"Novel aspects of scattering equations"

Liu, Zhengwen

ABSTRACT

The scattering equations, a system of algebraic equations connecting the space of kinematic invariants and the moduli space of punctured Riemann spheres, provide a new way to construct scattering amplitudes. In this novel framework, the tree-level S-matrix in many quantum field theories can be reformulated as a multiple integral that is entirely localized on the zeroes of the scattering equations. The aim of this thesis is to deepen our understanding of the physical and mathematical structures underlying the scattering equations and to broaden the scope for their applications. In particular, we analyze and extend the scattering equations to on-shell amplitudes in several effective field theories and form factors that carry off-shell momenta in gauge theory. We also study the asymptotic behavior of the scattering equations in various Regge kinematic regimes and derive the corresponding factorizations of amplitudes in gauge theory and gravity. Finally, we propose the physical homotopy ...

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Novel Aspects Of Scattering Equations

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Zhengwen Liu

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Prof. Claude Duhr (promoteur)	UCLouvain, Belgium
Prof. Jan Govaerts (président)	UCLouvain, Belgium
Dr. Andrea Giammanco	UCLouvain, Belgium
Prof. Philippe Ruelle	UCLouvain, Belgium
Prof. Jan Plefka	HU Berlin, Germany
Dr. Piotr Tourkine	CNRS, Sorbonne Université, France

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Abstract

The scattering equations, a system of algebraic equations connecting the space of kinematic invariants and the moduli space of punctured Riemann spheres, provide a new way to construct scattering amplitudes. In this novel framework, the tree-level S-matrix in many quantum field theories can be reformulated as a multiple integral that is entirely localized on the zeroes of the scattering equations. The aim of this thesis is to deepen our understanding of the physical and mathematical structures underlying the scattering equations and to broaden the scope for their applications. In particular, we analyze and extend the scattering equations to on-shell amplitudes in several effective field theories and form factors that carry off-shell momenta in gauge theory. We also study the asymptotic behavior of the scattering equations in various Regge kinematic regimes and derive the corresponding factorizations of amplitudes in gauge theory and gravity. Finally, we propose the physical homotopy continuation of the scattering equations and develop an efficient method to solve these equations.

Contents

1 Introduction

"Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses."

Julian Schwinger (1918 − 1994)

Scattering amplitudes (S-matrix elements), which are sometimes referred to as "*the most perfect microscopic structures in the universe*" [1], are the most fundamental quantities in quantum field theory (QFT). They allow us to make predictions for physical observables for particle scattering and decays that can be measured in high-energy experiments such as the Large Hadron Collider (LHC) at CERN. In particular, calculating scattering amplitudes efficiently has become crucially important in order to understand the properties of the Brout-Englert-Higgs boson more precisely and to search for new physics beyond the Standard Model (SM) since the start of the LHC. From a more theoretical perspective, scattering amplitudes have a remarkably rich structure. A good understanding of the mathematical structures of amplitudes often leads to a deeper understanding of quantum field theory, as well as new ideas enhancing our ability to compute amplitudes.

Traditionally, scattering amplitudes are calculated using Feynman diagrams in the framework of perturbative quantum field theory. Feynman diagrams provide a systematic procedure to generate amplitudes order by order (see standard textbooks, e.g. [2–4]). However, the Feynman diagram technique is not always efficient, because the number of diagrams increases rapidly with the number of external particles involved in a scattering process or the number of closed loops at higher orders. As a result, a large number of Feynman diagrams makes the calculation particularly difficult. However, the final results are usually simple. This implies that substantial simplifications and hidden structures for scattering amplitudes are invisible in Feynman diagrams.

A famous example is the Parke-Taylor formula of maximal helicity-violating (MHV) amplitudes[1] [5–7],

$$\mathcal{A}_n(g^- g^- g^+ \cdots g^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}, \tag{1.1}$$

which describes the tree-level interaction between two negative-helicity gluons and an arbitrary number of positive-helicity ones in non-Abelian gauge theory. A sum of a huge number of Feynman diagrams can be simplified into an astonishingly simple expression, as shown in (1.1). This surprising simplicity has been profoundly influential in an attempt to reformulate quantum field theory. More precisely, it has motivated theorists to look for alternative methods to calculate amplitudes without Feynman diagrams, and revolutionary advances have been made in this regard, during the past three decades. Significant among these developments is Witten's twistor string theory that provides a world-sheet description for tree-level scattering amplitudes in maximally supersymmetric Yang-Mills (SYM) [8]. In particular, by performing a half-Fourier transform from momentum space to twistor space, the MHV amplitude is supported on a curve of genus zero and degree one. This explains from a certain point of view why the MHV amplitudes take an exceptionally simple form.

Witten's prominent work has inspired the discovery of many alternative formulations and novel techniques of calculating scattering amplitudes, especially at tree level (see e.g. [9–12] for thorough reviews). These include Feynman-like MHV rules [13] and on-shell recursion relations [14, 15], which not only lead to new compact results but have changed the way we construct amplitudes from first principles instead of Feynman diagrams. Most relevantly, Roiban, Spradlin and Volovich (RSW) obtained in 2004 the compact formula that expresses any $\mathcal{N} = 4$ SYM amplitude as a localized integral over the moduli space of punctured Riemann spheres [16] via transforming Witten's formula back to momentum space. Similarly, two different formulas for tree-level am-

[1] In the case where at most one gluon has a negative helicity, the tree-level amplitude vanishes. The first non-zero amplitudes are then those where exactly two gluons carry negative helicity, and they are referred to as *maximal helicity-violating* (MHV) amplitudes.

plitudes in $\mathcal{N}=8$ supergravity (SUGRA) were proposed in [17, 18], and later derived from a new twistor string theory [19, 20] in 2012.

More recently, Cachazo, He and Yuan (CHY) in 2013 introduced a generalisation of the Witten-RSV formalism [8, 16] and analogs [17–21]. In the CHY formalism, the tree-level amplitude of n massless particles is reformulated as a multiple integral over the moduli space of Riemann spheres with n marked points for a variety of theories in generic spacetime dimensions [22–25]. The integral is completely localized on the zeroes of a set of theory-independent algebraic equations as follows:

$$f_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a \in \{1, 2, \ldots, n\}. \tag{1.2}$$

They are named as the *scattering equations*. This system of equations has a $\mathrm{SL}(2, \mathbb{C})$ redundancy. More details on them will be clarified later.

Various world-sheet models for the CHY formulalism have also been constructed based on so-called ambitwistor strings [26–30]. It is also particularly noteworthy that Geyer, Lipstein and Mason (GLM) obtained the new alternative formulation for amplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA from four-dimensional ambitwistor string models [28], like the Witten-RSV formula of $\mathcal{N}=4$ SYM amplitudes. The ambitwistor string theory has also provided a systematic approach to generalize the CHY formalism to loop level [31–37]. We suggest interested readers to [38] for a thorough review entitled "*Twistor theory at fifty: from contour integrals to twistor strings*".

The CHY formalism, twistor-string- and ambitwistor-string-inspired formulations share the same structure: a n-point on-shell amplitude can be calculated through a multiple contour integral over the moduli space of n-punctured spheres, where the contour is entirely determined by the zeros of some algebraic equations. Here we do not show the precise form for the equations that appear in the twistor-/ambitwistor-string formula of amplitudes, and we refer to them as the *four-dimensional scattering equations*. We also thus collectively refer to these formulations as the *scattering equation formalism*.

The discovery of the scattering equation formalism represents a major step towards reformulating scattering amplitudes and even QFT, "*quantum field the-*

ory without the quantum fields" [39]. A notable feature is that for a multiparticle process summing thousands, or even hundreds of thousands, of Feynman diagrams, is equal to a one-line contour integral. The advantages of this feature are obvious. First, having a compact formula may allow one to discover and study properties of amplitudes that are not visible in Feynman diagrams. Second, on a more practical level, the contour integral representation may allow one to find new and more efficient ways to evaluate amplitudes numerically. It was also noteworthy that the scattering equations have also established a link between quantum field theory, string theory and algebraic geometry.

This work aims at advancing our knowledge on the aspects of the scattering equations with a focus on both theoretical explorations and potential phenomenological applications.

We summarize the content of the remaining chapters as follows. Chapter 2 provides a minimal introduction to the scattering amplitudes in various theories under consideration in this work. After first presenting a brief review on the basic aspects of the scattering equations and Cachazo-He-Yuan formalism of scattering amplitudes in arbitrary-dimensional spacetime in Chapter 3, our discussion moves to four-dimensional spacetime in Chapter 4. We clarify that in the spinor-helicity formalism the scattering equations in four dimensions can be decomposed into "helicity sectors" and are equivalent to four-dimensional scattering equations appearing in the Witten-RSV or GLM formula. We also show how the two four-dimensional formulations are related to each other.

Chapter 5 focuses on some effective field theories with spontaneously broken symmetries in four dimensions, including the maximally supersymmetric Dirac-Born-Infeld-Volkov-Akulov (DBI-VA) theory, the $U(N)$ non-linear sigma model (NLSM) and a special Galileon theory. We propose new compact formulas for all tree-level amplitudes in these theories in four-dimensional spacetime. Moreover, we apply the formulas to derive various universal double-soft limits of amplitudes in these theories as well as in maximally gauge and gravity theories.

While new representations of tree-level amplitudes have been constructed in a large number of massless theories, it is expected that this success story can be repeated beyond on-shell amplitudes of massless particles. We extend the scattering equation formalism to form factors in Chapter 6. Unlike on-shell amplitudes, a n-point form factor is given by the overlap of a composite operator with off-shell momentum and n on-shell states with on-shell momenta. We develop new compact formulas for super form factors with chiral stress-tensor multiplet operator and bosonic form factors with scalar operators $\mathrm{Tr}(\phi^m)$ for arbitrary m in $\mathcal{N}=4$ SYM.

The representation of a tree-level scattering amplitude as a contour integral also makes manifest the properties of the amplitude in certain singular limits of kinematics. Indeed, the scattering equation formalism has been successfully applied to derive the factorized form of scattering amplitudes beyond leading order in single [40–43] and double [44–46] soft limits, as well as in the collinear limit [47]. One of the main aims of this work is to show that the scattering equations formalism also presents a very natural framework to study other kinematical limits of tree-level amplitudes, namely the so-called Regge limits of a 2-to-$(n-2)$ scattering where the final state particles are ordered in rapidity while having comparable transverse momenta. Of particular interest in this context is multi-Regge kinematics (MRK) where all produced gluons are strongly ordered in rapidity. By relaxing the strong ordering among the rapidities of the produced particles, one can define a tower of new kinematical limits, known as quasi-multi-Regge kinematics (QMRK).

In Chapter 7, we study the asymptotic behavior of the scattering equations in the QMRK. Through a numerical study of the solutions to the scattering equations in various quasi-multi-Regge regimes, we observe that in all cases the solutions present the same hierarchy as the rapidity ordering that defines the limit (if the $\mathrm{SL}(2,\mathbb{C})$ redundancy is fixed in a certain way). We conjecture that this feature holds in general, independently of the helicity configuration and the number of external legs.

While we do currently not have a rigorous mathematical proof of our conjecture, we show that the conjecture implies the correct factorization of amplitudes in gauge theory and gravity in certain quasi-multi-Regge limits. In Chapter 8, we show that in MRK our conjecture implies that the four-dimensional scattering equations have a unique solution in each "helicity configuration", and we determine this solution explicitly for arbitrary multiplicities. By localizing the contour integral to the unique solution in MRK, we derive the correct fully-factorized form of tree-level amplitudes for both Yang-Mills and

Einstein gravity. In Chapter 9, we show that when combined our conjecture with the CHY-type representation of gluon amplitudes, it reproduces the expected factorization in certain quasi-multi-Regge limits. While consistent with all known results for tree-level amplitudes, so far, the factorization of tree-level amplitudes in gauge theory and gravity in (quasi-)MRK remains partly conjectural and has only been proven to hold for arbitrary multiplicities for the simplest non-zero helicity amplitudes of gluons [48]. These give very strong support to the validity of our conjecture.

Due to the importance, it is crucial to solving the scattering equations. Notwithstanding efforts have been made to solve the scattering equations or evaluate the CHY formulas [23, 49–66], a good method is still missing. In Chapter 10, we develop an efficient technique to solve the scattering equations based on the numerical algebraic geometry. The cornerstone of our method is the concept of the physical homotopy continuation between different points in the space of kinematic invariants, which naturally induces a homotopy continuation of the scattering equations. As a result, the solutions of the scattering equations with different points in the kinematic space can be tracked from each other. Finally, with the help of soft limits, all solutions can be bootstrapped from the known solution for the four-particle scattering.

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- C. Duhr and Z. Liu, *Multi-Regge kinematics and the scattering equations*, *JHEP* 1901 (2019) 146;
- Z. Liu, *Gravitational scattering in the high-energy limit*, *JHEP* 1902 (2019) 112;
- Z. Liu and X. Zhao, *Bootstrapping solutions of scattering equations*, *JHEP* 1902 (2019) 071.

2 Preliminaries

This chapter provides a short introduction to scattering amplitudes in various theories under consideration in this work.

2.1 Yang-Mills theory

Yang-Mills theory is a gauge theory based on a non-Abelian Lie group G. Here we consider $G = SU(N)$ and follow ref. [67]. We start with a multiplet of fermionic (or scalar) fields

$$\psi(x) \; = \; \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}, \tag{2.1}$$

which transforms into one another under a *local gauge transformation*

$$\psi(x) \;\longrightarrow\; U(x)\psi(x), \quad U(x) = U\big(\alpha(x)\big) \equiv e^{i\alpha^a(x)t^a}, \tag{2.2}$$

where t^a are the fundamental representations of generators of $SU(N)$, and satisfy the following commutation relation

$$\big[t^a,\, t^b\big] \;=\; if^{abc}t^c. \tag{2.3}$$

It is easy to see that the derivative of the fermion field $\partial_\mu\psi(x)$ transforms differently from $\psi(x)$ itself under the local gauge transformation (2.2). Let us introduce a *gauge covariant derivative* D_μ

$$D_\mu \;\equiv\; \partial_\mu - igA^a_\mu t^a \;=\; \partial_\mu - ig\boldsymbol{A}_\mu, \quad \boldsymbol{A}_\mu \equiv A^a_\mu t^a, \tag{2.4}$$

such that $D_\mu\psi(x)$ has the same transformation behavior with $\psi(x)$, i.e.,

$$D_\mu\psi(x) \longrightarrow U(x)\big(D_\mu\psi(x)\big). \tag{2.5}$$

This leads to the transformation behavior of the gauge field $\boldsymbol{A}^\mu$,

$$\boldsymbol{A}_\mu \longrightarrow \boldsymbol{A}_\mu^U = U\boldsymbol{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \tag{2.6}$$

under the local gauge transformation. Using the covariant derivative D_μ, we can define the *gauge field strength* as follows:

$$\boldsymbol{F}_{\mu\nu} \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - ig\big[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu\big] = F^a_{\mu\nu}t^a, \tag{2.7}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{2.8}$$

From (2.6) it is easy to verify the gauge field strength behaviors as

$$\boldsymbol{F}_{\mu\nu} \longrightarrow \boldsymbol{F}^U_{\mu\nu} = U(x)\boldsymbol{F}_{\mu\nu}U(x)^{-1} \tag{2.9}$$

under the local gauge transformation. Now it is very clear to construct an local gauge invariant Lagrangian [68]

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{2.10}$$

where one takes the normalization convention $\mathrm{Tr}(t^at^b) = \frac{1}{2}\delta^{ab}$.

In addition to pure gauge field, the complete Lagrangian can also include the matter (e.g. fermionic) fields ψ,

$$\mathcal{L} = \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{matter}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{2.11}$$

When the gauge group takes $SU(3)$, this is the theory of strong interactions between quarks and gluons, i.e. *quantum chromodynamics* (QCD). When $G = U(1)$, it is reduced to the Lagrangian of quantum electrodynamics.

When the coupling constant is small enough, the scattering amplitudes may be calculated perturbatively. For example, an amplitude with n gluons in pure Yang-Mills or QCD can be expanded as a perturbative series with respect to the coupling constants g,

$$\mathscr{A}_n \;=\; g^{n-2}\mathscr{A}_n^{(0)} + g^n\mathscr{A}_n^{(1)} + g^{n+2}\mathscr{A}_n^{(2)} + \cdots \,. \tag{2.12}$$

In this expansion, each $\mathscr{A}_n^{(\ell)}$ can be given by a sum of Feynman diagrams with n external legs and ℓ loops in accordance with specific rules, so-called *Feynman rules*. It is straightforward to derive Feynman rules from a given Lagrangian, for example, Yang-Mills Lagrangian in eq. (2.10), using the standard textbook method, such as the path integral technique (c.f. any standard QFT textbook, e.g. [2–4]). This work is mostly focused on the leading order of the perturbation expansion, $\mathscr{A}_n^{(0)}$, which has no loops and is called the *tree-level amplitude*. For simplicity, we will omit the superscript $(\ell = 0)$ of $\mathscr{A}_n^{(0)}$ in the following.

In Yang-Mills theory, amplitudes depend not only on external kinematics but also on color information. From the Lagrangian (2.10), it is easy to see that there are two types of group theory objects that enter the amplitude, i.e. the structure constants f^{abc} and generator matrices t^a in the fundamental representation of $SU(N)$. It has been known that one can decompose a tree-level gluon amplitude into purely-kinematic part and color part which can be written in terms of either t^a or f^{abc}. More priviously, any tree-level amplitude of n gluons carrying colors $a_1, a_2, \ldots, a_n$, $(a_i = 1, \ldots, N^2-1)$, can be represented as [69,70]

$$\mathscr{A}_n \;=\; \sum_{\rho\in S_{n-1}} \mathrm{Tr}\left(T^{a_1}T^{a_{\rho_2}}\cdots T^{a_{\rho_n}}\right)\mathcal{A}_n(1,\rho_2,\ldots,\rho_n), \tag{2.13}$$

or [71,72]

$$\mathscr{A}_n \;=\; \sum_{\rho\in S_{n-2}} \left(F^{a_{\rho_2}}F^{a_{\rho_3}}\cdots F^{a_{\rho_{n-1}}}\right)_{a_1 a_n}\mathcal{A}_n(1,\rho_2,\ldots,\rho_{n-1},n), \tag{2.14}$$

where $T^a = \sqrt{2}t^a$ by QCD-literature convention and $(F^a)_{bc} = if^{abc}$, $\rho = (\rho_2, \ldots, \rho_n)$ or $\rho = (\rho_2, \ldots, \rho_{n-1})$ denotes the permutation of $\{2, \ldots, n\}$ or $\{2, \ldots, n-1\}$. In both representations (2.13) and (2.14), $\mathcal{A}_n$ are same and called *color-ordered amplitudes*, or *partial amplitudes*, which depend only on kinematic information, but not colors. Color-ordered amplitudes are gauge invariant. There also exist linear relations among partial amplitudes, such as cyclicity, Kleiss-Kuijf relations [72, 73] and fundamental Bern-Carrasco-Johansson relations [74] (see also e.g. [9, 11] for review).

The partial amplitudes have simpler analytic structures than the full amplitude [75]. They have only multi-particle poles where a sum of cyclically adjacent momenta goes on-shell. More precisely, in the limit $p_{1,m}^2 = (k_1+\cdots+k_m)^2 \to 0$, the amplitude $\mathcal{A}_n(1,\ldots,n)$ behaves as

$$\begin{aligned} &\mathcal{A}_n(1,\ldots,n) \\ &\quad\longrightarrow \sum_{h=\pm 1} \mathcal{A}_{m+1}\big(1,\ldots,m,-p^h\big)\frac{1}{p_{1,m}^2}\mathcal{A}_{n-m+1}\big(p^{-h},m{+}1,\ldots,n\big), \end{aligned} \tag{2.15}$$

where $p_{1,m}$ and h are the momentum and the helicity of the intermediate state respectively. This analytic property of amplitudes are very important, and they are often used to do consistency checks of the correctness of new results. It also plays a crucial role in deriving on-shell BCFW recursion relations [14,15]. It is worth underlining that the similar property also appears in amplitudes in other theories, which is directly related to *locality* in perturbation field theory.

In the case $m = 2$, $(k_1{+}k_2)^2 \to 0$ implies that two momenta k_1 and k_2 become parallel, or collinear. Let z describe the longitudinal momentum sharing of the two collinear momenta, and then one can parametrize the limit as

$$k_1 \| k_2 \qquad \Longleftrightarrow \qquad k_1 \to z\,p, \quad k_2 \to (1{-}z)\,p, \tag{2.16}$$

and the partial amplitude of gluons factorizes as

$$\begin{aligned} &\mathcal{A}_n(1,2,3,\ldots,n) \\ &\quad\xrightarrow{k_1\|k_2} \sum_{h=\pm 1} \mathtt{Split}_h(1,2)\,\mathcal{A}_{n-1}\big(p^{-h},3,\ldots,n\big), \end{aligned} \tag{2.17}$$

where $p = k_1 + k_2$ and h are momentum and helicity of the intermediate state respectively, $\mathtt{Split}$ is the tree-level *splitting amplitude* (or *splitting function)* that describes the splitting $g \to gg$ of a gluon into two collinear gluons [5, 6, 76–78]:

$$
\begin{aligned}
\texttt{Split}_-(1^+,2^+) &= \frac{1}{\langle 1\,2\rangle\,\sqrt{z(1-z)}}, \\
\texttt{Split}_+(1^+,2^+) &= 0, \\
\texttt{Split}_+(1^+,2^-) &= -\frac{(1-z)^2}{\langle 1\,2\rangle\,\sqrt{z(1-z)}}, \\
\texttt{Split}_-(1^+,2^-) &= \frac{z^2}{[1\,2]\,\sqrt{z(1-z)}}.
\end{aligned}
\tag{2.18}
$$

Besides the multi-particle factorization (including collinear limit), the partial amplitude of gluons also displays a universal behavior when one or more gluons become soft. Here we consider the case of the emission of a single soft gluon with momentum k_s. In the limit $k_s \to 0$, the partial amplitude takes the following factorziaed form [79]

$$
\begin{aligned}
&\mathcal{A}_n\big(\ldots,a,s,b,\ldots\big) \\
&\quad = \mathcal{S}^{\rm YM}(a,s,b)\,\mathcal{A}_{n-1}\big(\ldots,a,b,\ldots\big) + \mathcal{O}(k_s^0),
\end{aligned}
\tag{2.19}
$$

with

$$
\mathcal{S}^{\rm YM}(a,s,b) = -\frac{k_a\cdot\epsilon_s}{k_a\cdot k_s} + \frac{k_b\cdot\epsilon_s}{k_b\cdot k_s},
\tag{2.20}
$$

where ϵ_s denotes the the polarization vector of the soft gluon. In the spinor-helicity variables (c.f. Appendix A), they can written in a more familiar form

$$
\mathcal{S}^{\rm YM}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\,\langle s\,b\rangle}, \quad \mathcal{S}^{\rm YM}(a,s^-,b) = \frac{[a\,b]}{[a\,s][s\,b]}.
\tag{2.21}
$$

They are called *eikonal factors* and independent of the helicities of the two adjacent gluons a and b, reflecting the classical origin of soft radiation. The subleading term has also been derived in [80, 81]. The universal factorized hebavior of the scattering amplitude in the soft limit is also often called the *soft theorem* in the literature.

