Parton Shower

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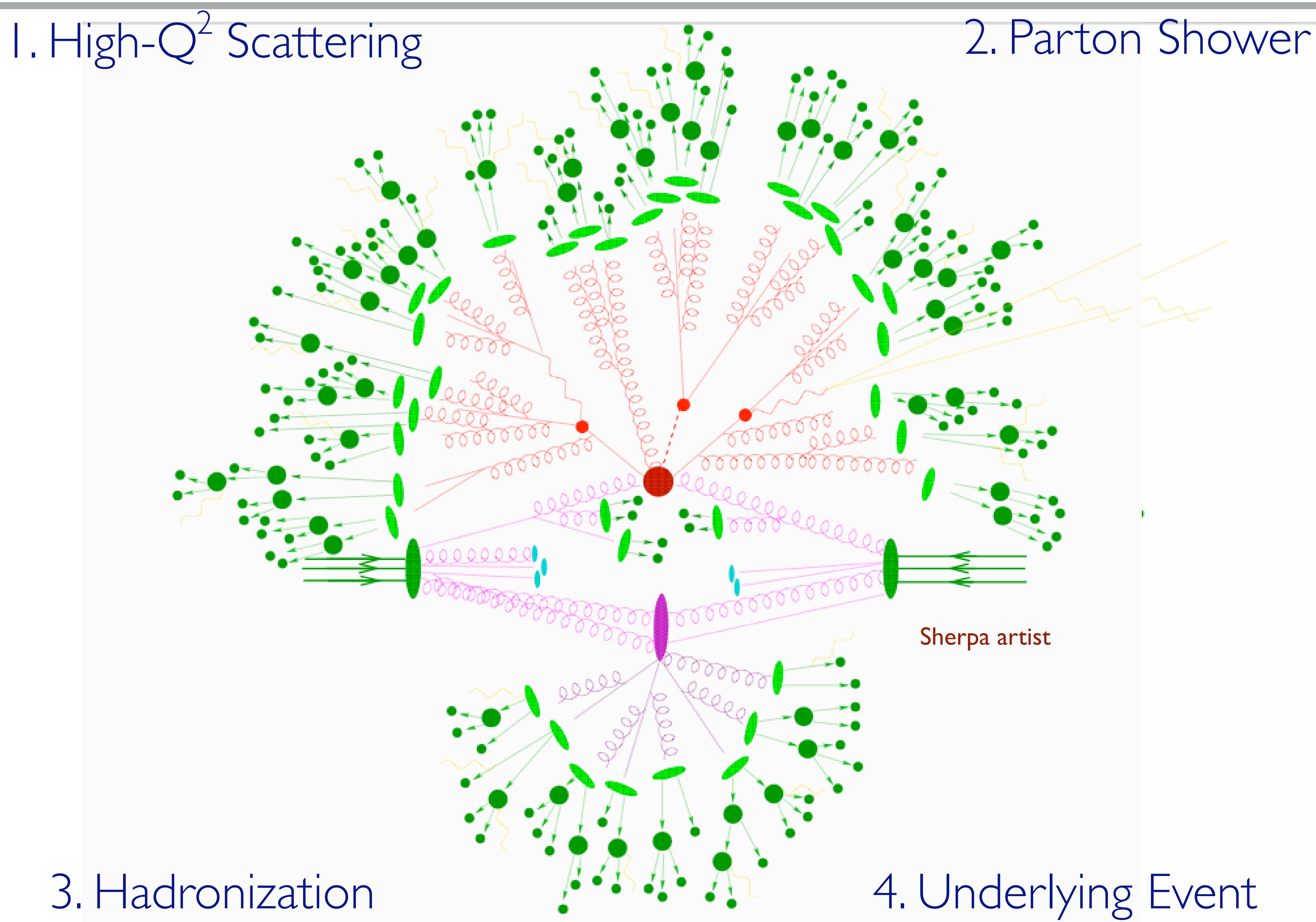
CP3/UCLouvain

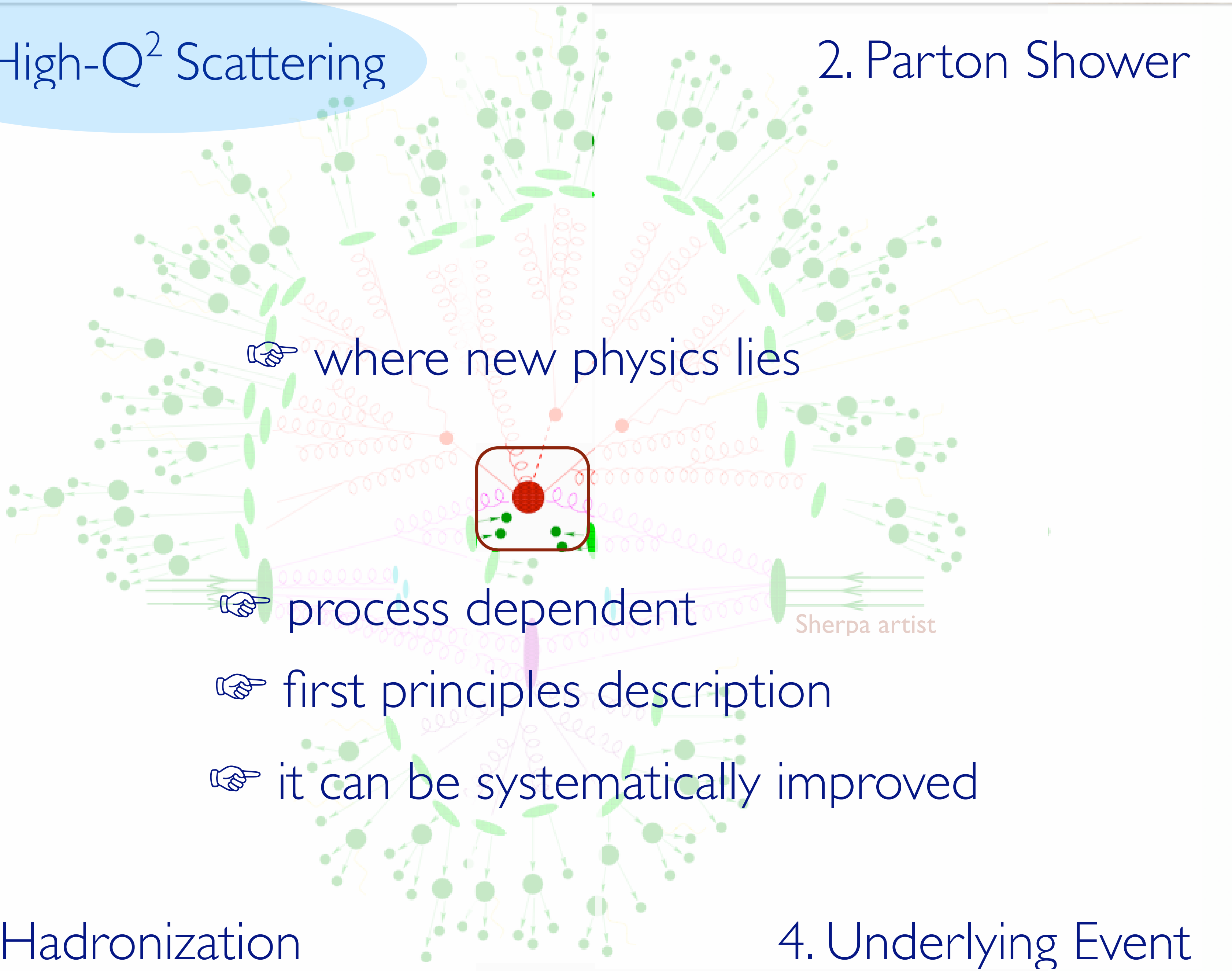
What are the MC for?



What are the MC for?

