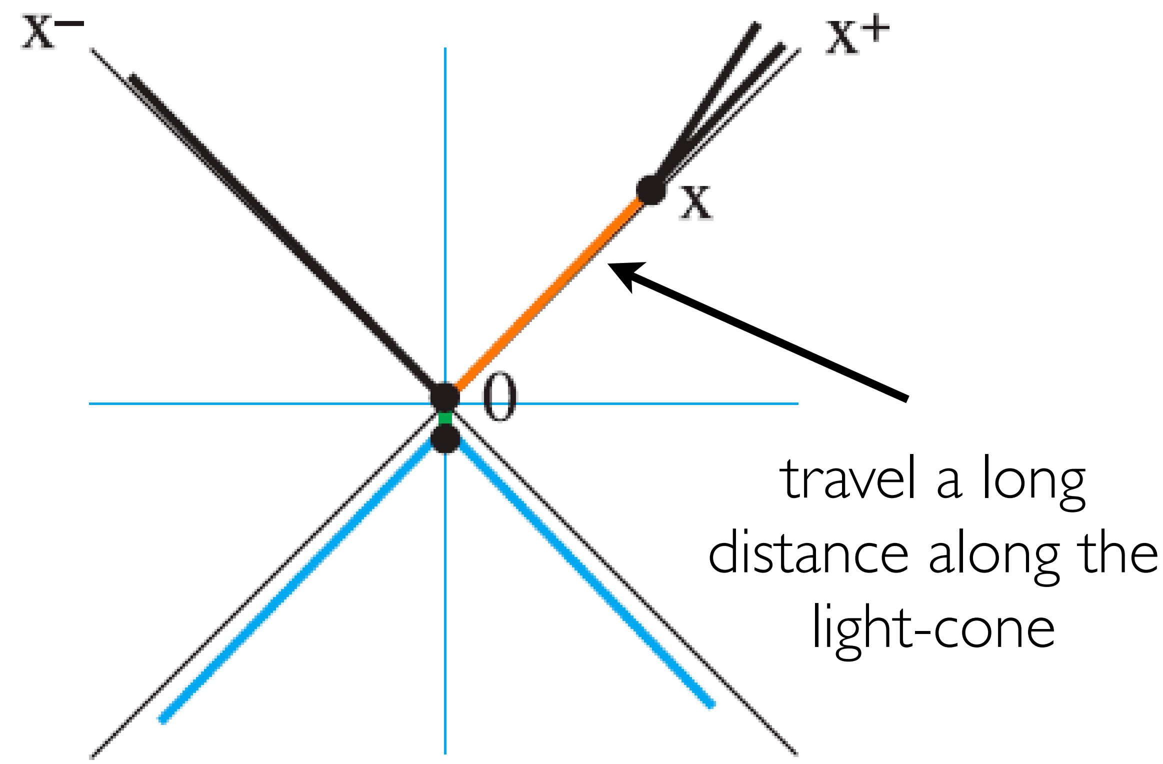
$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+p)^2 (k-\bar{p})^2}$$

also for soft and collinear or collinear configurations associated to the virtual partons with the region of integration of the loop momenta.

Space-time picture of IR singularities

The singularities can be understood in terms of light-cone coordinates. Take $p^\mu=(p^0, p^1, p^2, p^3)$ and define $p^\pm=(p^0 \pm p^3)/\sqrt{2}$. Then choose the direction of the + axis as the one of the largest between + and - . A particle with small virtuality travels for a long time along the x^+ direction.





$$k^+ \simeq \sqrt{s}/2$$ large

$$k^- \simeq (k^T + 2k^+k^-)\sqrt{s}/2$$ small

$$x^+ \simeq 1/k^-$$ large

$$x^- \simeq 1/k^+$$ small

Infrared divergences

Infrared divergences arise from interactions that happen a long time after the creation of the quark/antiquark pair.

When distances become comparable to the hadron size of ~1 Fermi, quasi-free partons of the perturbative calculation are confined/hadronized non-perturbatively.

We have seen that in total cross sections such divergences cancel. But what about for other quantities?

Obviously, the only possibility is to try to use the pQCD calculations for quantities that are not sensitive to the to the long-distance physics.

Can we formulate a criterium that is valid in general?

YES! It is called INFRARED SAFETY

Infrared-safe quantities

DEFINITION: quantities are that are insensitive to soft and collinear branching.

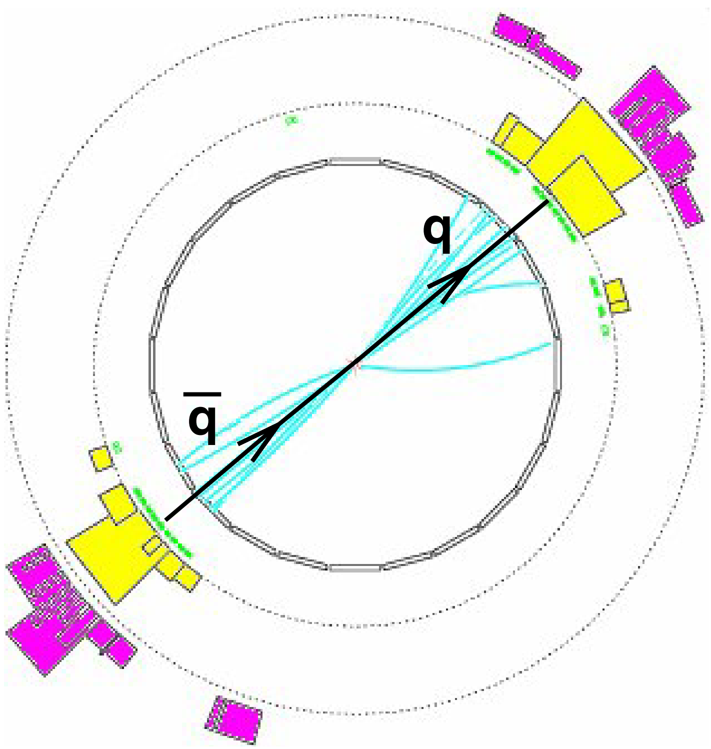
For these quantities, an extension of the general theorem (KLN) exists which proves that infrared divergences cancel between real and virtual or are simply removed by kinematic factors.

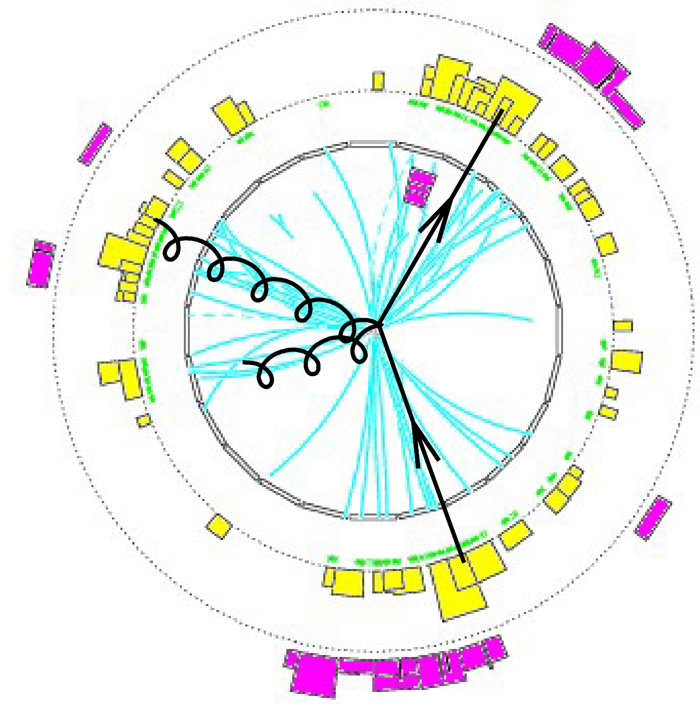
Such quantities are determined primarily by hard, short-distance physics. Long-distance effects give power corrections, suppressed by the inverse power of a large momentum scale (which must be present in the first place to justify the use of PT).

Examples:

1. Multiplicity of gluons is not IRC safe
2. Energy of hardest particle is not IRC safe
3. Energy flow into a cone is IRC safe

EVENT SHAPE VARIABLES





pencil-like → spherical

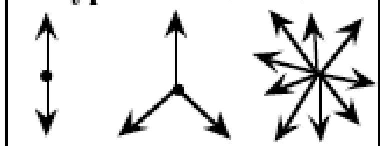
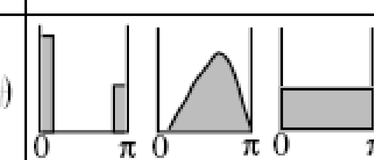
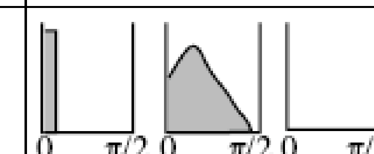
Event shape variables

The idea is to give more information than just total cross section by defining "shapes" of an hadronic event (pencil-like, planar, spherical, etc..)

In order to be comparable with theory it MUST be IR-safe, that means that the quantity should not change if one of the parton "branches" $p_k \to p_i + p_j$

Examples are: Thrust, Spherocity, C-parameters,...

Similar quantities exist for hadron collider too, but they much less used.

Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\Sigma_i \lvert \vec{p}_i \vec{n} \rvert}{\Sigma_i \lvert \vec{p}_i \rvert} \right)$	1	≥2/3	≥1/2	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	≤1/3	≤1/√2	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	≤1/2	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	≤1/3	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq \ldots \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\Sigma_i p_i^\alpha p_i^\beta}{\Sigma_i p_i^2}$	0	≤3/4	≤1	none (not infrared safe)
Aplanarity	$A = 1.5\, Q_1$	0	0	≤1/2	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = (\Sigma_i E_i^2 - \Sigma_i \vec{p}_i^{\,2})_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	≤1/3 ≤1/3	≤1/2 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\Sigma_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2\, \Sigma_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	≤1/(2√3) ≤1/(2√3)	≤1/(2√2) ≤1/(2√3)	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi - \frac{\Delta\chi}{2}}^{\chi + \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$				(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$				$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

Is the thrust IR safe?

$$T = \max_{\vec{n}} \frac{\sum_i \vec{p}_i \cdot \vec{n}}{\sum_i \vec{p}_i}$$

Contribution from a particle with momentum going to zero drops out.

Replacing one particle with two collinear ones does not change the thrust:

$$|(1-\lambda)\vec{p}_k \cdot \vec{u}| + |\lambda \vec{p}_k \cdot \vec{u}| = |\vec{p}_k \cdot \vec{u}|$$

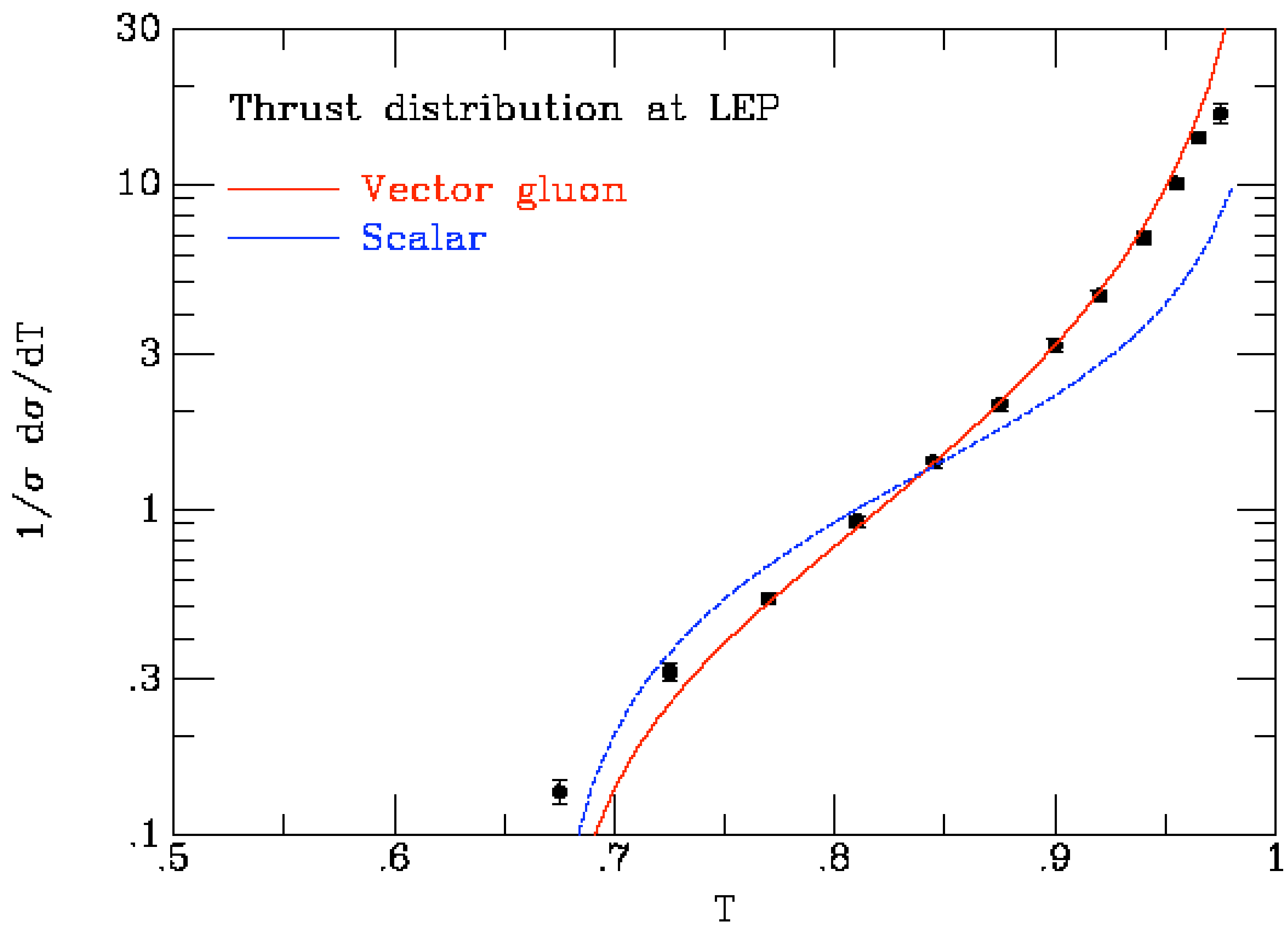
and

$$|(1-\lambda)\vec{p}_k| + |\lambda \vec{p}_k| = |\vec{p}_k|$$

Calculation of event shape variables: Thrust

The values of the different event-shape variables for different topologies are

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F\frac{\alpha_S}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right)-\frac{3(3T-2)(2-T)}{1-T}\right].$$



$O(\alpha_S{}^2)$ corrections (NLO) are also known. Comparison with data provide test of QCD matrix elements, through shape distribution and measurement of α_S from overall rate. Care must be taken around T=1 where

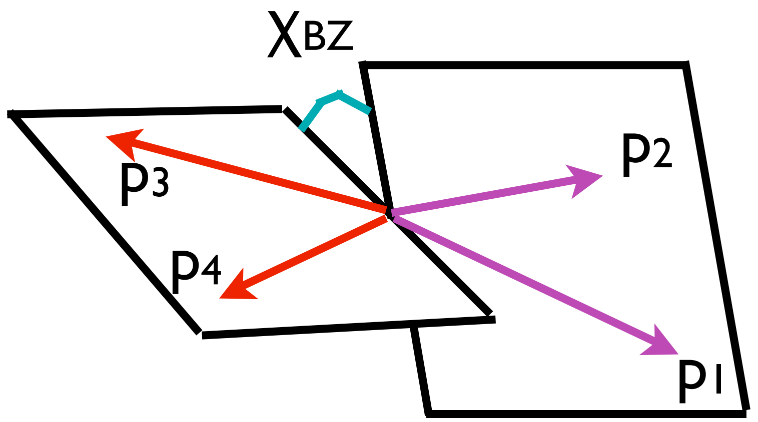
(a) hadronization effects become large and

(b) large higher order terms of the form $\alpha_S{}^N\,[\log^{2N-1}(1-T)]/(1-T)$ need to be resummed.

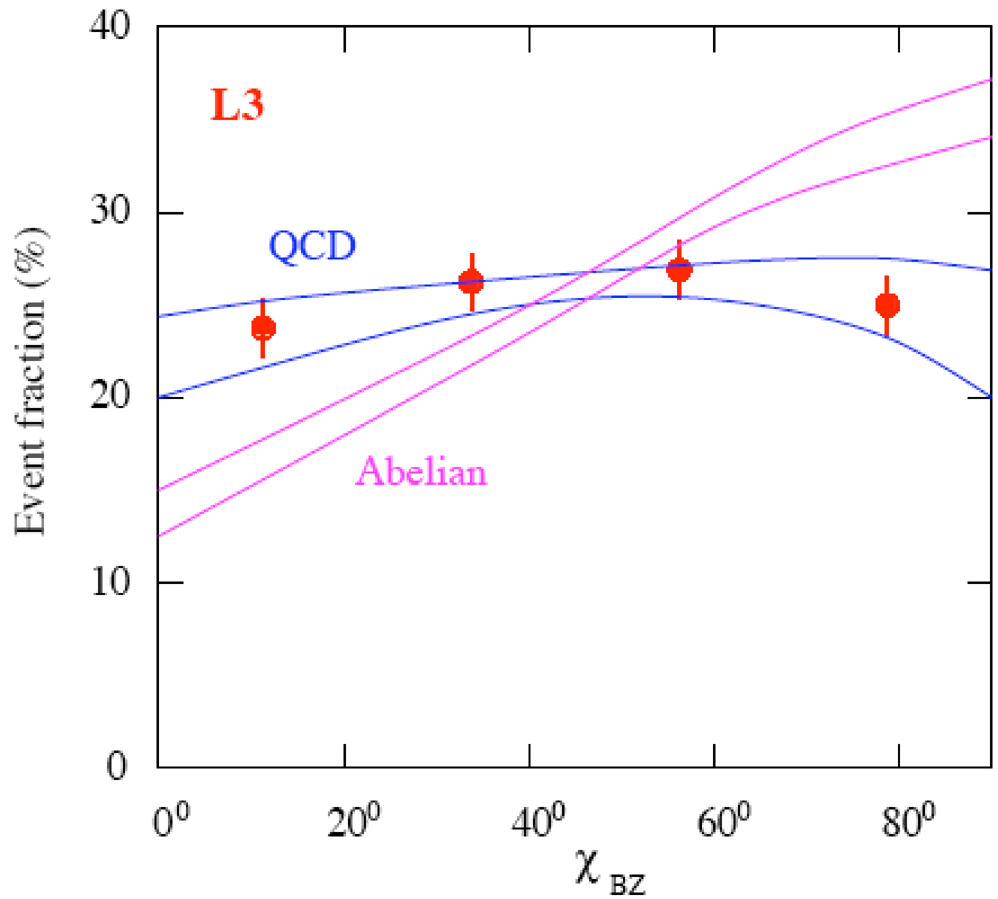
At lower T multi-jet matrix element become important.