2.2 Einstein gravity theory

In this section, we turn to introduce the Einstein gravity theory. Einstein's theory provides an elegant geometric description of the fundamental interaction of gravitation. Here we discuss gravity from a purely field theoretical point of view: Einstein gravity is equivalent to the quantum theory of a massless, self-interacting, spin-2 field [82]. Here we mainly follow ref. [83] (we also suggest interested readers to [84–89] for recent or classical review papers and lecture notes).

Our starting point is the Einstein-Hilbert action

$$S_{\text{Einstein-Hilbert}} = \int d^D x \sqrt{-g} \left(\frac{1}{2\kappa^2}(R - 2\Lambda) + \mathcal{L}_{\text{matter}} \right), \tag{2.22}$$

where $\mathcal{L}_{\text{matter}}$ describes the matter part appearing in the theory, Λ is the cosmological constant, $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor, R is the Ricci scalar, and $\kappa^2 = 8\pi G_N$ with G_N Newton's gravitational constant. One can derive the famous Einstein field equation from the Einstein-Hilbert action.

In the weak field limit, we may expand the metric field $g_{\mu\nu}(x)$ around flat spacetime, i.e.,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x), \tag{2.23}$$

where $h_{\mu\nu}(x)$ is the symmetric rank-2 tensor which spans a representation of the Lorentz group. We regard this fluctuating field as the *graviton field*. Plugging (2.23) into $\sqrt{-g}$ and R, one can get an infinite series in κ. In particular, the expansion of $\sqrt{-g}$ does not contain derivatives of the graviton field $h_{\mu\nu}$. This explains why the cosmological constant term does not affect the dynamics of the theory [90]. In the following, we consider only the pure graviton field by setting $\mathcal{L}_{\text{matter}}$ and Λ to zero.

We see the diffeomorphism invariance of the theory, which is sometimes considered as gauge symmetry from a QFT viewpoint. Let us consider an infinitesimal coordinate transformation:

$$x^\mu \longrightarrow \tilde{x}^\mu(x) = x^\mu + \varepsilon^\mu(x), \qquad \varepsilon \ll 1, \tag{2.24}$$

under which the metric transforms as a covariant rank-2 tensor

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x). \tag{2.25}$$

Using eq. (2.23) and eq. (2.24), one find that the metric transformation (2.25) is equivalent to the gauge transformation of the graviton field:

$$h_{\mu\nu} \longrightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\varepsilon_{\nu} - \partial_{\nu}\varepsilon_{\mu} + \mathcal{O}(\varepsilon^2). \tag{2.26}$$

Given the Lagrangian and its symmetries, using the standard textbook method, one can derive Feynman rules to compute amplitudes. Gravity has an infinite number of higher-point interaction vertices, and even the simplest cubic vertex has a much more complicated structure than Yang-Mills. Here we do not list Feynman rules since we never calculate any graviton amplitude from Feynman diagrams in this work. Instead, we understand graviton amplitudes from a modern perspective – *gravity as a double copy of Yang-Mills* [74, 91, 92].

More than three decades ago, Kawai, Lewellen and Tye (KLT) derived that a closed string amplitude can be written in terms of a sum of the square of open string amplitudes at tree level [91]. In field theory limit, it is reduced to a similar relation between graviton amplitudes and color-ordered gluon amplitudes. To be clear, let us see some lower-point examples

$$\begin{aligned}
\mathcal{M}_3 &= \mathcal{A}_3(123)\,\mathcal{A}_3(123), \\
\mathcal{M}_4 &= -s_{12}\mathcal{A}_4(1234)\mathcal{A}_4(1243), \\
\mathcal{M}_5 &= s_{12}s_{34}\mathcal{A}_5(12345)\mathcal{A}_5(14352) + s_{13}s_{24}\mathcal{A}_5(13245)\mathcal{A}_5(14253).
\end{aligned} \tag{2.27}$$

Here from each $\mathcal{M}_n$ a overall factor κ^{n-2} has been stripped off. More generally, for arbitrary multiplicity, one has [93]

$$\mathcal{M}_n = (-1)^{n+1} \sum_{\alpha,\beta \in S_{n-3}} \mathcal{A}_n(n-1, n, \alpha, 1)\, \mathcal{S}[\alpha|\beta]_{k_1} \mathcal{A}_n(1, \beta, n-1, n), \tag{2.28}$$

where $\mathcal{S}[\alpha|\beta]$ is called the *momentum kernel* defined as [94]

$$\mathcal{S}\big[i_1,\ldots,i_k|j_1,\ldots,j_k\big]_{k_1} = \prod_{t=1}^{k}\left(s_{i_t 1}+\sum_{q>t}^{k}\theta(i_t,i_q)s_{i_t i_q}\right), \tag{2.29}$$

where $\theta(i_t,i_q)$ is zero when the pair (i_t,i_q) has same ordering at both sets $\{i_1,\ldots,i_k\}$ and $\{i_1,\ldots,i_k\}$, but otherwise it is unity.

As a simple application, a combination of the soft (collinear) limit of color-ordered gluon amplitudes and the KLT relations may lead to the soft (collinear) factorization of graviton amplitudes [83, 92, 95, 96]. In the collinear limit (2.16), any n-graviton amplitude takes the following factorized form [83, 92]

$$\mathcal{M}_n(k_1,k_2) \xrightarrow{k_1\|k_2} \sum_{h=\pm 2}\mathcal{M}_{n-1}\big(p^{-h}\big)\,\mathtt{Split}_h^{\text{gravity}}(1,2), \tag{2.30}$$

where the gravity splitting amplitude is simply just a product of two gluon splitting amplitudes, i.e.,

$$\mathtt{Split}_h^{\text{gravity}}(1,2) = -s_{12}\,\mathtt{Split}_h(1,2)\times\mathtt{Split}_h(2,1). \tag{2.31}$$

In [92], it was argued that the gravity splitting amplitude doesn't receive higher loop corrections. Similarly, in the soft limit $k_n\to 0$, for graviton amplitudes one has [97]

$$\mathcal{M}_n(k_n) = \mathcal{S}^{\text{gravity}}\,\mathcal{M}_{n-1}+\mathcal{O}(k_n), \tag{2.32}$$

where the soft factor also can be obtained as the double copy of eikonal factors

$$\mathcal{S}^{\text{gravity}}(n^{\pm}) = \sum_{a=1}^{n-1}s_{an}\,\mathcal{S}^{\text{YM}}(x,n^{\pm},i)\,\mathcal{S}^{\text{YM}}(y,n^{\pm},i), \tag{2.33}$$

where x and y are arbitrary reference spinors. The leading soft factorization of graviton amplitudes is uncorrected to all loop orders, and is called "Weinberg theorem" [97, 98]. Very interestingly, the subleading [99–102] and subsub-leading [102] terms also exhibit universal structure in the soft limit.

In 2008, Bern, Carrasco and Johansson (BCJ) discovered a new realization of the double copy relation between gauge and gravity theories, known as "*color-*

kinematics duality" [74, 103]. We illustrate this remarkable duality using tree-level amplitudes. Like the color decomposition (2.14) of gluon amplitudes, one can reorganize gluon amplitudes in terms of cubic graphs

$$\mathscr{A}_n = g^{n-2} \sum_i \frac{c_i n_i}{D_i}, \tag{2.34}$$

where the sum is over all possible cubic graphs, D_i denote inverse propagators that can be straightforwardly read off from the graph, c_i and n_i denote corresponding color factors and kinematic numerators respectively. In such a decomposition, one needs that kinematic numerators n_i obey the same Jacobi relations and symmetry properties as the color factors c_i. Then the gravity amplitudes can be obtained by simply replacing the color factors of gluon amplitude with the numerator factors, i.e.,

$$\mathscr{M}_n = \kappa^{n-2} \sum_i \frac{n_i \tilde{n}_i}{D_i}. \tag{2.35}$$

This work focuses mostly on the scattering equations. We will show in the next chapters that the scattering equations also provide a new elegant description of the double copy relation between gauge and gravity amplitudes.

2.3 Effective field theories

As just shown in previous sections, the amplitude in gauge theory and gravity displays a universal factorized behavior in the single soft limit. In this section, we discuss some effective field theories (EFT), in which the scattering amplitudes vanish when emitting a single soft particle.

Dirac-Born-Infeld Let us begin with the Born-Infeld (BI) theory whose Lagrangian in four dimensions reads

$$\mathcal{L}_{\mathrm{BI}} = \ell^{-2}\left(1 - \sqrt{-\det\left(\eta_{\mu\nu} - \ell F_{\mu\nu}\right)}\right), \tag{2.36}$$

where ℓ is the coupling constant. The BI model is a non-linear generalization of Maxwell theory. The Dirac-Born-Infeld (DBI) theory is a generalization of the BI theory to include scalars, i.e.,

$$\mathcal{L}_{\mathrm{DBI}} = \ell^{-2}\left(1-\sqrt{-\det\left(\eta_{\mu\nu}-\ell^2\partial_\mu\phi^I\partial_\nu\phi^I-\ell F_{\mu\nu}\right)}\right), \tag{2.37}$$

where I are flavor indices. Expanding the Lagrangian in ℓ yields an infinite series of local operators. For example, in four dimensions, the pure BI Lagrangian (2.36) can be expanded as [104]

$$\mathcal{L}_{\mathrm{BI}} = I_2+\ell^2 I_4\left[1+\ell^2 I_2+\ell^4\left(I_2^2-\frac{1}{2}I_4\right)+\mathcal{O}(\ell^6)\right], \tag{2.38}$$

where $I_2=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell Lagrangian, and all higher order terms are proportional to I_4 defined as follows:

$$I_4 := -\frac{1}{8}\left(F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}-\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right) = -\frac{1}{8}\left(F^{(+)}\right)^2\left(F^{(-)}\right)^2, \tag{2.39}$$

with

$$F^{\pm}\equiv F\pm *F,\quad *F^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{2.40}$$

Similarly, for the pure scalar sector of DBI, in the case of one flavor one has

$$\mathcal{L}_{\mathrm{DBI\ scalar}} = \frac{1}{2}(\partial\phi)^2+\frac{\ell^2}{2!}\left(\frac{(\partial\phi)^2}{2}\right)^2+\frac{3\,\ell^4}{3!}\left(\frac{(\partial\phi)^2}{2}\right)^3+\mathcal{O}(\ell^6). \tag{2.41}$$

NLSM Next, we consider the low-energy effective theory of the Goldstone bosons corresponding to the spontaneous symmetry breaking $U(N)\times U(N)\to U(N)$. It is referred to as $U(N)$ non-linear sigma model (NLSM) and governed by [105, 106]

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right), \tag{2.42}$$

where F is a constant with mass dimension 1 in four dimensions, and $\phi=2\phi^a t^a$ with t^a the fundamental representations of generators of $U(N)$. In the so-called Cayley parameterization, U can be written as an infinite series

$$U = \mathbb{1} + 2\sum_{m=1}^{\infty}\left(\frac{i}{2F}\phi\right)^{m}. \tag{2.43}$$

Plugging U into the Lagrangian (2.42), one can extract Feynman rules order by order and find that only terms with the even number of fields are non-vanishing. Like Yang-Mills, the amplitude of scalars in the NLSM has a similar color decomposition, as shown in eq. (2.13).

sGal Let us also see a special Galileon theory (sGal). Galileon theories are effective field theories of scalars in the decoupling limit of massive gravity [107, 108] and Dvali-Gabadadze-Porrati model [109]. The Lagrangian of the general Galileon theory reads

$$\mathcal{L}_{\text{Galileon}} = \frac{1}{2}(\partial\phi)^2 + \sum_{m=3}^{\infty} c_m\,\mathcal{L}^{(m)}, \tag{2.44}$$

with

$$\mathcal{L}^{(m)} := \sum_{\alpha\in S_m}(-1)^{\alpha} g^{\mu_1\nu_{\alpha_1}} g^{\mu_2\nu_{\alpha_2}}\cdots g^{\mu_m\nu_{\alpha_m}}\,\Phi_{\mu_1\nu_1}\Phi_{\mu_2\nu_2}\cdots\Phi_{\mu_m\nu_m}, \tag{2.45}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ denotes the permutation of $\{1, \ldots, m\}$ and $(-1)^{\alpha}$ is its sign, and $\Phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$. The special Galileon theory is a special case of Galileon theory that has an enhanced soft limit compared to the generic Galileon [110, 111]. Such enhanced soft limit fixes the coefficients c_m in the Lagrangian (2.44). Here we do not give the explicit form of c_m (c.f. [110–112] for detail), but just to emphasize that c_m vanishes for odd m in the sGal.

Volkov-Akulov Lastly, we consider a fermionic theory – Volkov-Akulov (VA) model [113, 114]

$$\mathcal{L}_{\text{VA}} = -\frac{1}{2}\rho^{-2}\det\left(1 + i\rho^2\psi\sigma^{\mu}\overleftrightarrow{\partial}_{\mu}\bar{\psi}\right), \tag{2.46}$$

where ψ is the Weyl fermion which describes the Goldstino, and ρ is a constant with mass dimension -2. The VA model is the low-energy effective theory of Goldstinos associated with the spontaneous breaking of supersymmetry.

In summary, a notable feature is that all these effective theories have only non-zero amplitudes with even number of external legs. This is consistent with they have Adler's zeros [115, 116]. More precisely, in the single soft limit $k_s \to 0$,

$$\mathcal{A}_{2n} = \mathcal{O}\big(k_s^m\big), \tag{2.47}$$

where m is a non-negative integer characterizing the soft behavior of theory, which has been used to classify scalar effective field theories in [111, 117].

Unlike Yang-Mills and gravity, it is currently not very clear how to apply standard techniques such as on-shell recursions [14, 15] to efficiently calculate amplitudes in these effective field theories. In Chapter 5, we develop new representations for tree-level amplitudes in these theories based on the scattering equations and show that these new formulas can be nicely used to produce various double soft theorems.

3 The scattering equations in arbitrary-dimensional spacetime

This chapter provides a short review on the basics of the scattering equations and the Cachazo-He-Yuan (CHY) representation of tree-level scattering amplitudes in arbitrary spacetime dimensions.

3.1 The scattering equations from a rational map

Let us start with a rational map from the moduli space[1] of Riemann spheres with n marked points $\mathfrak{M}_{0,n}$ to the space of momenta for n massless particle scattering [22, 23]

$$k_a^\mu = \frac{1}{2\pi i} \oint\limits_{|z-\sigma_a|=\epsilon} dz\, \omega^\mu(z) \tag{3.1}$$

with

$$\omega^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a} = \frac{P^\mu(z)}{\prod_{a=1}^{n}(z-\sigma_a)}, \tag{3.2}$$

where $\sigma_a \in \mathbb{CP}^1$ denote the marked points in the $\mathfrak{M}_{0,n}$. It is clear that $P^\mu(z)$ is a polynomial of degree $n-2$ for each Lorentz index $\mu = 0, \ldots, D-1$.

In order to determine the positions of the marked points, one has to impose additional conditions on the map. One can show that $P(\sigma_a)^2 = 0$ for all

[1] The moduli space of Riemann spheres with n marked points, denoted by $\mathfrak{M}_{0,n}$, is defined as the quotient of the configuration space of n distinct points $(\sigma_1, \ldots, \sigma_n) \in (\mathbb{CP}^1)^n$ modulo Möbius transformations.

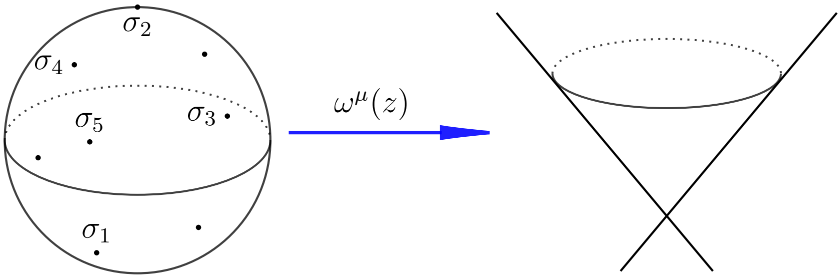


Figure 3.1: The rational map from the Riemann sphere to the null cone.

marked points because of the fact that all particles are massless and on-shell. It is natural to impose a requirement that the $P^\mu(z)$ remains null for any $z \in \mathbb{CP}^1$ in the sphere, as shown in Figure 3.1. Indeed, we show that this leads to a set of constraints that fix all marked points consistently below. Consider the following meromorphic function

$$w(z)^2 \equiv w_\mu(z)w^\mu(z) = \sum_a \frac{1}{z-\sigma_a} \sum_{b\neq a} \frac{2k_a \cdot k_b}{\sigma_a - \sigma_b}, \tag{3.3}$$

which has only simple pole at each $z = \sigma_a$ and no pole at $z = \infty$ due to momentum conservation. Then by using the residue theorem, one can immediately get the constraints to determine the marked points σ_a, i.e.,

$$0 = \frac{1}{4\pi i} \oint\limits_{|z-\sigma_a|=\varepsilon} dz\, w(z)^2 = \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}, \quad a \in \{1, \ldots, n\}, \tag{3.4}$$

which are named the *scattering equations* [22], and are the core object of this work. We refer to them as the D-dimensional scattering equations since they are valid for arbitrary spacetime dimension D.

Before discussing the properties of these equations in subsequent sections, here we would like to give some historical remarks on the scattering equations. These equations have made an appearance in previous literature in different contexts [118–124]. Cachazo, He and Yuan rediscovered them in order to develop new representations for S-matrices in arbitrary-dimensional spacetime, as will be shown in the last section of this chapter. Very recently, the scattering equations have also been generalized to different contexts, e.g. [125–128].

3.2 The properties of the scattering equations

In this section we discuss the properties of the scattering equations defined in (3.4) or (1.2). For convenience, let us rewrite them here

$$f_a(\sigma) \,=\, \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} \,=\, 0, \qquad a = \{1, 2, \ldots, n\}, \tag{3.5}$$

with $\sigma_{ab} \equiv \sigma_a - \sigma_b$. As will be detailed below, this system of algebraic equations owns a global $\mathrm{SL}(2, \mathbb{C})$ symmetry, and thus only $n-3$ out of the n equations are independent.

Let us define the Möbius transformation as

$$\sigma_a \,\longrightarrow\, \tilde{\sigma}_a \,=\, \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta}, \qquad a = \{1, 2, \ldots, n\}, \tag{3.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ sastifying $\alpha\delta - \beta\gamma = 1$. The system of the scattering equations in (3.5) is invariant under Möbius transformations. Namely, if $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a solution of the system of the scattering equations, then $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ is also its solution. More precisely, this can be seen from a simple calculation

$$\begin{aligned} f_a(\tilde{\sigma}) \,&\equiv\, \sum_{b\neq a} \frac{k_a \cdot k_b}{\tilde{\sigma}_a - \tilde{\sigma}_b} \\ &=\, \frac{(\gamma\sigma_a + \delta)^2}{\alpha\delta - \beta\gamma} \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} - \frac{\gamma(\gamma\sigma_a + \delta)}{\alpha\delta - \beta\gamma} \sum_{b\neq a} k_a \cdot k_b \\ &=\, (\gamma\sigma_a + \delta)^2\, f_a(\sigma), \end{aligned} \tag{3.7}$$

where one has used momentum conservation in the second line. This exactly shows that the a-th equation, $f_a = 0$, just picks up a factor $(\gamma\sigma_a + \delta)^2$ under the $\mathrm{SL}(2, \mathbb{C})$ transformation (3.6). In fact, there exist three extra relations among the scattering equations, i.e.,

$$\sum_{a=1}^{n} f_a \,=\, \sum_{a=1}^{n} \sigma_a\, f_a \,=\, \sum_{a=1}^{n} \sigma_a^2\, f_a \,=\, 0. \tag{3.8}$$

Consequently, only $n-3$ of the n equations in (3.5) are independent linearly. Moreover, we can fix three of the variables σ_a using $\mathrm{SL}(2, \mathbb{C})$ redundancy.

For instance, we can fix three variables as $(\sigma_1, \sigma_2, \sigma_3) \to (0, \infty, 1)$ via the following Möbius transformation

$$\sigma_i \longrightarrow \frac{(\sigma_i - \sigma_1)(\sigma_3 - \sigma_2)}{(\sigma_i - \sigma_2)(\sigma_3 - \sigma_1)}. \tag{3.9}$$

In the following, we show the Möbius invariance of the scattering equations from the viewpoint of Lie algebras. In [51], Dolan and Goddard show that the scattering equations (3.5) are equivalent to a system of homogeneous polynomial equations defining as:

$$\tilde{h}_m(z) = 0, \quad 2 \leq m \leq n-2, \tag{3.10}$$

with

$$\tilde{h}_m(z) := \sum_{I \subset \{1,\ldots,n\}, |I|=m} s_I \sigma_I, \quad s_I \equiv \sum_{i<j \in I} s_{ij}, \quad \sigma_I \equiv \prod_{i \in I} \sigma_i. \tag{3.11}$$

Let us define the generators of the $\mathfrak{sl}(2,\mathbb{C})$ algebra that act on a polynomial ring $\mathbb{C}[\sigma]$ [129]

$$\begin{aligned} L_0 &= -\sum_{a=1}^{n} \sigma_a \frac{\partial}{\partial \sigma_a} + \frac{n}{2}, \\ L_1 &= \sum_{a=1}^{n} \sigma_a - \sigma_a^2 \frac{\partial}{\partial \sigma_a}, \\ L_{-1} &= -\sum_{a=1}^{n} \frac{\partial}{\partial \sigma_a}. \end{aligned} \tag{3.12}$$

It is easy to verify that they satisfy the following commutation relations:

$$[L_1, L_{-1}] = 2L_0, \quad [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \tag{3.13}$$

In the following we show that the polynomial system of the scattering equations in (3.11) forms a representation of the $\mathfrak{sl}(2,\mathbb{C})$.

$$\begin{aligned} L_1\tilde{h}_m &= (m-1)\,\tilde{h}_{m+1}, & L_1\tilde{h}_{n-2} &= 0, \\ L_{-1}\tilde{h}_m &= -(n-m-1)\,\tilde{h}_{m-1}, & L_{-1}\tilde{h}_2 &= 0, \\ L_0\tilde{h}_m &= \frac{1}{2}(n-2m)\,\tilde{h}_m. \end{aligned} \tag{3.14}$$

The Casimir invariant is

$$C = (L_0)^2 - \frac{1}{2}\big(L_1L_{-1} + L_{-1}L_1\big), \tag{3.15}$$

$$C\,\tilde{h}_m = \left(\frac{1}{2}n-1\right)\left(\frac{1}{2}n-2\right)\tilde{h}_m. \tag{3.16}$$

These explicitly show that the system of polynomials $\tilde{h}_m$ with $2 \leq m \leq n-2$ spans an irreducible $(n-3)$-dimensional representation of the $\mathfrak{sl}(2,\mathbb{C})$ [129].