1. High-Q^2 Scattering

2. Parton Shower



☞ where new physics lies

☞ process dependent

☞ first principles description

☞ it can be systematically improved

3. Hadronization

4. Underlying Event

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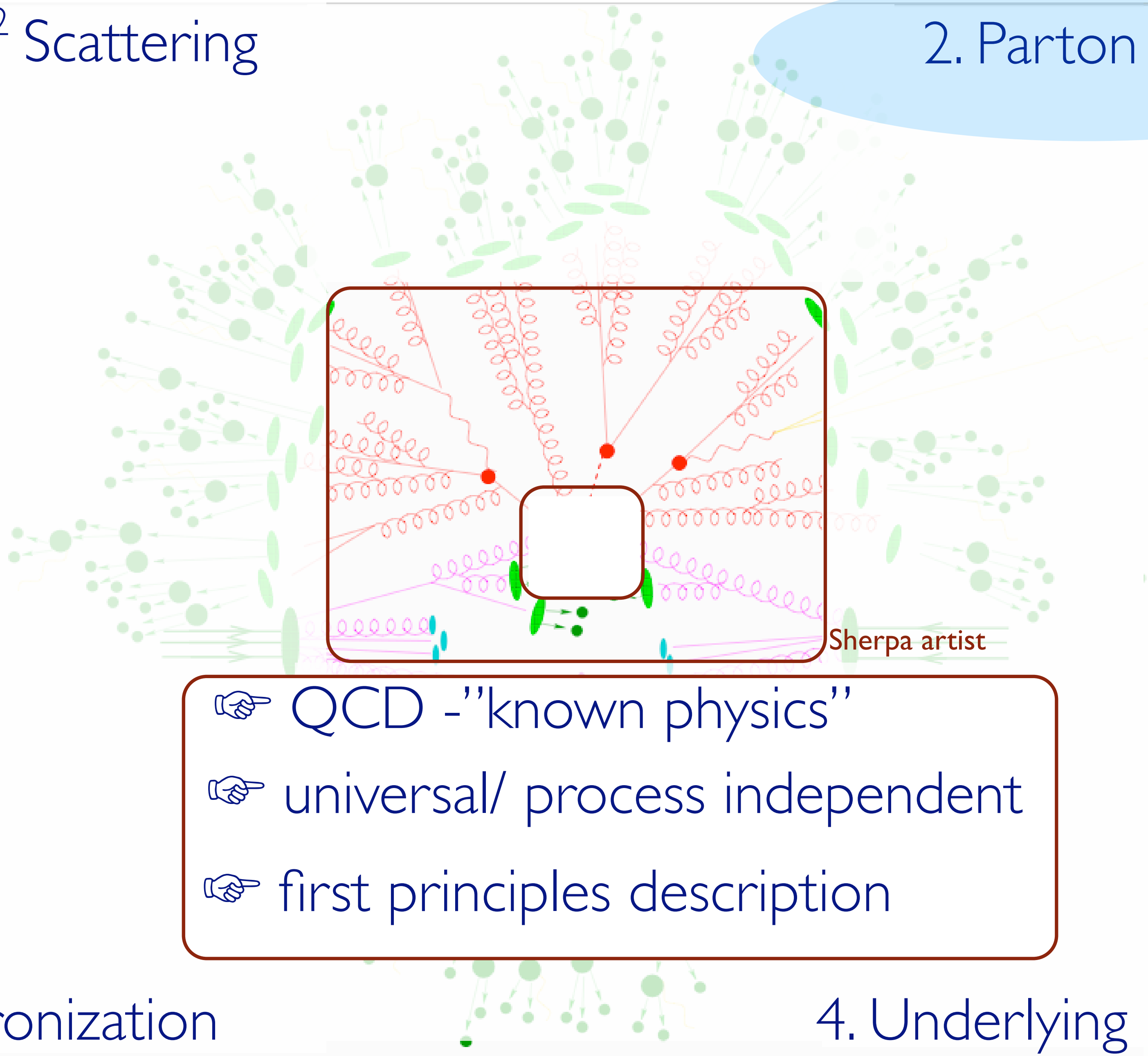
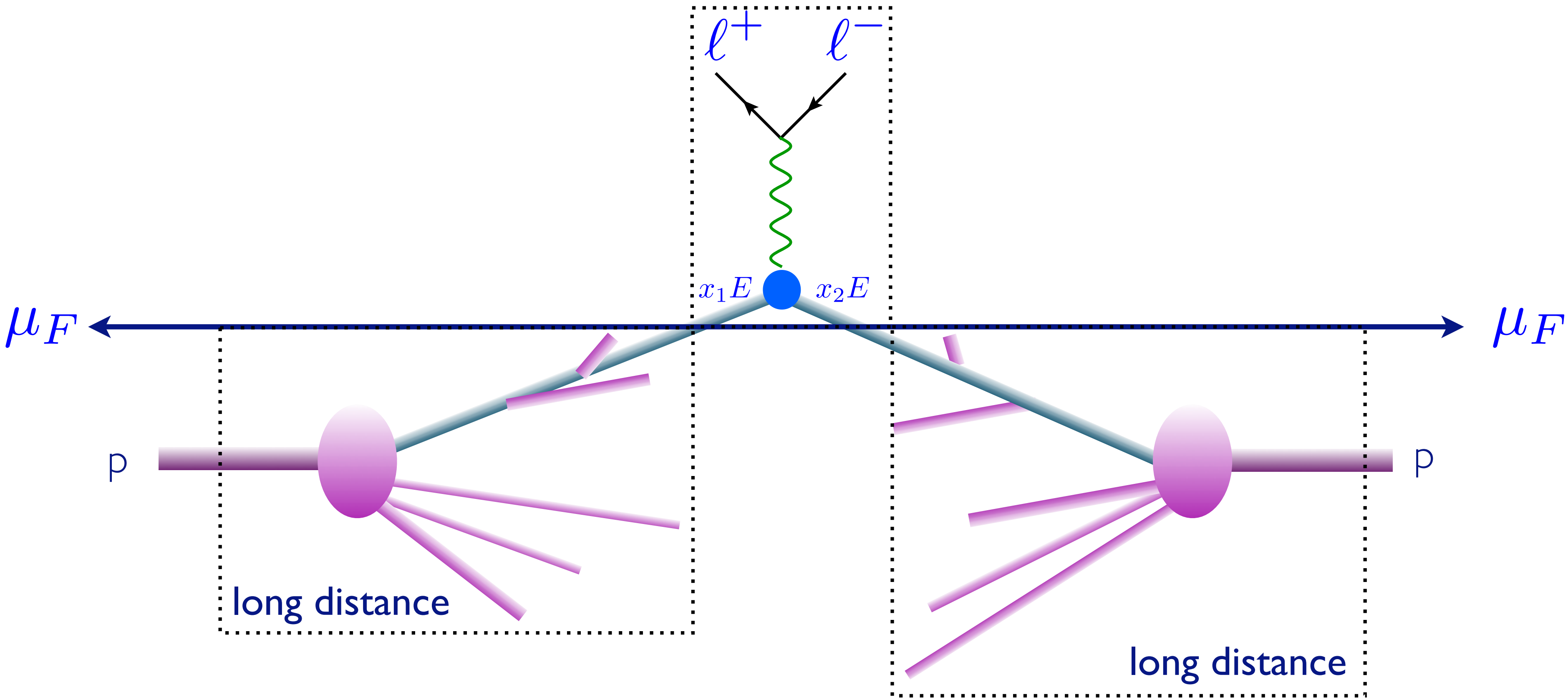


- ☞ QCD -"known physics"
- ☞ universal/ process independent
- ☞ first principles description

3. Hadronization

4. Underlying Event

Master formula for the LHC



$$\sum_{a,b} \int dx_1 dx_2 d\Phi_{\text{FS}}\, f_a(x_1, \mu_F) f_b(x_2, \mu_F)\, \hat{\sigma}_{ab\to X}(\hat{s}, \mu_F, \mu_R)$$

Phase-space integral

Parton density functions

Parton-level cross section

Parton shower

Goal

- We want to an **explicit** description of the SOFT radiation that are **ALREADY** included **implicitly in the LO events** (via the scale)

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  - Such radiations are already included within the event-generator

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Important

- Parton-Shower is **not ADDING radiation**
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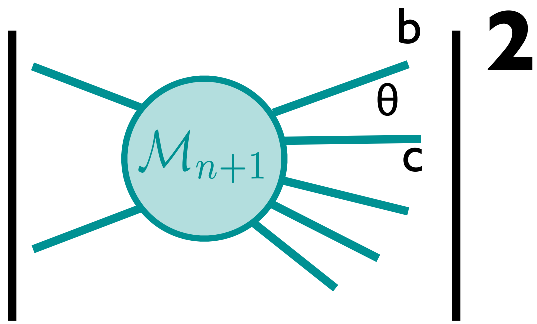
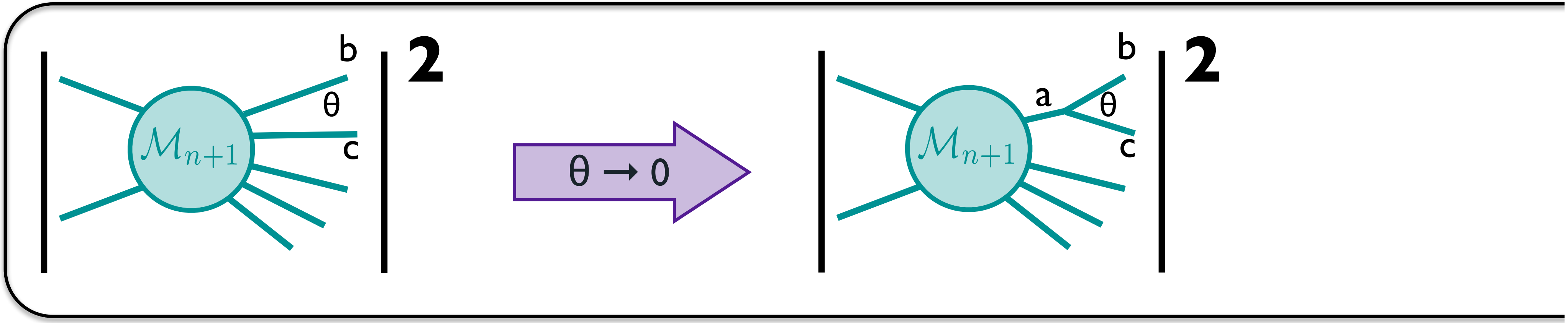
- We need to be able to describe an arbitrarily number of parton branchings, i.e. we need to 'dress' partons with radiation
- This effect should be **unitary:** the inclusive cross section shouldn't change when extra radiation is added

Collinear factorization



- Consider a process for which two particles are separated by a small angle θ.