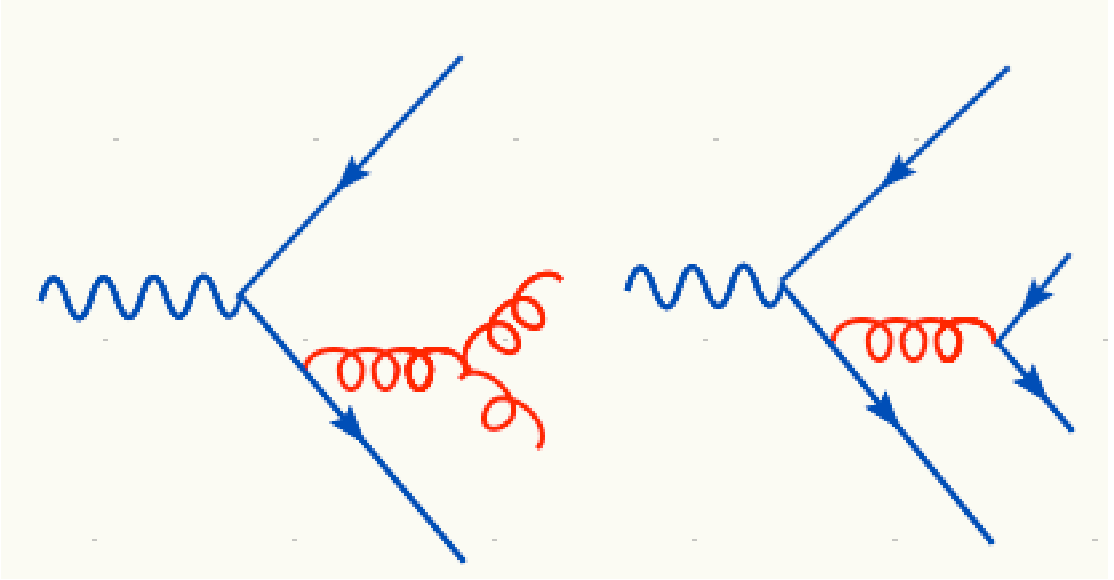
Intermezzo: how did we "see" the 3g vertex?

Angular correlations also provide interesting information about the properties of the matrix elements in QCD. One of these quantities is the so-called Bengtsson-Zerwas angle. It is the angle between planes of the two lowest and the two highest energy jets.



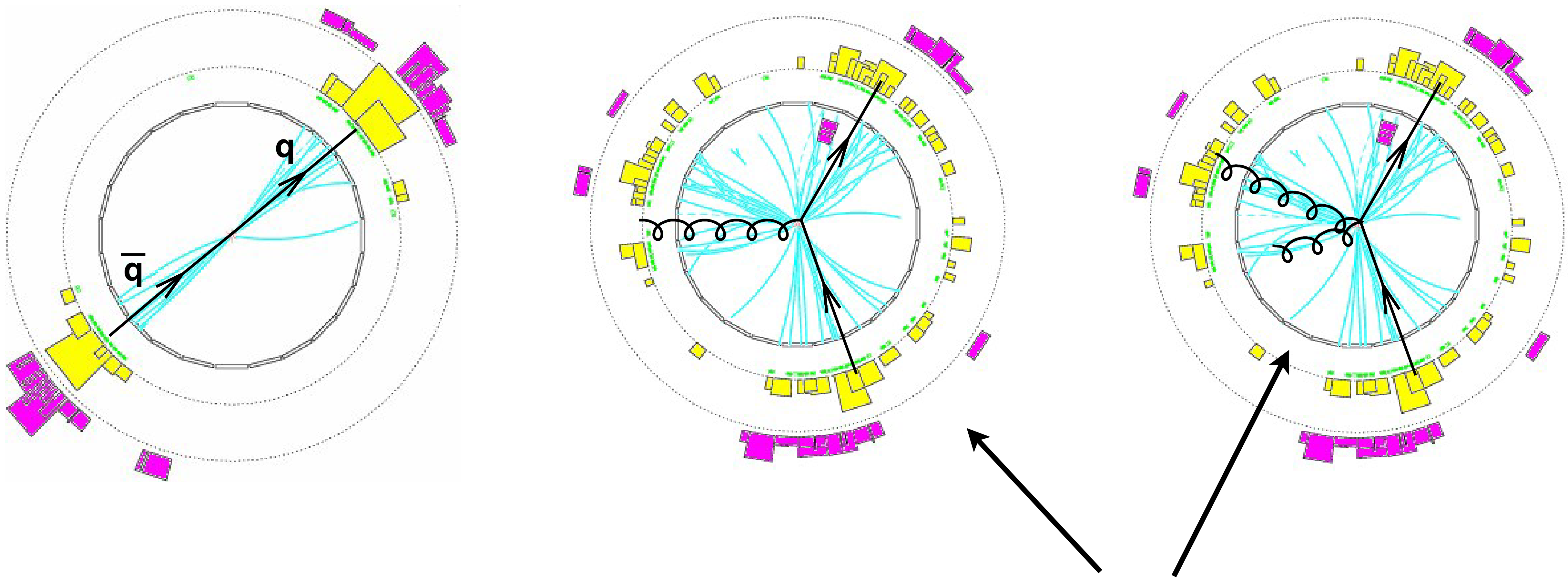
$$\cos\chi_{BZ} = \frac{(\boldsymbol{p}_1 \times \boldsymbol{p}_2)\cdot(\boldsymbol{p}_3 \times \boldsymbol{p}_4)}{|\boldsymbol{p}_1 \times \boldsymbol{p}_2|\,|\boldsymbol{p}_3 \times \boldsymbol{p}_4|} .$$





This quantity gives information on the presence and characteristics of the three-gluon vertex.

Jet algorithms



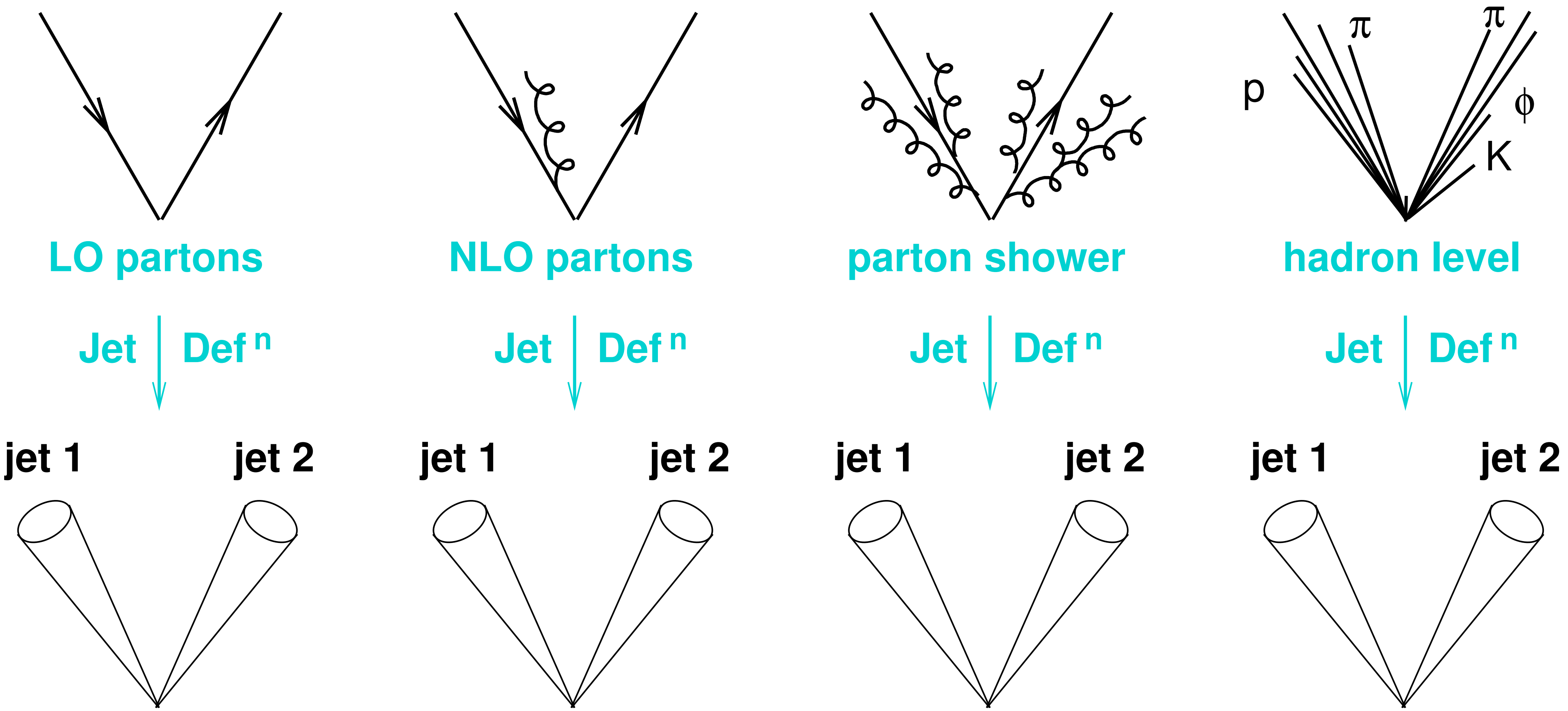
same event!!

2-jets 3-jets 4-jets

Jets are in the eye of the beholder!

Jet algorithms

A jet definition is a fully specified set of rules for projecting information from hundreds of hadrons, onto a handful of parton-like objects.



In the projection a lot of information is lost.

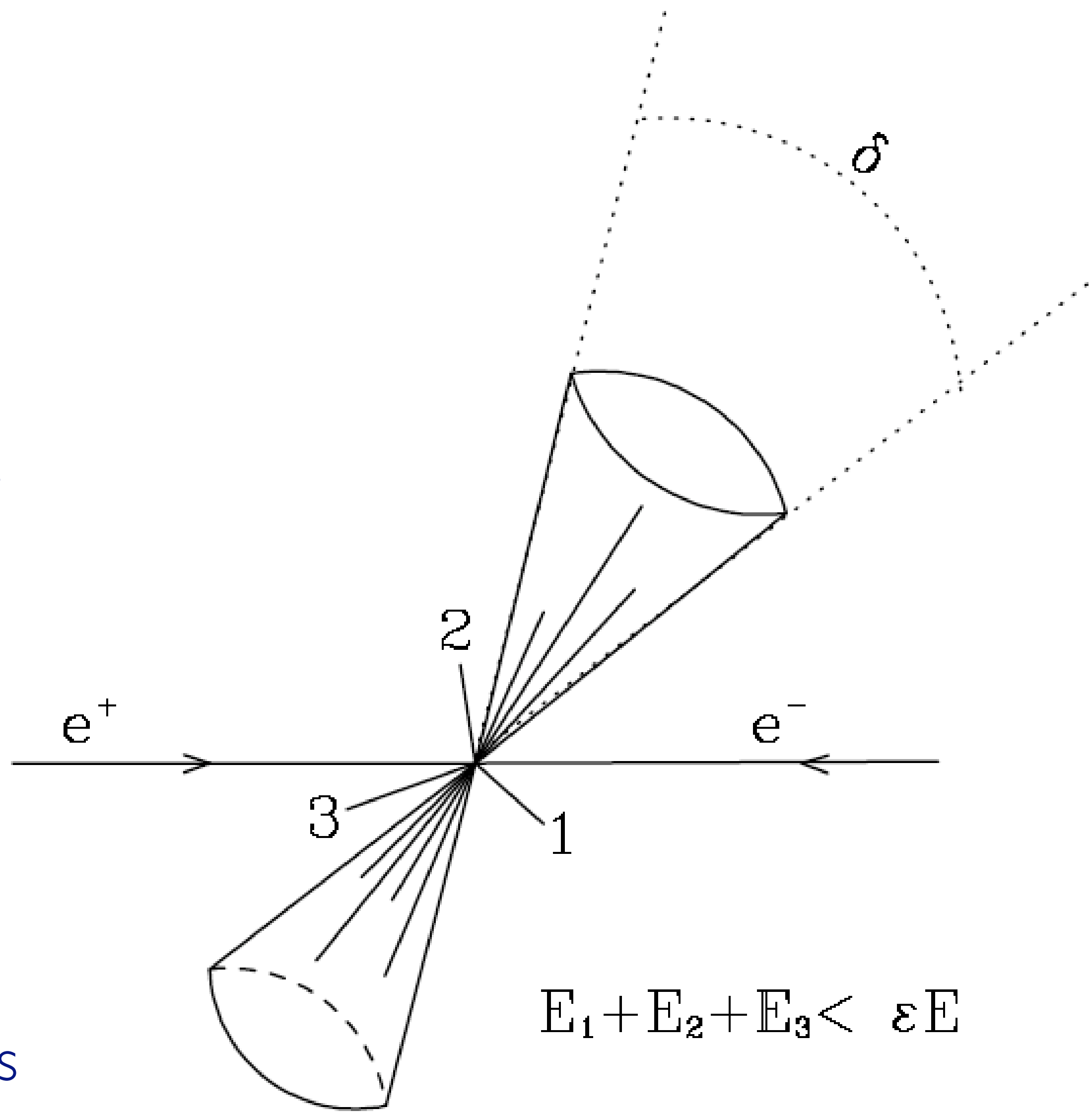
Projection to jets must be resilient to QCD effects

Jet algorithms

- The precise definition of a procedure how to cut be three-jet (and multi-jet) events is called "jet algorithm".

- Which jet algorithm to use for a given measurement/ experiment needs to be found out. Different algorithms have very different behaviors both experimentally and theoretically. Of course, it is important that a complete information is given on the jet algorithm when experimental data are to be compared with theory predictions!

- Weinberg-Sterman jets (intuitive definition):
  "An event is identified as a 2-jets if one can find 2 cones with opening angle δ that contain all but a small fraction εE of the total energy E".



Jets (top-down) at e-e+

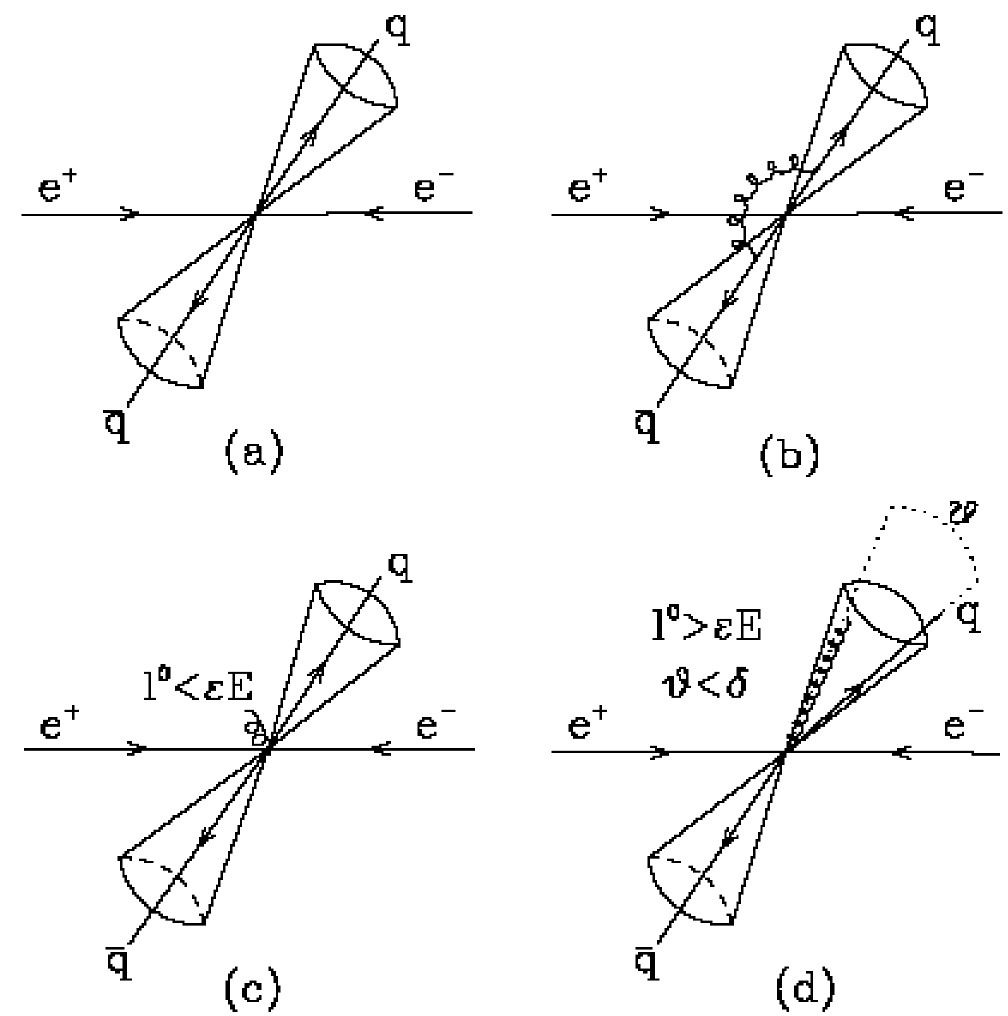
Let's see when the various contributions add up to the Sterman-Weinberg 2-jet cross section:

✷ The Born cross section contributes to the 2-jet cross section, INDEPENDENTLY of **ε** and **δ**.

✷The SAME as above for the virtual corrections.

✷The real corrections when k^0<**ε**E (soft).

✷The real corrections when k^0>**ε**E AND **θ**<**δ** (collinear).



$$\text{Born} + \text{Virtual} + \text{Real (a)} + \text{Real (b)} = \sigma^{\text{Born}} - \sigma^{\text{Born}} \frac{4\alpha_S C_F}{2\pi} \int_{\epsilon E}^{E} \frac{dk^0}{k^0} \int_{\delta}^{\pi-\delta} \frac{d\cos\theta}{1-\cos^2\theta}$$

$$= \sigma^{\text{Born}} \left(1 - \frac{4\alpha_S C_F}{2\pi} \log\epsilon \log\delta \right)$$

As long as **δ** and **ε** are not too small, we find that the fraction of 2-jet cross section is almost 1! At high energy most of the events are two-jet events. As the energy increases the jets become thinner.

A very simple jet iterative algorithm (bottom-up)

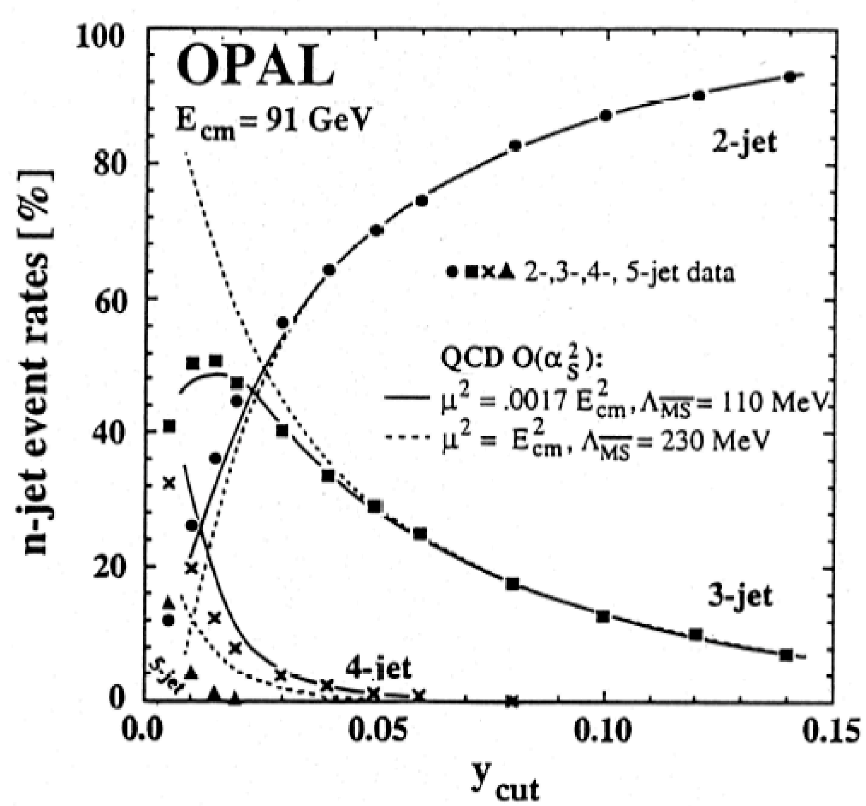
1. Consider $e^+e^- \to$ N partons
2. Consider all pairs i and j and calculate
   IF
   min $(p_i + p_j)^2 < y_{cut}\, s$
   THEN
   replace the two partons i,j by $p_{ij} = p_i + p_j$ and decrease N → N-1
3. IF N=1 THEN stop ELSE goto 2.
4. N = number of jets in the event using the "scale" y.