3.3 The solutions of the scattering equations

We discuss the solutions of the scattering equations in this section. Let us start by asking how many independent solutions do the scattering equations have. In order to answer this question, we first consider the polynomial equations (3.10). We fix two of n variables, e.g. $(\sigma_1, \sigma_n) \to (0, \infty)$, using the $\mathrm{SL}(2,\mathbb{C})$ redundancy according to [51]

$$h_m \equiv \lim_{\sigma_n\to\infty} \frac{\tilde{h}_{m+1}}{\sigma_n} = \sum_{I\subset\{2,\ldots,n-1\},|I|=m} s_{\{n\}\cup I}\,\sigma_I, \tag{3.17}$$

where $1 \leq m \leq n-3$. After fixing the two variables, the system still leaves a rescaling redundancy, i.e.,

$$\sigma_a \to \lambda\sigma_a \quad\Longrightarrow\quad h_m \to \lambda^m h_m, \quad \lambda\in\mathbb{C}^*. \tag{3.18}$$

The equations in (3.17) are equivalent to the scattering equations (3.5) and (3.10) up to fixing the two variables according to $(\sigma_1, \sigma_n) \to (0, \infty)$. A remarkable property is that each h_m is a homogeneous polynomial of degree m

and linear in the variables σ_2, ..., σ_{n-1}. Therefore one can conclude that the upper bound of the number of independent solutions to the scattering equations is $(n-3)!$ by Bézout's theorem. In fact, this number is exactly the number of independent solutions, as proven in [23]. Here we follow ref. [23] and give a brief review.

The idea is to consider the single soft limit. For example, we take $k_n \to \epsilon k_n$ with $\epsilon \to 0$. In this limit, the scattering equations (3.5) become

$$f_a = \sum_{\substack{b=1 \\ b\neq a}}^{n-1} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} + \epsilon \frac{k_a \cdot k_n}{\sigma_a - \sigma_n} = 0, \qquad a \in \{1, \ldots, n-1\}, \tag{3.19}$$

$$f_n = \epsilon \tilde{f}_n = 0, \quad \tilde{f}_n \equiv \sum_{b=1}^{n-1} \frac{k_n \cdot k_b}{\sigma_n - \sigma_b}. \tag{3.20}$$

We can easily observe that the first $n-1$ equations in (3.19) naturally get reduced to the scattering equations for $n-1$ particle scattering without soft momenta. We assume they have $\sharp(n-1)$ independent solutions. In addition, we find that the last equation, f_n in eq. (3.20), is invariant in the soft limit up to the soft parameter ϵ, namely $f_n = \epsilon \tilde{f}_n$. We can find the value of σ_n from equation $\tilde{f}_n = 0$. After fixing the $\mathrm{SL}(2, \mathbb{C})$ redundancy according to e.g. $(\sigma_1, \sigma_2, \sigma_3) \to (0, \infty, 1)$, $\tilde{f}_n = 0$ is equivalent to a polynomial equation of degree $n-3$ in σ_n, and thus has $(n-3)$ solutions. Therefore, the number of independent solutions to the complete n-point scattering equations is

$$\sharp(n) = (n-3)\,\sharp(n-1) = (n-3)!, \tag{3.21}$$

where we have used also the fact that the three-point scattering equations have a unique solution that is fully fixed by the Möbius symmetry.

This shows that the number of independent solutions of the scattering equations grows factorially with the number of external legs. In general, it is very challenging to exactly obtain all $(n-3)!$ solutions to be valid for generic kinematical data for arbitrary multiplicities. It is only possible to solve the scattering equations for special kinematics or to obtain some special solutions for arbitrary kinematics. In four dimensions, a special solution for any multiplicity is [118, 119, 122] (c.f. also refs. [50, 51])

$$\sigma_a = \frac{k_a^0 + k_a^3}{k_a^1 + ik_a^2}, \quad a = 1, \ldots, n \tag{3.22}$$

It is easy to check that the valves of σ_a given in (3.22) satisfy the scattering equations (3.5), and the complex conjugate of this solution also solves the equations. In the literature, this particular solution is commonly referred to as the *MHV solution* since it is responsible for MHV amplitudes shown in (1.1).

In Chapters 7, 8 and 9, we show the scattering equations simplifies vastly in Regge kinematics, where the final state particles are ordered in rapidity. In particular, in the multi-Regge kinematics, we can determine exactly all solutions to the scattering equations for any multiplicity. For generic kinematics, we also develop an efficient technique to solve the scattering equations numerically in Chapter 10.

3.4 From the scattering equations to scattering amplitudes

There are numerous different ways to compute tree-level amplitudes. In this work we are mostly interested in the scattering equation formalism, where an on-shell tree-level amplitude with n massless particles is expressed as a multiple integral over the moduli space $\mathfrak{M}_{0,n}$ [24, 25],

$$\mathcal{A}_n = \int \frac{\prod_{a=1}^n d\sigma_a}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(f_a\right) \mathcal{I}_n \,. \tag{3.23}$$

Since tree-level amplitudes are rational functions, the integrand is completely localized by the δ-functions whose arguments are the scattering equations. As shown in the previous section, three of the n equations are redundant because of the $\text{SL}(2,\mathbb{C})$ invariance of the scattering equations. We define

$$\prod_a{}' \delta\left(f_a\right) \equiv (\sigma_r - \sigma_p)(\sigma_p - \sigma_q)(\sigma_q - \sigma_r) \prod_{a \neq r,p,q} \delta\left(f_a\right) , \tag{3.24}$$

which is independent of the choice of $\{r, p, q\}$ [24, 25]. As shown exactly in eq. (3.7), each scattering equation f_a just picks up an overall factor $(\gamma\sigma_a + \delta)^2$ under the Möbius transformation (3.6). Thus it is easy to show that under the

Möbius transformation (3.6) the delta-functions defined in (3.24) behave as

$$\prod_a{}' \delta(f_a) \longrightarrow \left(\prod_{b=1}^{n} (\gamma\sigma_b + \delta)\right)^{-2} \prod_a{}' \delta(f_a). \tag{3.25}$$

In this case, we say the delta-functions have weight -2 under the $\mathrm{SL}(2,\mathbb{C})$ transformation.

Note that the invariant measure on the Möbius group can be defined as

$$\frac{d\sigma_i d\sigma_j d\sigma_l}{\sigma_{ij}\sigma_{jl}\sigma_{li}}, \tag{3.26}$$

where it is free to choose $\{i, j, l\}$. So we can verify that the measure in the CHY formula (3.23) transforms as under the Möbius transformation (3.6),

$$\frac{\prod_{a=1}^{n} d\sigma_a}{\text{vol } \mathrm{SL}(2,\mathbb{C})} \longrightarrow \left(\prod_{b=1}^{n} (\gamma\sigma_b + \delta)\right)^{-2} \frac{\prod_{a=1}^{n} d\sigma_a}{\text{vol } \mathrm{SL}(2,\mathbb{C})}. \tag{3.27}$$

Therefore the measure has weight "-2" under $\mathrm{SL}(2,\mathbb{C})$ transformation (3.6).

The last ingredient of the CHY formula (3.23) is the integrand function $\mathcal{I}_n(\sigma)$ which encodes the information on the dynamics of the theory. In order that the integral formula (3.23) remains invariant under the $\mathrm{SL}(2,\mathbb{C})$ transformation, the function $\mathcal{I}_n(\sigma)$ must have weight 4, namely

$$\mathcal{I}_n(\sigma) \longrightarrow \left(\prod_{a=1}^{n} (\gamma\sigma_a + \delta)\right)^{4} \mathcal{I}_n(\sigma). \tag{3.28}$$

CHY integrands for gauge theory and gravity

To be more precise, we take an example of the CHY integrands for gauge and gravity amplitudes.

Let us first define some building blocks. We introduce $2n \times 2n$ Skew-symmetric matrix as:

$$\Psi = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}, \tag{3.29}$$

where A, B and C are $n \times n$ antisymmetric matrices:

$$A_{ab} = \begin{cases} \dfrac{k_a \cdot k_b}{\sigma_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases}$$

$$C_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot k_b}{\sigma_{ab}}, & a \neq b, \\ -\displaystyle\sum_{c \neq a} \dfrac{\epsilon_a \cdot k_c}{\sigma_{ac}}, & a = b. \end{cases} \tag{3.30}$$

Here $\epsilon_i^\mu(k_i)$ denotes the polarization vector with momentum k_i. Then we can define[2]

$$\mathrm{Pf}'\,\Psi \equiv 2\frac{(-1)^{i+j}}{\sigma_{ij}}\,\mathrm{Pf}\left(\Psi^{ij}_{ij}\right), \tag{3.32}$$

where Ψ^{ij}_{ij} means two columns and two rows $1 \leq i, j \leq n$ have been deleted from the matrix Ψ. The reduced pfaffian $\mathrm{Pf}'\,\Psi$ defined above is permutation invariant and independent of the choice of labels $\{i, j\}$ [24]. Let us make some comments about the properties of this ingredient. First, it has weight 2 under the $\mathrm{SL}(2,\mathbb{C})$ transformation (3.6). Second, it is manifestly linear in each polarization vector ϵ_a. More physically, it is gauge invariant. This can be seen from the fact that a-th and $(a{+}n)$-th columns of Ψ become identical if we replace any ϵ_a by momentum k_a.

Another ingredient related to gauge and gravity amplitudes is the so-called Parke-Taylor factor

$$\mathcal{C}(\mu) \equiv \frac{1}{\sigma_{\mu_1\mu_2}\sigma_{\mu_2\mu_3}\cdots\sigma_{\mu_n\mu_1}}, \tag{3.33}$$

[2] The pfaffian of a skew-symmetric $2n \times 2n$ matrix N can be defined recursively as

$$\mathrm{Pf}(N) := \sum_{\substack{j=1\\ j\neq i}}^{2n} (-1)^{i+j+1+\theta(i-j)} N_{ij}\,\mathrm{Pf}(N^{ij}_{ij}), \quad \mathrm{Pf}(N_{0\times 0}) := 1, \tag{3.31}$$

where index i can be selected arbitrarily, $\theta(i{-}j)$ is the Heaviside step function, N^{ij}_{ij} denotes the matrix N with both the i-th and j-th rows and columns removed. The square of the Pfaffian of a matrix gives the determinant of the matrix, i.e. $\det(N) = \mathrm{Pf}(N)^2$. Moreover, pfaffians share many similar properties with determinants.

where the ordering $\mu = (\mu_1, \ldots, \mu_n)$ is a permutation of $\{1, \ldots, n\}$. It is easy to check that the Parke-Taylor factor has weight 2 under the Möbius transformation.

Given two ingredients defined in eqs. (3.32) and (3.33), the integrand for tree-level color-ordered amplitudes of gluons is [24]

$$\mathcal{I}_n^{(\mathrm{YM})}(1, \ldots, n) \;=\; \mathcal{C}(1, \ldots, n)\ \mathrm{Pf}'\,\Psi(\sigma, k, \epsilon). \tag{3.34}$$

The integrand for graviton amplitudes takes [24]

$$\mathcal{I}_n^{(\mathrm{GR})} \;=\; \det{}'\Psi \;\equiv\; \mathrm{Pf}'\,\Psi(\sigma, k, \epsilon)\,\mathrm{Pf}'\,\Psi(\sigma, k, \epsilon). \tag{3.35}$$

It is easy to see that the two integrands defined in (3.34) and (3.35) have the correct weight under $\mathrm{SL}(2, \mathbb{C})$ transformation. Furthermore, they satisfy the primary physical constraints, such as mass dimensionality, Lorentz invariance, gauge invariance and multilinearity in polarization vectors or tensors.

In addition, using these two ingredients we can define another function that has weight 4 under the $\mathrm{SL}(2, \mathbb{C})$ transformations,

$$\mathcal{I}_n^{(\phi^3)}(\mu|\rho) \;=\; \mathcal{C}(\mu)\,\mathcal{C}(\rho). \tag{3.36}$$

Here the use of superscript "ϕ^3" indicates that this function really describes ϕ^3 interactions [25]. More precisely, the CHY formula with the integrand (3.36) computes the double color-ordered amplitude of a massless cubic scalar theory in the adjoint of the color group $U(N) \times U(\widetilde{N})$.

An interesting observation is one has an elegant description for the double copy relation between amplitudes in Yang-Mills and gravity (denote as "GR $=$ YM $\otimes$ YM") [24, 25]. The scattering equations are universal for all theories, and we can obtain the CHY integrand for gravity by taking two copies of the CHY integrand for Yang-Mills, and divided by that of bi-adjoint ϕ^3 theory. More precisely,

$$\mathcal{I}_n^{(\mathrm{GR})} \;=\; \frac{\mathcal{I}_n^{(\mathrm{YM})}(\mu) \times \mathcal{I}_n^{(\mathrm{YM})}(\rho)}{\mathcal{I}_n^{(\phi^3)}(\mu|\rho)}. \tag{3.37}$$

More interestingly, the CHY representations for some other theories may also be derived from the double copy relation in a similar way [130] (see also Chapter 5).

Let us conclude this section with some remarks on the CHY representation of amplitudes. More generally, for the theory that admits a CHY representation, the integrand function takes a factorized form $\mathcal{I}_n = \mathcal{I}_n^{(L)}\mathcal{I}_n^{(R)}$. We refer to $\mathcal{I}_n^{(L)}$ or $\mathcal{I}_n^{(R)}$ as the "half integrand" that transforms with weight 2 under the Möbius transformation. This makes it possible to construct CHY representations for more theories with fewer ingredients. So far, the CHY integrands for a variety of theories have been obtained, such as Yang-Mills, Einstein gravity, bi-adjoint scalar, Yang-Mills-scalar, Einstein-Maxwell-scalar, Born-Infeld, Dirac-Born-Infeld, Non-linear sigma model and a special Galileon theory [23–25,125,130–137].

4 The scattering equations in four-dimensional spacetime

In this chapter, we show that in four dimensions in the spinor-helicity formalism, the scattering equations can be decomposed into different "helicity sectors". We also review two types of four-dimensional formulations of amplitudes that were derived from Witten's twistor string and ambitwistor string models respectively and clarify the equivalence between them.

4.1 The helicity scattering equations

Let us recall the map from the n-punctured spheres to the space of momenta

$$P^\mu(z) \;=\; \left(\sum_{a=1}^{n}\frac{k_a^\mu}{z-\sigma_a}\right)\left(\prod_{b=1}^{n}(z-\sigma_b)\right). \tag{4.1}$$

$P^\mu(z)$ is a *null* vector. In four dimensions, it can be expressed in spinor variables (c.f. Appendix A) as follows:

$$P^{\alpha\dot\alpha}(z) \;=\; \lambda^\alpha(z)\tilde\lambda^{\dot\alpha}(z), \tag{4.2}$$

where $\lambda^\alpha(z)$ and $\tilde\lambda^{\dot\alpha}(z)$ are spinor-valued polynomials in z of degree d and $\tilde d$ respectively, $d,\tilde d \in \{1,\dots,n{-}3\}$ and $d+\tilde d = n{-}2$. As will be explained later, the polynomials $\lambda(z)$ and $\tilde\lambda(z)$ may be constructed as follows:

$$\begin{aligned}\lambda^\alpha(z) \;&=\; \left(\prod_{a\in\mathfrak{N}}(z-\sigma_a)\right)\sum_{I\in\mathfrak{N}}\frac{t_I\lambda_I^\alpha}{z-\sigma_I},\\ \tilde\lambda^{\dot\alpha}(z) \;&=\; \left(\prod_{a\in\mathfrak{P}}(z-\sigma_a)\right)\sum_{i\in\mathfrak{P}}\frac{t_i\tilde\lambda_i^{\dot\alpha}}{z-\sigma_i},\end{aligned} \tag{4.3}$$

where $\mathfrak{N}$ is any subset of $\{1,\ldots,n\}$ with length $2 \leq k \leq n-2$ and $\mathfrak{P}$ is the corresponding complement. Therefore, $\lambda(z)$ has degree $d = k-1$, while $\tilde{\lambda}(z)$ has degree $\tilde{d} = n-k-1$. In (4.3), we have also introduced n additional variables $t_a \in \mathbb{CP}^1$ with $a \in \{1,\ldots,n\}$. As will be immediately seen below, these variables can be fixed by additional equations consistently.

Plugging (4.3) into (4.2), one finds [132, 138]

$$\begin{aligned} P^{\alpha\dot{\alpha}}(z) &= \lambda^\alpha(z)\tilde{\lambda}^{\dot{\alpha}}(z) \\ &= \prod_{a=1}^{n}(z-\sigma_a)\left(\sum_{I\in\mathfrak{N}}\frac{t_I\lambda_I^\alpha}{z-\sigma_I}\right)\left(\sum_{i\in\mathfrak{P}}\frac{t_i\tilde{\lambda}_i^{\dot{\alpha}}}{z-\sigma_i}\right) \\ &= \prod_{a=1}^{n}(z-\sigma_a)\sum_{I\in\mathfrak{N}}\sum_{i\in\mathfrak{P}}t_I t_i\lambda_I^\alpha\tilde{\lambda}_i^{\dot{\alpha}}\left(\frac{1}{z-\sigma_I}-\frac{1}{z-\sigma_i}\right)\frac{1}{\sigma_I-\sigma_i} \\ &= \prod_{a=1}^{n}(z-\sigma_a)\left(\sum_{I\in\mathfrak{N}}\frac{\lambda_I^\alpha}{z-\sigma_I}\sum_{i\in\mathfrak{P}}\frac{\tilde{\lambda}_i^{\dot{\alpha}}}{(I\,i)}+\sum_{i\in\mathfrak{P}}\frac{\tilde{\lambda}_i^{\dot{\alpha}}}{z-\sigma_i}\sum_{I\in\mathfrak{N}}\frac{\lambda_I^\alpha}{(i\,I)}\right), \end{aligned} \tag{4.4}$$

where $(a\,b) := (\sigma_a-\sigma_b)/(t_a t_b)$ and we used partial fraction decomposition in the third line. A simple comparison with the original definition of the map $P(z)$ in (4.1) gives

$$\tilde{\lambda}_I^{\dot{\alpha}} - \sum_{i\in\mathfrak{P}}\frac{\tilde{\lambda}_i^{\dot{\alpha}}}{(I\,i)} = 0, \quad I\in\mathfrak{N}; \qquad \lambda_i^\alpha - \sum_{I\in\mathfrak{N}}\frac{\lambda_I^\alpha}{(i\,I)} = 0, \quad i\in\mathfrak{P}. \tag{4.5}$$

We see that in four dimensions the scattering equations fall into different sectors characterized by $k = |\mathfrak{N}| \in \{2,\ldots,n-2\}$. As expected, we obtain $2n$ equations that can be used to determine σ's as well as n extra t_a variables.

The two spinor-valued maps in (4.3) have introduced an overall rescaling redundancy, i.e.

$$\{\lambda(z),\tilde{\lambda}(z)\} \longrightarrow \{c\lambda(z),c^{-1}\tilde{\lambda}(z)\} \;\text{ with }\; c\in\mathbb{C}^* \tag{4.6}$$

leaves $P(z) = \lambda(z)\tilde{\lambda}(z)$ invariant. Thus the system of spinor-valued equations in (4.5) has a global $\mathrm{GL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$ redundancy. To be clear, let us perform a $\mathrm{GL}(2,\mathbb{C})$ transform as follows [139]:

$$\begin{aligned}
\sigma_a &\longrightarrow \tilde{\sigma}_a = \frac{\alpha\sigma_a+\beta}{\gamma\sigma_a+\delta}, \qquad && a\in\{1,\ldots,n\} \\
t_i &\longrightarrow \tilde{t}_i = \frac{(\alpha\delta-\beta\gamma)\,t_i}{\gamma\sigma_i+\delta}, && i\in\mathfrak{P} \\
t_I &\longrightarrow \tilde{t}_I = \frac{t_I}{\gamma\sigma_I+\delta}, && I\in\mathfrak{N}
\end{aligned} \tag{4.7}$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ sastifying $\alpha\delta-\beta\gamma\neq 0$. It is then apparent that any $c_{iI} = 1/(i\,I)$ with $i\in\mathfrak{P}$ and $I\in\mathfrak{N}$ is variant under the transformation defined in (4.7). As a consequence, we can fix 4 of $2n$ variables using the $\mathrm{GL}(2,\mathbb{C})$ redundancy. For example, in the case of $\{1,2\}\subseteq\mathfrak{N}$, by performing the transformation (4.7) with

$$\alpha = -\frac{1}{t_1}, \quad \beta = \frac{\sigma_1}{t_1}, \quad \gamma = -\frac{1}{t_2}, \quad \delta = \frac{\sigma_2}{t_2}, \tag{4.8}$$

we can fix four variables $(\sigma_1,\sigma_2,t_1,t_2)$ as

$$\sigma_1 = 0, \quad t_1 = -1, \quad t_2 = \sigma_2 \to \infty. \tag{4.9}$$

Accordingly, only $2n-4$ of the $2n$ equations in (4.5) are independent. In fact, these equations imply momentum conservation [139, 140]. In other words, we can identify four redundant equations with the momentum conservation constraint. More precisely, for any $\{I,J\}\subseteq\mathfrak{N}$, one has

$$\delta^2\!\left(\tilde{\lambda}_I^{\dot\alpha} - \sum_{i\in\mathfrak{P}}\frac{\tilde{\lambda}_i^{\dot\alpha}}{(I\,i)}\right)\delta^2\!\left(\tilde{\lambda}_J^{\dot\alpha} - \sum_{i\in\mathfrak{P}}\frac{\tilde{\lambda}_i^{\dot\alpha}}{(J\,i)}\right) = \langle I\,J\rangle^2\,\delta^4\!\left(\sum_{a=1}^{n}\lambda_a^{\alpha}\tilde{\lambda}_a^{\dot\alpha}\right). \tag{4.10}$$

Similarly, for for any $\{i,j\}\subseteq\mathfrak{P}$, one has

$$\delta^2\!\left(\lambda_i^{\alpha} - \sum_{I\in\mathfrak{N}}\frac{\lambda_I^{\alpha}}{(i\,I)}\right)\delta^2\!\left(\lambda_j^{\alpha} - \sum_{I\in\mathfrak{N}}\frac{\lambda_I^{\alpha}}{(j\,I)}\right) = [i\,j]^2\,\delta^4\!\left(\sum_{a=1}^{n}\lambda_a^{\alpha}\tilde{\lambda}_a^{\dot\alpha}\right). \tag{4.11}$$

We put the detailed derivation in Appendix B.

The equations in (4.5) were originally derived from the four-dimensional ambitwistor string model by Geyer, Lipstein and Masin (GLM) in [28]. They have also appeared much earlier in the context of open twistor string theory [139]. As mentioned in the introduction chapter, there was another contour integral representation for supersymmetric gauge theory amplitudes in four-dimensional spacetime, which was derived from Witten's twistor string [8] by Roiban, Spradlin and Volovich (RSV) [16]. A crucial ingredient of the Witten-RSV formalism is a set of equations as follows

$$\begin{aligned} \sum_{a=1}^{n} t_a \sigma_a^m \tilde{\lambda}_a^{\dot{\alpha}} &= 0, \qquad m = 0, 1, \ldots, d, \\ \lambda_a^{\alpha} - t_a \sum_{m=0}^{d} \rho_m^{\alpha} \sigma_a^m &= 0, \qquad a = 1, \ldots, n. \end{aligned} \tag{4.12}$$

Compared to the system of equations (4.5), there are more equations and more variables, ρ_m^{α} ($m = 0, \ldots, d$), are involved besides σ_a and t_a. Like the equations (4.5), the system of equations in (4.12) is characterised by an integer $d = k{-}1$. Unlike (4.5), the equations in (4.12) are manifestly permutation invariant on particle labels $\{1, 2, \ldots, n\}$.