Collinear factorization



- Consider a process for which two particles are separated by a small angle θ.
- In the limit of θ → 0 the contribution is coming from a single parent particle going on shell: therefore its branching is related to time scales which are very long with respect to the hard subprocess.

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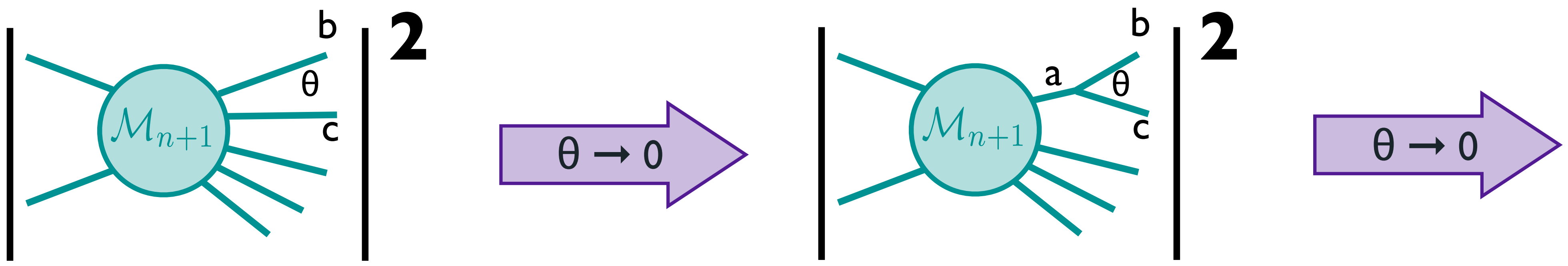

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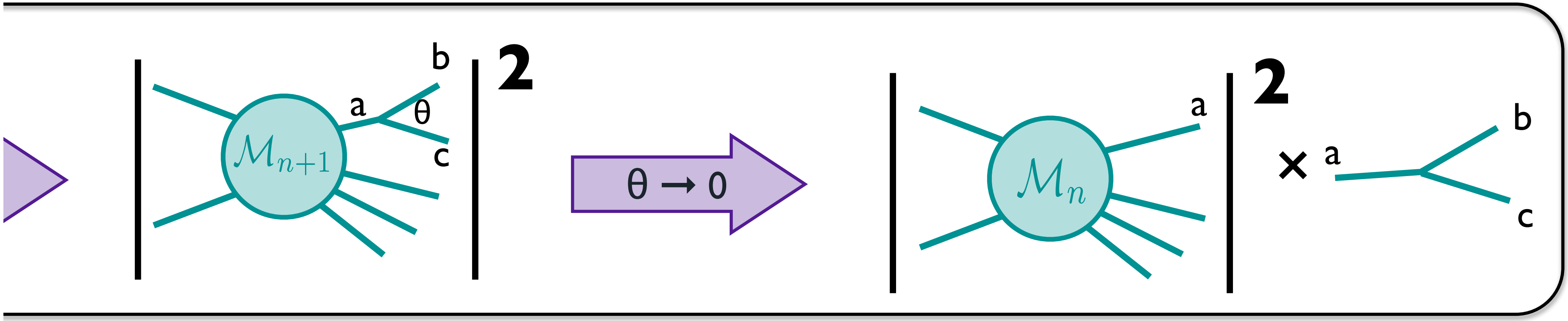

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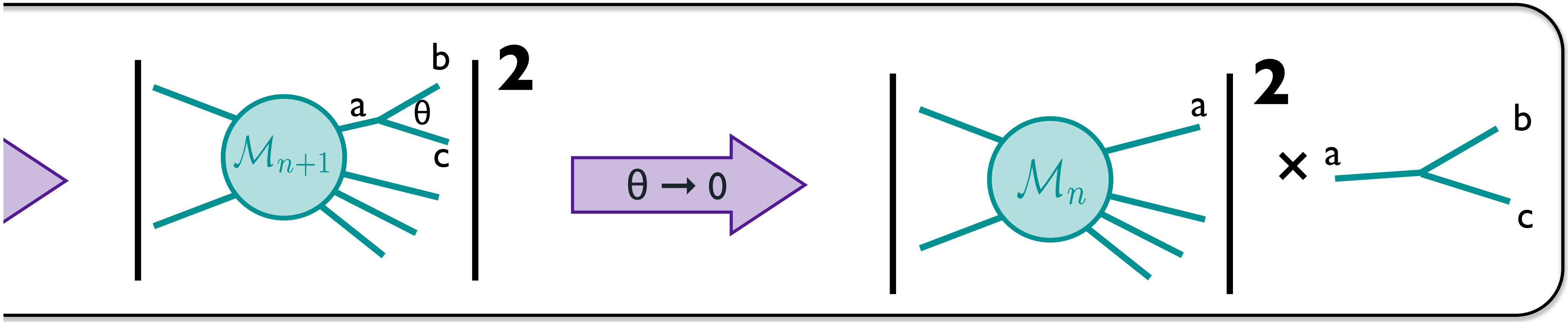
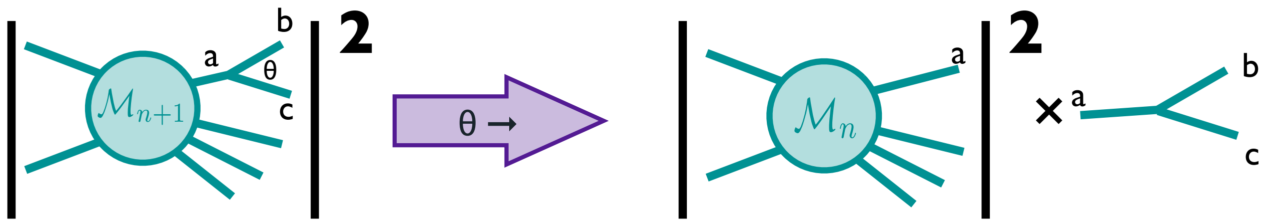


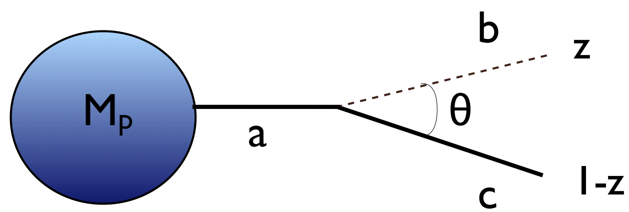
- Consider a process for which two particles are separated by a small angle θ.
- In the limit of θ → 0 the contribution is coming from a single parent particle going on shell: therefore its branching is related to time scales which are very long with respect to the hard subprocess.
- The inclusion of such a branching cannot change the picture set up by the hard process: the whole emission process must be writable in this limit as the simpler one times a branching probability.

Collinear factorization



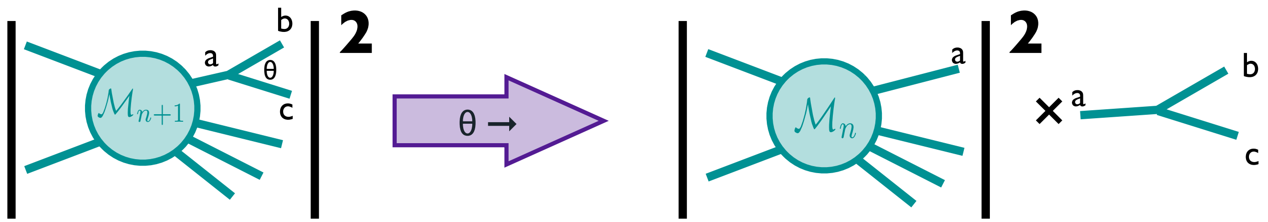
- The process factorizes in the collinear limit. This procedure it universal!

$$\frac{1}{(p_b+p_c)^2} \simeq \frac{1}{2E_bE_c(1-\cos\theta)} = \frac{1}{t}$$



$z = E_b/E_a$

Collinear factorization



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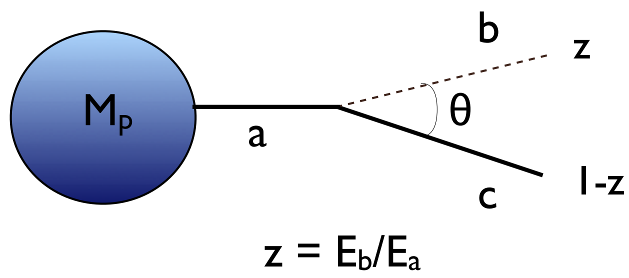