In our example
$0<x_1,x_2<1-y$, $x_1+x_2>1+y$
$y<1/3$

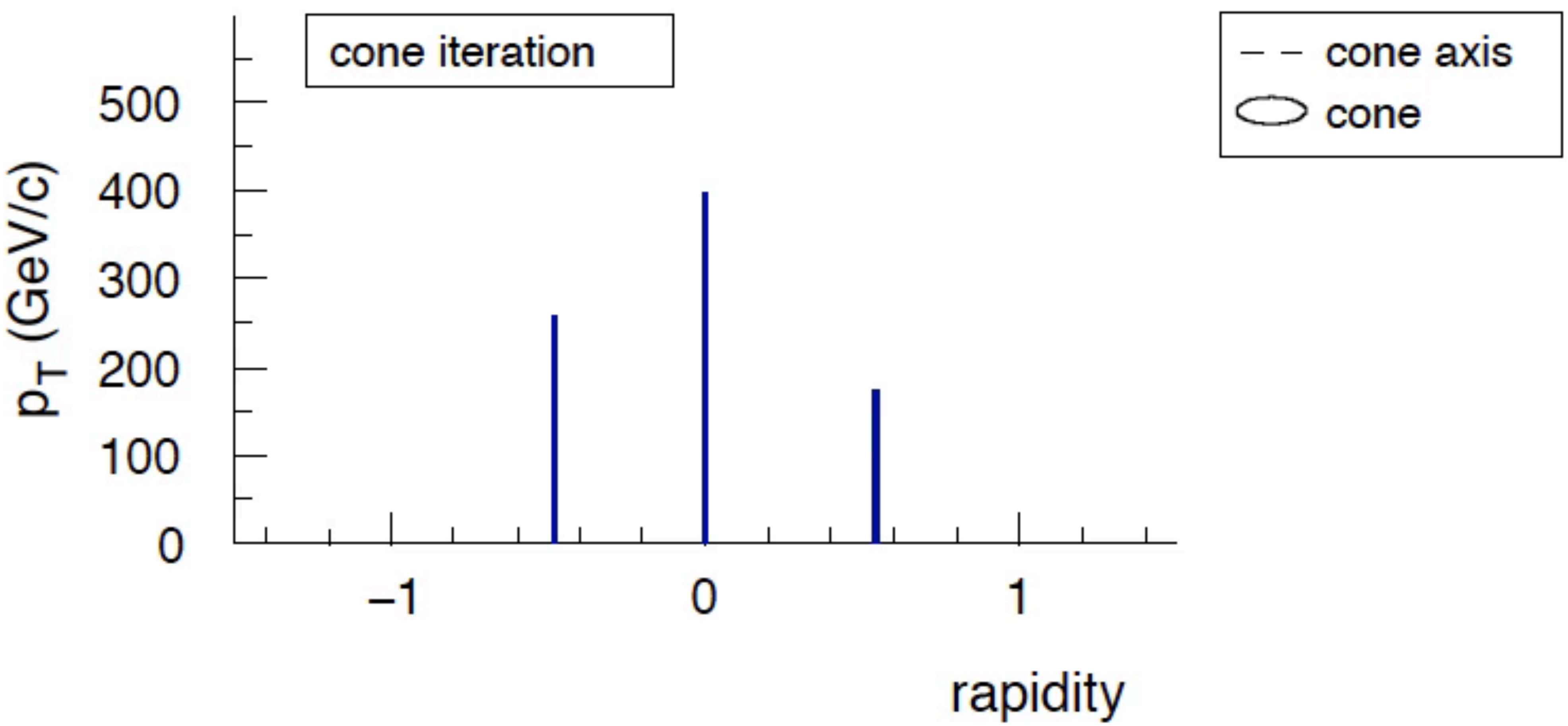
The result of the algo can be calculated analytically at NLO:

$$\sigma_{2j} = \sigma^{\rm Born}\left(1 - \frac{\alpha_S C_F}{\pi}\log^2 y + \ldots\right)$$

$$\sigma_{3j} = \sigma^{\rm Born}\frac{\alpha_S C_F}{\pi}\log^2 y + \ldots$$

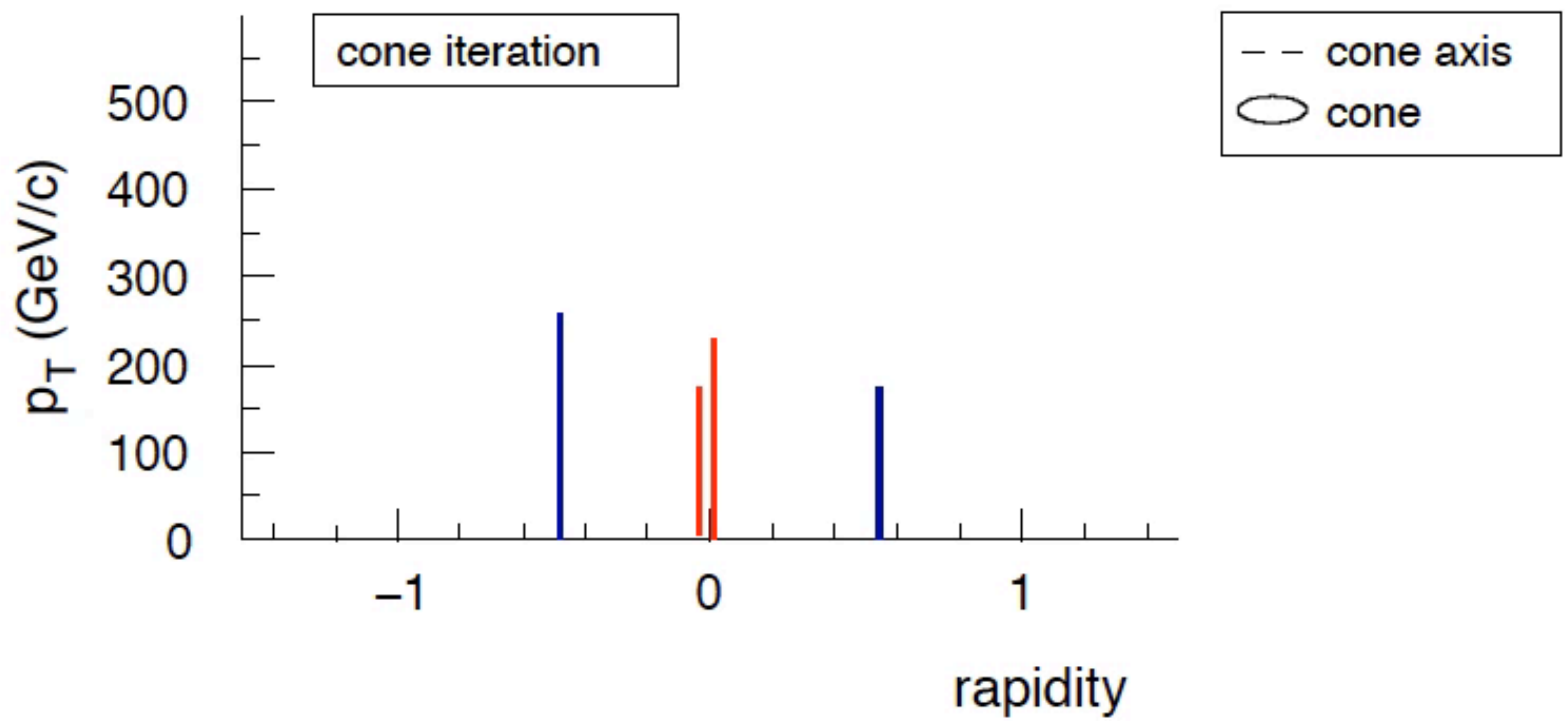


Infrared safety and jet algo's



- Take hardest particle as seed for cone axis
- Draw cone around seed
- Sum the momenta use as new seed direction, iterate until stable
- Convert contents into a "jet" and remove from event

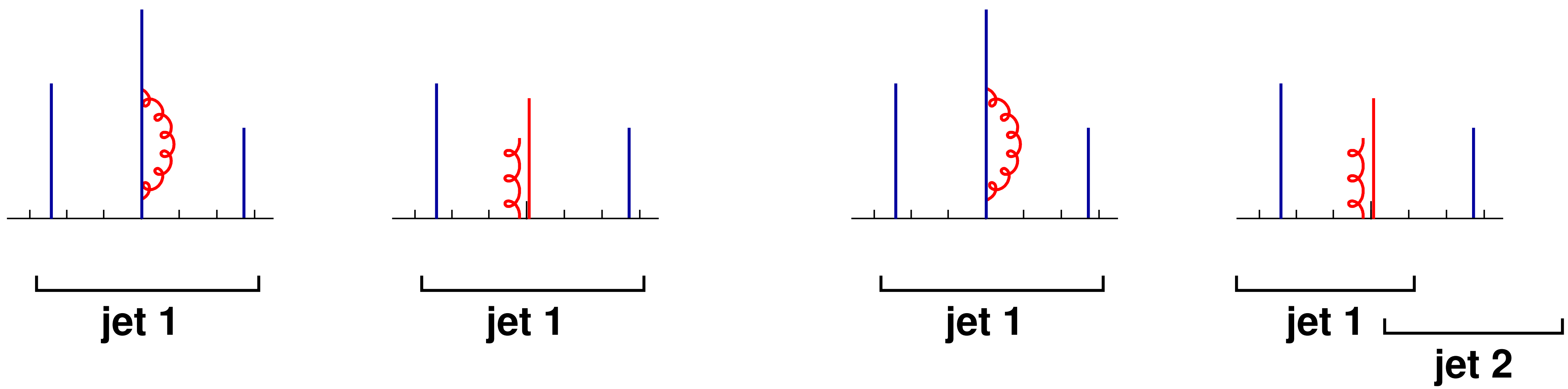
Infrared safety and jet algo's



GavinSalam®

Infrared safety and jet algo's

Collinear Safe | **Collinear Unsafe**



$\alpha_s^n \times (-\infty)$ $\alpha_s^n \times (+\infty)$ | $\alpha_s^n \times (-\infty)$ $\alpha_s^n \times (+\infty)$

Infinities cancel | **Infinities do not cancel**

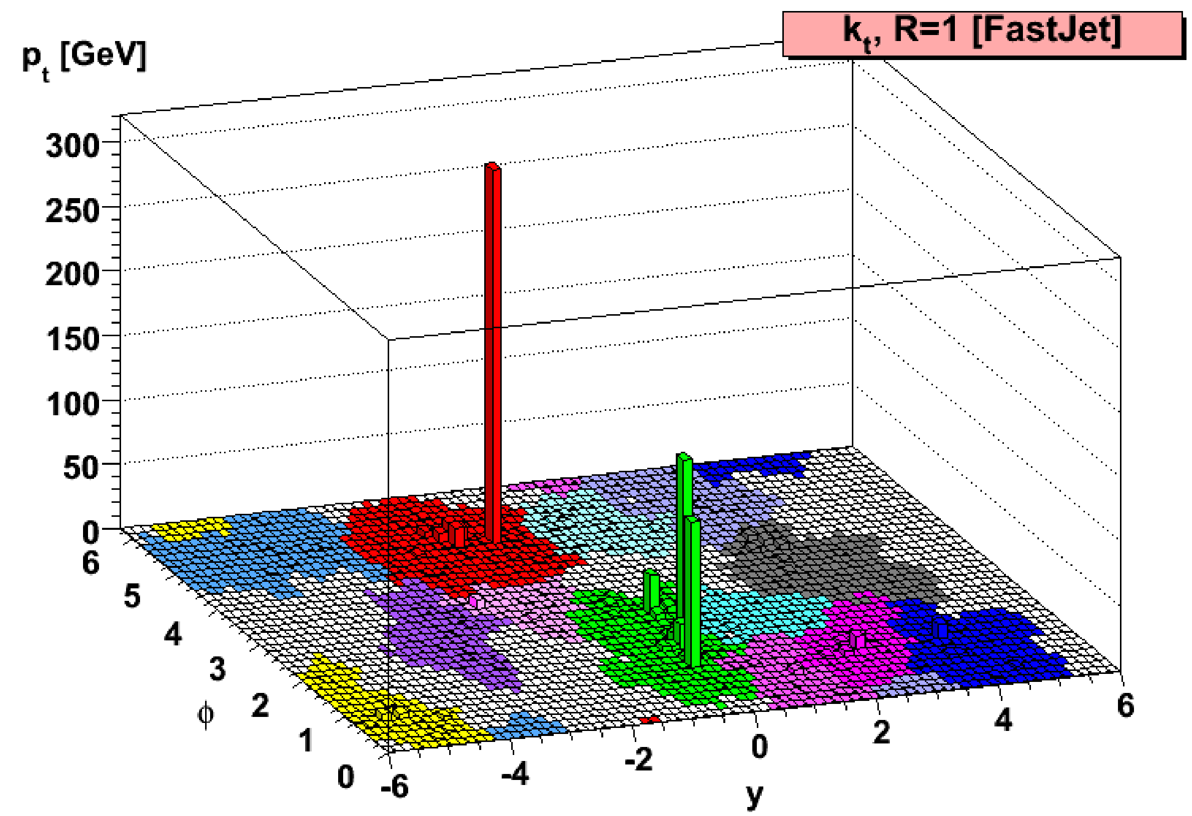
Invalidates comparison with perturbation theory results

K_T ALGORITHM AT HADRON COLLIDERS

Measure (dimensionful):

$$d_{ij} = \min(p_{ti}^2, p_{tj}^2)\frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = p_{ti}^2$$



The algorithm proceeds by searching for the smallest of the d_{ij} and the d_{iB}.

If it is a then d_{ij} particles i and j are recombined* into a single new particle.

If it is a d_{iB} then i is removed from the list of particles, and called a jet.

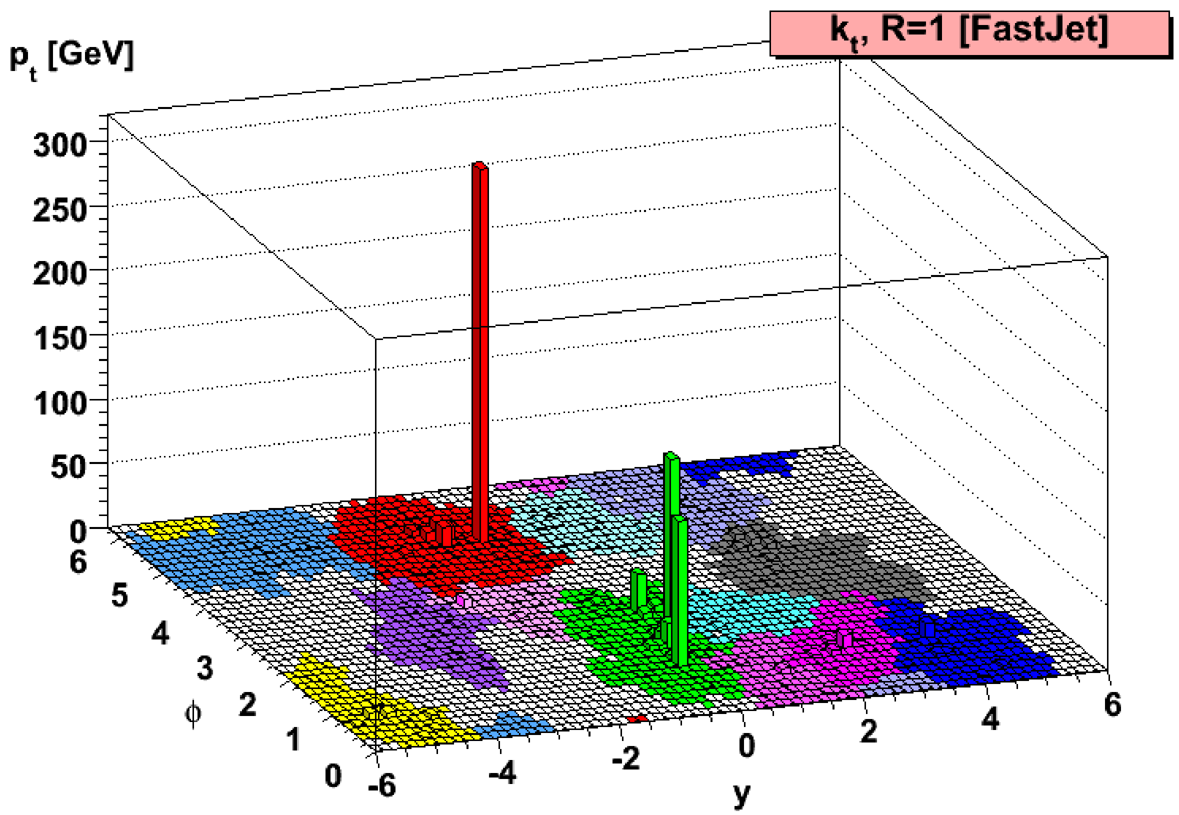
This is repeated until no particles remain.

kT algorigthm "undoes" the QCD shower

*a 4-momenta recombination scheme is needed (E-scheme)

GavinSalam®

kT ALGORITHM AT HADRON COLLIDERS



Comments:

As with cone algorithms, arbitrarily soft particles can form jets. It is therefore standard to place a p_T^{MIN} cutoff on the jets one uses for `hard' physics.

TWO PARAMETERS R, P_T^{MIN}

R in the kT algorithm plays a similar role to R in cone algorithms: if two particles i and j are within R of each other, i.e., $\Delta R_{ij} < R$, then $d_{ij} < d_{iB}, d_{jB}$ and so i and j will prefer to recombine rather than forming separate jets.