The number of independent solutions for the scattering equations in both (4.5) and (4.12) in sector k is given by the Eulerian number $\left\langle {n-3 \atop k-2} \right\rangle$ [22, 141]. It is well-known that [142]

$$\sum_{k=2}^{n-2} \left\langle {n-3 \atop k-2} \right\rangle = (n{-}3)! \tag{4.13}$$

in agreement with the fact that the D-dimensional scattering equations in (3.5) have $(n{-}3)!$ independent solutions. As we will show in the following, for both gauge and gravity theories exactly the sector-k (or $d = k{-}1$) equations are needed for the amplitudes in helicity sector k (e.g. those with k negative-helicity gluons or gravitons). Because of this reason, we refer to them as also *(four-dimensional) helicity scattering equations*. Moreover, we call each division of particle labels into $\mathfrak{N}$ and $\mathfrak{P}$ the "*helicity configuration*" of the helicity scattering equations in (4.5).

Equivalence between (4.5) and (4.12)

A natural and important question is about the equivalence between the two forms of helicity scattering equations, (4.5) and (4.12). We discuss the relation between them in the following. This part is based on the paper [132].

It was first pointed out in [143] that the equations in (4.12) can be viewed as the constraints on σ's through those on the so-called Veronese form of the Grassmannian (i.e. $k \times n$ matrix up to $\mathrm{GL}(k)$ transformation). From (4.12), we see that the form of the matrix (the "C-matrix") reads [143]

$$C_{m+1,a} \,=\, t_a \sigma_a^m \,, \quad \text{for } m = 0, \ldots, d, \quad a = 1, \ldots, n. \tag{4.14}$$

Viewing λ_a^α, $\tilde{\lambda}_a^{\dot{\alpha}}$ both as $n \times 2$ matrices, and ρ_m^α as $2 \times k$ matrix, the Witten-RSV scattering equations (4.12) become

$$C \cdot \tilde{\lambda} \,=\, 0, \quad \lambda^{\mathrm{T}} - \rho \cdot C \,=\, 0. \tag{4.15}$$

Here the dot "$\cdot$" and symbol "T" denote matrix multiplication and transpose respectively. Geometrically speaking, this means that the k-plane C is orthogonal to 2-plane $\tilde{\lambda}$, and it contains the 2-plane λ. For our purpose it is actually more convenient to rewrite the latter constraints as the statement that the orthogonal complement of C, denoted by $C^\perp$ (which is a $(n-k)$-plane or a $(n-k) \times n$ matrix), is orthogonal to the λ-plane. Thus (4.12) becomes

$$C \cdot \tilde{\lambda} = 0 \,, \qquad C^\perp \cdot \lambda = 0 \,. \tag{4.16}$$

To relate this form to the equations in (4.5) simply requires a $\mathrm{GL}(k)$ transformation such that a $k \times k$ submatrix of C becomes the identity. For convenience, without loss generality, we try to transform the first k columns and rows of the C into the identity, i.e.,

$$\mathcal{C} \,=\, L \cdot C \,=\, \left(\mathbb{1}_{k\times k} \,|\, \mathcal{C}_{k\times(n-k)}\right). \tag{4.17}$$

The $k \times k$ matrix L has been exactly worked out [132]

$$L_{IJ} \,=\, \left(t_I \prod_{K\neq I} \sigma_{IK} \right)^{-1} \Sigma_{IJ}, \tag{4.18}$$

where Σ_{IJ} is the degree $k-J$ terms of the following polynomial:

$$\Sigma_I = \prod_{K\neq I}(1-\sigma_K). \tag{4.19}$$

Substituting eq. (4.18) with (4.19) into eq. (4.17) gives

$$\mathcal{C}_{I\,a} = \sum_{J=1}^{k} L_{IJ}C_{J\,a} = \left(t_I \prod_{L\neq I}\sigma_{IL}\right)^{-1}\left(t_a \prod_{J\neq I}\sigma_{aJ}\right) \tag{4.20}$$

A simple algebra shows that $\mathcal{C}_{IJ} = \delta_{IJ}$, as expected. The remaining part has been previously spelled out as the so-called *link-representation form* [140, 144]:

$$\mathcal{C}_{I\,i} = \frac{t_i \prod_{J\neq I}\sigma_{iJ}}{t_I \prod_{K\neq I}\sigma_{IK}}, \quad k < i \leq n. \tag{4.21}$$

Note that after the fixing it is trivial to write $\mathcal{C}^{\perp}$ (see below). By performing the transformation

$$\tilde{t}_i = t_i\beta_i,\ \tilde{t}_I = \frac{1}{t_I\beta_I}, \ \text{ with } \beta_i = \prod_J \sigma_{iJ},\ \beta_I = \prod_{K\neq I}\sigma_{IK}, \tag{4.22}$$

we can absorb an overall factor in (4.21), and the variables $\mathcal{C}_{Ii}$ become

$$\mathcal{C}_{Ii} = \frac{\tilde{t}_I\,\tilde{t}_i}{\sigma_{I\,i}}. \tag{4.23}$$

Let us spell out the constraints $\mathcal{C}\cdot\tilde{\lambda} = \mathcal{C}^{\perp}\cdot\lambda = 0$ in this gauge-fixed form:

$$\begin{aligned} &\big(\mathbb{1}_{k\times k}\,|\,\mathcal{C}_{k\times(n-k)}\big)\cdot\tilde{\lambda} = 0_{k\times 2},\\ &\Big(\big(-\mathcal{C}^{\mathrm{T}}\big)_{(n-k)\times k}\,\Big|\,\mathbb{1}_{(n-k)\times(n-k)}\Big)\cdot\lambda = 0_{(n-k)\times 2}\,, \end{aligned} \tag{4.24}$$

which are exactly the GLM helicity scattering equations (4.5) up to rename t_a as $\tilde{t}_a$. Therefore, we have shown precisely that the GLM scattering equations (4.5) are nothing but the gauge-fixed or link-representation form of the Witten-RSV scattering equations (4.12).

4.2 From helicity scattering equations to helicity amplitudes

In the previous section, we have shown that in four-dimensional spacetime in the spinor-helicity formalism the scattering equations are reduced to two forms of helicity scattering equations, given in (4.5) and (4.12) respectively. We turn to the formulas for the scattering amplitudes based these equations in this section. As a warm-up, we focus on the maximally supersymmetric Yang-Mill and Einstein gravity theories in this section. We systematically discuss the effective field theories, including DBI, NLSM and a special Galileon theory, in the next chapter.

On-shell superspace

In order to proceed, let us first provide a very brief introduction to the on-shell superspace. $\mathcal{N}=4$ super Yang-Mills has $2^4 = 16$ on-shell degrees of freedom: two gluons, 4 gluinos and 8 scalars, which transform into each other according to supersymmetric generators. By introducing Grassmann odd variables η^A $(A=1,2,3,4)$ which transform in a fundamental representation of the $SU(4)$ R-symmetry, we can assemble the 16 states into a single on-shell sperfield [145, 146]

$$\begin{aligned}\Phi^{(\mathrm{SYM})}(\eta) &= g^+ + \eta^A\Gamma_A + \frac{1}{2!}\eta^A\eta^B\phi_{AB} \\ &\quad + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D + \eta^1\eta^2\eta^3\eta^4\, g^- .\end{aligned} \tag{4.25}$$

It turns out that it is convenient to assign the helicity $h=1/2$ to each η^A such that the on-shell superfield Φ has a uniform helicity $h=1$. We define the space parameterized by $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$ as the on-shell super(-momentum) space where the on-shell superfield Φ_i lives in. In the on-shell superspace, one can define the supersymmetry generators as

$$q_i^{\alpha A} \equiv \lambda_i^\alpha \eta_i^A, \quad \bar{q}_{i,A}^{\dot{\alpha}} \equiv \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A}, \tag{4.26}$$

which satisfy the SUSY algebra:

$$\{q_i^{\alpha A},\, q_{i,B}^{\dot{\alpha}}\} = \delta_B^A\, \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}. \tag{4.27}$$

In the on-shell superspace, we define the superamplitude as

$$\mathscr{A}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) \equiv \Big\langle \Phi_1(\lambda_1, \tilde{\lambda}_1, \eta_1) \cdots \Phi_n(\lambda_n, \tilde{\lambda}_n, \eta_n) \Big\rangle , \tag{4.28}$$

which is invariant under the supersymmetry generators in (4.26). Its external legs are on-shell superfields and, thus it packages all component amplitudes into a single object. According to the on-shell superfield (4.25), a superamplitude can be expanded as a sum of polynomials in η^A

$$\mathscr{A}_n(\eta_i) = \sum_{k=2}^{n-2} \mathscr{A}_{n,k}(\eta_i), \tag{4.29}$$

where $\mathscr{A}_{n,k}$ has Grassmann degree $4k$ and is referred to as the superamplitude of sector k since it incorporates all n-point N^{k-2}MHV amplitudes. Component helicity amplitudes with gluons, fermions or scalars can be read off from the η-expansion of the superamplitude.

Similarly, the 256 on-shell states of the supermultiplet in $\mathcal{N} = 8$ supergravity can be organized into a single on-shell field:

$$\begin{aligned} \Phi^{(\mathrm{SG})}(\eta) = {} & h^+ + \eta^A \lambda_A + \frac{1}{2!}\eta^A\eta^B v_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \chi_{ABC} \\ & + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D S_{ABCD} + \cdots , \end{aligned} \tag{4.30}$$

where $A = 1, \ldots, 8$ are fundamental indices of the $SU(8)$ R-symmetry.

Superamplitudes

Given the superspace, supersymmetry dictates that we include fermionic delta functions for η's in the same form as those for $\tilde{\lambda}$'s. Now we can write down the formulas of superamplitudes in $\mathcal{N} = 4$ SYM or $\mathcal{N} = 8$ SUGRA and see how the measures and integrands of these two forms transform between each other. Let us start with the GLM form

$$\mathscr{A}_{n,k} = \int d\mu_{n,k}^{(\mathcal{N})}\, \mathcal{I}_{n,k}^{(\mathrm{GLM})}(\tilde{t}_i, \tilde{t}_I) , \tag{4.31}$$

where for convenience, without loss generality, we take $\mathfrak{N} = \{1, \ldots, k\}$

$$d\mu_{n,k}^{(\mathcal{N})} := \frac{\prod_{a=1}^{n} d^2\tilde{\sigma}_a}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{I=1}^{k} \delta^{2|\mathcal{N}} \left((\tilde{\lambda}_I|\eta_I) - \sum_{i=k+1}^{n} \frac{\tilde{t}_I \tilde{t}_i}{\sigma_{Ii}} (\tilde{\lambda}_i|\eta_i) \right) \times \prod_{i=k+1}^{n} \delta^2 \left(\lambda_i - \sum_{I=1}^{k} \frac{\tilde{t}_i \tilde{t}_I}{\sigma_{iI}} \lambda_I \right), \tag{4.32}$$

with $d^2\tilde{\sigma}_a \equiv d\sigma_a d\tilde{t}_a/\tilde{t}_a^3$. In (4.31), $\mathscr{A}_{n,k}$ is considered to be n-point superamplitude of k-sector in $\mathcal{N}=4$ SYM or $\mathcal{N}=8$ SUGRA. As we will see why shortly, we have indicated the explicit dependence of the rational-form integrand on $\tilde{t}_i, \tilde{t}_I$. Performing the transformation in (4.22) and keeping track of the Jacobians, we get

$$\mathscr{A}_{n,k} = \int d\Omega_{n,k}^{(\mathcal{N})} (V_k)^{4-\mathcal{N}} \left(\prod_{i=k+1}^{n} \beta_i^{-2} \right) \left(\prod_{I=1}^{k} \beta_I^2 \, t_I^4 \right) \mathcal{I}_{n,k}^{(\mathrm{GLM})}\big(\beta_i t_i, \tfrac{1}{\beta_I t_I}\big), \tag{4.33}$$

where the $\mathrm{GL}(k)$ transformation is performed and we have defined the Jacobian

$$V_k = \left(\prod_{I} t_I \right) \left(\prod_{J \neq K} \sigma_{JK} \right), \tag{4.34}$$

as well as the Witten-RSV-form measure with $\mathcal{N}$ supersymmetries

$$d\Omega_{n,k}^{(\mathcal{N})} := \frac{\prod_{a=1}^{n} d^2\sigma_a}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{m=0}^{d} \delta^{2|\mathcal{N}} \left(\sum_{a=1}^{n} t_a \sigma_a^m (\tilde{\lambda}_a|\eta_a) \right) \times \int d^{2k}\rho \prod_{a=1}^{n} \delta^2 \left(t_a \sum_{m=0}^{d} \rho_m \sigma_a^m - \lambda_a \right), \tag{4.35}$$

where $d = k-1$ and $d^2\sigma_a \equiv d\sigma_a dt_a/t_a^3$. From (4.33), we find that the integrand with Witten-RSV scattering equations (4.12) is related to the one with the GLM scattering equations (4.5) in a simple way:

$$\mathcal{I}_{n,k}^{(\mathrm{RSV})}(t_a) = (V_k)^{4-\mathcal{N}} \left(\prod_{i=k+1}^{n} \beta_i^{-2} \right) \left(\prod_{I=1}^{k} \beta_I^2 \, t_I^4 \right) \mathcal{I}_{n,k}^{(\mathrm{GLM})}\big(\beta_i t_i, \tfrac{1}{\beta_I t_I}\big). \tag{4.36}$$

To summarize, we have exactly shown how the two types of formulas for amplitudes that are based on two kinds of four-dimensional helicity scattering equations are related to each other. These are part of the main results of this work, which have been published in [132].

Superamplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity

Now we show the precise formulas for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA amplitudes by writing out explicit integrand $\mathcal{I}_{n,k}$.

From (4.36), a remarkable property is the formulas with two forms of the helicity scattering equations have identical integrands for $\mathcal{N}=4$ SYM. This integrand is just the Parke-Taylor factor [28]

$$\mathrm{PT}(\mu) \equiv \frac{1}{(\mu_1\mu_2)(\mu_2\mu_3)\cdots(\mu_n\mu_1)}. \tag{4.37}$$

Thus for color-ordered superamplitudes in $\mathcal{N}=4$ SYM, we have

$$\mathscr{A}_{n,k}^{\mathrm{SYM}}(1,\ldots,n) = \int d\mu_{n,k}^{(\mathcal{N}=4)} \frac{1}{(12)(23)\cdots(n1)} \tag{4.38}$$

$$= \int d\Omega_{n,k}^{(\mathcal{N}=4)} \frac{1}{(12)(23)\cdots(n1)}. \tag{4.39}$$

For $\mathcal{N}=8$ SUGRA amplitudes, the formula with the GLM scattering equations reads [28]:

$$\mathscr{M}_{n,k}^{\mathrm{SUGRA}} = \int d\mu_{n,k}^{(\mathcal{N}=8)} \det{}'\mathbb{H}_k \det{}'\overline{\mathbb{H}}_{n-k}\,, \tag{4.40}$$

where $\det'$ denotes the minor with any one column and one row removed (since the rows and columns add up to zero), and $\mathbb{H}$ and $\overline{\mathbb{H}}$ are $k\times k$ and $(n-k)\times$

$(n-k)$ matrices of the form:

$$\begin{aligned}
\mathbb{H}_{ab} &= \frac{\langle a\, b\rangle}{(a\, b)} \text{ for } a \neq b, \quad \mathbb{H}_{aa} = -\sum_{\substack{b\in\mathfrak{N}\\ b\neq a}} \mathbb{H}_{ab}, \quad a,b \in \mathfrak{N};\\
\overline{\mathbb{H}}_{ab} &= \frac{[a\, b]}{(a\, b)} \text{ for } a \neq b, \quad \overline{\mathbb{H}}_{aa} = -\sum_{\substack{b\in\mathfrak{P}\\ b\neq a}}^{n} \overline{\mathbb{H}}_{ab}, \quad a,b \in \mathfrak{P}.
\end{aligned} \tag{4.41}$$

Note that the integrand $\det{}'\mathbb{H}_k \det{}'\overline{\mathbb{H}}_{n-k}$ is not permutation invariant, but when we rewrite the formula with the Witten-RSV form of scattering equations, the integrand obtained from (4.36) becomes those in [17, 18], which are permutation invariant. Henceforth for simplicity, we will only write the formula with the scattering equations (4.5) explicitly, the formula using Witten-RSV form can get from (4.36).

Interestingly, the formula for double-partial amplitudes in the bi-adjoint ϕ^3 theory has been obtained with the help of the Witten-RSV scattering equations in [147]. By (4.36) we can translate it into a formula with the GLM scattering equations (4.5):

$$\mathscr{A}_n^{(\phi^3)}(\mu|\nu) = \sum_{k=2}^{n-2} \int d\mu_{n,k}^{(0)} \frac{\mathrm{PT}(\mu)\, \mathrm{PT}(\nu)}{\det{}'\mathbb{H}_k \det{}'\overline{\mathbb{H}}_{n-k}}, \tag{4.42}$$

where we have Parke-Taylor factors with orderings μ, ν and the determinants appeared in the gravity formula (4.40). It is interesting to see that the formula for the bi-adjoint ϕ^3 theory in (4.42) is more complicated than $\mathcal{N}=4$ SYM as well as $\mathcal{N}=8$ SUGRA, especially in that one has to sum over all sectors. Each k sector gives contributions ("scalar blocks" [147]) with unphysical poles which only cancel each other in the sum over sectors.

As we have shown for general-dimensional case in Chapter 3, the formula for gravity can also be derived from the double-copy of Yang-Mills, divided by ϕ^3. From the observation of [147], a nice feature of the four-dimensional formulas is that this double-copy procedure works for each k-sector individually: one can easily derive (4.40) from (4.38) and (4.42) for each k [147].

We end this chapter by evaluating amplitudes by the scattering equations in the simplest sector. At the MHV sector, $\left\langle {n-3 \atop 0} \right\rangle = 1$, thus the helicity scattering equations have only one independent solutions. By using $\mathrm{GL}(2,\mathbb{C})$ redundancy to fix four variables according to (4.9), then we can explicitly write down this unique solution

$$(i\,1) = \frac{\langle 1\,2\rangle}{\langle i\,2\rangle}, \quad (i\,2) = \frac{\langle 1\,2\rangle}{\langle 1\,i\rangle}, \quad (i\,j) = \frac{\langle 1\,2\rangle^3 \langle i\,j\rangle}{\langle 1\,i\rangle \langle 2\,i\rangle \langle 1\,j\rangle \langle 2\,j\rangle}. \tag{4.43}$$

Plugging this solution into the formula (4.38) gives the MHV superamplitude of $\mathcal{N}=4$ SYM:

$$\mathscr{A}_{n,2}(1,2,\ldots,n) = \frac{\delta^4(p)\,\delta^8(q)}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n1\rangle}, \tag{4.44}$$

where p and q denote total momenta and super-momenta respectively,

$$p^{\alpha\dot{\alpha}} \equiv \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}, \quad q^{\alpha A} \equiv \sum_{i=1}^{n} \lambda_i^{\alpha} \eta_i^{A}. \tag{4.45}$$

The $\eta_1^1\eta_1^2\eta_1^3\eta_1^4\eta_2^1\eta_2^2\eta_2^3\eta_2^4$ component of (4.44) gievs the MHV gluon amplitude, as shown in (1.1). Similarly, by evaluating the gravity formula (4.40) in the MHV sector in a same way, we can obtain the MHV superamplitude in $\mathcal{N}=8$ SUGRA [148, 149].

5 EFT S-matrices from the scattering equations

Being parallel to $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA discussed in the previous chapter, we develop new representations for the four-dimensional scattering amplitudes in some effective field theories, including maximally supersymmetric Dirac-Born-Infeld-Volkov-Akulov theory (DBI-VA), the $U(N)$ non-linear sigma model (NLSM) and a special Galileon (sGal) theory in this chapter.

While the formulas for amplitudes in the DBI, NLSM and sGal have been proposed based on the D-dimensional scattering equations in [130], we develop new representations for these theories using the four-dimensional helicity scattering equations in this chapter. Our motivation is multifold. First, a significant simplification appears in the helicity amplitudes in these theories in four dimensions since only the middle sector $k = n/2$ is needed for amplitudes in these theories, as pointed out in [130]. Second, we find that the formula for DBI amplitudes begs to be put in the supersymmetric form in four dimensions. This is parallel to the cases of gauge and gravity formulas in four dimensions, which take nice forms as we include the supermultiplet and write them in a manifestly supersymmetric manner [16–18], as have been shown in the previous chapter. The formula naturally leads us to find the maximally supersymmetric completion of the usual DBI theory. Third, the new four-dimensional formulas allow us to nicely derive the universal factorized form of amplitudes in the case of the emission of two soft particles (known as "double soft theorem"), in particular in the supersymmetric extension of the DBI theory. Soft theorems play particularly important roles in these EFTs. For example, the famous Adler's zero means that the emission of a single soft Goldstone boson gives vanishing amplitude [115, 150], and double-soft emission probes the coset algebra structure of the vacuum (c.f. [151] for double-soft-scalar emission in $\mathcal{N}=8$ SUGRA).

This chapter follows the paper [132]. In Section 2.3, we have provided a brief introduction to the theories under consideration in this chapter. The rest of the chapter is organized as follows. We present new formulas for all tree amplitudes in the maximally supersymmetric extension of the DBI theory based on the four-dimensional scattering equations; then we use the double copy relations to derive the formulas for amplitudes in the NLSM and the sGal in the next section. We use the four-dimensional formulas to derive various double soft emissions of amplitudes in various theories in Section 5.2.

5.1 New formulas for EFT amplitudes

Let us begin with the on-shell superfield of the $\mathcal{N}=4$ supersymmetric completion of the DBI theory

$$\begin{aligned}\Phi^{\text{(DBI-VA)}}(\eta) &= \gamma^+ + \eta^A\psi_A + \frac{1}{2!}\eta^A\eta^B S_{AB} \\ &\qquad + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\psi}^D + \eta^1\eta^2\eta^3\eta^4\,\gamma^-\,,\end{aligned} \tag{5.1}$$

which shares the same structure with $\mathcal{N}=4$ SYM but has very different particle contents. Gauge vectors, called “BI photon” or “photon” for short, singlets in the $SU(4)$ R-symmetry, are from the DBI theory (2.37). Scalars $S_{BA}=-S_{AB}$ are described by the DBI Lagrangian (2.37). The fermionic sector is known to coincide with the Volkov-Akulov (VA) theory (2.46). In the on-shell superfield (5.1), fermions ψ_A and scalars S_{AB} carry fundamental indices of $SU(4)$ R-symmetry.

It is well known that for photon scattering in BI theory, only helicity-conserved amplitudes with even multiplicity are non-vanishing. By supersymmetry this generalizes to the superamplitude, thus we will only have the middle sector $k=n/2$ for even n. Let us write out the measure with the helicity scattering equations in eq. (4.5)

$$d\mu^{(\mathcal{N})}_{n,\frac{n}{2}} = \frac{d^{2n}\sigma}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{i=n/2+1}^{n} \delta^2\Big(\lambda_i - \sum_{I=1}^{n/2} \frac{1}{(i\,I)}\,\lambda_I\Big)$$
$$\times \prod_{I=1}^{n/2} \delta^{2|\mathcal{N}}\Big((\tilde\lambda_I|\eta_I) - \sum_{i=\frac{n}{2}+1}^{n} \frac{1}{(I\,i)}\,(\tilde\lambda_i|\eta_i)\Big). \tag{5.2}$$

We will omit the subscript $k = n/2$ of the measure in this chapter.