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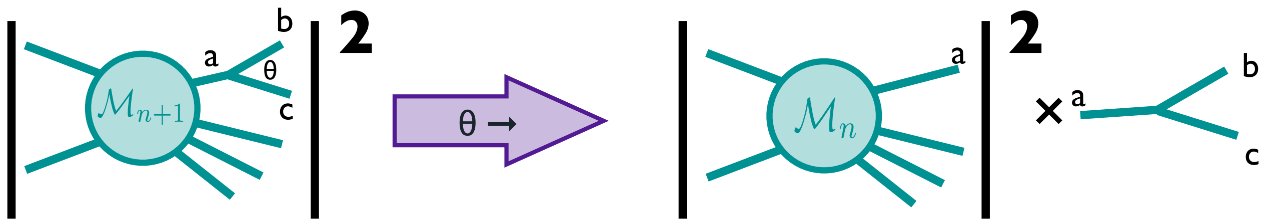

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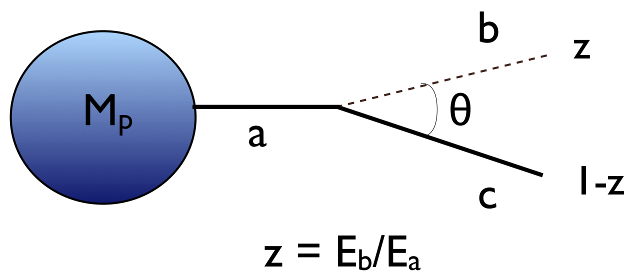
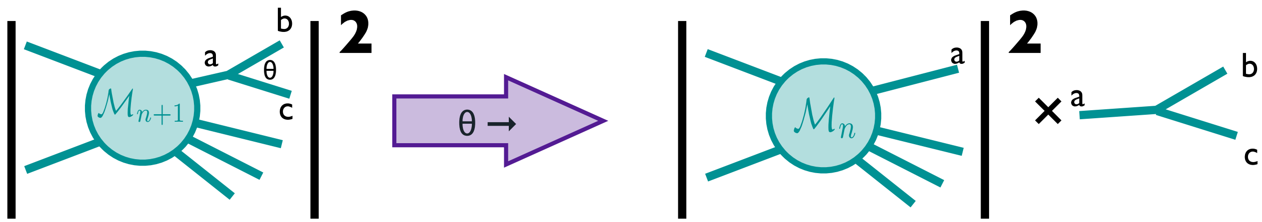


soft and collinear

divergencies

Collinear factorization



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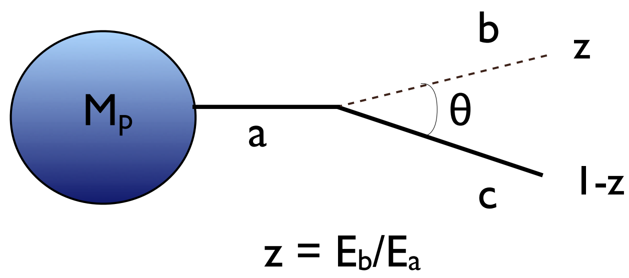

soft and collinear

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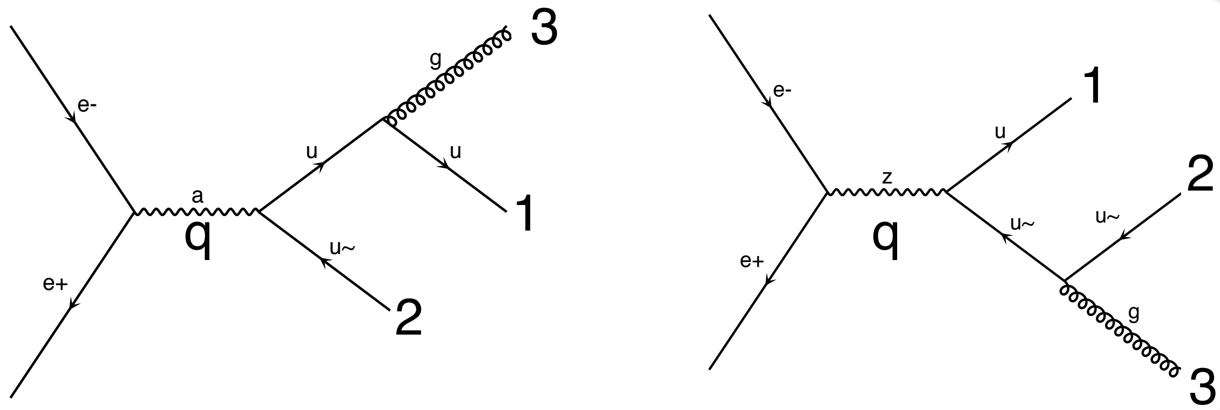
Collinear factorization:

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

when θ is small.

First Example

$$e^+e^- \to q\bar{q}g$$



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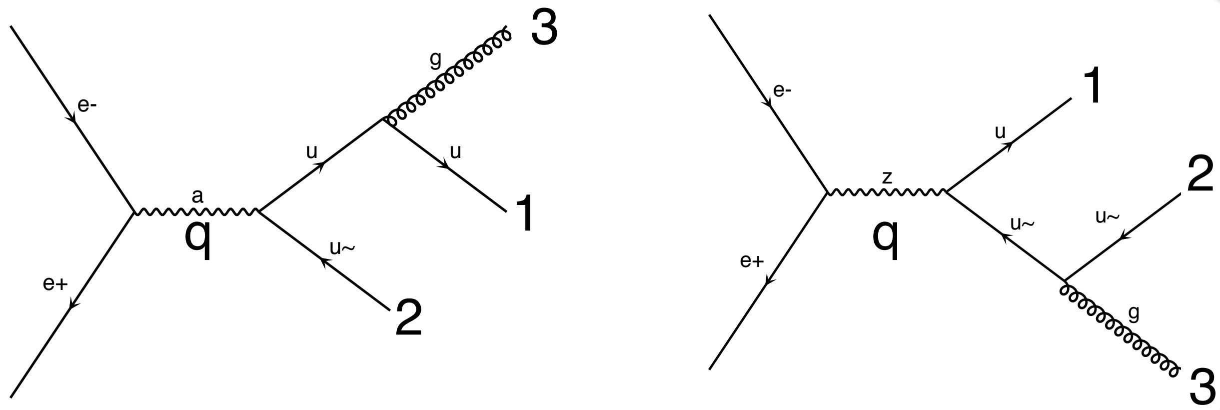

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

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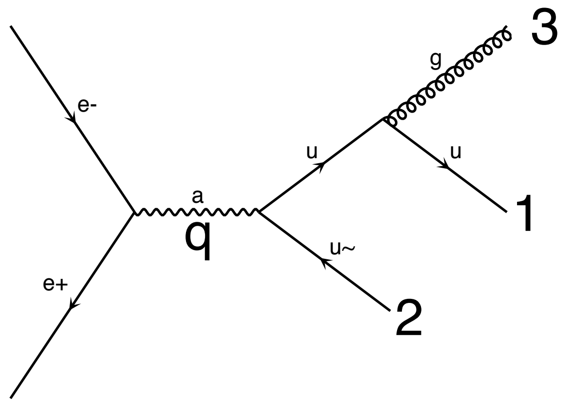

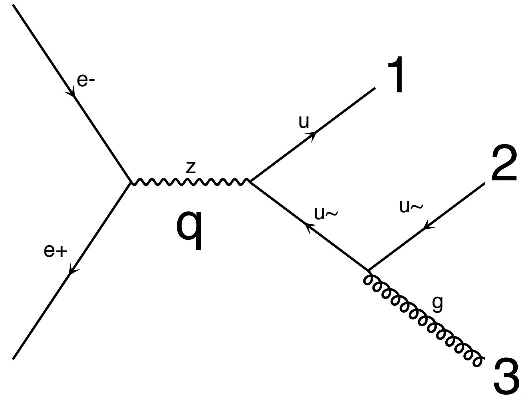

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- Divergent at $x_1 = 1$ and $x_2 = 1$
    - Soft Divergencies
    - Collinear Divergencies

$$(1-x_1) = \frac{x_2 x_3}{2}(1 - cos\theta_{23})$$

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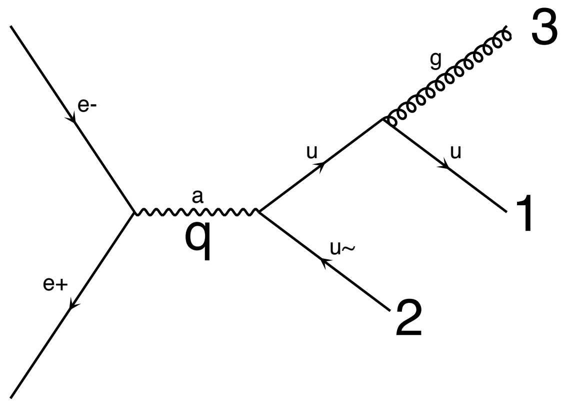