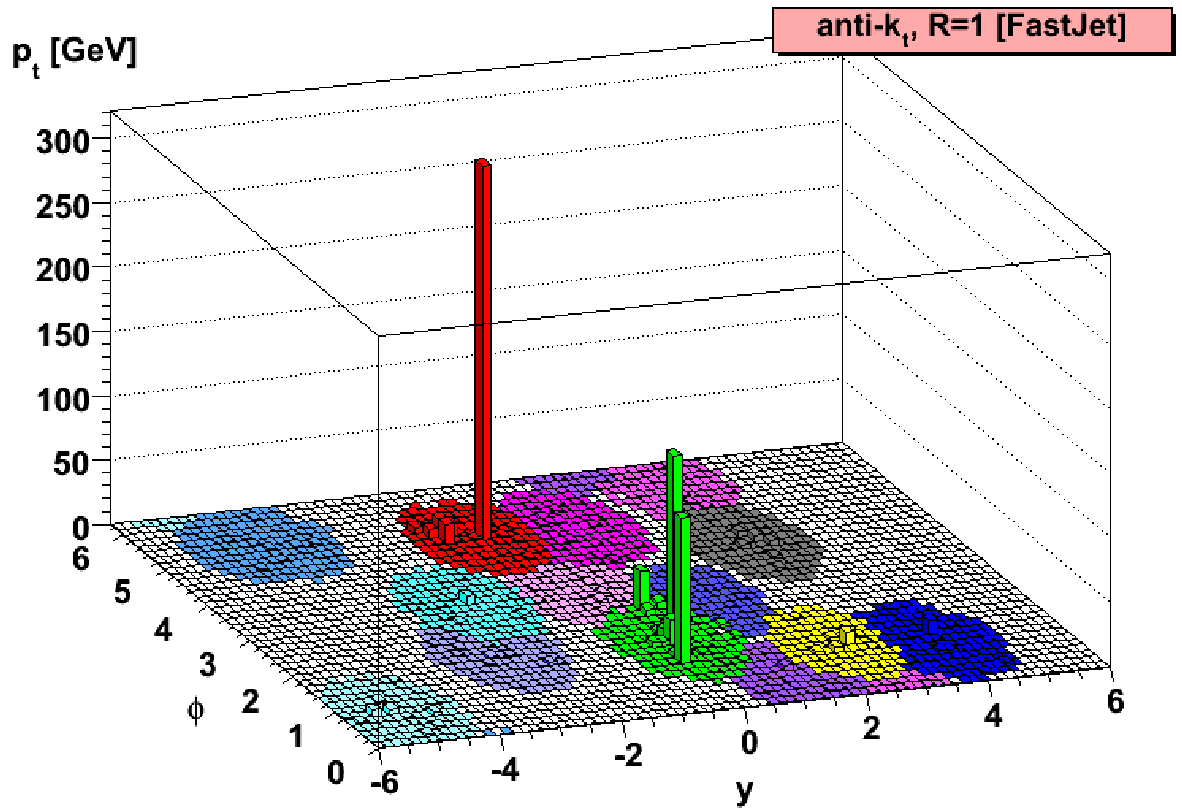
For the kT algorithm, the jets have irregular edges, because many of the soft particles cluster together early in the recombination sequence

ANTI-KT ALGORITHM

Measure (dimensionful):

$$d_{ij} = \frac{1}{\max(p_{ti}^2, p_{tj}^2)} \frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = \frac{1}{p_{ti}^2}$$



Objects that are close in angle prefer to cluster early, but that clustering tends to occur with a hard particle (rather than necessarily involving soft particles). This means that jets `grow' in concentric circles out from a hard core, until they reach a radius R, giving circular jets.

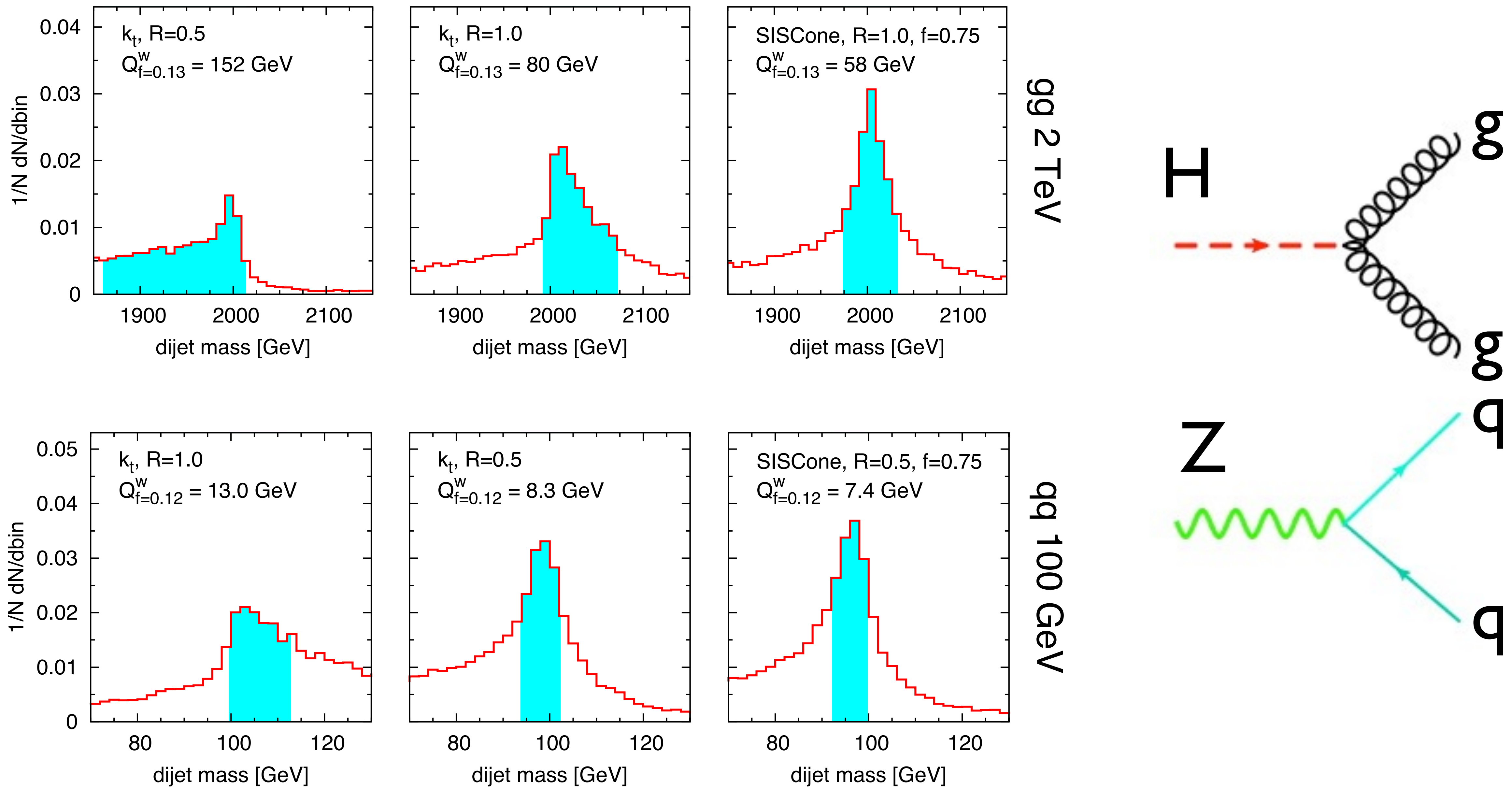
Unlike cone algorithms the `anti-kT' algorithm is collinear (and infrared) safe.

This, (and the fact that it has been implemented efficiently in FastJet), has led to be the default jet algorithm at the LHC.

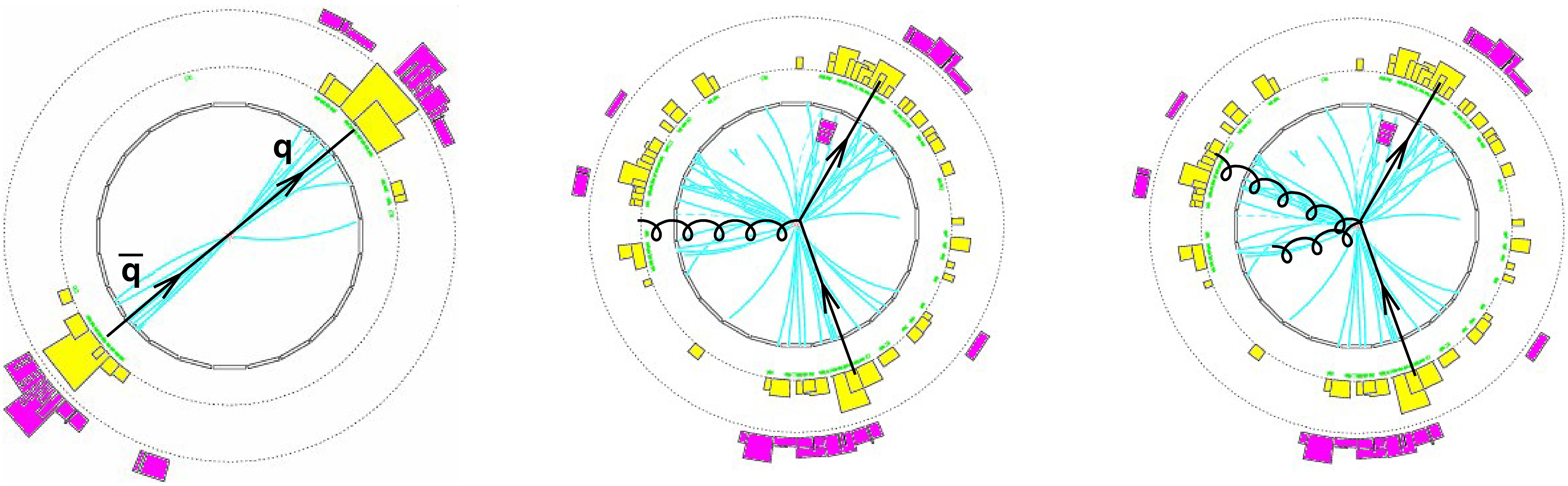
It's a handy algorithm but it does not provide internal structure information.

WHAT JET ALGO SHOULD I CHOOSE?

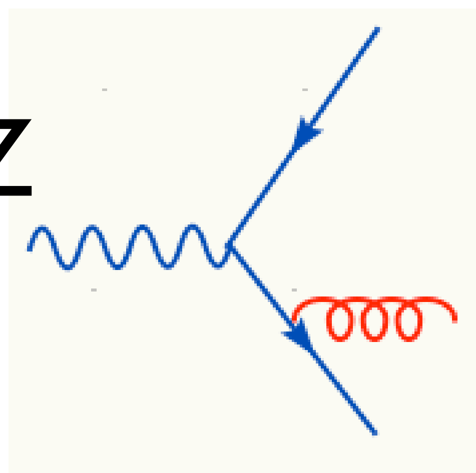
It depends on what are you looking (Singlet or colored, resonance decaying to gg, qq, bb) for and which observable you want to accurately measure : see a sharp peak or measure the position of the peak...



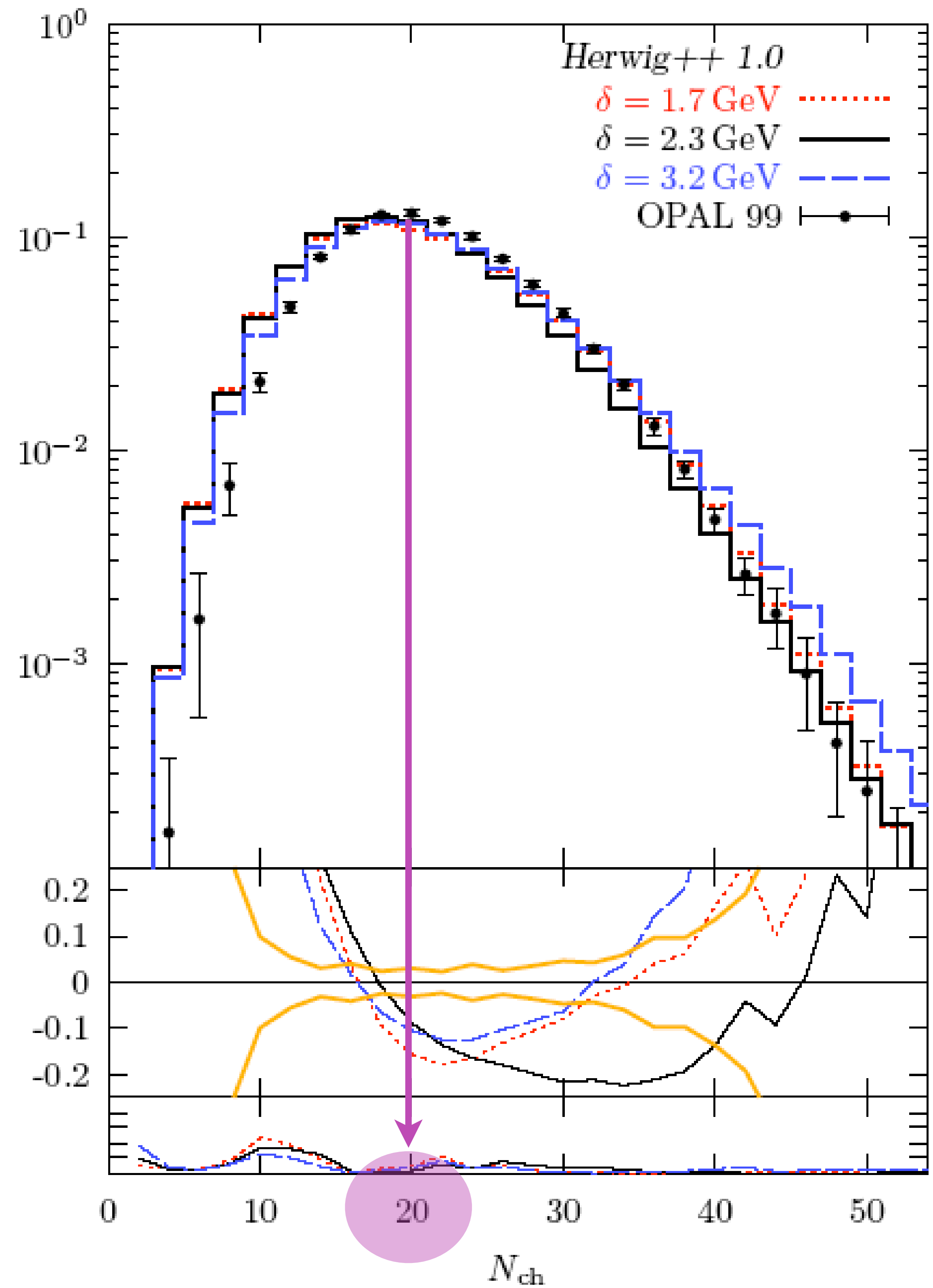
More exclusive quantities



Number of particles in the final state?
Number of particles per jet?
Jet mass?


γ*,Z

?


Herwig++ 1.0
δ = 1.7 GeV
δ = 2.3 GeV
δ = 3.2 GeV
OPAL 99
10^0
10^−1
10^−2
10^−3
0.2
0.1
0
-0.1
-0.2
0
10
20
30
40
50
N_ch

Summary

1. We have studied the problem of infrared divergences in the calculation of the fully inclusive cross section, with the help of the soft limit.

2. We have introduced the concept of an Infrared Safe quantity, i.e. an observable which is both computable at all orders in pQCD and has a well defined counterpart at the experimental level.

3. We have discussed more exclusive quantities, from shape functions to fully exclusive quantities and compared them with e+ e- data. We have introduced the method of exponentiation.

4. We have introduced the idea of jet algorithms (top-down and bottom-up) and discussed the most recent algorithms.

QCD REDUX

PART III

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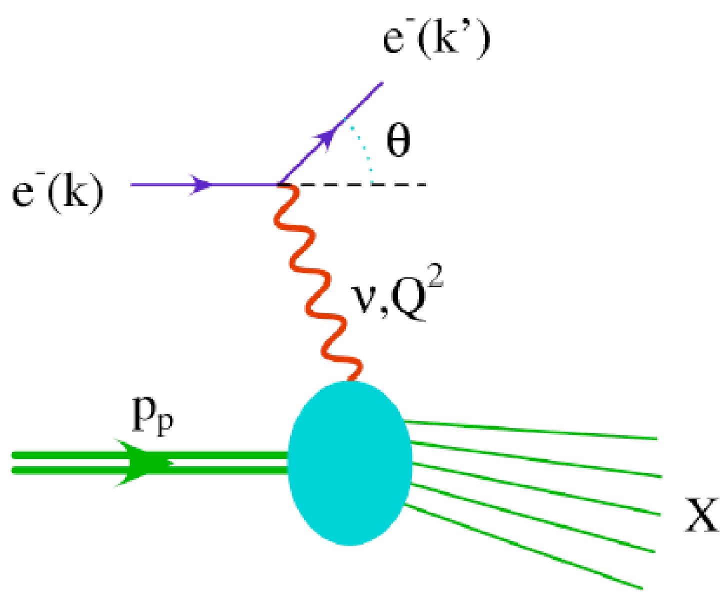
QCD in the initial state

1. DIS and the parton model
2. DIS with pQCD
3. The idea of factorization
4. Q^2 Evolution and PDF's
5. pp collisions

Deep-inelastic scattering: towards the parton model



$$s = (P+k)^2$$ cms energy²

$$Q^2 = -(k-k')^2$$ momentum transfer²

$$x = Q^2/2(P\cdot q)$$ scaling variable

$$\nu = (P\cdot q)/M = E - E'$$ energy loss

$$y = (P\cdot q)/(P\cdot k) = 1 - E'/E$$ rel. energy loss

$$W^2 = (P+q)^2 = M^2 + \frac{1-x}{x}Q^2$$ recoil mass

"deep inelastic" : $Q^2 >> 1$ GeV²

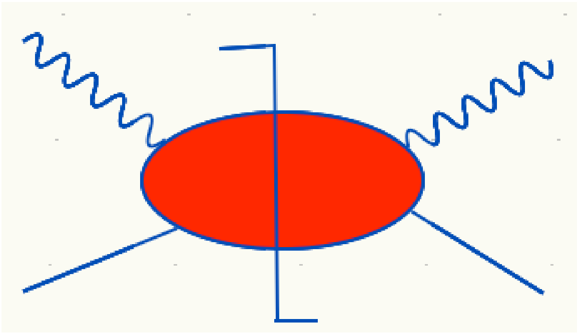
"scaling limit": $Q^2 \to \infty$, x fixed

The idea is that by measuring all the kinematics variables of the outgoing electron one can study the structure of the proton in terms of the probe characteristics, Q2,x,y...Very inclusive measurement from the QCD point of view.