It turns out that we only need one more ingredient for writing down the formulas for amplitudes in all the three theories. We define an $n \times n$ antisymmetric matrix X_n with entries

$$X_{aa} = 0, \quad X_{ab} = \frac{2k_a \cdot k_b}{(a\,b)} \text{ for } a \neq b. \tag{5.3}$$

It has two null vectors and we define the reduced pfaffian and determinant as

$$\mathrm{Pf}'X_n := \frac{(-)^{i+j}}{(i\,j)}\,\mathrm{Pf}\,X^{ij}_{ij}\,, \quad \det{}'X_n := \big(\mathrm{Pf}'X_n\big)^2. \tag{5.4}$$

One can show that the rank of the matrix X_n is less than $n-2$ when we plug in the solutions of four-dimensional scattering equations in any sector except the middle sector $k = n/2$ [130]. Thus $\det' X_n$ is only non-vanishing for the sector $k = n/2$, which already suggests strongly that it should appear in the formula for $\mathcal{N} = 4$ super-DBI-VA. The formula for the complete tree-level S-matrix in this theory reads:

$$\mathcal{M}^{\text{super-DBI-VA}}_n = \int d\mu^{(4)}_n \det{}'X_n. \tag{5.5}$$

As shown in [130], one has double-copy relations for the special Galileon theory and super-DBI-VA:

$$\mathrm{BI} = \mathrm{YM} \otimes \mathrm{NLSM}, \quad \text{super-DBI-VA} = \mathrm{SYM} \otimes \mathrm{NLSM}, \tag{5.6}$$
$$\mathrm{sGal} = \mathrm{NLSM} \otimes \mathrm{NLSM}, \tag{5.7}$$

where in the first line the second relation follow from the first one by supersymmetry. From these relations, it has become clear that the formula for NLSM and sGal must take the form

$$\mathcal{M}_n^{\text{NLSM}}(1,2,\ldots,n) = \int d\mu_n^{(0)} \frac{1}{(12)(23)\cdots(n1)} \frac{\det' X_n}{H_n}, \tag{5.8}$$

$$\mathcal{M}_n^{\text{sGal}} = \int d\mu_n^{(0)} \frac{\left(\det' X_n\right)^2}{H_n}, \tag{5.9}$$

where we have defined $H_n := \det'\mathbb{H}_{n/2}\det'\overline{\mathbb{H}}_{n/2}$. It should be noted that the formula in (5.8) computes the color-ordered scalar amplitude in the BLSM. Unlike the bi-adjoint ϕ^3 theory, these scalar amplitudes are only non-vanishing for the $k = n/2$ sector of the solutions to four-dimensional scattering equations. This can be explained from the appearance of $\det' X_n$, as already noticed in [130]. The double-copy relations (5.6) also specify to the middle sector in four dimensions, where only the term $k = n/2$ in (4.42) is needed [147].

We have very strong evidence for the new formulas, (5.5), (5.8), (5.9), by comparing with their general-dimension CHY formulas, or by studying their factorization properties directly. More explicitly, we have computed numerically up to six points and verify that they give correct amplitudes. For example, by directly evaluating (5.5) for $n = 4$ we find

$$\mathcal{M}_4^{\text{super-DBI-VA}} = \delta^4(p)\,\delta^{0|8}(q)\,\frac{[3\,4]^2}{\langle 1\,2\rangle^2}, \tag{5.10}$$

where

$$p^{\alpha\dot\alpha} := \sum_{a=1}^{4} \lambda_a^\alpha \tilde\lambda_a^{\dot\alpha}, \qquad q^{\dot\alpha A} := \sum_{a=1}^{4} \tilde\lambda_a^{\dot\alpha}\eta_a^A. \tag{5.11}$$

Similarly we have checked six-scalar amplitudes in all three theories, as well as six-photon amplitudes [152], two-fermions-four-photon and two-scalar-four-photon amplitudes [153], as well as six-fermion amplitudes[1] in $\mathcal{N} = 4$ super DBI-VA.

[1]The six-fermion amplitude in Volkov-Akulov theory was obtained in [154], and very recently reproduced in [135] using a formula similar to our (5.5).

5.2 Double soft theorems

In this section, as both consistency checks and important applications of our new formulas proposed in the previous section, we derive the double soft theorems in $\mathcal{N}=4$ super-DBI-VA, NLSM, sGal. We also discuss some double limits in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA [28].

As shown in [44], in the simultaneous double soft limit, there are two types of solutions to the scattering equations – those non-degenerate ones, i.e. all σ's are distinct from each other, and a *unique degenerate* solution with the two σ's of the soft legs coincide. We find the same conclusion for the solutions of four-dimensional scattering equations (4.5).

The key observation [44] is that, when the contribution of the degenerate solution dominates over that of non-degenerate ones in the double soft limit, one can derive double soft theorems by evaluating the formula for the degenerate solution only. Here we will see that it is indeed the case for all superamplitudes in $\mathcal{N}=4$ super-DBI-VA involving the emission of a pair of soft photons, fermions or scalars.

Let us start with a $(n+2)$-point amplitude with even n in $\mathcal{N}=4$ super-DBI-VA theory,

$$\mathcal{M}_{n+2} = \int d\mu^{(4)}_{n+2}\,\det{}'X_{n+2}, \tag{5.12}$$

and here we write the measure $d\mu^{(4)}_{n+2}$ as,

$$\begin{aligned}
&\frac{d^{2(n+2)}\sigma}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\prod_{I=1}^{n/2}\delta^{2|4}\Big((\tilde\lambda_I|\eta_I)-\sum_{i=n/2+1}^{n}\frac{(\tilde\lambda_i|\eta_i)}{(I\,i)}-\frac{(\tilde\lambda_p|\eta_p)}{(I\,p)}\Big)\\
&\times\delta^{2|4}\Big((\tilde\lambda_q|\eta_q)-\sum_{i=n/2+1}^{n}\frac{(\tilde\lambda_i|\eta_i)}{(q\,i)}-\frac{(\tilde\lambda_p|\eta_p)}{(q\,p)}\Big)\\
&\times\prod_{i=n/2+1}^{n}\delta^2\Big(\lambda_i-\sum_{I=1}^{n/2}\frac{\lambda_I}{(i\,I)}-\frac{\lambda_q}{(i\,q)}\Big)\delta^2\Big(\lambda_p-\sum_{I=1}^{n/2}\frac{\lambda_I}{(p\,I)}-\frac{\lambda_q}{(p\,q)}\Big),
\end{aligned}\tag{5.13}$$

where $I = n+2, 1, ..., n/2$ and $i = n/2+1, ..., n, n+1$. For the sake of brevity, here and in the rest of this section we denote the indices $n+1$ and $n+2$ as p and q respectively.

To be concrete, we perform anti-holomorphic and holomorphic soft limits for the external legs p and q respectively, and introduce a small real parameter ϵ to control this simultaneous double soft limit:

$$\tilde{\lambda}_p \to \epsilon \tilde{\lambda}_p, \quad \lambda_q \to \epsilon \lambda_q, \tag{5.14}$$

while λ_p, η_p and $\tilde{\lambda}_q, \eta_q$ stay finite [151]. In this limit, we have $(a\, b) \sim \mathcal{O}(1)$ for non-degenerate solutions, while for the degenerate solution, $(p\, q) \sim \mathcal{O}(\epsilon)$.

Now we can study the scaling behavior of the formula in ϵ for both degenerate and non-degenerate solutions. In the double soft limit (5.14), the bosonic part of measure (5.13) behaves as $d\mu_{n+2}^{(0)} \sim \mathcal{O}(\epsilon)$ for the degenerate solution and $d\mu_{n+2}^{(0)} \sim \mathcal{O}(\epsilon^0)$ for non-degenerate solutions while $\det' X_{n+2} \sim \mathcal{O}(\epsilon^4)$ and $\det' X_{n+2} \sim \mathcal{O}(\epsilon^2)$ for the degenerate solution and non-degenerate ones respectively.

We also need to consider the scaling behavior from fermionic delta functions in the measure (5.13), which strongly depends on the $SU(4)$ flavors of the soft particles. Let first recall the on-shell superfield (5.1) and the following fermionic δ-function in the measure (5.13)

$$\delta^{0|4}\left(\eta_q - \sum_{i=\frac{n}{2}+1}^{n} \frac{\eta_i}{(q\, i)} - \frac{\eta_p}{(q\, p)}\right) \prod_{I=1}^{n/2} \delta^{0|4}\left(\eta_I - \sum_{i=\frac{n}{2}+1}^{n} \frac{\eta_i}{(I\, i)} - \frac{\eta_p}{(I\, p)}\right). \tag{5.15}$$

While it is obvious that for any pair of soft particles, it is $\mathcal{O}(1)$ for non-degenerate solutions in the limit (5.14), the case for the degenerate solution is more subtle. One needs to distinguish between two cases: (i) when the two soft particles form a $SU(4)$ flavor-singlet, i.e. (γ^+, γ^-) photon pair, $(\psi_A, \bar{\psi}^A)$ fermion pair, or (S_{AB}, S^{AB}) scalar pair[2], and (ii) when they do not form a singlet, e.g. $(\psi_A, \bar{\psi}^B)$ or (S_{AD}, S^{BD}).

[2] In the on-shell superfield with $\mathcal{N}$ supersymmetries, fundamental indices of the $SU(\mathcal{N})$ R-symmetry are raised and lowered using the Levi-Civita symbol. For example in $\mathcal{N} = 4$ super-DBI-VA, $S^{AB} = \frac{1}{2}\epsilon^{ABCD} S_{CD}$.

- For the first case, the leading-order contribution comes from picking out all η_p, η_q from the last fermionic delta function of (5.15), and the remainder becomes exactly fermionic delta functions for n-point formula. The last fermionic delta function evaluates to $1/(p\,q)^{2-2s}$ which behaves as $\mathcal{O}(\epsilon^{2s-2})$, where "$s$" denotes the spin of the soft pair.
- For the second case, we also have one η_p from other fermionic delta functions, and the factor becomes $1/(p\,q)^{1-2s}$.

When combining with the bosonic measure and integrand, for both cases, the contribution from degenerate solution always dominates. The second case is sub-leading compared to the first case, so we refer to the latter as the "leading-order" double-soft theorems and the former as the "sub-leading" ones. We first discuss the leading-order case and postpone the very interesting discussion of the subleading case to the end of this subsection.

It is convenient to introduce the change of variable for the degenerate solution [44]

$$\sigma_p \;=\; \rho - \epsilon\,\frac{\xi}{2}, \quad \sigma_q \;=\; \rho + \epsilon\,\frac{\xi}{2}, \tag{5.16}$$

with $\sigma_{qp} = \epsilon\,\xi \sim \mathcal{O}(\epsilon)$, and we have $d\sigma_p d\sigma_q \;=\; \epsilon\, d\rho\, d\xi$. In these variables, the integrand, $\det{}'X$, can be written as

$$\det{}'X_{n+2} \;=\; \epsilon^2\,\frac{s_{pq}^2 t_p^2 t_q^2}{\xi^2}\det{}'X_n + \mathcal{O}(\epsilon^4), \tag{5.17}$$

and we can write the complete measure involving a pair of soft particles of spin s in a unified form:

$$d\mu_{n+2}^{(4)} \;=\; \epsilon\left(-\frac{\epsilon\xi}{t_p t_q}\right)^{-2(1-s)} \frac{dt_p dt_q}{t_p^3 t_q^3}\, d\rho d\xi\, \delta^2\big(\bar{\mathcal{E}}_q^{\dot\alpha}\big)\delta^2\big(\mathcal{E}_p^{\alpha}\big)\, d\mu_n^{(4)} + \mathcal{O}(\epsilon^{2s})$$

with

$$\bar{\mathcal{E}}_q^{\dot\alpha} \equiv \tilde\lambda_q - \sum_{i=n/2+1}^{n} \frac{1}{(q\,i)}\,\tilde\lambda_i^{\dot\alpha} - \frac{t_p t_q}{\xi}\,\tilde\lambda_p^{\dot\alpha}, \tag{5.18}$$

$$\mathcal{E}_p^{\alpha} \equiv \lambda_p^{\alpha} - \sum_{I=1}^{n/2} \frac{1}{(p\,I)}\,\lambda_I^{\alpha} + \frac{t_p t_q}{\xi}\,\lambda_q^{\alpha}. \tag{5.19}$$

Our task is to perform the integral over t_p, t_q, ξ and ρ by using the four additional delta functions above. For this purpose it is convenient to rewrite these delta functions as

$$\begin{aligned}\delta^2(\bar{\mathcal{E}}_q^{\dot\alpha}) &= \frac{1}{t_p t_q [pq]}\delta\left(1 - \sum_{i=\frac{n}{2}+1}^{n} \frac{[p\,i]}{[pq]}\frac{t_q t_i}{\sigma_{qi}}\right)\delta\left(\sum_{i=\frac{n}{2}+1}^{n} \frac{[qi]}{[qp]}\frac{t_i}{t_p\sigma_{qi}} + \frac{1}{\xi}\right),\\ \delta^2(\mathcal{E}_p^{\alpha}) &= \frac{-1}{t_p t_q\,\langle pq\rangle}\delta\left(1 - \sum_{I=1}^{n/2} \frac{\langle qI\rangle}{\langle qp\rangle}\frac{t_p t_I}{\sigma_{pI}}\right)\delta\left(\sum_{I=1}^{n/2} \frac{\langle pI\rangle}{\langle pq\rangle}\frac{t_I}{t_q\sigma_{pI}} - \frac{1}{\xi}\right).\end{aligned} \tag{5.20}$$

It is clear now that from the RHS of (5.20), we can use the two delta functions without ξ to fix t_p, t_q:

$$t_p^{-1} = \sum_{I=1}^{n/2} \frac{\langle qI\rangle}{\langle qp\rangle}\frac{t_I}{\sigma_{pI}}, \qquad t_q^{-1} = \sum_{i=n/2+1}^{n} \frac{[pi]}{[pq]}\frac{t_i}{\sigma_{qi}}. \tag{5.21}$$

After integrating out t_p, t_q, the formula in the double soft limit (5.14) becomes

$$\begin{aligned}\mathcal{M}_{n+2}^{(s)} &= (-1)^{1-2s}\epsilon^{1+2s}\int d\mu_n^{(4)}\det{}' X_n\\ &\quad\times\int d\rho d\xi\,\frac{s_{pq}}{(t_p t_q)^{2s}\,\xi^{4-2s}}\delta(f_1)\delta(f_2) + \mathcal{O}(\epsilon^{2+2s}),\end{aligned} \tag{5.22}$$

where we used the superscript (s) for the spin of the soft pair. Here we also denote

$$\begin{aligned} f_1 &= \sum_{i=\frac{n}{2}+1}^{n} \frac{[qi]}{[qp]} \frac{t_i}{t_p} \frac{1}{\sigma_{qi}} + \frac{1}{\xi} = -\frac{1}{s_{pq}} \sum_{i=\frac{n}{2}+1}^{n} \sum_{I=1}^{n/2} \frac{[i|q|I\rangle\, t_I t_i}{\sigma_{pI}\sigma_{qi}} + \frac{1}{\xi}, \\ f_2 &= \sum_{I=1}^{n/2} \frac{\langle pI\rangle}{\langle pq\rangle} \frac{t_I}{t_q} \frac{1}{\sigma_{pI}} - \frac{1}{\xi} = -\frac{1}{s_{pq}} \sum_{i=\frac{n}{2}+1}^{n} \sum_{I=1}^{n/2} \frac{[i|p|I\rangle\, t_I t_i}{\sigma_{pI}\sigma_{qi}} - \frac{1}{\xi}, \end{aligned} \tag{5.23}$$

and in the second equality we have plugged in the solution for t_p, t_q given in (5.21).

Now the problem of integrating over ρ and ξ resembles that in deriving double soft theorems in arbitrary dimensions in [44], and we recall the transformation of the delta functions,

$$\delta(f_1)\delta(f_2) = -2\delta(f_1+f_2)\delta(f_1-f_2)\,. \tag{5.24}$$

The key point here is to note that $f_1 \pm f_2$ can be simplified to particularly nice form as a sum over $\{1,\ldots,n\}$. Let us make a partial fraction decomposition for $1/(\sigma_{pI}\sigma_{qi})$, then f_2+f_2 can be written as

$$\begin{aligned} f_1+f_2 &= -\frac{1}{s_{pq}} \sum_{i=n/2+1}^{n} \sum_{I=1}^{n/2} \left(\frac{1}{\rho-\sigma_i} - \frac{1}{\rho-\sigma_I}\right) \frac{[i|(p+q)\,|I\rangle}{(i\,I)} \\ &= -\frac{1}{s_{pq}} \left\{ \sum_{i=n/2+1}^{n} \frac{1}{\rho-\sigma_i} \sum_{I=1}^{n/2} \frac{[i|(p+q)\,|I\rangle}{(i\,I)} \right. \\ &\qquad \left. + \sum_{I=1}^{n/2} \frac{1}{\rho-\sigma_I} \sum_{i=n/2+1}^{n} \frac{[i|(p+q)\,|I\rangle}{(I\,i)} \right\}. \end{aligned} \tag{5.25}$$

By the scattering equations (4.5), the two inner sums simply give $[i|p+q|i\rangle$ and $[I|p+q|I\rangle$ respectively, and then we obtain

$$f_1+f_2 = \frac{1}{s_{pq}} \sum_{a=1}^{n} \frac{2k_a\cdot(p+q)}{\rho-\sigma_a}. \tag{5.26}$$

The same technique works for $f_1 - f_2$, and one obtains immediately the solution for ξ from $f_1 - f_2 = 0$ as follows

$$\begin{aligned}\xi^{-1} &= \frac{1}{2s_{pq}} \sum_{i=n/2+1}^{n} \sum_{I=1}^{n/2} \left(\frac{1}{\rho - \sigma_i} - \frac{1}{\rho - \sigma_I} \right) \frac{[i|(p-q)\,|I\rangle}{(I\,i)} \\ &= \frac{1}{s_{pq}} \sum_{a=1}^{n} \frac{k_a \cdot (p-q)}{\rho - \sigma_a}. \end{aligned} \tag{5.27}$$

Similarly, from eq. (5.21) and eq. (5.16) one can get a similar result for $t_p t_q$:

$$\begin{aligned} t_p^{-1} t_q^{-1} &= \frac{1}{s_{pq}} \sum_{i=n/2+1}^{n} \sum_{I=1}^{n/2} \left(\frac{1}{\rho - \sigma_i} - \frac{1}{\rho - \sigma_I} \right) \frac{[pi]\,\langle Iq\rangle}{(i\,I)} \\ &= \frac{1}{s_{pq}} \sum_{a=1}^{n} \frac{[pa]\,\langle aq\rangle}{\rho - \sigma_a}. \end{aligned} \tag{5.28}$$

Now we can package everything together. First we localize the ξ-integral by $\delta(f_1 - f_2)$, and regard the ρ-integral as a contour integral with contour $\mathcal{C}$ encircling the zeroes of $f_1 + f_2 = 0$,

$$\begin{aligned}\mathcal{M}_{n+2}^{(s)} &= (-1)^{1-2s} \epsilon^{1+2s} \int d\mu_n^{(4)} \det{}' X_n \\ &\quad \times \oint_{\mathcal{C}} \frac{d\rho}{2\pi i} \frac{s_{pq}\,(t_p t_q)^{-2s}\,\xi^{-2(1-s)}}{f_1 + f_2} + \mathcal{O}(\epsilon^{2+2s})\,. \end{aligned} \tag{5.29}$$

Plugging eqs. (5.26), (5.27), (5.28) into eq. (5.29) immediately gives

$$\begin{aligned}\mathcal{M}^{(s)}{}_{n+2} &= (-\epsilon)^{1+2s} \int d\mu_n^{(4)} \det{}' X_n \\ &\times \oint_{\mathcal{C}} \frac{d\rho}{2\pi i} \frac{\left(\sum_{a=1}^{n} \frac{[p|a|q\rangle}{\rho - \sigma_a}\right)^{2s} \left(\sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{\rho - \sigma_b}\right)^{2(1-s)}}{\sum_{c=1}^{n} \frac{k_c \cdot (p+q)}{\rho - \sigma_c}} + \mathcal{O}(\epsilon^{2+2s}). \end{aligned} \tag{5.30}$$

This integral do not receive the contribution from a simple pole at $\rho = \infty$ due to momentum conservation in the numerator. Thus we only need to consider simple poles at $\rho = \sigma_a$ with $a = 1, 2, \ldots, n$ and obtain by the residue theorem

$$\mathcal{M}_{n+2}^{(s)} = \epsilon^{1+2s} \sum_{a=1}^{n} \frac{\big(k_a \cdot (q-p)\big)^{2-2s} [p|a|q\rangle^{2s}}{2k_a \cdot (p+q)} \mathcal{M}_n^{(s)} + \mathcal{O}(\epsilon^{2+2s}). \quad (5.31)$$

It is highly non-trivial that the combinations appeared, f_1+f_2, f_1-f_2 and $t_p t_q$, all become a sum over $a = 1, \ldots, n$, which is what we need to derive the nice soft theorems (5.31). The key for this to happen is the use of scattering equations (4.5). Note that these theorems now directly hold for superamplitudes in four dimensions, i.e. hard particles can be any particles in supermultiplet (5.1).

The double soft photon limit ($s = 1$) and double soft scalar limit ($s = 0$) in the DBI theory are obtained using CHY representations in [44], while the double fermion limit for $s = \frac{1}{2}$ without flavors is conjectured by studying six-fermion amplitudes in Volkov-Akulov theory [155]. Here we have shown that these seemingly different double soft theorems can be unified for superamplitudes in $\mathcal{N}=4$ super DBI-VA and this unified form (5.31) certainly deserves further study.

Sub-leading theorems in $\mathcal{N} = 4$ super-DBI-VA

Now we turn to the case that the two soft particles are not in a flavor singlet of SU(4), and for simplicity, we consider $(\psi_A, \bar{\psi}^B)$ fermion-pair, and (S_{AD}, S^{BD}) scalar-pair.