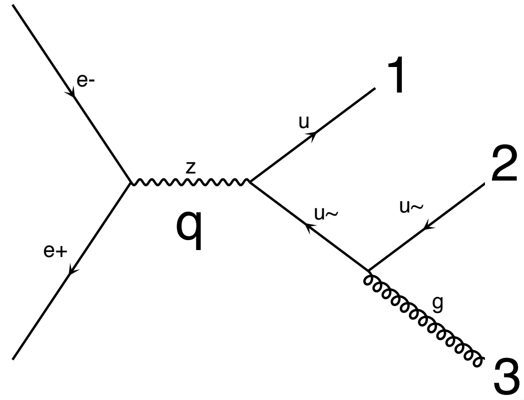

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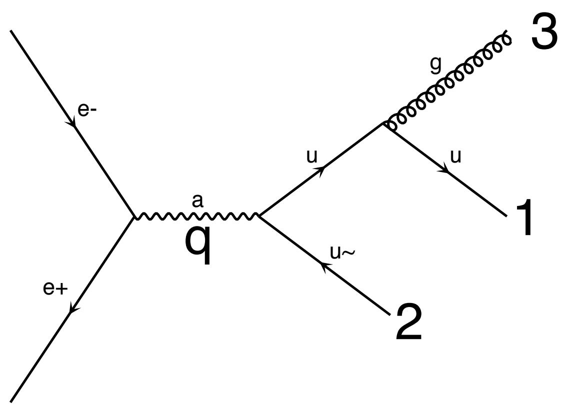

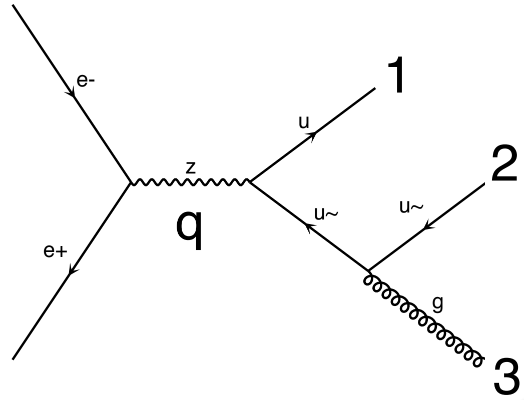

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- Collinear limit
  - Split our integral in two

$$\frac{2\, dcos\theta_{13}}{sin^2\theta_{13}} = \frac{dcos\theta_{13}}{1-cos\theta_{13}} + \frac{dcos\theta_{13}}{1+cos\theta_{13}}$$
$$\approx \frac{d\cos\theta_{13}}{(1-\cos\theta_{13})} + \frac{d\cos\theta_{23}}{(1-\cos\theta_{23})}$$
$$\approx \frac{d\theta_{13}^2}{\theta_{13}^2} + \frac{d\theta_{23}^2}{\theta_{23}^2}$$

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$$d\sigma = \sigma_0 \sum_{\text{jets}} C_F \frac{\alpha_s}{2\pi}\frac{d\theta^2}{\theta^2}\, dz \frac{1+(1-z)^2}{z}$$

☞ z fraction of energy

☞ **Generic Formula**

Parton Shower basics

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \;\simeq\; |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

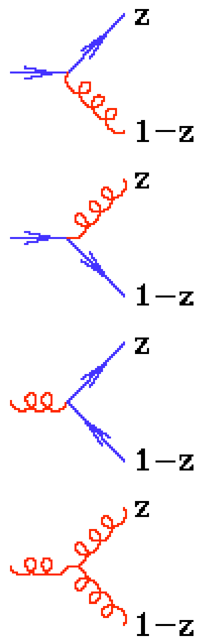
The spin averaged (unregulated) splitting functions for the various types of branching are (Altarelli-Parisi):

$$\hat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)}\right],$$

$$\hat{P}_{gq}(z) = C_F \left[\frac{1+(1-z)^2}{z}\right],$$

$$\hat{P}_{qg}(z) = T_R \left[z^2+(1-z)^2\right],$$

$$\hat{P}_{gg}(z) = C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z(1-z)\right].$$



$C_F = \frac{4}{3}, C_A = 3, T_R = \frac{1}{2}.$

Parton Shower basics

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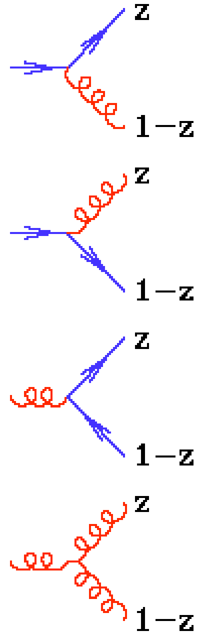


$$C_F = \tfrac{4}{3}, C_A = 3, T_R = \tfrac{1}{2}.$$

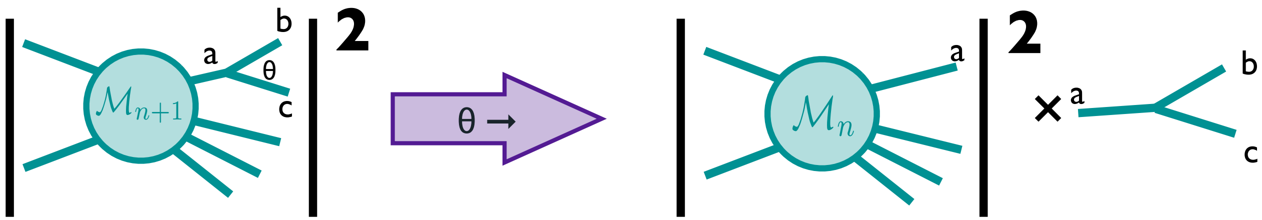
Comments:

* Gluons radiate the most

* There are soft divergences in z=1 and z=0.

* P_{qg} has no soft divergences.

Collinear factorization



- The process factorizes in the collinear limit. This procedure it universal!

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \;\simeq\; |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Notice that what has been roughly called 'branching probability' is actually a singular factor, so one will need to make sense precisely of this definition.

- At the leading contribution to the (n+1)-body cross section the Altarelli-Parisi splitting kernels are defined as:

$$P_{g\to qq}(z) = T_R\left[z^2 + (1-z)^2\right], \qquad P_{g\to gg}(z) = C_A\left[z(1-z) + \frac{z}{1-z} + \frac{1-z}{z}\right],$$

$$P_{q\to qg}(z) = C_F\left[\frac{1+z^2}{1-z}\right], \qquad P_{q\to gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right].$$

Collinear factorization

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- **t** can be called the 'evolution variable' (will become clearer later): it can be the virtuality **m^2** of particle a or its **p_T^2** or **$E^2\theta^2$** …

$$d\theta^2/\theta^2 = dm^2/m^2 = dp_T^2/p_T^2$$

$$m^2 \simeq z(1-z)\theta^2 E_a^2$$
$$p_T^2 \simeq zm^2$$

  - It represents the hardness of the branching and tends to 0 in the collinear limit.
  - Different choice of 'evolution parameter' in different Parton-shower code

Collinear factorization

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

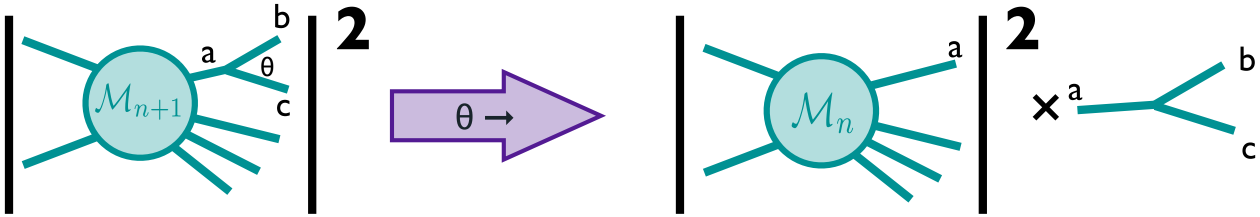
- **z** is the "energy variable": it is defined to be the energy fraction taken by parton **b** from parton **a.** It represents the energy sharing between **b** and **c** and tends to 1 in the soft limit (parton c going soft)
- **Φ** is the azimuthal angle. It can be chosen to be the angle between the polarization of a and the plane of the branching.

Argument of α_S

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Each choice of argument for α_S is equally acceptable at the leading-logarithmic accuracy. However, there is a choice that allows one to resum certain classes of subleading logarithms.
- The more natural choices is to evaluated it at scale "t"
  - Can be proof to be a good choice since it allows to include sub-logarithm contributions.
  - Each radiation evaluates alpha_s at his own scale
    - Different from fixed order computation where all value use the renormalisation scale.

Collinear factorization



- Cross section factorization in the collinear limit. This procedure it universal!

- The process factorizes in the collinear limit. This procedure it universal!

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- This is an amplitude squared: naively one would maybe expect **1/t²** dependence. Why is the square not there?

  - It's due to angular-momentum conservation.
    E.g., take the splitting **q ⟶ qg**: helicity is conserved for the quarks, so the final state spin differs by one unity with respect to the initial one. The scattering happens in a p-wave (orbital angular momentum equal to one), so there is a suppression factor as **t ⟶ 0**.