Deep-inelastic scattering: towards the parton model

* Divide phase-space factor into a leptonic and a hadronic part:

$$d\Phi = \frac{d^3k'}{(2\pi)^3 2E'} d\Phi_X = \frac{ME}{8\pi^2} y\, dy\, dx\, d\Phi_X$$



* Separate also the square of the Feynman amplitude, by defining:

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{Q^4} L^{\mu\nu} h_{X\mu\nu}$$

* The leptonic tensor can be calculated explicitly:

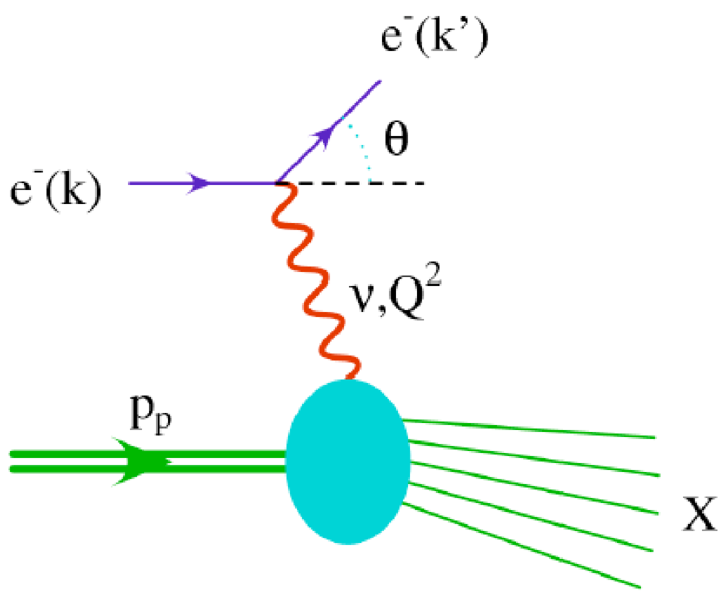
$$L^{\mu\nu} = \frac{1}{4}\mathrm{tr}[\not{k}\gamma^\mu \not{k}'\gamma^\nu] = k^\mu k'^\nu + k'^\mu k^\nu - g^{\mu\nu} k\cdot k'$$

* Combine the hadronic part of the amplitude and phase space into "hadronic tensor" and use just Lorentz symmetry and gauge invariance to write

$$W^{\mu\nu} = \sum_X \int d\Phi_X h_{X\mu\nu}$$

$$W_{\mu\nu}(p,q) = \left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + \left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) \frac{1}{p\cdot q} F_2(x,Q^2)$$

Deep-inelastic scattering: towards the parton model



$$\sigma^{ep\to eX} = \sum_X \frac{1}{4ME}\int d\Phi\, \frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2$$

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1+(1-y)^2]F_1(x,Q^2) + \frac{1-y}{x}\left[F_2(x,Q^2) - 2xF_1(x,Q^2)\right]\right\}$$

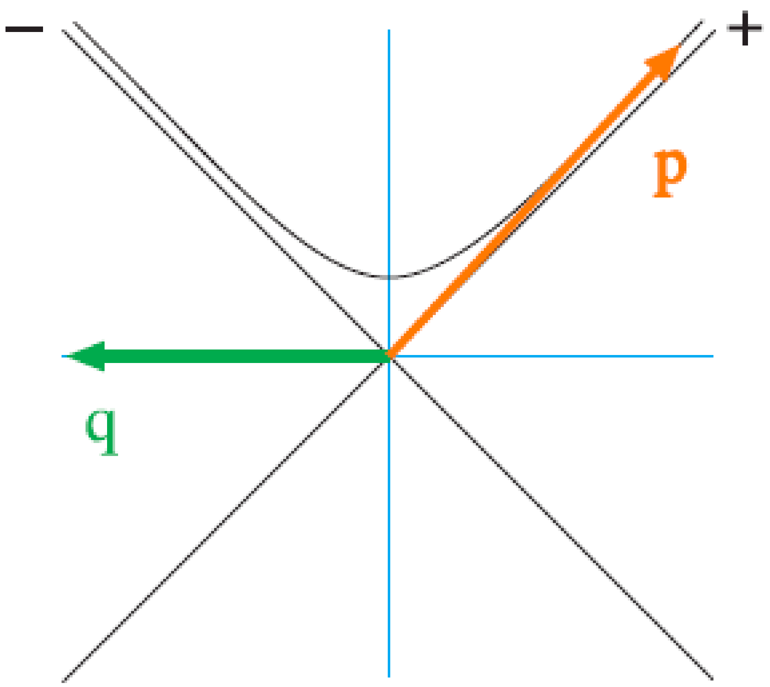
Comments:

* Different y dependence can differentiate between F_1 and F_2
* The first term represents the absorption of a transversely polarized photon, the second of a longitudinal one.
* Bjorken scaling ⇒ F_1 and F_2 obey scaling themselves, i.e. they do not depend on Q.

Deep-inelastic scattering: towards the parton model

We want to "watch" the scattering from a frame where the physics is clear. Feynman suggested that what happens can be best understood in a reference frame where the proton moves very fast and $Q >> m_h$ is large.

4-vector	hadron rest frame	Breit frame
$(p^+, p^-, \vec{p}_T)$	$\frac{1}{\sqrt{2}}(m_h, m_h, \vec{0})$	$\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{x m_h^2}{Q}, \vec{0})$
$(q^+, q^-, \vec{q}_T)$	$\frac{1}{\sqrt{2}}(-m_h x, \frac{Q^2}{m_h x}, \vec{0})$	$\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$

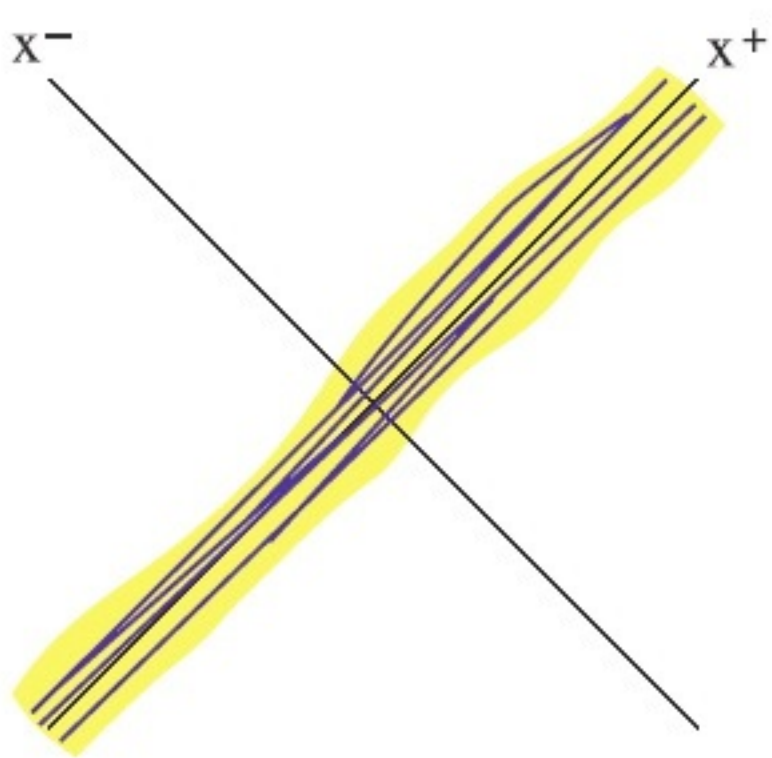


You can check that a Lorentz transformation acts on a light-cone formulation of the four-momentum:

$$(a^+, a^-, \vec{a}) \to (e^{\omega} a^+, e^{-\omega} a^-, \vec{a}) \quad \text{with} \quad e^{\omega} = Q/(x m_h)$$

DEEP-INELASTIC SCATTERING: TOWARDS THE PARTON MODEL

Now let's see how the proton looks in this frame, and in the light-cone space coordinates (suitable for describing relativistic particles).



Lorentz transformation divides out the interactions. Hadron at rest has separation of order:

$\Delta x^{+} \sim \Delta x^{-} \sim 1/m$,

while in the moving hadron has:

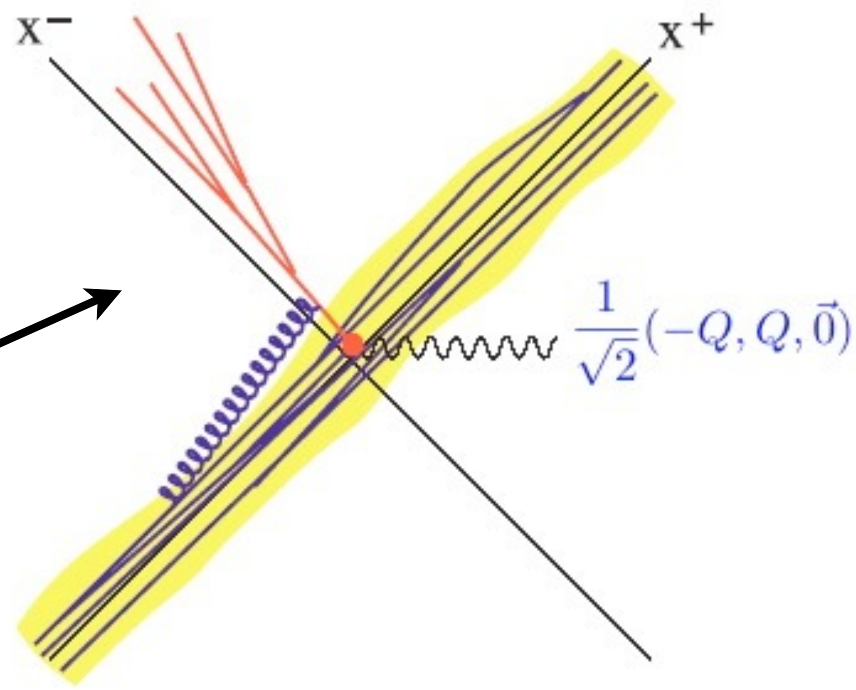
$\Delta x^{+} \sim 1/m \times Q/m = Q/m^2$ LARGE

$\Delta x^{-} \sim 1/m \times m/Q = 1/Q$, SMALL

Deep-inelastic scattering: towards the parton model

And now let the virtual photon hit the fast moving hadron:

Struck quark kicked into the x-direction



Moving hadron has:

$\Delta x^+ \sim Q/m^2$,

interaction with photon $q^- \sim Q$ is localized within

$\Delta x^+ \sim 1/Q$,

thus quarks and gluons are like partons and effectively free.

In this frame the time scale of a typical parton-parton interaction is much larger than the hard interaction time.

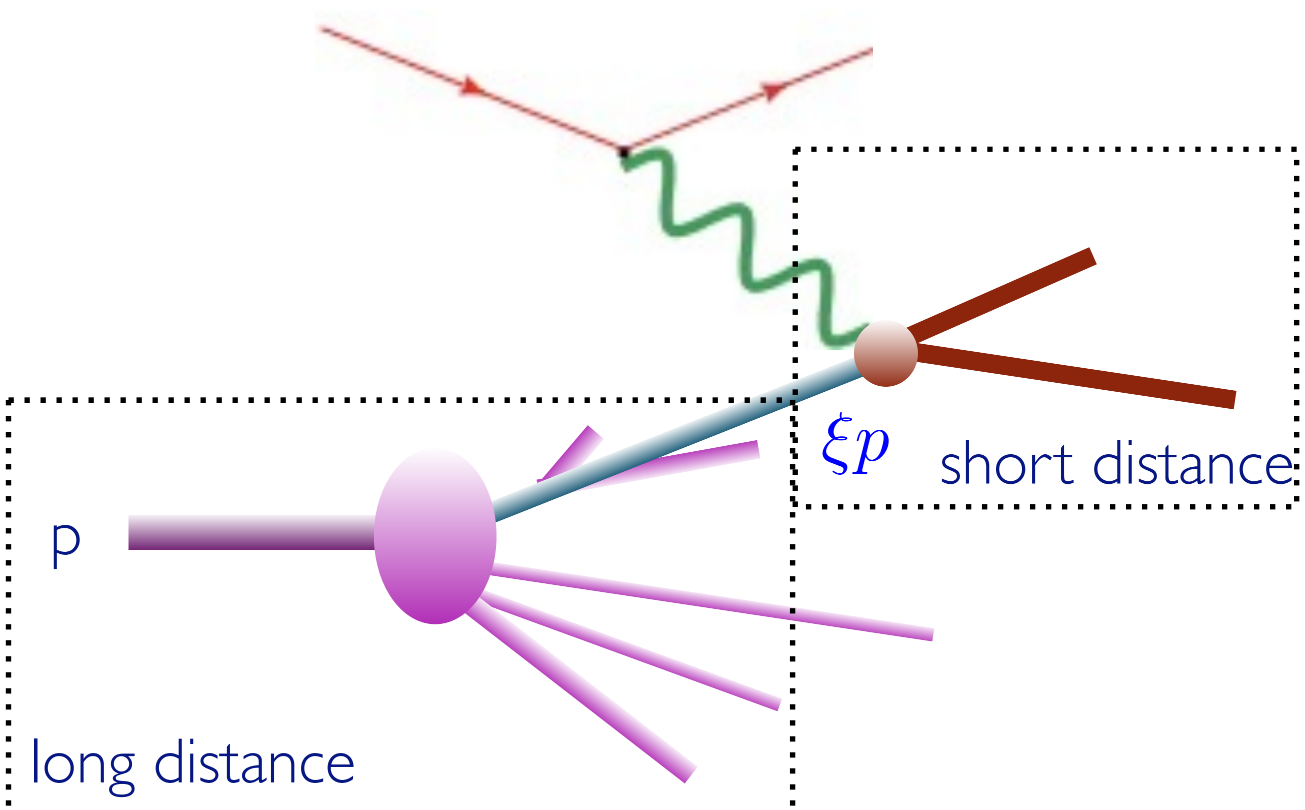
So we can picture the hadron as an incoherent flux of partons $(p^+,p^-,p^\perp)_i$, each carrying a fraction $0<\xi_i = p_i^+/p^+<1$ of the total available momentum.

Deep-inelastic scattering: towards the parton model

The space-time picture suggests the possibility of separating short- and long-distance physics ⇒ factorization! Turned into the language of Feynman diagrams DIS looks like:

$$\frac{d^2\sigma}{dx dQ^2} = \int_0^1 \frac{d\xi}{\xi} \sum_i f_i(\xi) \frac{d^2\hat{\sigma}}{dx dQ^2}\left(\frac{x}{\xi}, Q^2\right)$$

where



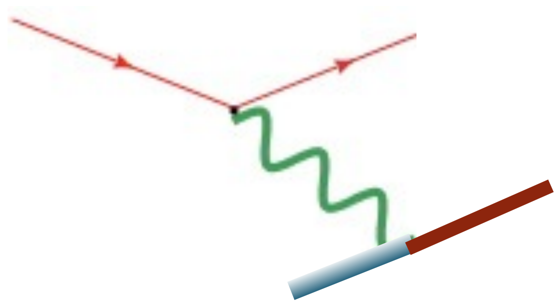
$f_{i/h}(\xi)$ is the probability to find a parton with flavor i in an hadron h carrying a light-cone momentum $\xi p+$

$\frac{d^2\hat{\sigma}}{dx dQ^2}$ is the cross section for electron-parton scattering

Deep-inelastic scattering: towards the parton model

We can now explain scaling within the parton model:

Let's take the LO computation we performed for e+e- → qq, cross it (which also mean to be careful with color), and use it the DIS variables to express the differential cross section in dQ^2



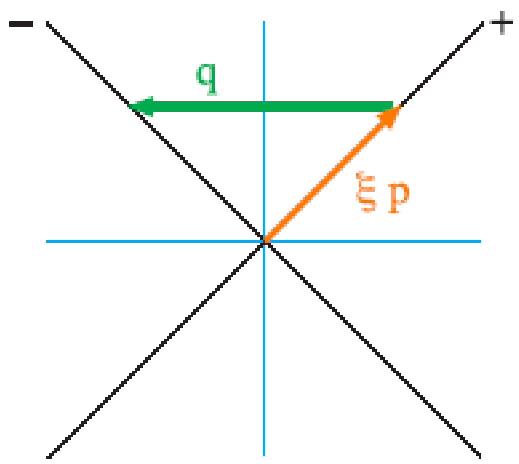
$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1+(1-y)^2\right]$$

Notice that the outgoing quark is on its mass shell:

$$\xi p^+ + q^+ = 0$$

$$p^+ = Q/(x\sqrt{2})$$

$$q^+ = -Q/\sqrt{2}$$



$$\frac{d^2\hat{\sigma}}{dQ^2 dx} = \frac{4\pi\alpha^2}{Q^4}\frac{1}{2}\left[1+(1-y)^2\right]\delta(x-\xi)$$

This implies that $\xi = x$ at LO!

Deep-inelastic scattering: towards the parton model

We can now compare with our "inclusive" description of DIS in terms of structure functions (which, BTW, are physical measurable quantities),

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1+(1-y)^2]F_1(x,Q^2) + \frac{1-y}{x}\left[F_2(x,Q^2) - 2xF_1(x,Q^2)\right]\right\}$$

with our parton model formulas:

$$\frac{d^2\sigma}{dx dQ^2} = \int_0^1 \frac{d\xi}{\xi} \sum_i f_i(\xi) \frac{d^2\sigma}{d\hat{x} dQ^2}(\frac{x}{\xi}, Q^2) \quad \text{with} \quad \frac{d^2\hat{\sigma}}{dQ^2 dx} = \frac{4\pi\alpha^2}{Q^4}\frac{1}{2}\left[1+(1-y)^2\right] e_q^2\, \delta(x-\xi)$$

we find (be careful to distinguish x and ξ)

$$F_2(x) = 2xF_1 = \sum_{i=q,\bar{q}} \int_0^1 d\xi\, f_i(\xi)\, x e_q^2 \delta(x-\xi) = \sum_{i=q,\bar{q}} e_q^2\, x f_i(x)$$

* So we find the scaling is true: no dependence on Q^2.
* q and qbar enter together : no way to distinguish them with NC. Charged currents are needed.
* $F_L(x) = F_2(x) - 2 F_1(x)$ vanishes at LO (Callan-Gross relation), which is a test that quarks are spin 1/2 particles! In fact if the quarks where scalars we would have had $F_1(x) = 0$ and $F_2=F_L$.