For convenience, let us first rewrite the fermionic δ-function (5.15) here

$$\delta^{0|4}\Bigg(\eta_q - \sum_{i=\frac{n}{2}+1}^{n} \frac{\eta_i}{(q\, i)} - \frac{\eta_p}{(q\, p)}\Bigg) \prod_{I=1}^{n/2} \delta^{0|4}\Bigg(\eta_I - \sum_{i=\frac{n}{2}+1}^{n} \frac{\eta_i}{(I\, i)} - \frac{\eta_p}{(I\, p)}\Bigg),$$

and take a closer look. Unlike the single-flavor case, here we pick η_p^A from one of those δ-functions with η_I, and the remaining three η's, $(\eta_q^3)_B$ for $s = \frac{1}{2}$ or $\eta_p^D (\eta_q^2)_{BD}$ for $s = 0$, from the first δ-function. The operation of extracting η_p^A from those δ-functions amounts to taking derivative $\partial/\partial\eta_I$ with a factor $1/(I\, p)$ and a sum over I. Furthermore, an additional η from the last δ-function must come from the sum $\sum_i \eta_i/(q\, i)$. To be more precise, by projecting upon the relevant terms in the η_p and η_q, one finds the fermionic part of the measure contributing to the leading soft limits,

$$-\left(-\frac{\epsilon\xi}{t_p t_q}\right)^{2s-1}\sum_{i=\frac{n}{2}+1}^{n}\sum_{I=1}^{n/2}\frac{1}{(q\,i)(I\,p)}\eta_i^B\frac{\partial}{\partial\eta_I^A}\,\delta^{(2n)}(\mathcal{F}_n)+\mathcal{O}(\epsilon^{2s}),\qquad(5.32)$$

where we denote the product of fermionic δ-functions, and the $SU(4)$ generator on the leg a as:

$$\delta^{(2n)}(\mathcal{F}_n)\equiv\prod_{I=1}^{n/2}\delta^{0|4}\Big(\eta_I-\sum_{i=n/2+1}^{n}\frac{\eta_i}{(I\,i)}\Big),\qquad(R_a)^B_{\ A}\equiv\eta_a^B\frac{\partial}{\partial\eta_a^A}.\qquad(5.33)$$

Performing a partial fraction decomposition for 1/(q i)(I p), then (5.32) can be written as

$$-(-\epsilon)^{2s-1}\frac{(t_pt_q)^{2(1-s)}}{\xi^{1-2s}}\Bigg\{\sum_{I=1}^{n/2}\frac{1}{\rho-\sigma_I}\sum_{i=n/2+1}^{n}\frac{\eta_i^B\partial_{\eta_I^A}}{(i\,I)}$$

$$+\sum_{i=n/2+1}^{n}\frac{1}{\rho-\sigma_i}\sum_{I=1}^{n/2}\frac{\eta_i^B\partial_{\eta_I^A}}{(I\,i)}\Bigg\}\delta^{(2n)}(\mathcal{F}_n)+\mathcal{O}(\epsilon^{2s})\qquad(5.34)$$

Like the bosonic case, further simplification can be done for the two terms in the curly brackets by using the following relations of Grassmann variables:

$$\sum_{i=n/2+1}^{n}\frac{1}{(I\,i)}\eta_i^A=\eta_I^A,\qquad\sum_{I=1}^{n/2}\frac{1}{(i\,I)}\frac{\partial}{\partial\eta_I^A}=\frac{\partial}{\partial\eta_i^A}.\qquad(5.35)$$

The first set of equations are sometimes referred to "fermionic scattering equations". In the second set of equations, derivatives w.r.t. Grassmann variables act on the fermionic δ-functions $\delta^{(2n)}(\mathcal{F}_n)$ defined in (5.33), and these equations can be derived from the first set of equations. Finally, eq. (5.32) becomes

$$-(-\epsilon)^{2s-1}\frac{(t_pt_q)^{2(1-s)}}{\xi^{1-2s}}\sum_{a=1}^{n}\frac{(R_a)^B_{\ A}}{\rho-\sigma_a}\,\delta^{(2n)}(\mathcal{F}_n)+\mathcal{O}(\epsilon^{2s}).\qquad(5.36)$$

Note eq. (5.36) $\sim\mathcal{O}(\epsilon^{2s-1})$ as we claimed, which means that the double-soft behavior is sub-leading for non-singlet soft pair, compared to the singlet pair. However, recall behavior of $d\mu^{(0)}$ and $\det{}'X$, it is still the case that the degenerate solution is dominant at this order, see also table 5.2 at the end of the section. By eq. (5.36) and repeating the exact same derivation gives for $\mathcal{M}_{n+2}^{(s)}$

$$-(-\epsilon)^{2+2s} s_{pq} \int d\mu_n^{(0)} \det{}' X_n$$
$$\times \oint_{\mathcal{C}} \frac{d\rho}{2\pi i} \frac{\left(\sum_{a=1}^{n} \frac{[p|a|q\rangle}{\rho-\sigma_a}\right)^{2s} \left(\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{\rho-\sigma_b}\right)^{1-2s} \sum_{c=1}^{n} \frac{(R_c)^B{}_A}{\rho-\sigma_c} \delta^{(2n)}(\mathcal{F}_n)}{\sum_{d=1}^{n} \frac{k_d\cdot(p+q)}{\rho-\sigma_d}} + \mathcal{O}(\epsilon^{3+2s}).$$

Similarly, performing the ρ-integral by encountering simple poles at $\rho = \sigma_a$ yields

$$\mathcal{M}_{n+2}^{(s)} = -\epsilon^{2+2s} s_{pq} \sum_{a=1}^{n} \frac{(k_a \cdot (q-p))^{1-2s} \, [p|a|q\rangle^{2s}}{2k_a \cdot (p+q)} \eta_a^B \frac{\partial}{\partial \eta_a^A} \mathcal{M}_n^{(s)} + \mathcal{O}(\epsilon^{3+2s}) \tag{5.37}$$

for two soft fermions $(\psi_A, \bar{\psi}^B)$ emission ($s = \frac{1}{2}$) and two soft scalars (S_{AD}, S^{BD}) emission ($s = 0$) respectively. The result bears striking similarity with the double soft scalar theorem in $\mathcal{N} = 8$ SUGRA [151] (see [156–158] for recent works on double soft behavior in $\mathcal{N}=4$ SYM). In that case, the theorem directly probes the coset structure ($E_{7(7)}/SU(8)$) of the vacua, and we hope that our results here, which has a similar structure, can be useful for studying the coset structure of $\mathcal{N}=4$ super-DBI-VA theory.

More double-soft theorems

Having established all double-soft theorems in super-DBI-VA, we now briefly discuss double soft theorems for NLSM, sGal, as well as those in $\mathcal{N}=4$ SYM and $\mathcal{N} = 8$ SUGRA. For color-ordered amplitudes in SYM and NLSM, we will focus on the case that the soft particles are adjacent.

All we need are the behavior of the Parke-Taylor factor and that for $\det{}'\mathbb{H} \det{}'\overline{\mathbb{H}}$, in the double soft limit. In the double soft limit (5.14), for non-degenerate solutions, the Parke-Taylor factor has leading order behavior of $\mathcal{O}(1)$, while for the degenerate solution, it is straightforward to get

$$\begin{aligned}
&\frac{1}{(12)\cdots(np)(pq)(q1)} \\
&\qquad = \frac{1}{(12)\cdots(n1)} \times \frac{t_p^2 t_q^2}{\epsilon\,\xi}\left(\frac{1}{\rho-\sigma_n} - \frac{1}{\rho-\sigma_1}\right) + \mathcal{O}(\epsilon^0). \qquad (5.38)
\end{aligned}$$

Similarly, in the double limit, $\det'\mathbb{H}\det'\overline{\mathbb{H}} \sim \mathcal{O}(\epsilon^2)$ for non-degenerate solutions, while for degenerate solution we have

$$\begin{aligned}
&\det{}'\mathbb{H}_{k+1}\det{}'\overline{\mathbb{H}}_{n-k+1} \\
&\qquad = \epsilon^2\left(-\sum_{I=1}^{k}\mathbb{H}_{qI}\right)\det{}'\mathbb{H}_k\left(-\sum_{i=k+1}^{n}\mathbb{H}_{pi}\right)\det{}'\overline{\mathbb{H}}_{n-k} + \mathcal{O}(\epsilon^3) \\
&\qquad = -\epsilon^2\, t_p t_q \sum_{a=1}^{n}\frac{[p|a\,|q\rangle}{\rho-\sigma_a}\det{}'\mathbb{H}_k\,\det{}'\overline{\mathbb{H}}_{n-k} + \mathcal{O}(\epsilon^3) \\
&\qquad = -\epsilon^2\, s_{pq}\det{}'\mathbb{H}_k\,\det{}'\overline{\mathbb{H}}_{n-k} + \mathcal{O}(\epsilon^3), \qquad (5.39)
\end{aligned}$$

where the same trick as the case for $f_1 \pm f_2$ and $t_p t_q$ is nicely used again. Of course, it also holds for $k = n/2$ with n even, namely $H_{n+2} = -\epsilon^2 s_{pq} H_n + \mathcal{O}(\epsilon^3)$ in the same limit.

We summarize the soft scaling behavior in ϵ for all the (bosonic) building blocks in table 5.1.

Building Block	$\mathcal{O}$(d)	$\mathcal{O}$(nd)
$d\mu^{(0)}$	1	0
$\det'X$	2	4
Parke-Taylor factor	-1	0
$\det'\mathbb{H}\det'\overline{\mathbb{H}}$	2	2

Table 5.1: Leading scaling behavior in soft parameter ϵ of the building blocks in the double soft limit (5.14). Here "d" and "nd" stand for the degenerate and non-degenerate solutions respectively.

For $U(N)$ NLSM and the special Galileon theory, let us recall the formula for their amplitudes:

$$\mathcal{M}_{n+2}^{\text{NLSM}}(1,\ldots,n,p,q) = \int d\mu_{n+2}^{(0)} \frac{1}{(12)\cdots(np)(pq)(q1)} \frac{\det' X_{n+2}}{H_{n+2}}, \quad (5.40)$$

$$\mathcal{M}_{n+2}^{\text{sGal}} = \int d\mu_{n+2}^{(0)} \frac{\left(\det' X_{n+2}\right)^2}{H_{n+2}}. \quad (5.41)$$

By power counting of the soft parameter ϵ for building blocks, again we find the soft scalar limits at leading order only receive the contribution from the degenerate solution. The same derivation as for super-DBI-VA gives the leading double soft scalar theorems:

$$\mathcal{M}_{n+2}(1,\ldots,n,p,q) = \epsilon^m \mathcal{S}\,\mathcal{M}_n(1,\ldots,n) + \mathcal{O}(\epsilon^{m+1}), \quad (5.42)$$

where $m = 0$ for NLSM and $m = 3$ for sGal, and soft factors are given respectively by

$$\mathcal{S}^{\text{(NLSM)}} = \frac{k_n\cdot(p-q)}{2k_n\cdot(p+q)} + \frac{k_1\cdot(q-p)}{2k_1\cdot(q+p)}, \quad (5.43)$$

$$\mathcal{S}^{\text{(sGal)}} = s_{pq}\sum_{a=1}^{n} \frac{\left(k_a\cdot(p-q)\right)^2}{2k_a\cdot(p+q)}\,, \quad (5.44)$$

which coincide with the leading-order results of [44]. Note that single and double scalar emissions in NLSM were also investigated in [106, 159, 160].

Finally, we make a classification of double soft theorems for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA. Unlike the case for the other three theories, the degenerate solution does not always dominate for leading double soft limit in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA, as listed in table 5.2. For $\mathcal{N}=4$ SYM, the degenerate solution still dominates for the following three cases, giving double-soft theorems:

$$\mathcal{A}_{n+2}\big(\ldots,\Gamma_A(p),\bar{\Gamma}^A(q)\big) \quad (5.45)$$
$$= \frac{1}{\epsilon}\frac{1}{s_{pq}}\left(\frac{[p|k_n|q\rangle}{2k_n\cdot(p+q)} - \frac{[p|k_1|q\rangle}{2k_1\cdot(p+q)}\right)\mathcal{A}_n + \mathcal{O}(\epsilon^0),$$

$$\begin{aligned}&\mathcal{A}_{n+2}\big(\ldots,\phi_{AB}(p),\phi^{AB}(q)\big) \\ &\quad= \frac{1}{\epsilon^2}\frac{1}{s_{pq}}\left(\frac{k_n\cdot(p-q)}{2k_n\cdot(p+q)}-\frac{k_1\cdot(p-q)}{2k_1\cdot(p+q)}\right)\mathcal{A}_n+\mathcal{O}(\epsilon^{-1}),\end{aligned} \tag{5.46}$$

$$\begin{aligned}&\mathcal{A}_{n+2}\big(\ldots,\phi_{AD}(p),\phi^{BD}(q)\big) \\ &\quad= \frac{1}{\epsilon}\left(\frac{(R_n)^B_A}{2k_n\cdot(p+q)}-\frac{(R_1)^B_A}{2k_1\cdot(p+q)}\right)\mathcal{A}_n+\mathcal{O}(\epsilon^{0}).\end{aligned} \tag{5.47}$$

Similarly for $\mathcal{N}=8$ SUGRA, we find that for the following cases of double-soft particles in the supermultiplet (4.30), the degenerate solution dominates and we have the corresponding double-soft theorems

$$\begin{aligned}&\mathcal{M}_{n+2}\big(\ldots,v_{AB}(p),\bar{v}^{AB}(q)\big) \\ &\quad= \frac{\epsilon}{p\cdot q}\sum_{a=1}^{n}\frac{[p|a|q\rangle^2}{2k_a\cdot(p+q)}\mathcal{M}_n+\mathcal{O}(\epsilon^2),\end{aligned} \tag{5.48}$$

$$\begin{aligned}&\mathcal{M}_{n+2}\big(\ldots,\chi_{ABC}(p),\bar{\chi}^{ABC}(q)\big) \\ &\quad= -\frac{1}{s_{pq}}\sum_{a=1}^{n}\frac{k_a\cdot(p-q)\,[p|a|q\rangle}{2k_a\cdot(p+q)}\mathcal{M}_n+\mathcal{O}(\epsilon),\end{aligned} \tag{5.49}$$

$$\begin{aligned}&\mathcal{M}_{n+2}\big(\ldots,\chi_{ADE}(p),\bar{\chi}^{BDE}(q)\big) \\ &\quad= \epsilon\sum_{a=1}^{n}\frac{[p|a|q\rangle}{2k_a\cdot(p+q)}(R_a)^B_A\,\mathcal{M}_n+\mathcal{O}(\epsilon^2),\end{aligned} \tag{5.50}$$

$$\begin{aligned}&\mathcal{M}_{n+2}\big(\ldots,\phi_{ABCD}(p),\phi^{ABCD}(q)\big) \\ &\quad= \frac{1}{\epsilon}\frac{1}{s_{pq}}\sum_{a=1}^{n}\frac{\big(k_a\cdot(p-q)\big)^2}{2k_a\cdot(p+q)}\mathcal{M}_n+\mathcal{O}(\epsilon^0),\end{aligned} \tag{5.51}$$

$$\begin{aligned}&\mathcal{M}_{n+2}\big(\ldots,\phi_{ADEF}(p),\phi^{BDEF}(q)\big) \\ &\quad= -\sum_{a=1}^{n}\frac{k_a\cdot(p-q)}{2k_a\cdot(p+q)}(R_a)^B_A\,\mathcal{M}_n+\mathcal{O}(\epsilon).\end{aligned} \tag{5.52}$$

Now we have obtained, from formulas with the four-dimensional scattering equations, all these universal double-soft theorems, among which some are new, and others are known previously. The most famous one is the double soft-scalar theorem (5.52) in $\mathcal{N}=8$ SUGRA [151], and more recently, double soft graviphotino (spin-1/2) theorems in supergravity were studied in four dimensions as well as three dimensions in [155, 161]. The double soft graviton emission was also studied in arbitrary dimensions using the CHY formula in [162]. In $\mathcal{N}=4$ SYM, double scalar theorems (5.47) were obtained using BCFW recursions in [156, 158], and from string theory in [157]; double gluino/scalar theorems, (5.45) and (5.46), were given in [163] from MHV diagrams.

5.3 Discussions

Let us conclude this chapter with some discussions. While there has been considerable progress for loop-level CHY formulas in general dimensions [31–34, 37, 164, 165], it would be very interesting to do so for supersymmetric theories in four dimensions (see [166] for a conjecture for $\mathcal{N}=8$ SUGRA). It would be interesting to see what is special about these effective field theories in four dimensions.

Just as double-scalar theorems in $\mathcal{N}=8$ SUGRA probing the coset structure of $E_{7(7)}$ symmetries, the double-fermion theorems in super-DBI-VA may reveal the structures of non-linearly realized supersymmetries of the theory. Indeed, significant progress has been made in the effect of non-linear (super)symmetries on the S-matrix [167, 168] based on our results. It would also be very fascinating to study sub-leading theorems similar to those in [44], which involve bosonic derivatives (rather than fermionic ones in this work). Perhaps by combining these two types of sub-leading theorems, one can associate them to possible hidden symmetries and structures.

theory	soft particle pair	$\mathcal{O}$(d)	$\mathcal{O}$(nd)	"d" dominant
$\mathcal{N}=4$ DBI-VA	$(\gamma^+,\, \gamma^-)$	3	4	✓
	$(\psi_A,\, \bar\psi^A)$	2	4	✓
	$(\psi_A,\, \bar\psi^B)$	3	4	✓
	$(\phi_{AB},\, \phi^{AB})$	1	4	✓
	$(\phi_{AD},\, \phi^{BD})$	2	4	✓
NLSM	$(\phi,\, \phi)$	0	2	✓
sGal	$(\phi,\, \phi)$	3	6	✓
$\mathcal{N}=4$ SYM	$(g^+,\, g^-)$	0	0	
	$(\psi_A,\, \bar\psi^A)$	-1	0	✓
	$(\psi_A,\, \bar\psi^B)$	0	0	
	$(\phi_{AB},\, \phi^{AB})$	-2	0	✓
	$(\phi_{AD},\, \phi^{BD})$	-1	0	✓
$\mathcal{N}=8$ SUGRA	$(h^+,\, h^-)$	3	2	
	$(\psi_A,\, \bar\psi^A)$	2	2	
	$(\psi_A,\, \bar\psi^B)$	3	2	
	$(v_{AB},\, \bar v^{AB})$	1	2	✓
	$(v_{AD},\, \bar v^{BD})$	2	2	
	$(\chi_{ABC},\, \bar\chi^{ABC})$	0	2	✓
	$(\chi_{ADE},\, \bar\chi^{BDE})$	1	2	✓
	$(\phi_{ABCD},\, \phi^{ABCD})$	-1	2	✓
	$(\phi_{ADEF},\, \phi^{BDEF})$	0	2	✓

Table 5.2: Leading scaling in ϵ of the formulas of scattering amplitudes in the double limit (5.14). For soft particle pairs with flavors indices, one demands $A \neq B$ which corresponds to two soft particles do not form a $SU(\mathcal{N})$ flavor singlet. Here the tick ✓ denotes that the degenerate solution is dominant at leading order, and in these cases, we give the double-soft theorems.

6 From the scattering equations to form factors

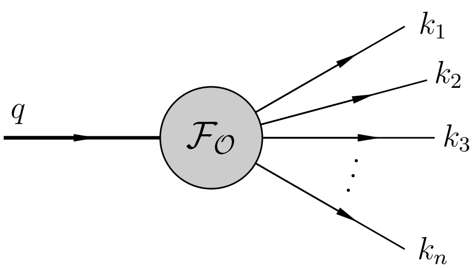
The study of the scattering equation formalism for on-shell scattering amplitudes has received considerable attention during the past decade. Based on the scattering equations, new representations of tree-level amplitudes have been proposed for a large number of theories in arbitrary, four, and six dimensions, as shown in previous chapters.

It is interesting and important to extend the modern methods for on-shell amplitudes to off-shell quantities. Form factors provide a bridge between on-shell amplitudes and purely off-shell correlation functions. Thus they are perfect for testing the applicability of on-shell techniques to off-shell generalizations. Recent years have witnessed considerable advances in the study of form factors in $\mathcal{N}=4$ SYM using on-shell methods and integrability, both at weak coupling [169–202] and strong coupling [203, 204] (see [205, 206] for a review). In particular, already at tree-level, form factors inherited remarkable structures from amplitudes, such as recursion relations, Grassmannian and polytope pictures (see [207] for a review); all these ideas are intimately related to twistor-inspired prescription for amplitudes. Given the success of twistor-string and scattering-equation-based construction for amplitudes, it is natural to study their applications to form factors especially in $\mathcal{N}=4$ SYM, which is the subject of this chapter.

This chapter is based on the paper [208] and is outlined as follows. Section 6.1 provides a short introduction to form factors under consideration in this work, including super form factors of the chiral part of the stress-tensor multiplet operator and bosonic form factor with scalar operators $\mathcal{O}_m \equiv \mathrm{Tr}\,[(\phi_{12})^m]$ in $\mathcal{N}=4$ SYM. We then present new representations for these form factors with the help of the four-dimensional scattering equations in Section 6.2. We validate our results by evaluating the new formulas for some special examples in Section 6.3.

6.1 Introduction

A n-point form factor is given by the overlap of a composite operator with off-shell momentum q, and n on-shell states with on-shell momenta $k_1, k_2, \ldots, k_n$, as illustrated below.



To be exact, a form factor with operator $\mathcal{O}$ is defined as

$$\begin{aligned} \mathcal{F}_{\mathcal{O}}(1, \ldots, n) &= \int \frac{d^4x}{(2\pi)^4} e^{-iq\cdot x} \langle 1, \ldots, n|\mathcal{O}(x)|0\rangle \\ &= \delta^4\big(P\big) \langle 1, \ldots, n|\mathcal{O}(0)|0\rangle \,, \end{aligned} \tag{6.1}$$

where $P = \sum_{i=1}^{n} k_i - q$ and the number of fields in $\mathcal{O}$ cannot exceed the number of external legs, n.

We consider bosonic form factors with scalar operators of $\mathcal{O}_m \equiv \mathrm{Tr}\,[(\phi_{12})^m]$, where ϕ_{12} is scalar in (4.25). For $m = 2$, $\mathcal{O}_2 = \mathrm{Tr}\,\big[(\phi_{12})^2\big]$ is the bottom component of the chiral part of the stress-tensor multiplet operator $\mathcal{T}_2$. For general m, $\mathcal{O}_m$ is the bottom component of half-BPS operators dual to Kaluza-Klein modes in supergravity [180]. Since the $\mathcal{O}_m$ is composed of m scalar fields, thus there must be m external states to be m external states of scalars to contract with $\mathcal{O}_m$. For simplicity, we consider the form factor with the operator

$\mathcal{O}_m$ and m scalars and $n-m$ gluons with arbitrary helicities:

$$\mathcal{F}_{\mathcal{O}_m,n} \equiv \delta^4(P)\ \langle g(k_1)\cdots\phi_{12}(k_{i_1})\cdots\phi_{12}(k_{i_m})\cdots g(k_n)\,|\,\mathcal{O}_m(0)\,|\,0\rangle\,. \tag{6.2}$$

In on-shell superspace, the super form factor for the operator $\mathcal{T}_2$ is defined as

$$\mathscr{F}_{\mathcal{T}_2,n} \;:=\; \langle \Phi_1\cdots\Phi_n\,|\,\mathcal{T}(0)\,|\,0\rangle\,, \tag{6.3}$$

where external legs $\Phi_i = \Phi_i(p_i,\eta_i)$ are $\mathcal{N}=4$ on-shell superfields (4.25). $\mathcal{T}_2 = \mathcal{T}(x,\theta^+,u)$ is the chiral part of the stress-tensor multiplet operator which has the form in harmonic superspace[1]

$$\begin{aligned}
&\mathcal{T}\big(x,\theta^+,\bar\theta_-=0,u\big) \qquad\qquad (6.4)\\
&= \mathrm{Tr}\left(\phi^{++}\phi^{++}\right) + i\,2\sqrt{2}\,\theta^{+a}_\alpha\,\mathrm{Tr}\left(\psi^{+\alpha}_a\phi^{++}\right)\\
&\quad + \theta^{+a}_\alpha\epsilon_{ab}\theta^{+b}_\beta\,\mathrm{Tr}\left(\psi^{+c(\alpha}\psi^{+\beta)}_c - i\sqrt{2}F^{\alpha\beta}\phi^{++}\right)\\
&\quad - \theta^{+a}_\alpha\epsilon^{\alpha\beta}\theta^{+b}_\beta\,\mathrm{Tr}\left(\psi^{+\gamma}_{(a}\psi^{+}_{b)\gamma} - g\sqrt{2}\big[\phi^{+C}_{(a},\bar\phi_{C\,+b)}\big]\phi^{++}\right)\\
&\quad - \frac{4}{3}(\theta^+)^{3\,a}_{\ \alpha}\,\mathrm{Tr}\left(F^\alpha_\beta\psi^{+\beta}_a + ig\big[\phi^{+B}_a,\bar\phi_{BC}\big]\psi^{C\alpha}\right) + \frac{1}{3}(\theta^+)^4\mathcal{L}(x).
\end{aligned}$$

Its $(\theta^+)^0$ component is just the scalar operator, while the $(\theta^+)^4$ component is the chiral form of the $\mathcal{N}=4$ on-shell Lagrangian:

$$\begin{aligned}
\mathcal{L} \;=\; \mathrm{Tr}\bigg(&-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{2}\,g\,\psi^{\alpha A}\big[\phi_{AB},\psi^B_\alpha\big]\\
&-\frac{1}{8}g^2\big[\phi^{AB},\phi^{CD}\big]\big[\phi_{AB},\phi_{CD}\big]\bigg).
\end{aligned} \tag{6.5}$$

Note that only the chiral half of the $\mathcal{N}=4$ multiplet, i.e. $F^{\alpha\beta}$, ψ^A_α and ϕ_{AB}, is needed for $\mathcal{T}_2$.