  - In fact, a factor **1/t** is always cancelled in an explicit computation

To Remember

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \;\simeq\; |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_{\mathrm{S}}}{2\pi} P_{a\to bc}(z)$$

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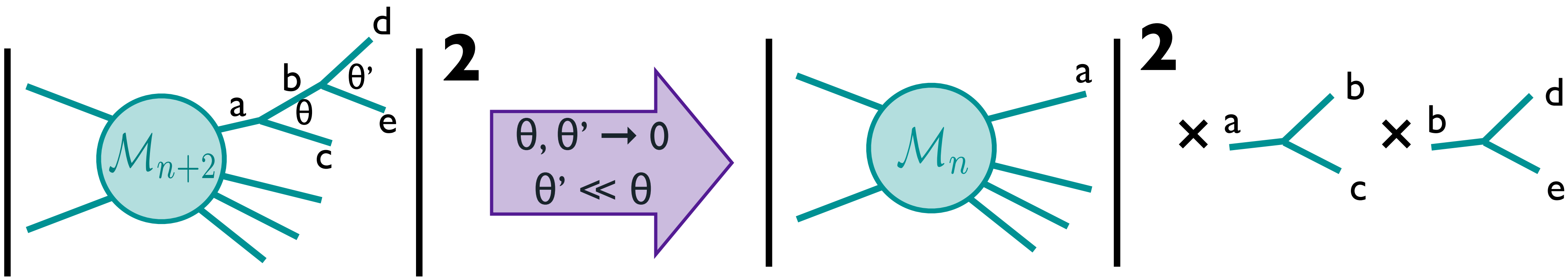
- **alpha_s** need to be evaluated at the scale t
- **P** is the splitting Kernel (control the soft behaviour)

Multiple emission

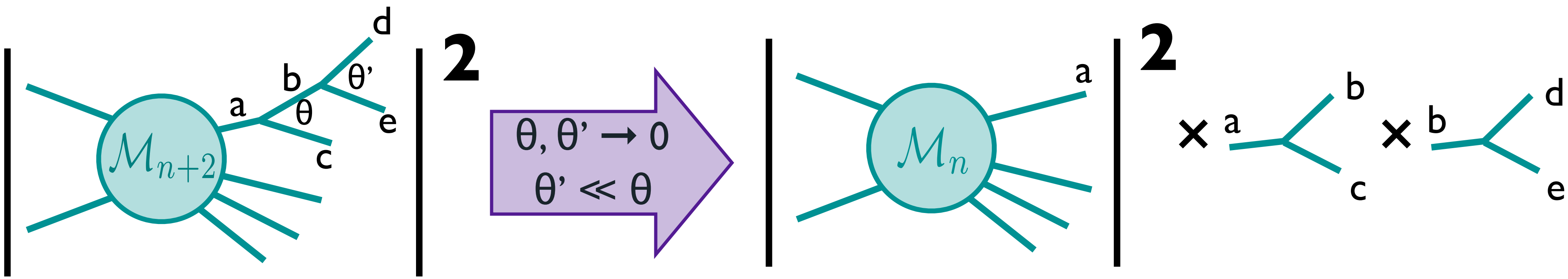


- Now consider $\mathbf{M_{n+1}}$ as the new core process and use the recipe we used for the first emission in order to get the dominant contribution to the (n+2)-body cross section: add a new branching at angle much smaller than the previous one:

$$|\mathcal{M}_{n+2}|^2 d\Phi_{n+2} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$
$$\times \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi} \frac{\alpha_S}{2\pi} P_{b\to de}(z')$$

- This can be done for an arbitrary number of emissions. The recipe to get the leading collinear singularity is thus cast in the form of an iterative sequence of emissions whose probability does not depend on the past history of the system: a 'Markov chain'. No interference!!!

Multiple emission



- The dominant contribution comes from the region where the subsequently emitted partons satisfy the strong ordering requirement: θ » θ' » θ''...
  For the rate for multiple emission we get

$$\sigma_{n+k} \propto \alpha_S^k \int_{Q_0^2}^{Q^2} \frac{dt}{t} \int_{Q_0^2}^{t} \frac{dt'}{t'} \cdots \int_{Q_0^2}^{t^{(k-2)}} \frac{dt^{(k-1)}}{t^{(k-1)}} \propto \sigma_n \left(\frac{\alpha_S}{2\pi}\right)^k \log^k(Q^2/Q_0^2)$$

  where Q is a typical hard scale and Q_0 is a small infrared cutoff that separates perturbative from non perturbative regimes.

- Each power of α_s comes with a logarithm. The logarithm can be easily large, and therefore it can lead to a breakdown of perturbation theory.

Absence of interference

- The collinear factorization picture gives a branching sequence for a given leg starting from the hard subprocess all the way down to the non-perturbative region.
- Suppose you want to describe two such histories from two different legs: these two legs are treated in a completely uncorrelated way. And even within the same history, subsequent emissions are uncorrelated.
- The collinear picture completely misses the possible interference effects between the various legs. The extreme simplicity comes at the price of quantum inaccuracy.
- Nevertheless, the collinear picture captures the leading contributions: it gives an excellent description of an arbitrary number of (collinear) emissions:
  - It is a "resummed computation"
  - It bridges the gap between fixed-order perturbation theory and the non-perturbative hadronization.

Sudakov Form Factor

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$$P_{no-branching}(Q^2, t) = \lim_{N\to\infty}\prod_{i=0}^{N}\left(1 - \frac{\delta t}{t_i}\frac{\alpha_S}{2\pi}\int dz\hat{P}(z)\right)$$

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$$\simeq e^{-\int_t^{Q^2}\frac{dt'}{t'}\int dz\frac{\alpha_S}{2\pi}\hat{P}(z)} \equiv e^{-\int_t^{Q^2}dp(t')}$$

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$$\boxed{\Delta(Q^2, t)}$$

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➡Property: $\Delta(A,B) = \Delta(A,C)\,\Delta(C,B)$

Parton shower

- The Sudakov form factor is the heart of the parton shower. It gives the probability that a parton does not branch between two scales
- Using this no-emission probability the branching tree of a parton is generated.
- Define **dP_k** as the probability for k ordered splittings from leg a at given scales

$$
\begin{aligned}
dP_1(t_1) &= \Delta(Q^2, t_1)\, dp(t_1)\Delta(t_1, Q_0^2), \\
dP_2(t_1, t_2) &= \Delta(Q^2, t_1)\, dp(t_1)\, \Delta(t_1, t_2)\, dp(t_2)\, \Delta(t_2, Q_0^2)\Theta(t_1 - t_2), \\
\ldots &= \ldots \\
dP_k(t_1, \ldots, t_k) &= \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l)\Theta(t_{l-1} - t_l)
\end{aligned}
$$

- **Q_0^2** is the hadronization scale (~1 GeV). Below this scale we do not trust the perturbative description for parton splitting anymore.

Unitarity

$$dP_k(t_1, ..., t_k) \quad = \quad \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l)\Theta(t_{l-1} - t_l)$$

- The parton shower has to be unitary (the sum over all branching trees should be 1). We can explicitly show this by integrating the probability for **k** splittings:

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$$P_k \equiv \int dP_k(t_1, ..., t_k) = \Delta(Q^2, Q_0^2) \frac{1}{k!} \left[\int_{Q_0^2}^{Q^2} dp(t) \right]^k, \quad \forall k = 0, 1, ...$$

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- Hence, the total probability is conserved

singularities

- We have shown that the showers is unitary. However, how are the IR divergences cancelled explicitly? Let's show this for the first emission:
  Consider the contributions from (exactly) 0 and 1 emissions from leg a:

$$\frac{d\sigma}{\sigma_n} = \Delta(Q^2, Q_0^2) + \Delta(Q^2, Q_0^2) \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Expanding to first order in α_s gives

$$\frac{d\sigma}{\sigma_n} \simeq 1 - \sum_{bc} \int_{Q_0^2}^{Q^2} \frac{dt'}{t'} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z) + \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Same structure of the two latter terms, with opposite signs: cancellation of divergences between the approximate virtual and approximate real emission cross sections.

- The probabilistic interpretation of the shower ensures that infrared divergences will cancel for each emission.

Final-state parton showers

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5. For each emitted particle, iterate steps 2-4 until branching stops.

Veto Algorithm

1. find overestimate of the branching probability

$$\bar{P}(z) \geq \hat{\boldsymbol{P}}(z), \quad \bar{z}_{min} \leq z_{min}(t), \; z_{max}(t) \leq \bar{z}_{max}, \quad \bar{\alpha}_S \geq \alpha_S(t)$$

$$g(t) = \frac{\bar{\alpha}}{2\pi t} \int_{\bar{z}_{min}}^{\bar{z}_{max}} \bar{P}(z) \geq \int \frac{\alpha_S}{2\pi} \frac{1}{t} \hat{P}(z) = p(t)$$

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$$R = \bar{\Delta}(Q^2, t) \equiv e^{-\int_t^{Q^2} g(t') dt'}$$

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Standard unweighting does not work!