Deep-inelastic scattering: towards the parton model

Probed at scale Q, sea contains all quarks flavours with mq less than Q.
For Q ~1 we expect

$$
\begin{aligned}
u(x) &= u_V(x) + \bar{u}(x) \\
d(x) &= d_V(x) + \bar{d}(x) \\
s(x) &= \bar{s}(x)
\end{aligned}
$$

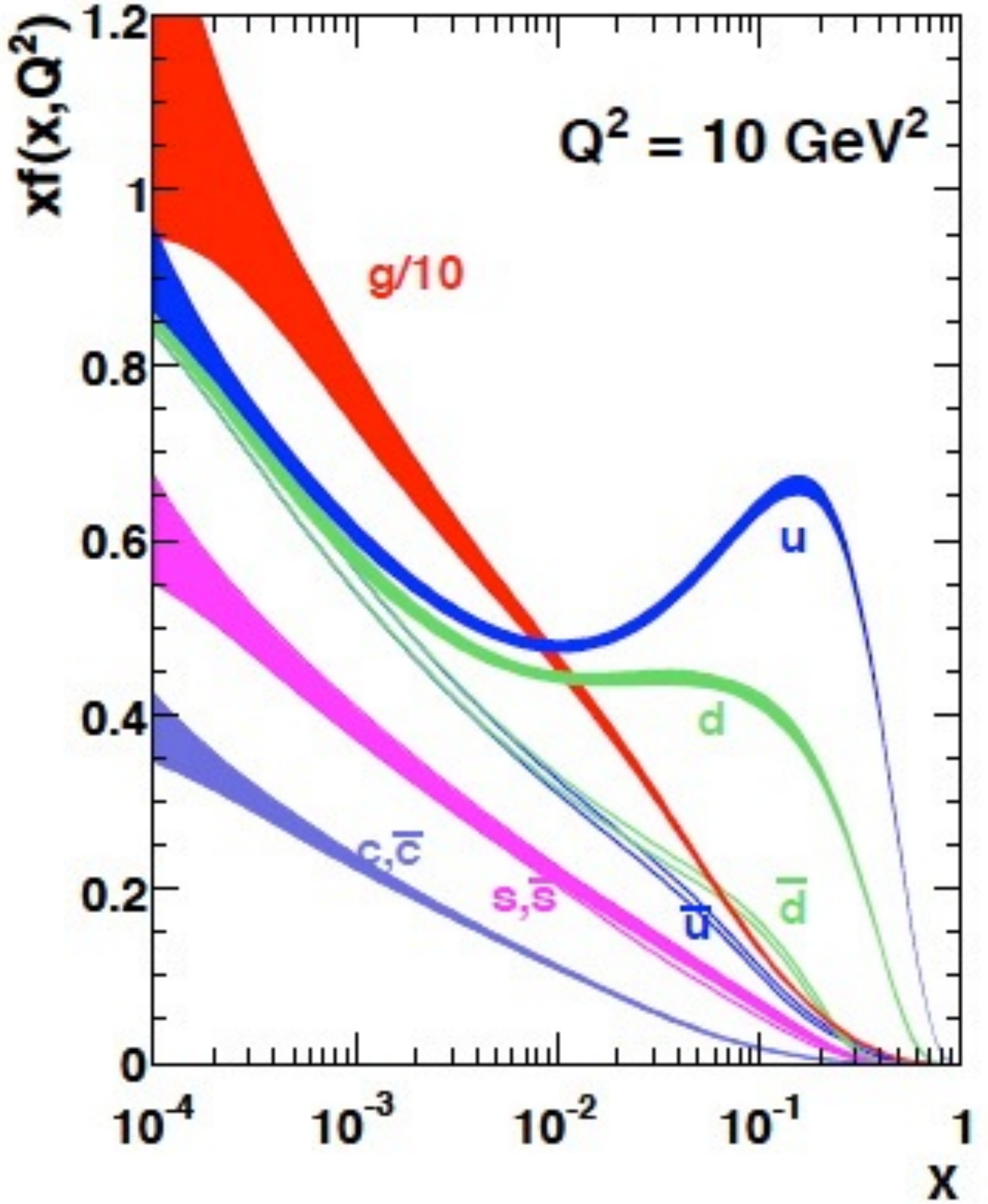
$$\int_0^1 dx\, u_V(x) = 2\,, \quad \int_0^1 dx\, d_V(x) = 1\,.$$

And experimentally one finds

$$\sum_q \int_0^1 dx\, x[q(x) + \bar{q}(x)] \simeq 0.5\,.$$

Thus quarks carry only about 50% of proton's momentum. The rest is carried by gluons. Although not directly measured in DIS, gluons participate in other hard scattering processes such as large-pt and prompt photon production.

Quark and gluon distribution functions



Comments:

The sea is NOT SU(3) flavor symmetric.

The gluon is huge at small x

There is an asymmetry between the ubar and dbar quarks in the sea.

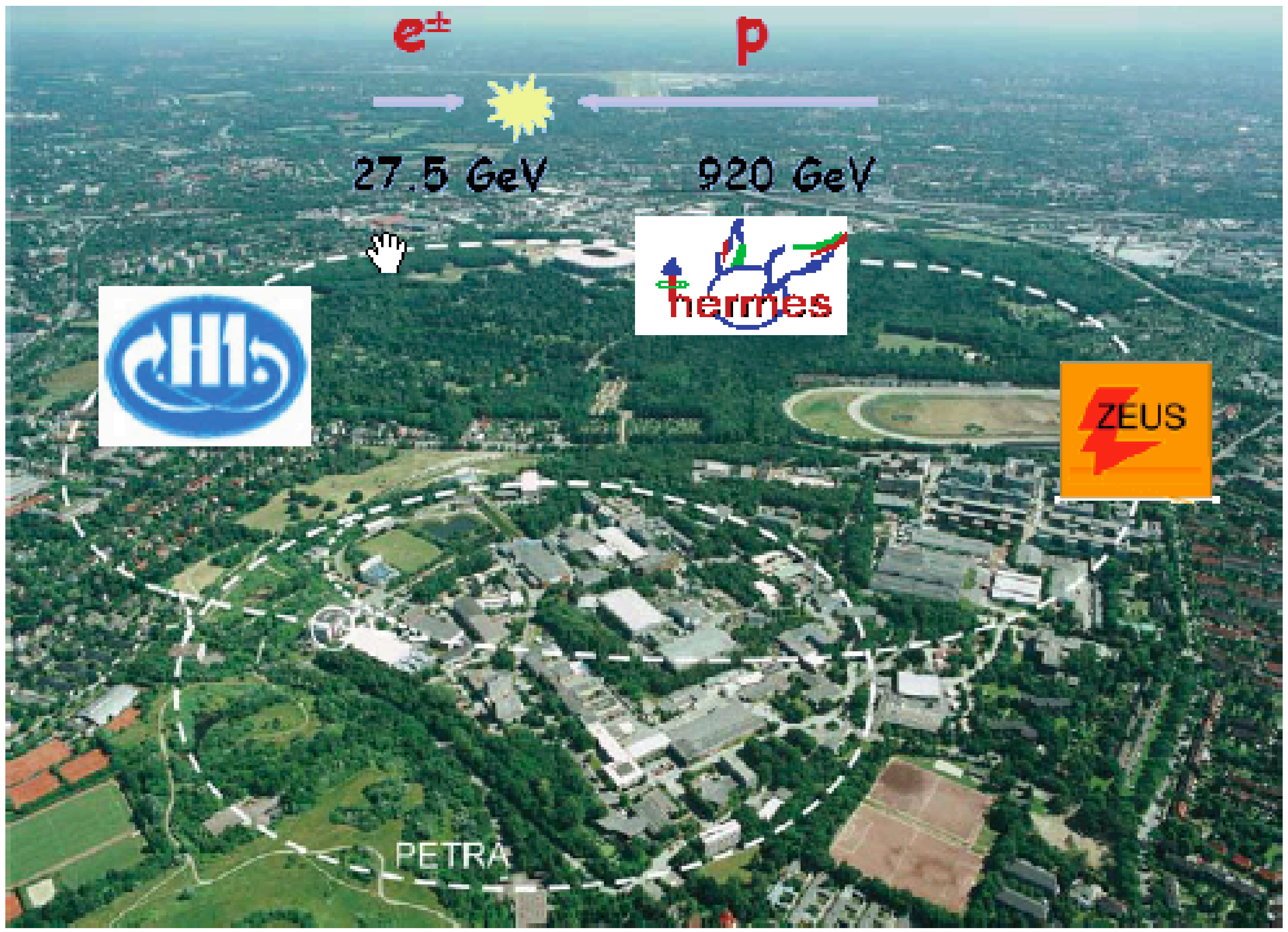
Note that there are uncertainty bands!!

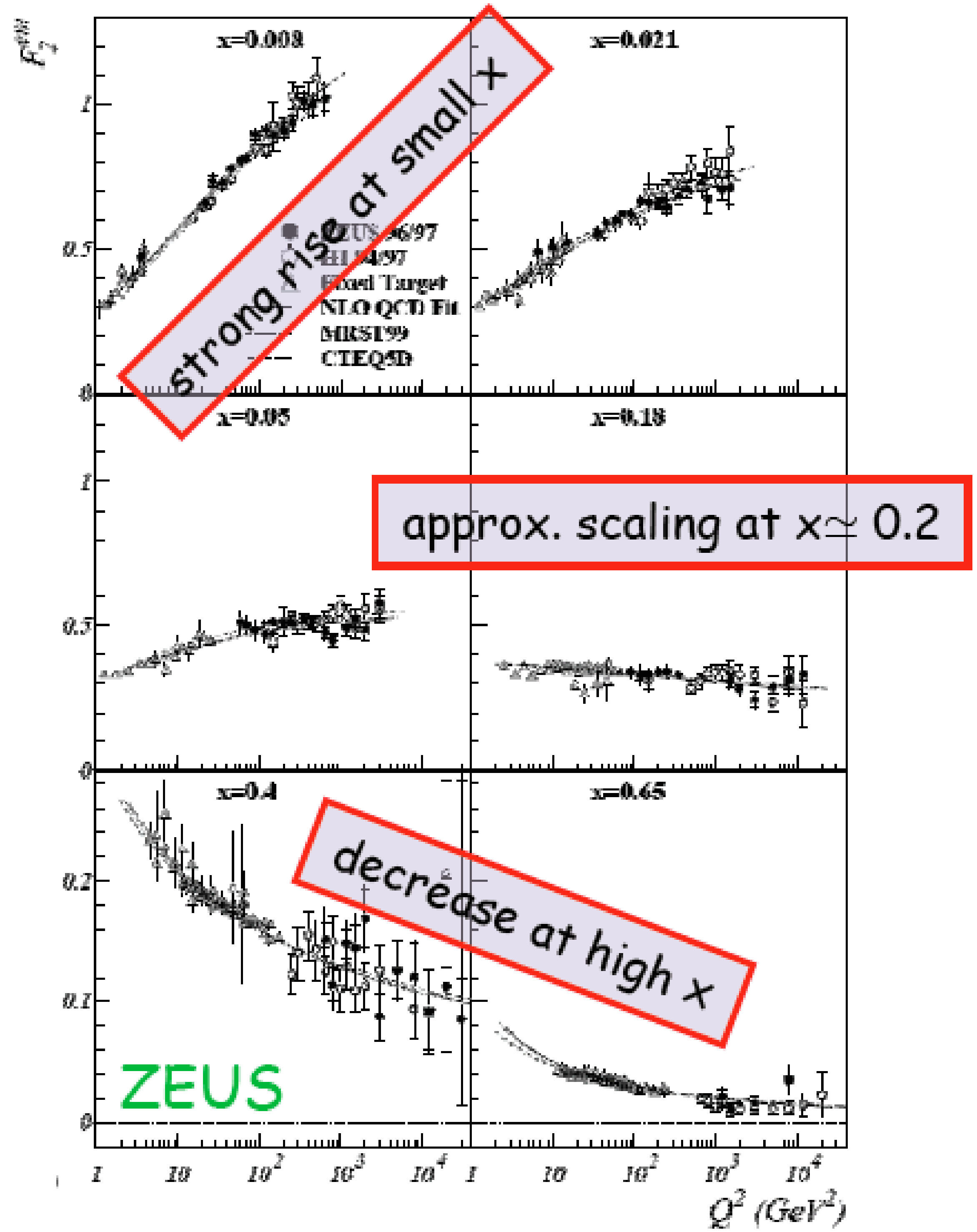
Questions:

1. What has QCD to say about the naïve parton model?
2. Is the picture unchanged when higher order corrections are included?
3. Is scaling exact?

Scaling violations

first ep collider



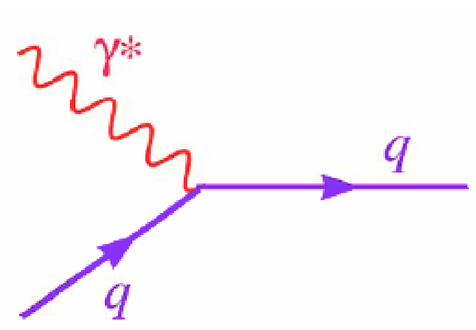


At HERA scaling violations were observed!

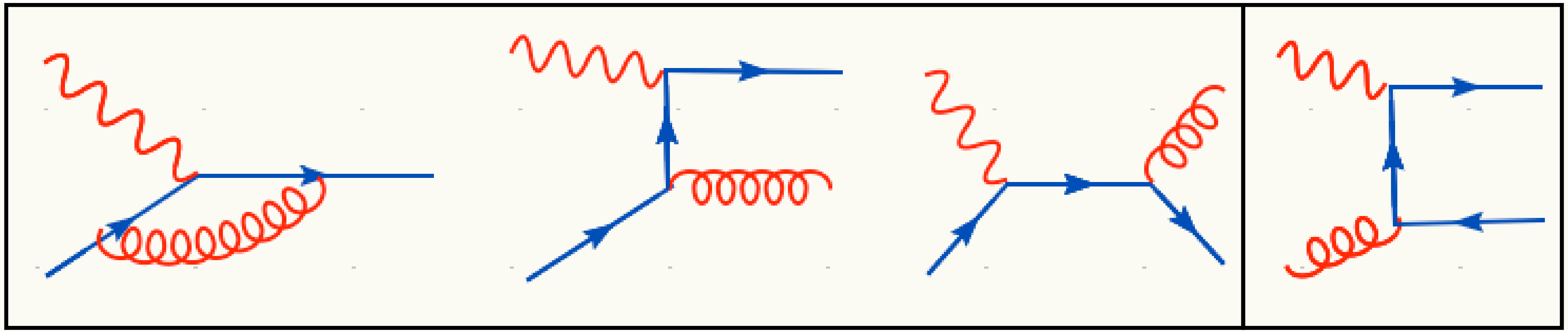
Deep-inelastic scattering in QCD

We got a long way without even invoking QCD. Let's do it now.

The first diagram to consider is the same as in the parton model:



At NLO we find again both real and virtual corrections:



α_S corrections to the LO process   photon-gluon fusion

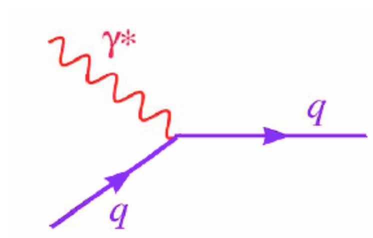
Our experience so far: have to expect IR divergences!
In order to make the intermediate steps of the calculation finite, we introduce a regulator, which will be removed at the end.

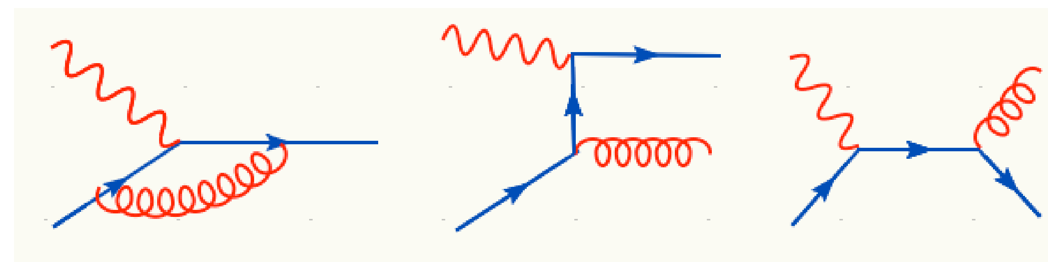
Dimensional regularization is the best choice to perform serious calculations.
However for illustrative purposes other regulators (that cannot be easily used beyond NLO) are better suited. We'll use here a small quark/gluon mass.

Deep-inelastic scattering in QCD

Once we compute the diagrams we indeed find that UV and soft divergences all cancel, but for a collinear divergence arising when the emitted gluon becomes collinear to the incoming quark:

$$\frac{d^2\hat{\sigma}}{dx\,dQ^2}\Big|_{F_2} \equiv \hat{F}_2^q$$



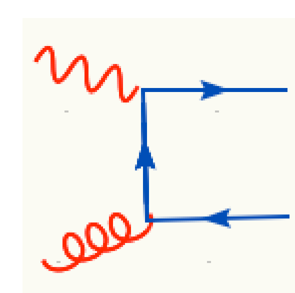


$$= e_q^2 x \left[\delta(1-x) + \frac{\alpha_S}{4\pi}\left[P_{qq}(x)\log\frac{Q^2}{m_g^2} + C_2^q(x)\right]\right]$$

$$\frac{d^2\hat{\sigma}}{dx\,dQ^2}\Big|_{F_2} \equiv \hat{F}_2^g$$

IR cutoff

$$= \sum_q e_q^2 x \left[0 + \frac{\alpha_S}{4\pi}\left[P_{qg}(x)\log\frac{Q^2}{m_q^2} + C_2^g(x)\right]\right]$$



The presence of large logs is a clear sign that we have a residual infrared sensitivity that we have to deal with!

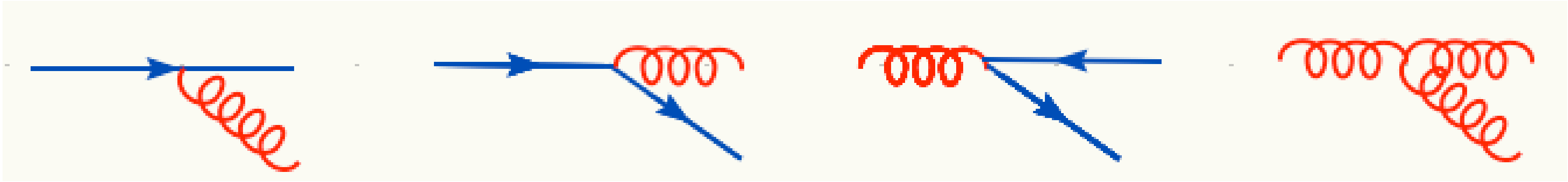
Important observations:

1. Large logarithms of Q^2/m^2 or ($1/\varepsilon$ in dim reg) incorporate ALL the RESIDUAL long-distance physics left after summing over all real and virtual diagram. This terms are of a collinear nature.

2. The coefficients $P_{ij}(x)$ that multiply the log's are UNIVERSAL and calculable in perturbative QCD.

They are called SPLITTING FUNCTIONS and their physical meaning is easy to give:

$P_{ij}(x)$ give the probability that a parton j splits collinearly into a parton i + something else carrying a momentum fraction x of the original parton j.



Pqq(x) Pgq(x) Pqg(x) Pgg(x)

Deep-inelastic scattering in QCD

So the natural question is: what is it that is going wrong? Do we have IR sensitiveness in a physical observable? Well not yet!!

To obtain the physical cross section we have to convolute our partonic results with the parton densities, as we have learned from the parton model.

For instance:

$$F_2^q(x,Q^2) = x \sum_{i=q,\bar{q}} e_q^2 \left[f_{i,0}(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{i,0}(\xi) \left[P_{qq}(\frac{x}{\xi}) \log \frac{Q^2}{m_g^2} + C_2^q(\frac{x}{\xi}) \right] \right]$$

And now comes the magic: as long as the divergences are universal and do not depend on the hard scattering functions but only on the partons involved in the splitting, we can reabsorb the dependence on the IR cutoff (once for all!) into $f_{q,0}(x)$:

$$f_q(x,\mu_f) \equiv f_{q,0}(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q,0}(\xi) P_{qq}(\frac{x}{\xi}) \log \frac{\mu_f^2}{m_g^2} + z_{qq}$$

"Renormalized" parton densities: we have factorized the IR collinear physics into a quantity that we cannot calculate but it is universal. So how does the final result looks like?

Deep-inelastic scattering in QCD

The structure function is a MEASURABLE object, therefore, at all orders, it cannot depend on the choice of scales.
This will lead exactly to the same concepts of renormalization group invariance that we found in the UV.