[1]In harmonic superspace: $\theta^{+a}_\alpha = \theta^A_\alpha u^{+a}_A$, $\theta^{-a'}_\alpha = \theta^A_\alpha u^{-a'}_A$ with $a,a'=1,2$, where $(u^{+a}_A, u^{-a'}_A)$ is the normalized harmonic matrix of SU(4). It is useful to work with harmonic variables for the super form factor of chiral operator. For more details about $\mathcal{N}=4$ harmonic superspace, c.f. [209, 210]. (see also e.g. [171, 180, 205]).

6.2 New representations for form factors

Evidently, the scattering equations shown in Chapter 3 and Chapter 4 are not applicable for form factors since off-shell momenta are involved. Therefore, as a first step towards introducing contour integral representation for form factors, we have to find new scattering equations that contain off-shell momenta.

The scattering equations for chiral form factors

A natural idea is to modify the existing scattering equations to incorporate the off-shell momentum. We first decompose q into two null vectors k_x and k_y, i.e.,

$$q = -(k_x + k_y), \qquad q^2 = 2k_x \cdot k_y. \tag{6.6}$$

There is a minus sign on the right-hand side of the first equality because we take a convention of considering all momenta outgoing. A physical interpretation may be that the operator with q is effectively described as two auxiliary on-shell legs with momenta k_x, k_y. It is thus natural to assign two additional punctures σ_x^α and σ_y^α for them. For general chiral form factors, a modification of the scattering equations in (4.5) was given in [133]

$$\begin{aligned}
&\bar{\mathcal{E}}_x = \tilde{\lambda}_x - \sum_{i\in\mathfrak{P}} \frac{\tilde{\lambda}_i}{(x\,i)} = 0, \quad \bar{\mathcal{E}}_y = \tilde{\lambda}_y - \sum_{i\in\mathfrak{P}} \frac{\tilde{\lambda}_i}{(y\,i)} = 0,\\
&\bar{\mathcal{E}}_I = \tilde{\lambda}_I - \sum_{i\in\mathfrak{P}} \frac{\tilde{\lambda}_i}{(I\,i)} = 0, \quad \mathcal{E}_i = \lambda_i - \sum_{I\in\mathfrak{N}} \frac{\lambda_I}{(i\,I)} - \frac{\lambda_x}{(i\,x)} - \frac{\lambda_y}{(i\,y)} = 0,
\end{aligned} \tag{6.7}$$

where $i \in \mathfrak{P}$, $I \in \mathfrak{N}$ and $|\mathfrak{N}| = k \in \{0, \ldots, n-2\}$. We refer to these equations as the *chiral off-shell scattering equations* of sector k. The anti-chiral case is obtained by parity. The equations in (6.7) have been used to construct the amplitude with one massive Higgs boson and any number of gluons, which is equivalent to the form factor with operator $H\,\mathrm{Tr}(F^2)$. In the following, we show they also work for $\mathcal{O}_m$ and $\mathcal{T}_2$.

Form factor with $\mathcal{O}_2$

Let us begin with the form factor with the scalar operator $\mathcal{O}_2 = \mathrm{Tr}\left[(\phi_{12})^2\right]$:

$$\mathcal{F}_{\mathcal{O}_m,n} \equiv \delta^4(P)\ \langle g(k_1)\cdots\phi_{12}(k_i)\cdots\phi_{12}(k_j)\cdots g(k_n)\,|\,\mathcal{O}_m(0)\,|\,0\rangle\,. \quad (6.8)$$

Our proposal for this form factor takes a very similar form with gluon amplitudes:

$$\mathcal{F}^{(\mathrm{N}^k\mathrm{MHV})}_{\mathcal{O}_2,n} = \delta^4(P)\int d\mu^{(0)}_{n,k}\,\frac{\mathcal{I}_2(i,j;x,y)}{(1\,2)\cdots(n\,1)}, \quad (6.9)$$

where the measure in k sector (note $|\mathfrak{N}| = k$ and $|\mathfrak{P}| = n-k$) is defined as

$$d\mu^{(0)}_{n,k} = \left(\prod_{i=1}^{n} d^2\sigma_i\right)\prod_{I\in\mathfrak{N}}\delta^2(\bar{\mathcal{E}}^{\dot\alpha}_I)\prod_{i\in\mathfrak{P}}\delta^2(\mathcal{E}^{\alpha}_i). \quad (6.10)$$

It is beautiful that the punctures x, y do not enter the Parke-Taylor factor. In addition, the integrand contains an additional factor that depends only on x, y (operator $\mathcal{O}_2$) and i, j (external scalars)

$$\mathcal{I}_2(i,j;x,y) = -\frac{\langle xy\rangle^2\ (i\,j)^2}{(i\,x)^2(i\,y)^2(j\,x)^2(j\,y)^2}. \quad (6.11)$$

A remarkable property is that the function $\mathcal{I}_2(i,j;x,y)$ is independent of information of external gluon legs such that it has the exact same form for all helicity sectors. In (6.9), one has fixed the four variables σ^α_x, σ^α_y and pull out their four delta functions as $\delta^4(P)$ when removing $\mathrm{GL}(2,\mathbb{C})$ redundancy.

Now we may generalize our formula (6.9) along two directions – one is to supersymmetrize it to for the super form factor (6.3), and the other is to extend it to bosonic form factors (6.2).

Super form factor with $\mathcal{T}_2$

Let us address the super form factor first. In addition to the bosonic part, (6.10), $\mathcal{N}=4$ SUSY also requires fermionic delta functions which can be viewed as the superpartner of the scattering equations. Our notation is that γ^α_{+a} stands for the Grassmann variable conjugate to θ^{+a}_α for $\mathcal{T}_2$ (there is no γ^- because we only consider the chiral part of the stress tensor), and we introduce

$$\eta_{\pm a,i} := \bar{u}^{A}_{\pm a}\eta_{A,i} \tag{6.12}$$

for each on-shell external leg i. Thus the supermomentum Q^{α}_{A} can be written as:

$$Q^{\alpha}_{+a} = \gamma^{\alpha}_{+a} - \sum_{i=1}^{n} \lambda^{\alpha}_{i}\eta_{+a,i}\,, \qquad Q^{\alpha}_{-a} = \sum_{i=1}^{n} \lambda^{\alpha}_{i}\eta_{-a,i}\,. \tag{6.13}$$

Note that we need to split the indices of γ_{+} into two parts by projecting it along two directions,

$$\eta_{+a,x} \equiv \frac{\lambda_{y,\alpha}\gamma^{\alpha}_{+a}}{\langle y\, x\rangle}, \qquad \eta_{+a,y} \equiv \frac{\lambda_{x,\alpha}\gamma^{\alpha}_{+a}}{\langle x\, y\rangle}, \tag{6.14}$$

which imply that $\gamma_{+} = \lambda_x\eta_{+,x} + \lambda_y\eta_{+,y}$. The fermionic delta functions for η's then take the same form as the delta functions for $\tilde{\lambda}$. By combining the bosonic and fermionic part, we obtain the supersymmetric measure in the k-sector:

$$\begin{aligned} d\mu^{(4)}_{n,k} := d\mu^{(0)}_{n,k} \prod_{I=1}^{k} \delta^{0|2}\left(\eta_{+,I} - \sum_{i\in\mathfrak{P}} \frac{\eta_{+,i}}{(I\, i)}\right)\delta^{0|2}\left(\eta_{-,I} - \sum_{i\in\mathfrak{P}} \frac{\eta_{-,i}}{(I\, i)}\right) \\ \times\, \delta^{0|2}\left(\frac{\lambda_{y,\alpha}\gamma^{\alpha}_{+}}{\langle y\, x\rangle} - \sum_{i\in\mathfrak{P}} \frac{\eta_{+,i}}{(x\, i)}\right)\delta^{0|2}\left(\frac{\lambda_{x,\alpha}\gamma^{\alpha}_{+}}{\langle x\, y\rangle} - \sum_{i\in\mathfrak{P}} \frac{\eta_{+,i}}{(y\, i)}\right) \\ \times\, \delta^{0|2}\left(\sum_{i\in\mathfrak{P}} \frac{\eta_{-,i}}{(x\, i)}\right)\delta^{0|2}\left(\sum_{i\in\mathfrak{P}} \frac{\eta_{-,i}}{(y\, i)}\right). \end{aligned} \tag{6.15}$$

To write down the complete formula, let us take a closer look at formula (6.9) again. An interesting observation is that the function $\mathcal{I}_2(i,j;x,y)$ (6.11) can be understood as coming from the η-projection when we evaluate the fermionic delta functions of the supersymmetric formula. To conclude, all we need for the super form factor (6.3) is simply the Parke-Taylor factor

$$\mathscr{F}^{(\mathrm{N}^k\mathrm{MHV})}_{\mathcal{T}_2,n} = \delta^4(P)\int d\mu^{(4)}_{n,k}\,\frac{\langle x\, y\rangle^4}{(1\,2)\cdots(n\,1)}. \tag{6.16}$$

As we have mentioned, our new formula is very similar to that for superamplitude in $\mathcal{N}=4$ SYM – the only modification is to include two on-shell legs x, y in a supersymmetric way, to represent the operator $\mathcal{T}_2$.

As discussed in Chapter 4, it is straightforward to translate the formula to a manifestly permutation-invariant form, similar to the Witten-RSV formula for SYM amplitudes [8, 16]. By applying the same procedure to (6.16), we obtain an equivalent formula for the super form factor with $\mathcal{T}_2$

$$\mathscr{F}_{\mathcal{T}_2,n}^{(\mathrm{N}^k\mathrm{MHV})} = \int d\Omega_{n,k}^{(4)} \frac{\langle x\, y\rangle^2}{(1\,2)\cdots(n\,1)} \tag{6.17}$$

with the supersymmetric measure $d\Omega_{n,k}^{(4)}$ similar to the Witten-RSV form

$$\begin{aligned} &\frac{d^{2n+4}\sigma}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \int d^{2k+4}\rho \prod_{i=1,\ldots,n,x,y} \delta^2\left(t_i \sum_{m=0}^{k+1} \rho_m \sigma_i^m - \lambda_i\right) \\ &\times \prod_{m=0}^{k+1} \delta^{2|2}\left(\left(t_x\sigma_x^m\tilde{\lambda}_x + t_y\sigma_y^m\tilde{\lambda}_y|0\right) + \sum_{i=1}^{n} t_i\sigma_i^m\left(\tilde{\lambda}_i|\eta_{-,i}\right)\right) \\ &\times \prod_{m=0}^{k+1} \delta^{2|2}\left(t_x\sigma_x^m\left(\tilde{\lambda}_x|\eta_{+,x}\right) + t_y\sigma_y^m\left(\tilde{\lambda}_y|\eta_{+,y}\right) + \sum_{i=1}^{n} t_i\sigma_i^m\left(\tilde{\lambda}_i|\eta_{+,i}\right)\right) \end{aligned} \tag{6.18}$$

where $d^{2n+4}\sigma := d^2\sigma_x d^2\sigma_y \prod_{i=1}^n d^2\sigma_i$.

Form factors with $\mathcal{O}_m$

Now we consider the bosonic form factor with the operator $\mathcal{O}_m$ with $m > 2$. A prior it is not apparent what to do for such operators with $m > 2$ fields: we could either represent the operator using two on-shell legs as in (6.7) or using m of them. We show that there is very strong evidence for the universality of scattering equations for form factors with $\mathcal{O}_m$ below.

The key observation is that no matter how many fields in the operator, we only need two additional legs/punctures x, y, and $d\mu_{n,k}$ (including delta functions imposing (6.7)) is exactly the same as in the $m = 2$ case.

For $\mathcal{O}_m$, all we need is a simple modification of Parke-Taylor factor which takes into account the positions of the m on-shell scalars. This supports our general conjecture that all the information of the operator and external states is encoded in the integrand of the formula. In other words, what we find is that similar to $m = 2$ case, now we have a function $\mathcal{I}_m$ that incorporates the information of $\mathcal{O}_m$ and m on-shell scalars. Our proposal is the following formula

$$\mathcal{F}^{(\mathrm{N}^k\mathrm{MHV})}_{\mathcal{O}_m,n} = \delta^4(P) \int d\mu^{(0)}_{n,k} \frac{\mathcal{I}_m(i_1,\ldots,i_m;x,y)}{(1\,2)\cdots(n\,1)} \tag{6.19}$$

with the measure $d\mu^{(0)}_{n,k}$ is exactly same with the one for $\mathcal{O}_2$ (6.10), and

$$\mathcal{I}_m(i_1,\ldots,i_m;x,y) = \langle x\,y\rangle^m \frac{(i_1\,i_2)\cdots(i_m\,i_1)}{\prod_{\alpha=1}^m (i_\alpha\,x)^2(i_\alpha\,y)^2}, \tag{6.20}$$

which goes back to (6.11) when $m = 2$. The function $\mathcal{I}_m$ takes a compact form and incorporates the information of $\mathcal{O}_m$ and m external scalars, while being independent of information of external gluon legs. Unlike $\mathcal{O}_2$, when $m > 2$ it is still not clear how to supersymmetrize (6.19) to obtain new formulas for super form factors with $\mathcal{T}_m$ which is a generalization of $\mathcal{T}_2$ and has $\mathcal{O}_m$ as its bottom component.

6.3 Examples

As consistency checks for both formulas (6.16) and (6.19), we evaluate them and compare with known results for some form factors. These include MHV and maximally-non-MHV super form factors for all multiplicities, as well as some more complicated bosonic form factors.

MHV

First we consider the MHV sector which corresponds to $k = 0$ in our notation. The MHV scattering equations have only a unique solution after fixing GL(2)-redundancy, as shown in Chapter 4. It is convenient to fix the locations of the two "additional" punctures σ_x, σ_y in a similar way to (4.9), then the unique solution for the MHV equations is given by:

$$(i\,x) = \frac{\langle x\,y\rangle}{\langle i\,y\rangle}, \quad (i\,y) = \frac{\langle x\,y\rangle}{\langle x\,i\rangle}, \quad (i\,j) = \frac{\langle x\,y\rangle^3 \langle i\,j\rangle}{\langle x\,i\rangle\langle y\,i\rangle\langle x\,j\rangle\langle y\,j\rangle}. \tag{6.21}$$

Here the two-braket $(i\,j)$ are natural variables. Note that the integral measure can be written nicely in terms of the $2n$ 2-brackets (ix), (iy)

$$\prod_{i=1}^{n} d^2\sigma_i \;=\; \prod_{i=1}^{n} d(ix)\, d(iy), \tag{6.22}$$

and all the delta functions can be rewritten in terms of them,

$$\delta^2\left(\lambda_i - \frac{\lambda_x}{(i\,x)} - \frac{\lambda_y}{(i\,y)}\right) = \frac{\langle xy\rangle^3}{\langle ix\rangle^2 \langle iy\rangle^2}\, \delta\left((ix) - \frac{\langle i\,y\rangle}{\langle x\,y\rangle}\right) \delta\left((iy) - \frac{\langle x\,i\rangle}{\langle x\,y\rangle}\right). \tag{6.23}$$

Plugging the solution (6.21) and the Jacobian for measure and delta-functions, (6.22), (6.23), into (6.19) and (6.16) respectively, we obtain immediately the correct result for the bosonic form factor

$$\mathcal{F}^{\mathrm{MHV}}_{\mathcal{O}_m,n} \;=\; \delta^4(P)\frac{\langle i_1\, i_2\rangle\, \langle i_2\, i_3\rangle \cdots \langle i_m\, i_1\rangle}{\langle 1\,2\rangle\, \langle 2\,3\rangle \cdots \langle n 1\rangle}, \tag{6.24}$$

and similarly for the super form factor

$$\mathscr{F}^{\mathrm{MHV}}_{\mathcal{T}_2,n} \;=\; \frac{\delta^4(P)\, \delta^4\left(\gamma_+ \;-\; \sum_{i=1}^{n} \lambda_i \eta_{+,i}\right) \delta^4\left(\sum_{i=1}^{n} \lambda_i \eta_{-,i}\right)}{\langle 1\,2\rangle\, \langle 2\,3\rangle \cdots \langle n 1\rangle}. \tag{6.25}$$

Maximally non-MHV

Similarly, it is also easy to get the $\overline{\mathrm{MHV}}$ (usually called maximally non-MHV) form factor with $\mathcal{T}_2$ by simply flipping the helicities of spinors for the MHV formula, i.e.,

$$\mathscr{F}^{\overline{\mathrm{MHV}}}_{\mathcal{T}_2,n} \;=\; \int d\bar{\mu}_n\, \frac{\langle x\, y\rangle^2\, [x\, y]^2}{(1\,2)\cdots(n\,1)} \tag{6.26}$$

with

$$d\bar{\mu}_n = \left(\prod_{i=1}^{n} d^2\sigma_i\right) \prod_{I=1}^{n} \delta^2\left(\tilde{\lambda}_I - \frac{\tilde{\lambda}_x}{(I\,x)} - \frac{\tilde{\lambda}_y}{(I\,y)}\right) \\ \times \prod_{I=1}^{n} \delta^{0|2}\left(\eta_{+,I} - \frac{\eta_{+,x}}{(I\,x)} - \frac{\eta_{+,y}}{(I\,y)}\right) \delta^{0|2}\big(\eta_{-,I}\big). \tag{6.27}$$

The solution for this case is nothing but solution (6.21) by replacing $\langle ij\rangle \to [ij]$. Imposing the unique solution and the Jacobian for the bosonic delta-functions, one can immediately obtain

$$\mathscr{F}^{\overline{\text{MHV}}}_{\mathcal{T}_2,n} = \delta^4(P)\frac{q^4}{[1\,2][2\,3]\cdots[n\,1]} \\ \times \prod_{I=1}^{n} \delta^{0|2}\left(\eta_{+,I} - \frac{[I\,y]}{[x\,y]}\eta_{+,x} - \frac{[x\,I]}{[x\,y]}\eta_{+,y}\right) \delta^{0|2}\big(\eta_{-,I}\big). \tag{6.28}$$

Its $(\eta_x)^0(\eta_y)^0$ component is the famous Sudakov form factor.

NMHV and NNMHV form factors

We have also evaluated our formulas for more involved examples where one needs to sum over multiple solutions to scattering equations. For the form factor of $\mathcal{O}_2$, we have evaluated it numerically for four-point NMHV, five-point NMHV and NNMHV, as well as six-point NMHV and NNMHV cases. This is done by first solving (6.7) and then summing over the resulting 4, 11, 11, 26 and 66 solutions respectively. In addition, for the form factor of $\mathcal{O}_3$, we have checked up to five-point NMHV case. These results can be computed from MHV rules or recursion relations as in [171], and in all these cases, we find perfect agreement.

More interestingly, by following the same procedure of [140, 211], we can evaluate the following two NMHV form factors analytically, and verified that they give correct results [171, 180]:

$$F_{\mathcal{O}_2}(\phi_{12}, \phi_{12}, g^-, g^+; q) \tag{6.29}$$
$$= \frac{1}{\langle 41\rangle [23] s_{34}} \left(\frac{\langle 14\rangle \langle 23\rangle [24]^2}{s_{234}} + \frac{[14][23] \langle 13\rangle^2}{s_{134}} - \langle 13\rangle [24] \right),$$

$$F_{\mathcal{O}_3}(\phi_{12}, \phi_{12}, \phi_{12}, g^-; q) = \frac{[31]}{[34][41]}. \tag{6.30}$$

To summarize, our news formulas (6.16) and (6.19) have reproduced correct results for lower-point cases as well as the MHV form factors with any number of external states. Remarkably, all final results depend only on q but not on individual momenta k_x, k_y, as expected. These checks provide very strong evidence that our new formulas are correct.

6.4 Discussions

We have initiated the investigation on the scattering-equation-based prescription for tree-level form factors in $\mathcal{N} = 4$ SYM. Based on an off-shell extension of the scattering equations, we obtained the new formula (6.16) for forms factors with the chiral part of stress-tensor multiplet operator $\mathcal{T}_2$, which has $\mathcal{O}_2 = \mathrm{Tr}\left[(\phi_{12})^2\right]$ as its bottom component. More interestingly, we studied generalizations to half-BPS operator $\mathcal{O}_m$ for arbitrary m which are generally no longer bilinear, and obtained again very compact formula (6.19). Our results strongly support the universality of the scattering equations (6.7) for chiral form factors.

In [143], an interesting "duality" has been proposed, which relates the scattering equation formalism and $G(k, n)$ Grassmannian description for amplitudes in $\mathcal{N}=4$ SYM. By repeated use of the Global residue theorem, one can rewrite the Witten-RSV formula as particular sums of residues of the Grassmannian contour integral, which give tree amplitudes in the so-called link representation. The same conclusion holds for our formulas for form factors. In general, the formula must admit link representation which is given by particular contours for the integrals over $G(k + 2, n + 2)$ (or $G(k, n + 2)$ for the anti-chiral form factors), where x, y do not enter the cyclic measure. This agrees with the results in [187] and [201], and we expect a careful investigation can pick out the precise contour that gives the tree form factor. Generic residues of the Grassmannian contour integral correspond to on-shell diagrams, which have manifest symmetry properties and significance for loops. It would be very tempting to see if our formulas shed new lights on these ideas for form factors.

We expect that our results can be generalized to form factors for general operators in $\mathcal{N}=4$ SYM at tree level. In [212], a closed formula for all MHV form factors with most general operators was given, and it is likely that with the scattering-equation-based prescription that can be generalized to any N^{k}MHV sector. It would also be very interesting to compare with these recent works on $\mathcal{N}=4$ SYM form factors using twistor space method [213, 214], which can help us understand better off-shell quantities in twistor space. It is plausible that the idea of connected prescriptions and twistor strings can be applied to correlation functions in $\mathcal{N}=4$ SYM (see [215]). Last but not least, most results have been restricted to four dimensions, but scattering-equation-based construction can be useful for studying form factors in general dimensions as well. Given the success of scattering equations at the loop level, a natural next step would be applying them to loop-level form factors.