3. Special selection: Veto Algorithm

Veto Algorithm

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Check if this is bigger or lower!

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   - We want to compensate the over-estimate of the choice of the scale by not re-generate above that scale if the scale is rejected

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➡ $\boxed{\text{if } \dfrac{p(t_{i+1})}{g(t_{i+1})} < R'_{i+1}}$ return t_{i+1}

Check if this is bigger or lower!

Veto Algorithm

1. Idea

- We want to compensate the over-estimate of the choice of the scale by not re-generate above that scale if the scale is rejected

2. Algorithm

1. Start with i=0 and $t_0 = Q^2$

2. move to i+1 and generate t_{i+1} such that $R_{i+1} = \bar{\Delta}(t_i, t_{i+1})$

3. accept/reject with probability $\dfrac{p(t_{i+1})}{g(t_{i+1})}$

➡ if $\boxed{\dfrac{p(t_{i+1})}{g(t_{i+1})} < R'_{i+1}}$ return t_{i+1}

Check if this is bigger or lower!

➡ else go back to point 2

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Is is what we want?

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So this is what we want!

Final-state parton showers

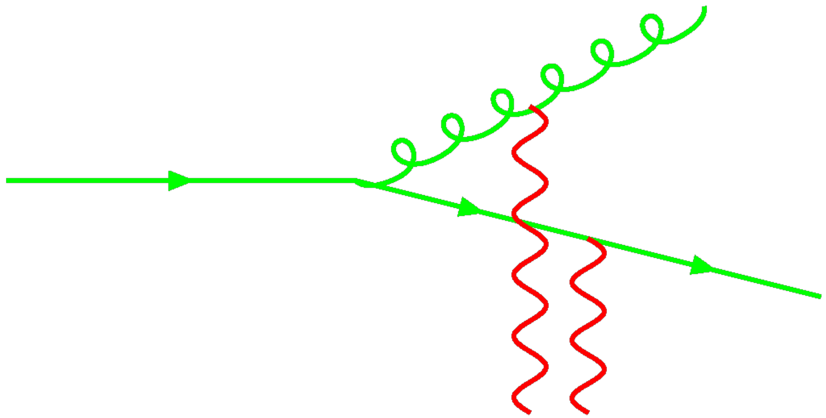
With the Sudakov form factor, we can now implement a final-state parton shower in a Monte Carlo event generator!

1. Start the evolution at the virtual mass scale t_0 (e.g. the mass of the decaying particle) and momentum fraction $z_0 = 1$
2. Given a virtual mass scale t_i and momentum fraction x_i at some stage in the evolution, generate the scale of the next emission t_{i+1} according to the Sudakov probability $\Delta(t_i,t_{i+1})$ by solving $\Delta(t_{i+1},t_i) = R$ where R is a random number (uniform on [0, 1]).
3. If $t_{i+1} < t_{cut}$ it means that the shower has finished.
4. Otherwise, generate $z = z_i/z_{i+1}$ with a distribution proportional to $(\alpha_s/2\pi)P(z)$, where $P(z)$ is the appropriate splitting function.
5. For each emitted particle, iterate steps 2-4 until branching stops.

Soft Limit

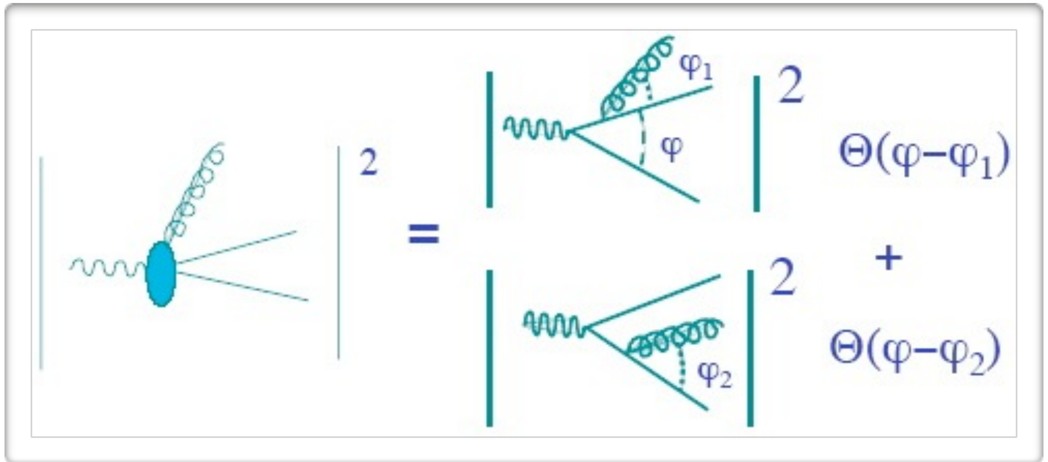
$$\Delta(Q^2,t) = \exp\left[-\sum_{bc}\int_t^{Q^2}\frac{dt'}{t'}dz\frac{d\phi}{2\pi}\frac{\alpha_S}{2\pi}P_{a\to bc}(z)\right]$$

- There is a lot of freedom in the choice of evolution parameter t. It can be the virtuality m^2 of particle a or its p_T^2 or $E^2\theta^2$ ... For the collinear limit they are all equivalent
- However, in the soft limit ($z \longrightarrow 0,1$) they behave differently
- Can we chose it such that we get the correct soft limit?
- Soft gluon comes from the full event!

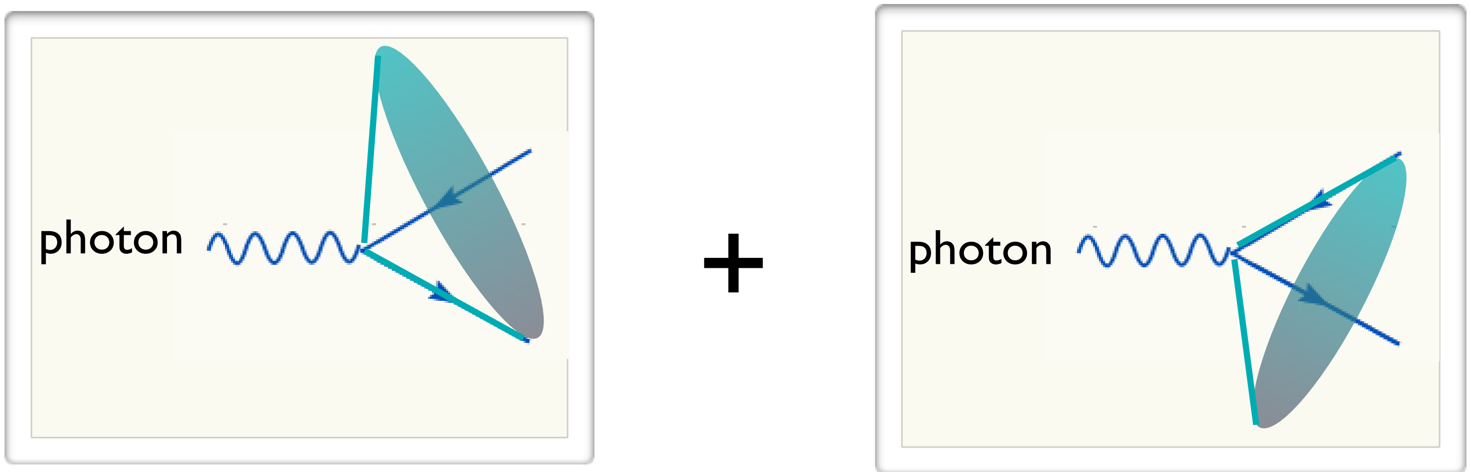


- Quantum Interference

Angular ordering

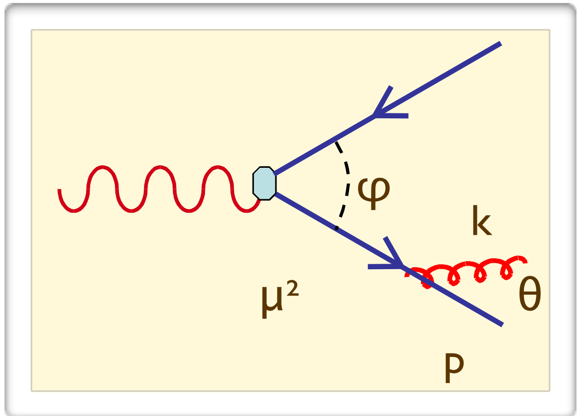


Radiation inside cones around the original partons is allowed (and described by the eikonal approximation), outside the cones it is zero (after averaging over the azimuthal angle)