The final result depends of course also on α_S and therefore to the choice of the renormalization scale.

$$F_2^q(x,Q^2) = x \sum_{i=q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2) \left[\delta(1-\frac{x}{\xi}) + \frac{\alpha_S(\mu_r)}{2\pi}\left[P_{qq}(\frac{x}{\xi})\log\frac{Q^2}{\mu_f^2} + (C_2^q - z_{qq})(\frac{x}{\xi})\right]\right]$$

Long distance physics is universally factorized into the parton distribution functions. These cannot be calculated in pQCD.They depend on μ_f in the exact way so as to cancel the overall dependence at all orders.

Short-distance (Wilson coefficient), perturbative calculable and finite. It depends on the factorization scale. It also depends on finite terms which define the factorization scheme.

Factorization

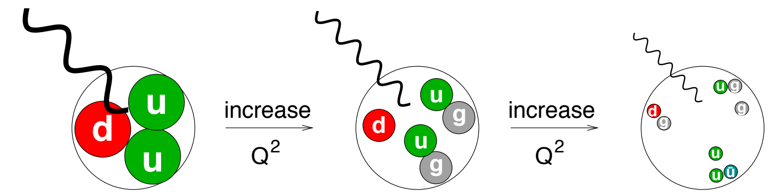
$$F_2^q(x,Q^2) = x \sum_{i=q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2) \left[\delta(1-\frac{x}{\xi}) + \frac{\alpha_S(\mu_r)}{2\pi} \left[P_{qq}(\frac{x}{\xi}) \log\frac{Q^2}{\mu_f^2} + (C_2^q - z_{qq})(\frac{x}{\xi})\right]\right]$$

Questions:

1. Can we exploit the fact that physical quantities have to be scale independent to gain information on the pdfs?

2. What exactly have we gained in hiding the large logs in the redefined pdf's? Aren't we just hiding the problem?

Evolution



$$F_2(x,Q^2) \sim \sum_i f_i(x,\mu_f) \otimes \hat{F}_2(x,\frac{Q}{\mu_f})$$

As a first step it is very convenient to transform the nasty convolution into a simple product. This can be done with the help of a Mellin transform:

$$f(N) \equiv \int_0^1 dx x^{N-1} f(x)$$

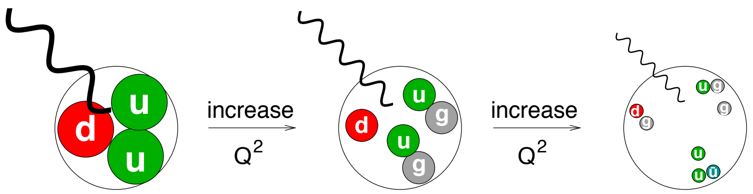
small/large x ⇔small/large N

Let us show that a Mellin transform turns a convolution into a simple product:

$$\int_0^1 dx x^{N-1} \left[\int_x^1 \frac{dy}{y} f(y) g(\frac{x}{y}) \right] = \int_0^1 dx x^{N-1} \int_0^1 dy \int_0^1 dz \delta(x - zy) f(y) g(z)$$

$$= \int_0^1 dy \int_0^1 dz (zy)^{N-1} f(y) g(z) = f(N) g(N)$$

Evolution



$$F_2(x,Q^2) \sim \sum_i f_i(x,\mu_f) \otimes \hat{F}_2(x, \frac{Q}{\mu_f})$$

Let's now apply it to F_2

$$\frac{dF_2(x,Q^2)}{d\log\mu_f} = 0$$

we get:

$$\frac{dq(N,\mu_f)}{d\log\mu_f}\hat{F}_2(N,\frac{\mu_f}{Q}) + q(N,\mu_f)\frac{d\hat{F}_2(N,\frac{\mu_f}{Q})}{d\log\mu_f} = 0$$

$$\frac{d\log\hat{F}_2(N,\frac{Q}{\mu_f})}{d\log\frac{Q}{\mu_f}} = \frac{d\log q(N,\mu_f)}{d\log\mu_f} = k$$

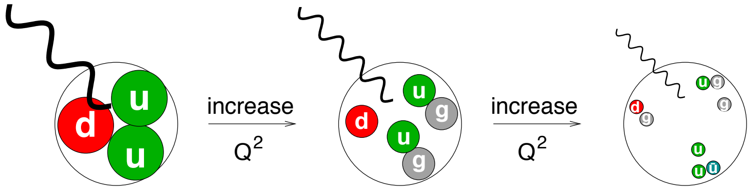
These are called anomalous dimensions and are just the Mellin transform of the corresponding splitting function

whose solution is:

$$q(N,\mu) = q(N,\mu_0)e^{k\log(\frac{\mu_f}{\mu_0})}$$

The pdf "evolves" with the scale!

Scaling violations



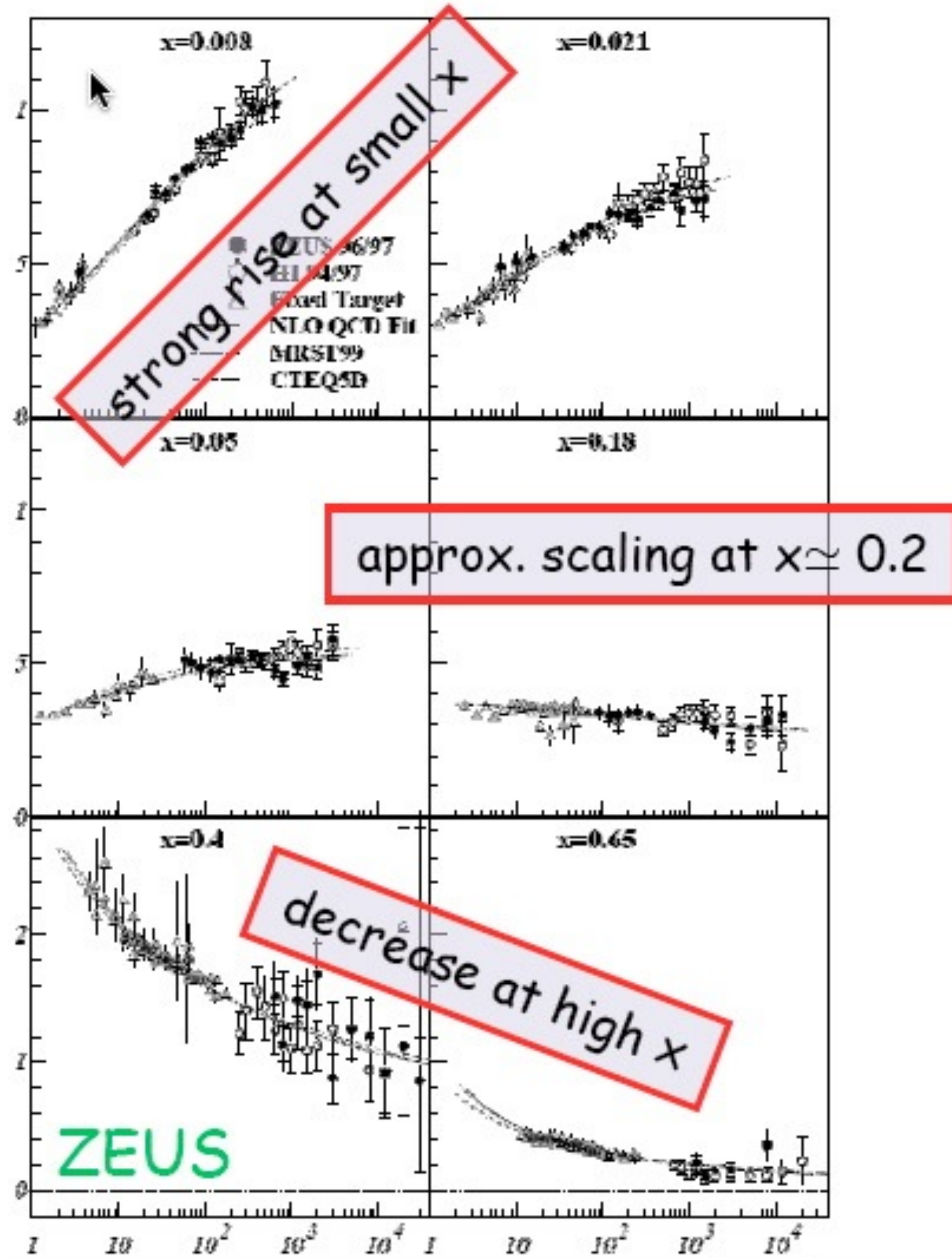
The solution for V can be rewritten in terms of t and α_S as follows:

$$q(N,t) = q(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

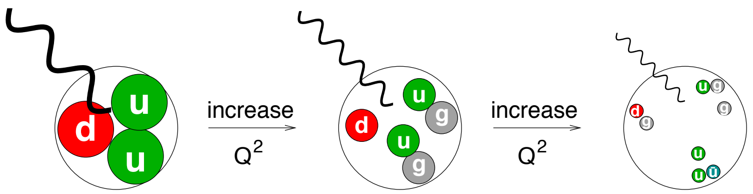
where

$$d_{qq}(N) = \gamma_{qq}^{(0)}/2\pi b_0$$

Now $d_{qq}(1)=0$ and $d_{qq}(N) <0$ for N>1. Thus as t increases V decreases at large x and increases at small x. Physically this is due to an increase in the phase space for gluon emission by quarks as t increases, leading to a loss of momentum.



Evolution



In fact the equations are a bit more complicated as quarks and gluons do mix.
It is convenient to introduce two linear combinations, the singlet **Σ** and the non-singlet q^{NS} to separate the piece that mixes with that that does not:

$$\Sigma(x,Q^2) = \sum_{i=1}^{n_f} (q_i(x,Q^2) + \bar{q}_i(x,Q^2))$$

this is coupled to the gluon

$$q^{\rm NS}(x,Q^2) = q_i(x,Q^2) - \bar{q}_j(x,Q^2)$$

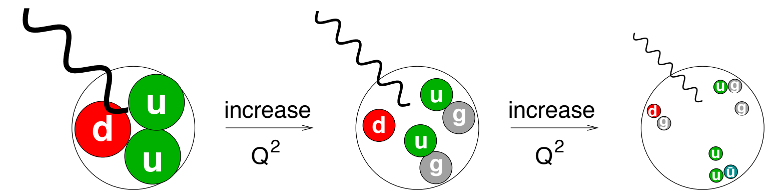
these evolve independently

The complete evolution equations (in Mellin space) to solve are:

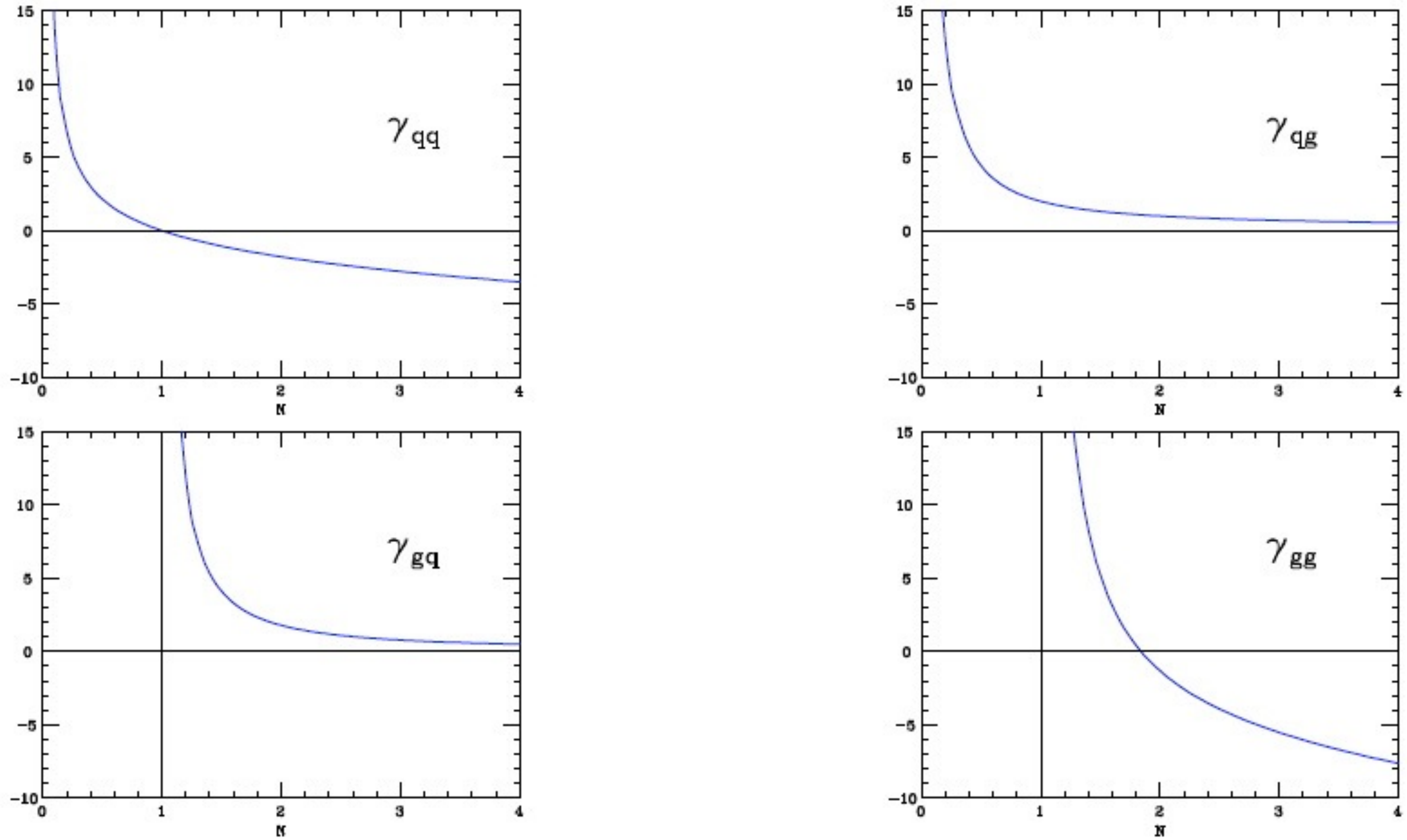
$$\frac{d}{dt}\Delta q^{\rm NS}(N,Q^2) = \frac{\alpha_S(t)}{2\pi}\gamma_{qq}^{\rm NS}(N,\alpha_S(t))\Delta q^{\rm NS}(N,Q^2)$$

$$\frac{d}{dt}\begin{pmatrix} \Delta\Sigma(N,Q^2) \\ \Delta g(N,Q^2) \end{pmatrix} = \frac{\alpha_S(t)}{2\pi}\begin{pmatrix} \gamma_{qq}^{\rm S} & 2n_f\gamma_{qg}^{\rm S} \\ \gamma_{gq}^{\rm S} & \gamma_{gg}^{\rm S} \end{pmatrix}\begin{pmatrix} \Delta\Sigma(N,Q^2) \\ \Delta g(N,Q^2) \end{pmatrix}$$

StefanoForte®

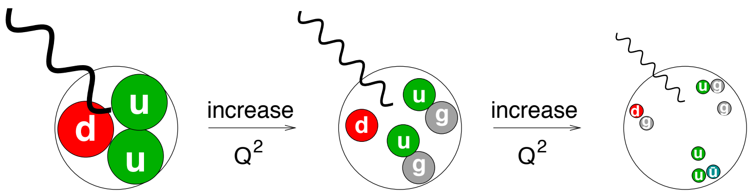


Evolution

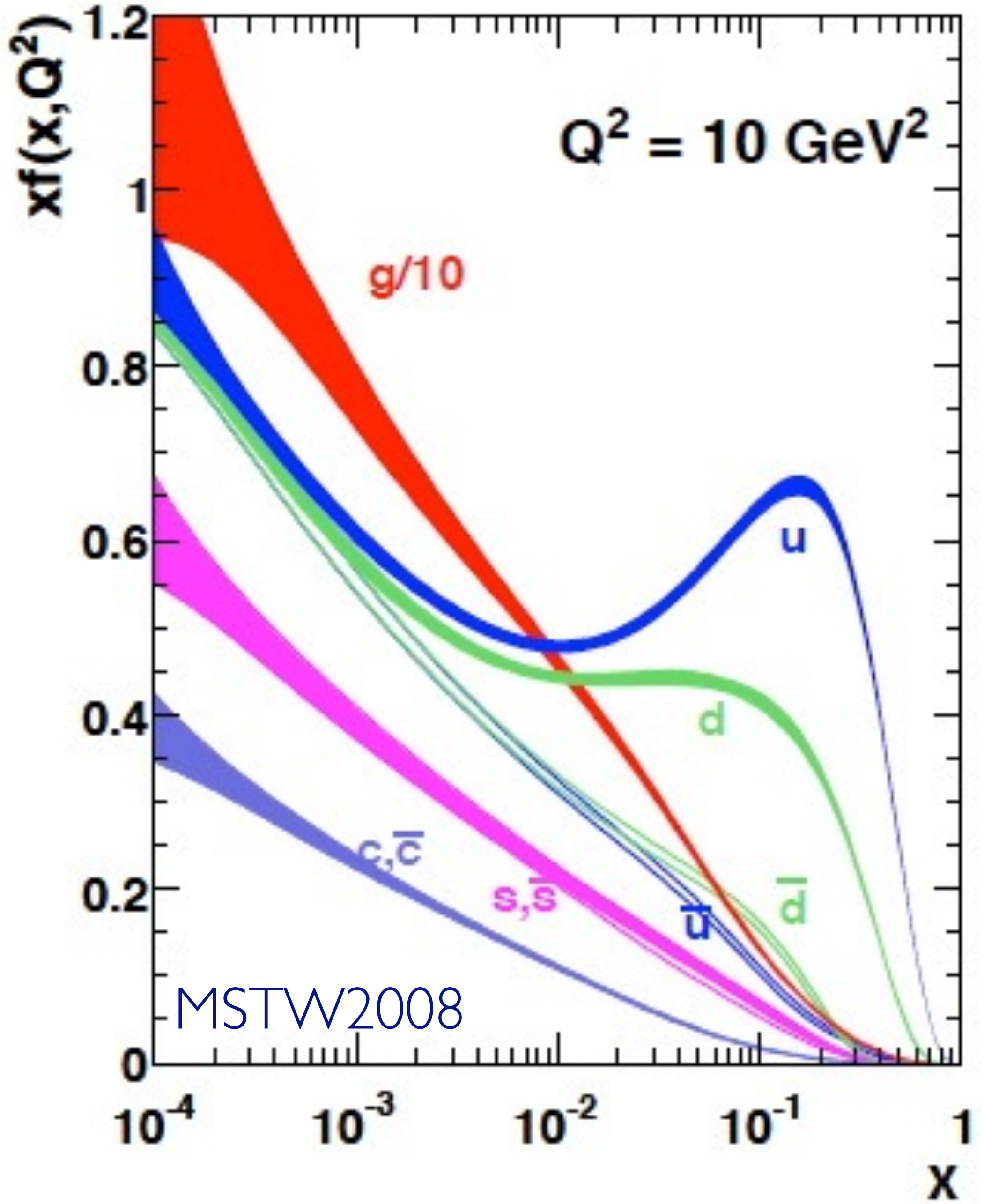


- As Q^2 increases, pdf's decrease at large x and increase at small x due to radiation and momentum loss.
- Gluon singularity at N=1 ⇒ it grows more at small x.
- $\gamma_{qq}(1)=0$ ⇒ number of quarks conserved.