Intuitive explanation

MichelangeloMangano®



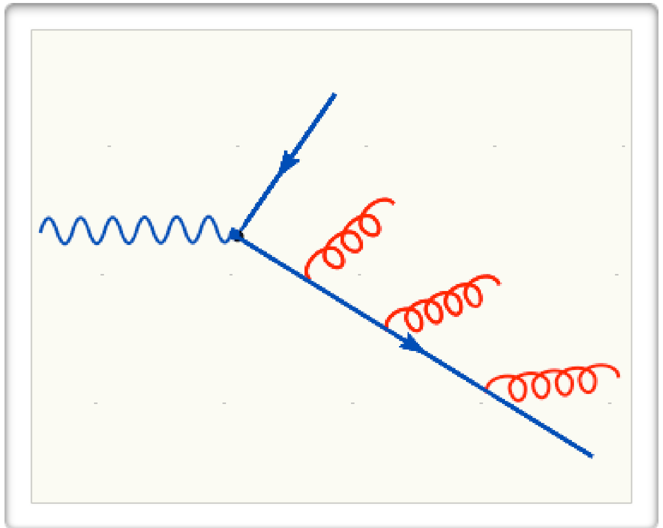
- Lifetime of the virtual intermediate state:
  $\tau < \gamma/\mu = E/\mu^2 = 1/(k_0\theta^2) = 1/(k_\perp\theta)$
- Distance between q and qbar after τ:
  $d = \varphi\tau = (\varphi/\theta)\ 1/k_\perp$

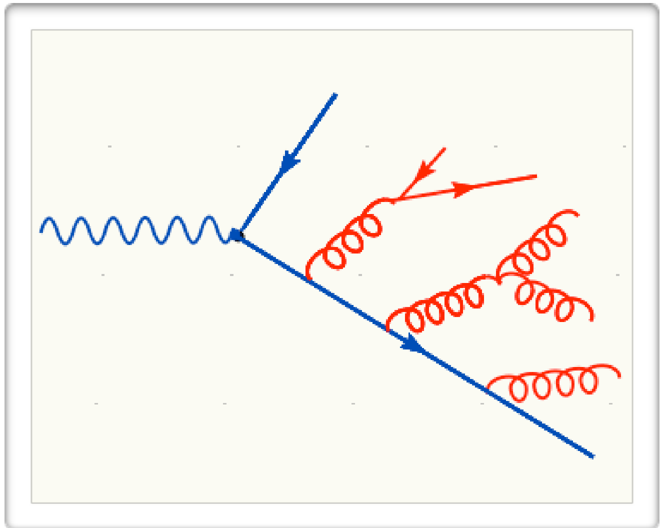
$$\mu^2 = (p+k)^2 = 2E\, k_0\, (1-\cos\theta)$$
$$\sim E\, k_0\, \theta^2 \sim E\, k_\perp\, \theta$$

If the transverse wavelength of the emitted gluon is longer than the separation between q and qbar, the gluon emission is suppressed, because the q qbar system will appear as colour neutral (i.e. dipole-like emission, suppressed)

Therefore $d>1/k_\perp$, which implies $\theta < \varphi$.

Angular ordering





- The construction can be iterated to the next emission, with the result that the emission angles keep getting smaller and smaller.
- One can generalize it to a generic parton of color charge Q_k splitting into two partons i and j, $Q_k=Q_i+Q_j$. The result is that inside the cones i and j emit as independent charges, and outside their angular-ordered cones the emission is coherent and can be treated as if it was directly from color charge Q_k.
- KEY POINT FOR THE MC!
- Angular ordering is automatically satisfied in θ ordered showers! (and easy to account for in p_T ordered showers).

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3. It is not an exclusive property of QCD (i.e., it is also present in QED) but in QCD produces very non-trivial effects, depending on how particles are color connected.

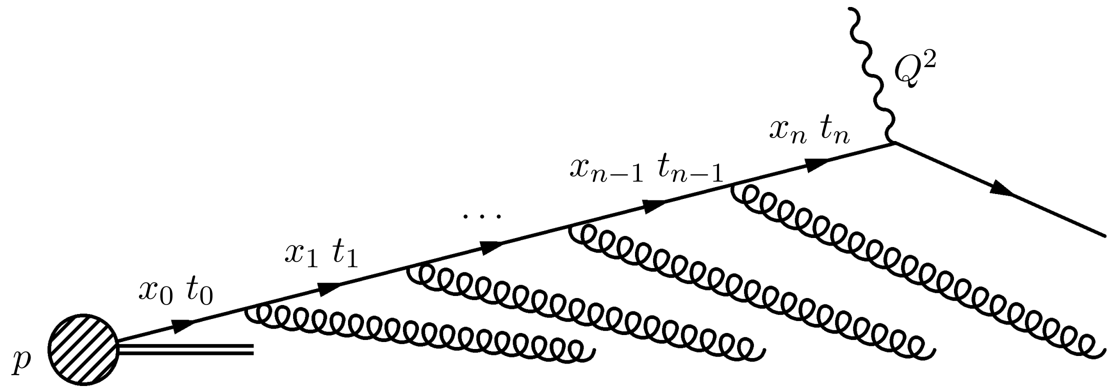
To Remember

- Sudakov Form-Factor: Probability of No-emission between two scale.

$$\Delta(Q^2, t) \simeq e^{-\int_t^{Q^2} \frac{dt'}{t'} dz \frac{\alpha_S}{2\pi} \hat{P}(z)} \equiv e^{-\int_t^{Q^2} dp(t')}$$

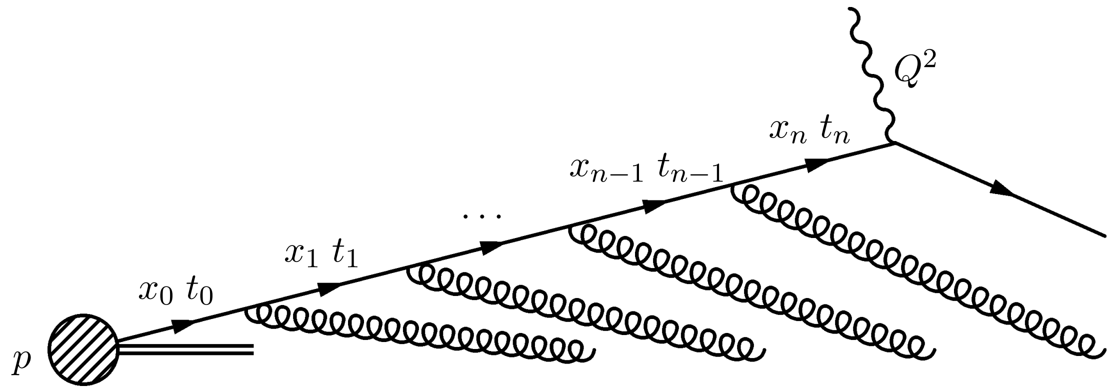
- Parton shower is unitary (and IR save)
- Parton shower is a Markov Chain
  - ➡One emission at the time
- Each interactions has its own scale for alphas
- Various choice for the evolution parameter

Initial-state



- So far, we have looked at final-state (time-like) splittings. For initial state, the splitting functions are the same
- However, there is another ingredient: the parton density (or distribution) functions (PDFs). Naively: Probability to find a given parton in a hadron at a given momentum fraction **$x = p_z/P_z$** and scale **t**.

Initial-state



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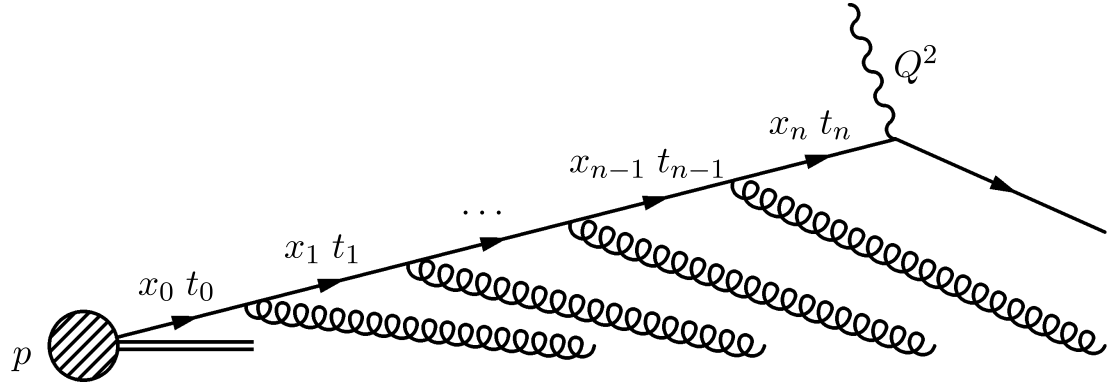

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- How do the PDFs evolve with increasing **t**?

$$t\frac{\partial}{\partial t} f_i(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi} P_{ij}(z) f_j\left(\frac{x}{z}, t\right) \qquad \text{DGLAP}$$

To Remember

- The parton shower dresses partons with radiation. This makes the inclusive parton-level predictions (i.e. inclusive over extra radiation) completely exclusive
    - In the soft and collinear limits the partons showers are exact, but in practice they are used outside this limit as well.
    - Partons showers are universal (i.e. independent from the process)
    - Building block of the parton shower is the Sudakov
- There is a cut-off in the shower (below which we don't trust perturbative QCD) at which a hadronization model takes over