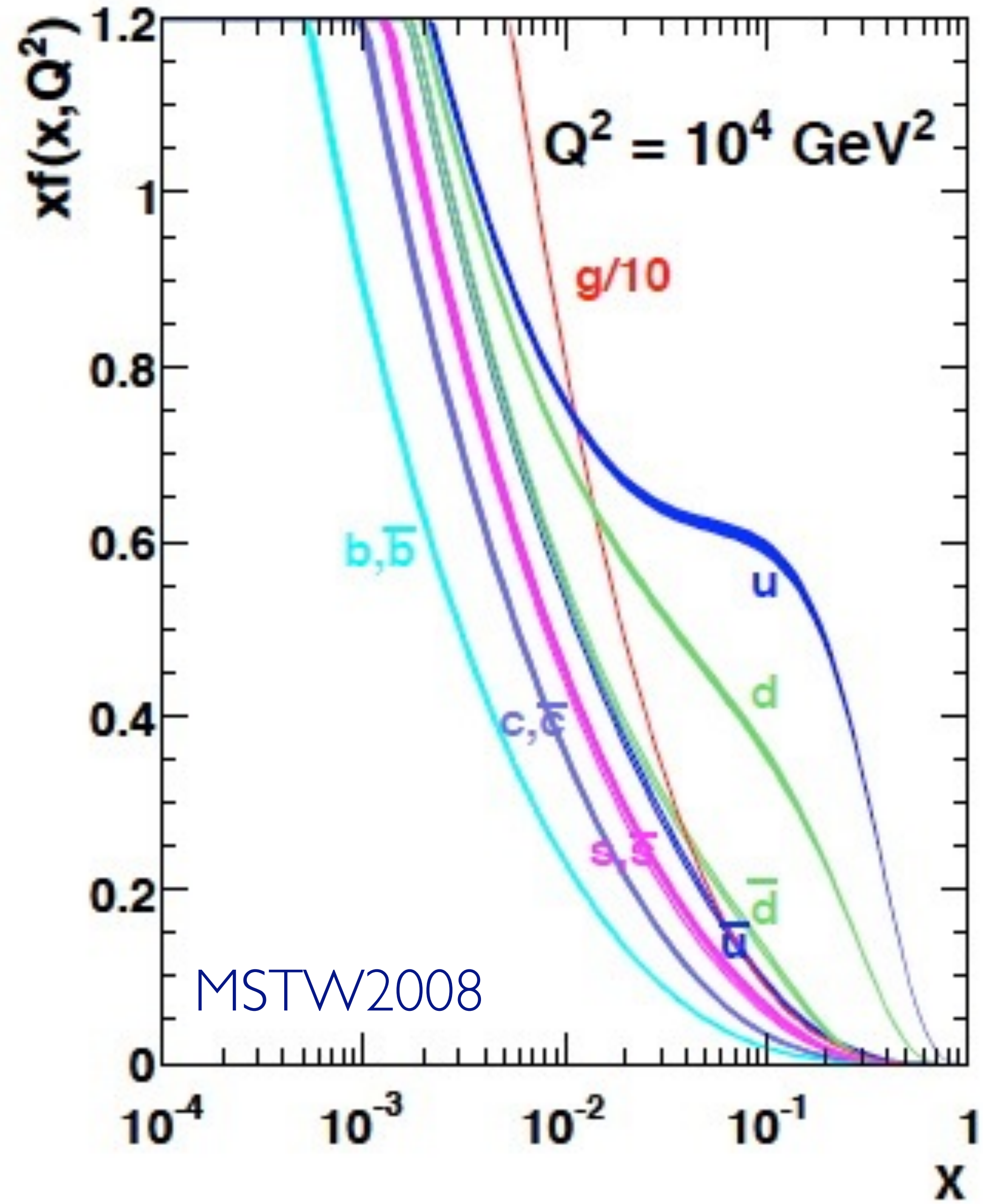
Evolution



d
u
u
increase
Q2
increase
Q2

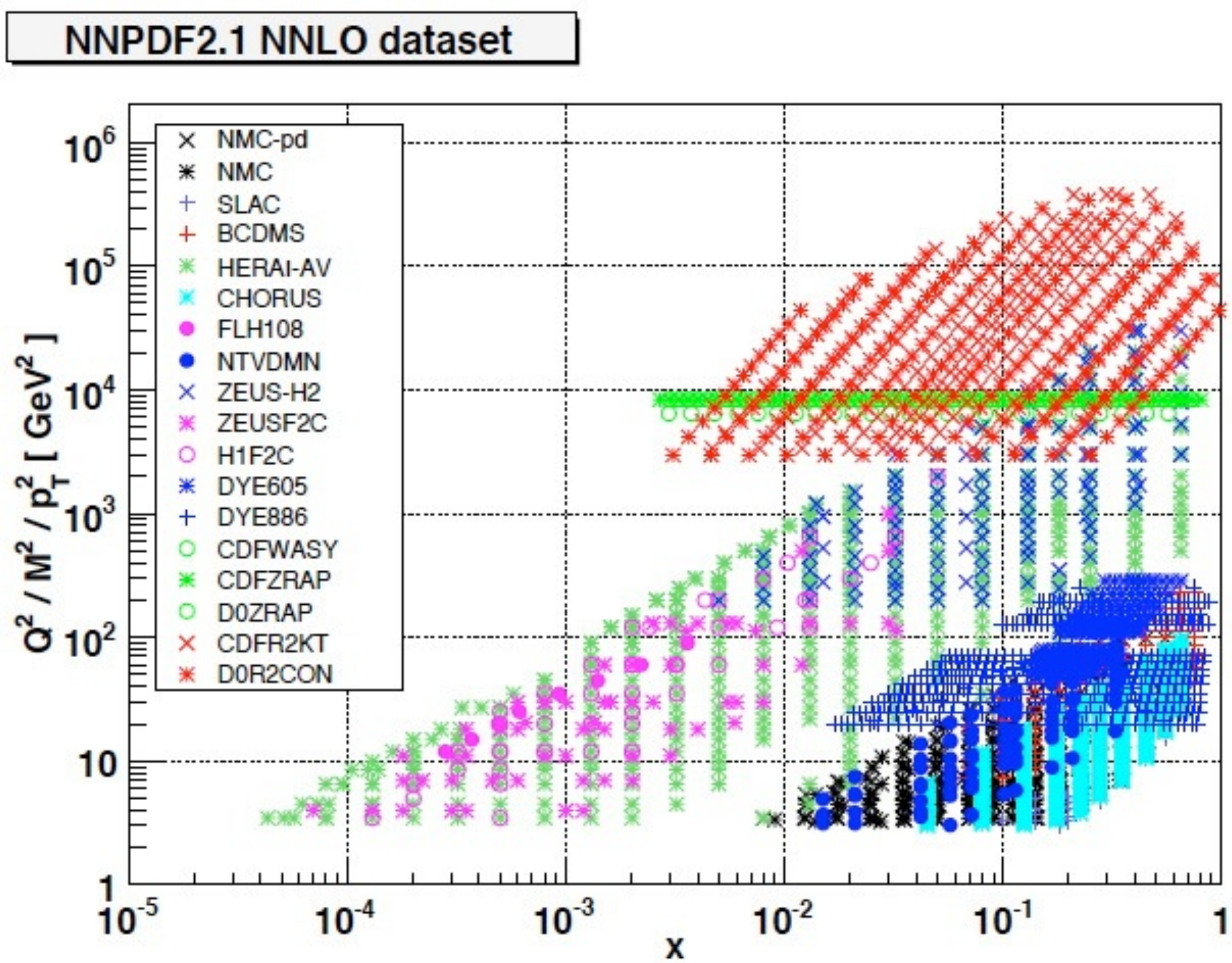


xf(x,Q2)
Q2 = 10 GeV2
g/10
u
d
c,c̄
s,s̄
ū
d̄
MSTW2008
x



xf(x,Q2)
Q2 = 104 GeV2
g/10
b,b̄
u
d
c,c̄
s,s̄
ū
d̄
MSTW2008
x

Modern PDF sets



There are now several collaborations providing PDF sets via a common interface (LHAPDF).

Three of them are global fits.

They provide uncertainties (be careful different procedures for each set!)

Several of them are now at NNLO and include HQ matched.

CTEQ6.6: GLOBAL, NLO, VFN, several α_s
MSTW08: GLOBAL, NNLO, VFN, several α_s
NNPDF2.1: GLOBAL, NNLO, VFN, several α_s

Plus other sets: Alekhin, HERAPDF, GRV/GJR...

Final strategy for QCD predictions

We now have a strategy to get a reliable result in perturbation theory:

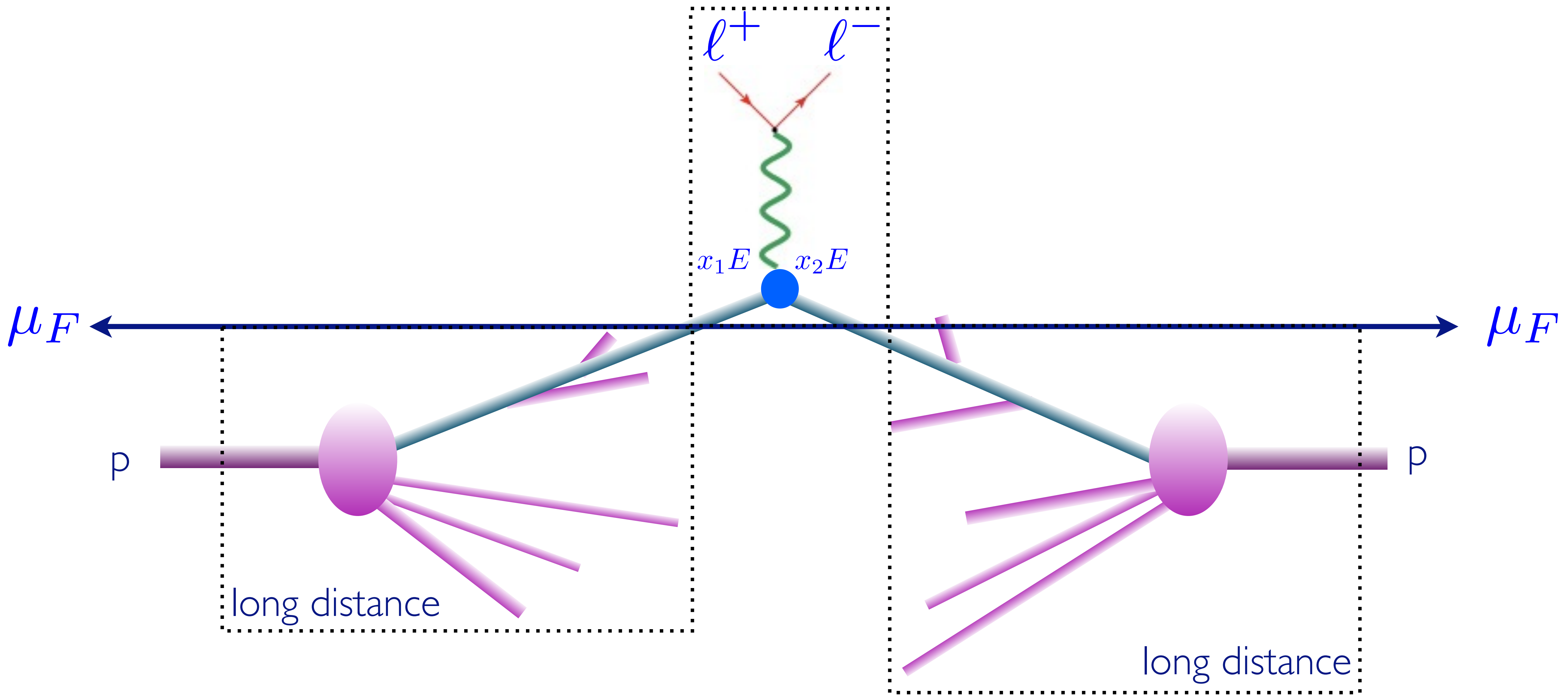
1. Calculate the short distance coefficient in pQCD corresponding to an observable. All divergences will cancel except those due to the collinear splitting of initial partons.

2. Re-absorbe such divergences in the pdf's and introduce a factorization scale.

3. Extract from experiment the initial condition for the pdf's at a given reference scale.

4. Evolve the pdf's at the scale of the process we are interested it. In so doing all large logs of the factorization scale over a small scale are resummed.

LHC master formula



$$\sigma_X = \sum_{a,b} \int_0^1 dx_1 dx_2 \, f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \times \hat{\sigma}_{ab \to X}(x_1, x_2, \alpha_S(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2})$$

Remark on our master formula

$$\sigma_X = \sum_{a,b} \int_0^1 dx_1 dx_2\, f_a(x_1,\mu_F^2) f_b(x_2,\mu_F^2) \times \hat{\sigma}_{ab\to X}(x_1, x_2, \alpha_S(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2})$$

- By calculating the short distance coefficient at tree-level we obtain the first estimate of rates for inclusive final states.

- Even at LO extra radiation is included: it is described by the PDF's in the initial state and by the definition of a final state parton, which at LO represents all possible final state evolutions.

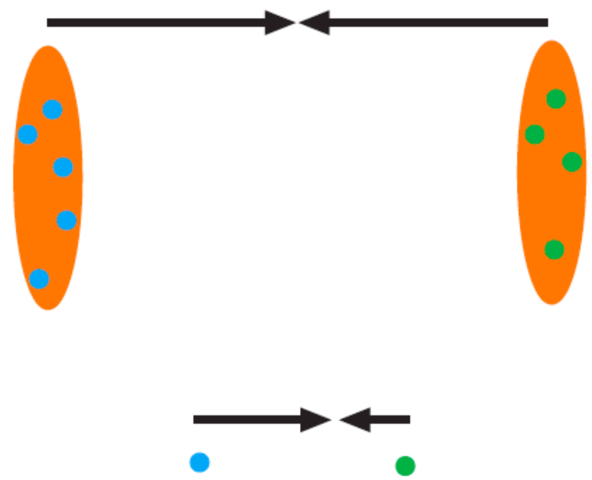
- Due to the above approximations a cross section at LO can strongly depend on the factorization and renormalization scales.

- Predictions can be systematically improved, at NLO and NNLO, by including higher order corrections in the short distance and in the evolution of the PDF's.

PP KINEMATICS

We describe the collision in terms of parton energies

$E_1 = x_1$ Ebeam
$E_2 = x_2$ Ebeam

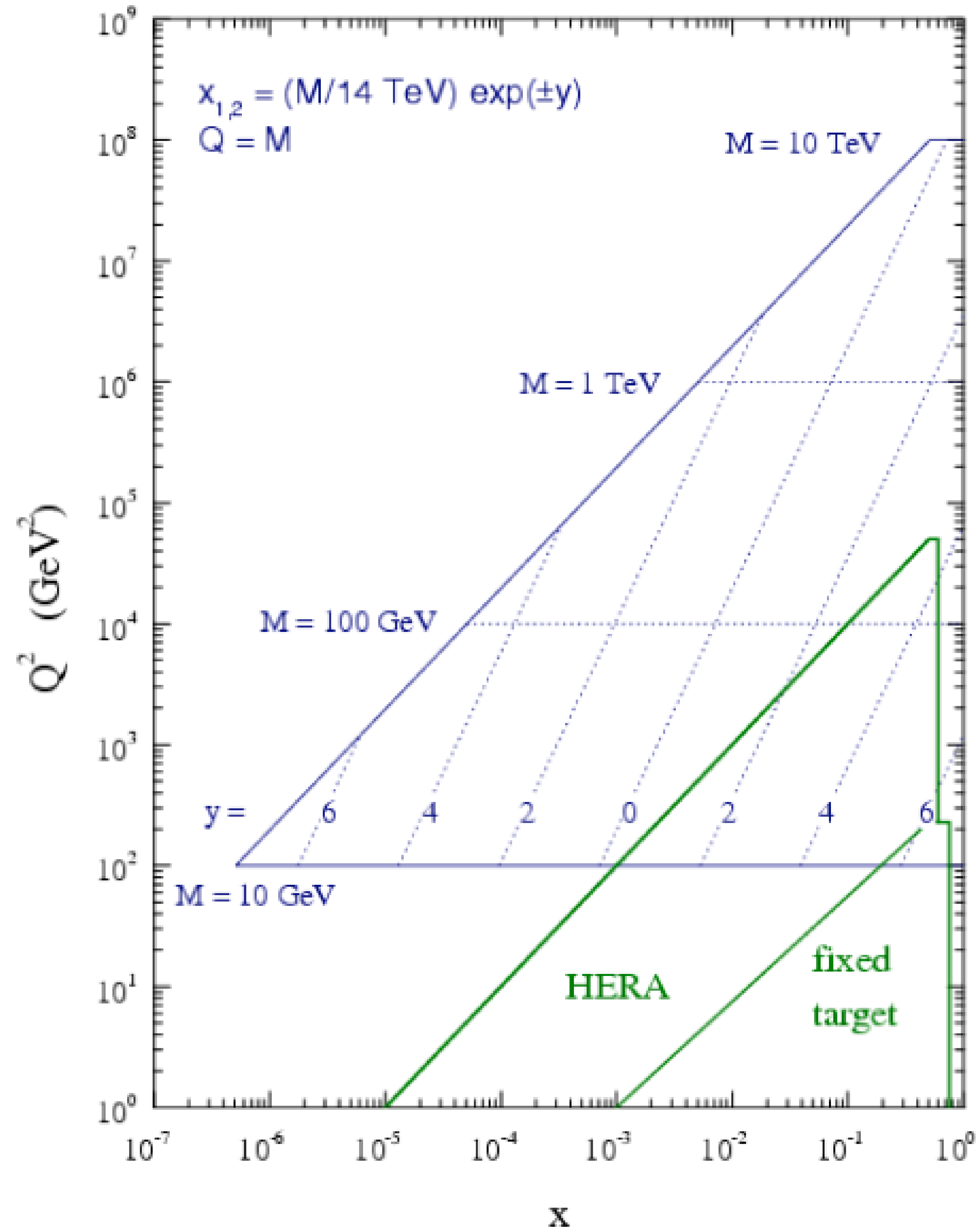


Obviously the partonic c.m.s. frame will be in general boosted. Let us say that the two partons annihilate into a particle of mass M.

$$M^2 = x_1 x_2 S = x_1 x_2 4 E^2_{\rm beam}$$

$$y = \frac{1}{2} \log \frac{x_1}{x_2}$$

$$x_1 = \frac{M}{\sqrt{S}} e^{y} \quad x_2 = \frac{M}{\sqrt{S}} e^{-y}$$



QCD IN THE INITIAL STATE

1. We have introduced the physics of Deep Inelastic Scattering and the associated kinematics. We interpreted scaling in the parton model framework, trying to give a description of the physics involved by choosing a suitable frame.

2. We have shown that the parton model survives to QCD corrections, which affect the scaling picture only with logarithmic corrections.

3. In order to make prediction in pQCD, we have introduced the idea of factorization, which stands as a pillar for all interesting applications of pQCD.

4. The idea is to separate short-distance physics from long-distance one.
The first is calculable in pQCD. The second in non-perturbative and therefore not calculable but universal. So it can be measured in one experiment and used in another.

5. We have introduced the DGLAP equations that regulate the evolution of the pdf with the scale and allow the resummation of large logs.