7 The scattering equations in Regge kinematics

The representation of a tree-level amplitude as a residue integral also makes manifest the properties of the amplitude in certain singular limits. As shown exactly in Chapter 5, the scattering equation method provides a beautiful framework to produce various double soft theorems in many theories (c.f. also [44, 132, 162]). Similarly, in the single-soft limit, by expanding the scattering equations as well as other ingredients of the formulas for amplitudes around the soft momentum, one can obtain various single soft theorems in many theories, for example the soft graviton theorem up to sub-sub-leading order [24, 40–43]. In the framework of the scattering equations, the collinear factorization of the amplitudes was also investigated up to sub-leading order in Yang-Mills, gravity and cubic scalar theories [47]. It was found that the solutions of the scattering equations can be interpreted as the zeros of the Jacobi polynomials in a two-parameter family of kinematics [49], and the amplitudes in Yang-Mills and gravity can be obtained explicitly for any multiplicity. More interestingly, the results in [49] have been used to study the black hole formation in a particular high-energy regime [216].

In this chapter and the next two chapters, we show that the scattering equations also present a very natural framework to study other kinematical limits of scattering amplitudes, namely the Regge limits of a 2-to-$(n-2)$ scattering where the final state particles are ordered in rapidity. It is well known that in multi-Regge kinematics any tree-level amplitude in gauge theory factorizes into a product of universal building blocks known as impact factors and Lipatov vertices [217–219], connected by t-channel propagators. More interestingly, in this kinematical regime, the tree-level amplitude in Einstein gravity also takes a similar fully-factorized form with gluon amplitudes, where the universal building blocks are the double copies of impact factors or Lipatov vertices. This is very surprising since gravity amplitudes are incredibly complicated in general kinematics. In more general quasi-multi-Regge kinematics (QMRK), the gauge theory amplitude is again expected to have a factorized form [220].

One of the goals of this work is to use the scattering-equation-based representation of amplitudes in both Yang-Mills and gravity to shed light on the quasi-Regge factorization of amplitudes. To achieve the goal relies on a systematic investigation of the asymptotic behavior of the scattering equations in MRK as well as various quasi-Regge regimes.

To be well-organized, we first perform a thorough analysis of the asymptotic behavior of the scattering equations in various Regge limits in this chapter. In the next chapter, we derive the factorization of the amplitude in both Yang-Mills and Einstein gravity in MRK. Then we extend the analysis to various quasi-multi-Regge limits, and we obtain CHY-type representations for the generalised impact factors and Lipatov vertices in Chapter 9.

7.1 Multi-Regge kinematics

In order to fix our notations and conventions, we give in this section a brief review on the quasi-Multi-Regge kinematics.

For a $2 \to (n-2)$ scattering, multi-Regge kinematics is defined as the regime where the final-state particles are strongly ordered in rapidity while having comparable transverse momenta, i.e.,

$$y_3 \gg y_4 \gg \cdots \gg y_n \quad \text{and} \quad |k_3^\perp| \simeq |k_4^\perp| \simeq \ldots \simeq |k_n^\perp|, \tag{7.1}$$

where $k_a^\perp$ denote the transverse momenta, and in four dimensions we define the complexified transverse momenta as $k_a^\perp = k_a^1 + ik_a^2$. Employing lightcone coordinates $k_a = (k_a^+, k_a^-, k_a^\perp)$ with $k_a^\pm = k_a^0 \pm k_a^3$, the strong ordering in rapidity is equivalent to a strong ordering in k^+-components as follows:

$$k_3^+ \gg k_4^+ \gg \cdots \gg k_n^+. \tag{7.2}$$

It is convenient to work in the center-of-momentum frame where two incoming particles are back-to-back on the z-axis,

$$k_1 \;=\; (0, -\chi; 0), \quad k_2 \;=\; (-\chi, 0; 0)\,, \quad \chi \equiv \sqrt{s}, \tag{7.3}$$

where s is the square of the center-of-mass energy, and we take a convention of considering all momenta outgoing.

In this kinematical regime, it is conjectured that the tree-level scattering amplitude in both Yang-Mills and Einstein gravity takes a surprisingly simple factorized form: any n particle amplitude is given by only one Feynman graph with effective vertices (see Figure 7.1).

Let us first see gauge theory amplitudes. In MRK every gluon amplitude factorizes into a set of universal building blocks describing the emission of gluons along a t-channel ladder [221, 222], i.e.,

$$\begin{aligned} &\mathcal{A}_n(1,\ldots,n) \\ &\simeq s\,\mathcal{C}(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^\perp|^2}\mathcal{V}(q_{n-1};n{-}1;q_n)\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1;n), \end{aligned} \tag{7.4}$$

where $q_a = \sum_{b=2}^{a-1} k_b$ with $4 \leq a \leq n$ are the momenta exchanged in the t-channel. We use the "$\simeq$" sign to denote equality up to terms that are power-suppressed in the limit. The quantities that appear in the right-hand side of eq. (7.4) are the *impact factors* [48, 217],

$$\begin{aligned} &\mathcal{C}(2^+;3^+) = \mathcal{C}(2^-;3^-) = \mathcal{C}(1^+;n^+) = \mathcal{C}(1^-;n^-) = 0, \\ &\mathcal{C}(2^-;3^+) = \mathcal{C}(2^+;3^-) = 1, \\ &\mathcal{C}(1^-;n^+) = \mathcal{C}(1^+;n^-)^* = \frac{k_n^{\perp *}}{k_n^\perp} \end{aligned} \tag{7.5}$$

and the *Lipatov vertices* [48, 218, 219],

$$\mathcal{V}\big(q_a;a^+;q_{a+1}\big) \;=\; \mathcal{V}\big(q_a;a^-;q_{a+1}\big)^* \;=\; \frac{q_a^{\perp *}\,q_{a+1}^\perp}{k_a^\perp}\,. \tag{7.6}$$

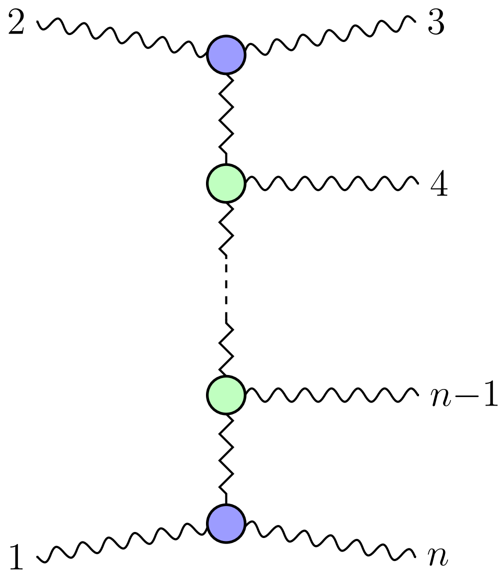


Figure 7.1: The factorized form of a tree-level amplitude in MRK

Helicity is conserved by the impact factors, which forces some of the helicity combinations in eq. (7.5) to vanish.

Now we turn to gravity amplitudes. In MRK, any n graviton amplitude takes the same factorized structure as in gauge theory [221, 222],

$$\mathcal{M}_n \simeq -s^2\,\mathcal{C}_g(2;3)\frac{-1}{|q_4^{\perp}|^2}\mathcal{V}_g(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^{\perp}|^2}\mathcal{V}_g(q_{n-1};n-1;q_n)\frac{-1}{|q_n^{\perp}|^2}\mathcal{C}_g(1;n). \tag{7.7}$$

We refer to $\mathcal{C}_g$ and $\mathcal{V}_g$ as the *gravitational impact factor* and *gravitational Lipatov vertex* respectively. They can be written as the double copy of gauge theory vertices. $\mathcal{C}_g$ is simply given by $\mathcal{C}^2$,

$$\mathcal{C}_g(2;3) = \big(\mathcal{C}(2;3)\big)^2, \quad \mathcal{C}_g(1;n) = \big(\mathcal{C}(1;n)\big)^2. \tag{7.8}$$

We see from (7.8) that helicity is conserved by the impact factors in gravity, like in gauge theory. The gravitational Lipatov vertex is defined as

$$\begin{aligned}\mathcal{V}_g\big(q_a; a^+; q_{a+1}\big) &= \mathcal{V}_g\big(q_a; a^-; q_{a+1}\big)^* \\ &= \frac{q_a^{\perp *}\big(k_a^\perp q_a^{\perp *} - k_a^{\perp *} q_a^\perp\big) q_{a+1}^\perp}{(k_a^\perp)^2} .\end{aligned} \tag{7.9}$$

It has also a double-copy construction, see Appendix C for detail. An interesting feature is that only two types of effective three-point vertices are needed for the multi-Regge limit of graviton scattering.

The (gravitational) impact factors and Lipatov vertices appearing in MRK are entirely determined by the four- and five-point amplitudes in both gauge theory and gravity. Since there are no non-MHV helicity configurations for four and five particles, we see that in MRK any amplitude is determined by MHV-type building blocks, independently of the helicity configuration. Although consistent with all explicit results for tree-level amplitudes, eq. (7.4) was only rigorously proven for arbitrary multiplicity for the simplest helicity configurations [48]. One of the aims of this work is to explore how far the scattering equations formalism can be used to shed light on the factorization of scattering amplitudes in Regge kinematics.

Starting from the multi-Regge limit in eq. (7.1), one can define a tower of new kinematic regimes, called quasi-multi-Regge kinematics (QMRK), by relaxing the hierarchy among some of the rapidities of the produced particles, e.g.,

$$y_3 \simeq \cdots \simeq y_k \gg y_{k+1} \simeq \cdots \simeq y_{n-1} \gg y_n \,, \tag{7.10}$$

while all transverse components are of the same size. In QMRK the gauge theory amplitude is conjectured to factorize in a way very similar to MRK in eq. (7.4). For example, in the limit in eq. (7.10), the amplitude is expected to factorize as follows:

$$\begin{aligned}&\mathcal{A}_n(1,\ldots,n) \\ &\simeq s\,\mathcal{C}(2;3,\cdots,k)\frac{-1}{|q_{k+1}^\perp|^2}\mathcal{V}\big(q_{k+1}; k{+}1,\cdots n{-}1; q_n\big)\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1;n).\end{aligned} \tag{7.11}$$

Here the two building blocks $\mathcal{C}(2;3,\cdots,k)$ and $\mathcal{V}(q_a;a,\cdots b;q_{b+1})$ are the generalized impact factor and Lipatov vertex respectively. Impact factors for the production of up to four particles have been computed in ref. [48, 71] from helicity amplitudes, and in ref. [223] using the effective action for high-energy processes in QCD [224]. The Lipatov vertices are known for the emission of up to four particles emitted in the center [48, 71, 225]. We do not discuss the quasi-multi-Regge limit of graviton amplitudes in this work.

7.2 The scattering equations in lightcone variables

As shown in the previous section, the multi-Regge limit is most naturally defined in terms of lightcone variables. It is thus natural to write the scattering equations in terms of lightcone coordinates. For general dimensional scattering equations, (3.5), only Mandelstam variables are needed:

$$s_{ij} \;=\; 2k_i\cdot k_j \;=\; k_i^+k_j^- + k_i^-k_j^+ - k_i^{\perp *}k_j^{\perp} - k_i^{\perp}k_j^{\perp *}. \tag{7.12}$$

The four-dimensional scattering equations employ the spinor variables. In this work we define two-component spinors (c.f. Appendix A) as [226]

$$\begin{aligned}
&\lambda_1 = -\tilde\lambda_1 = \begin{pmatrix} 0 \\ \sqrt{\chi} \end{pmatrix}, \quad \lambda_2 = -\tilde\lambda_2 = \begin{pmatrix} \sqrt{\chi} \\ 0 \end{pmatrix}, \\
&\lambda_a = \frac{1}{\sqrt{k_a^+}}\begin{pmatrix} k_a^+ \\ k_a^{\perp} \end{pmatrix}, \quad \tilde\lambda_a = \frac{1}{\sqrt{k_a^+}}\begin{pmatrix} k_a^+ \\ k_a^{\perp *} \end{pmatrix}, \quad 3\le a\le n.
\end{aligned} \tag{7.13}$$

While one can get equations with lightcone moementa by simply substituting (7.13) into the scattering equations (4.5), we show that one can obtain a nicer form by fixing the redundancy and rescaling variables and equations below. First, let us use the $\mathrm{GL}(2,\mathbb{C})$ redundancy to fix four variables according to eq. (4.9), i.e.,

$$\sigma_1 \;=\; 0, \qquad \sigma_2 \;=\; t_2 \;\to\; \infty, \qquad t_1 \;=\; -1. \tag{7.14}$$

Here we always use the convention where $\{1,2\}\subseteq\mathfrak{N}$, and we define $\overline{\mathfrak{N}}\equiv\mathfrak{N}\backslash\{1,2\}$. Moreover, we follow the convention that elements of $\mathfrak{P}$ and $\overline{\mathfrak{N}}$ are denoted by small and capital letters respectively, e.g. $i\in\mathfrak{P}$ and $I\in\overline{\mathfrak{N}}$.

Second, we perform a rescaling for the t_a variables as follows:

$$t_i = \tau_i \sqrt{\frac{k_i^+}{\chi}}, \qquad t_I = \tau_I \frac{\sqrt{\chi k_I^+}}{k_I^\perp}. \tag{7.15}$$

Then let us also perform a rescaling for the scattering equations according to:

$$\begin{aligned}
&\mathcal{S}_i^1 \equiv \frac{1}{\lambda_i^1}\,\mathcal{E}_i^1, \quad \mathcal{S}_i^2 \equiv \frac{\lambda_i^{\dot{1}}}{k_i^\perp}\,\mathcal{E}_i^2; \qquad \bar{\mathcal{S}}_I^{\dot{1}} \equiv \lambda_I^2\,\bar{\mathcal{E}}_I^{\dot{1}}, \quad \bar{\mathcal{S}}_I^{\dot{2}} \equiv \lambda_I^1\,\bar{\mathcal{E}}_I^{\dot{2}};\\
&\bar{\mathcal{S}}_1^{\dot{1}} \equiv \lambda_1^2\,\bar{\mathcal{E}}_1^{\dot{1}}, \quad \bar{\mathcal{S}}_1^{\dot{2}} \equiv \lambda_1^2\,\bar{\mathcal{E}}_1^{\dot{2}}; \qquad \bar{\mathcal{S}}_2^{\dot{1}} \equiv \lambda_2^1\,\bar{\mathcal{E}}_2^{\dot{1}}, \quad \bar{\mathcal{S}}_2^{\dot{2}} \equiv \lambda_2^1\,\bar{\mathcal{E}}_2^{\dot{2}}.
\end{aligned} \tag{7.16}$$

As a consequence, we obtain a set of equations that contain only the terms linear in k_a^+. Explicitly, they are

$$\begin{aligned}
&\mathcal{S}_i^1 = 1 + \tau_i - \sum_{I\in\mathfrak{N}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I}\frac{k_I^+}{k_I^\perp} = 0, \quad \mathcal{S}_i^2 = 1 + \frac{k_i^+}{k_i^\perp}\frac{\tau_i}{\sigma_i} - \frac{k_i^+}{k_i^\perp}\sum_{I\in\mathfrak{N}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I} = 0;\\
&\bar{\mathcal{S}}_I^{\dot{1}} = k_I^\perp - \sum_{i\in\mathfrak{P}} \frac{\tau_i\tau_I}{\sigma_I - \sigma_i} k_i^+ = 0, \qquad \bar{\mathcal{S}}_I^{\dot{2}} = k_I^{\perp^*} - \frac{k_I^+}{k_I^\perp}\sum_{i\in\mathfrak{P}} \frac{\tau_i\tau_I}{\sigma_I - \sigma_i} k_i^{\perp^*} = 0;\\
&\bar{\mathcal{S}}_1^{\dot{1}} = -\sum_{i\in\mathfrak{P}} \frac{\tau_i}{\sigma_i} k_i^+ = 0, \qquad \bar{\mathcal{S}}_1^{\dot{2}} = -\chi - \sum_{i\in\mathfrak{P}} \frac{\tau_i}{\sigma_i} k_i^{\perp^*} = 0;\\
&\bar{\mathcal{S}}_2^{\dot{1}} = -\chi - \sum_{i\in\mathfrak{P}} \tau_i k_i^+ = 0, \qquad \bar{\mathcal{S}}_2^{\dot{2}} = -\sum_{i\in\mathfrak{P}} \tau_i k_i^{\perp^*} = 0.
\end{aligned} \tag{7.17}$$

We would like to emphasize that no limit has been applied to these equations, and they are completely equivalent to the scattering equations in (4.5), up to fixing the $\mathrm{GL}(2,\mathbb{C})$ redundancy according to (7.14) and performing the rescaling according to (7.15) and (7.16).

7.3 The scattering equations in Regge kinematics

Given the preparations in previous sections, this section aims to study the behaviour of the scattering equations in various quasi-multi-Regge limits.

A warm-up: the MHV solution

Before we present in the next subsection a concise conjecture on the behaviour of the solutions in these limits, we find it instructive to analyse in detail the MHV sector, $k = 2$. We do this for two reasons. First, we have argued in the previous section that in MRK any amplitude is determined by MHV-type building blocks, so the MHV sector should capture a lot of information on the multi-Regge limit. Second, while in general it is very hard, if not impossible, to find exact analytic solutions to the scattering equations, in the MHV sector one can solve the equations exactly and study their behaviour analytically.

As discussed in Chapter 3, there is unique solution in the MHV sector:

$$\sigma_a = \frac{k_a^+}{k_a^\perp} \quad \text{for} \quad 4 \leq a \leq n, \tag{7.18}$$

here we use the $\mathrm{SL}(2,\mathbb{C})$ redundancy to fix three of the σ_a according to

$$\sigma_1 = 0\,, \qquad \sigma_2 = \infty\,, \qquad \sigma_3 = \frac{k_3^+}{k_3^\perp}. \tag{7.19}$$

The complex conjugate of this solution is the solution of the equations in the $\overline{\text{MHV}}$ sector $k = n-2$.

Let us now analyse what the MHV solution becomes in a quasi-multi-Regge limit. From eqs. (7.18), (7.19) and (7.10) it is easy to see that for the MHV solution (and also for its complex conjugate) the σ_a, $a > 2$, are of the same order of magnitude as the corresponding lightcone coordinates k_a^+, i.e. $\sigma_a = \mathcal{O}(k_a^+)$. More precisely, for the MHV solution in eqs. (7.18) and (7.19) we have

$$\mathrm{Re}(\sigma_a) = \mathcal{O}(k_a^+) \text{ and } \mathrm{Im}(\sigma_a) = \mathcal{O}(k_a^+), \quad 3 \leq a \leq n\,. \tag{7.20}$$

In other words, in QMRK the MHV solution admits the same strong ordering as the rapidities. In particular, in MRK the MHV solution admits a hierarchy very reminiscent of the MRK hierarchy in eq. (7.1),

$$|\,\mathrm{Re}(\sigma_3)| \gg \cdots \gg |\,\mathrm{Re}(\sigma_n)| \text{ and } |\,\mathrm{Im}(\sigma_3)| \gg \cdots \gg |\,\mathrm{Im}(\sigma_n)|. \tag{7.21}$$

Let us also look at the MHV solution of the four-dimensional scattering equations in eq. (4.5). They do not only depend on the variables σ_a, but also t_a. For concreteness, since we use the convention where $\{1,2\} \subseteq \mathfrak{N}$, in the MHV case we have $\mathfrak{N} = \{1,2\}$. We have given this solution in (4.43) in Chapter 4. Here we rewrite it in terms of lightcone momenta:

$$\sigma_a = \frac{k_a^+}{k_a^\perp}, \qquad t_a = -\sqrt{\frac{k_a^+}{\chi}}, \quad a \geq 3, \tag{7.22}$$

the other variables have been fixed by the $\mathrm{GL}(2,\mathbb{C})$ according to eq. (7.14) or eq. (4.9). We see that, as expected, the σ_a behave again according to eq. (7.20). The variables t_i instead behave like

$$t_a = \mathcal{O}\left(\sqrt{k_a^+/\chi}\right), \quad a \geq 3\,. \tag{7.23}$$

Main conjectures

Our previous analysis shows that in every quasi-multi-Regge limit, the MHV solution admits the same hierarchy as the rapidities of the produced particles. Here we extend this observation to the solutions of the scattering equations in other sectors. More precisely, we propose the following conjecture:

Main Conjecture: In QMRK, if the $\mathrm{SL}(2,\mathbb{C})$ redundancy is fixed according to eq. (7.19), then all solutions of the D-dimensional scattering equations (3.5) satisfy the same hierarchy as the ordering of the rapidities that defines the QMRK, i.e.,

$$\mathrm{Re}(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad \mathrm{Im}(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad 4 \leq a \leq n. \tag{7.24}$$

This holds for all σ_a solutions of the four-dimensional scattering equations after fixing the $\mathrm{GL}(2,\mathbb{C})$ redundancy according to eq. (7.14). A particular example is that in MRK, we conjecture

$$\begin{aligned} &|\,\mathrm{Re}(\sigma_3)| \gg |\,\mathrm{Re}(\sigma_4)| \gg \cdots \gg |\,\mathrm{Re}(\sigma_n)| \\ \text{and} \quad &|\,\mathrm{Im}(\sigma_3)| \gg |\,\mathrm{Im}(\sigma_4)| \gg \cdots \gg |\,\mathrm{Im}(\sigma_n)|. \end{aligned} \tag{7.25}$$

Furthermore, in QMRK for the four-dimensional scattering equations, the two sets of solutions $(t_i)_{i\in\mathfrak{P}}$ and $(t_I)_{I\in\mathfrak{N}}$ are individually ordered according to the same hierarchy as the rapidities (though there may not be any ordering between elements belonging to different sets).

We have performed a detailed numerical analysis of the scattering equations for a set of external momenta which approach various different quasi-multi-Regge regimes, both in their D and their four-dimensional guises. We can approach a given quasi-multi-Regge limit numerically by choosing an appropriate numerical hierarchy between the lightcone components of the external momenta. We have solved the D-dimensional scattering equations up to eight points, and the four-dimensional scattering equations for all helicity sectors up to seven external particles. All results of the numerical analyses agree with our conjecture.

We can formulate our conjecture more sharply and give the precise behaviour of the solutions of the four-dimensional scattering equations. We observe that in all cases, including for the non-MHV sectors $k > 2$, all solutions of the scattering equations in eq. (4.5) or (7.17) behave in a similar way as the MHV solution in eqs. (7.20) and (7.23). Based on this study, we have led to state the following conjecture:

Conjecture 4d: In QMRK, if $\{3, n\} \subseteq \mathfrak{P}$ and the $\mathrm{GL}(2, \mathbb{C})$ redundancy is fixed according to eq. (7.14), then all solutions of the four-dimensional scattering equations (4.5) behave as, with $3 \leq a \leq n$,

$$\mathrm{Re}(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad \mathrm{Im}(\sigma_a) = \mathcal{O}\left(k_a^+\right), \; t_a = \mathcal{O}\left(\sqrt{k_a^+ \chi^{-h_a}}\right), \tag{7.26}$$

where $h_a = 1$ when $a \in \mathfrak{P}$, otherwise $h_a = -1$.

We stress that the Conjecture 4d only holds in the case $\{3, n\} \subseteq \mathfrak{P}$. For any helicity amplitude in Yang-Mills and gravity, it is always able to take $\{3, n\} \subseteq \mathfrak{P}$ as a convention. The reason is pretty simple. As an example, let us see the formula for $\mathcal{N} = 4$ SYM amplitudes. In (4.38), it is completely free to divide n particle labels into two sets, $\mathfrak{N}$ and $\mathfrak{P}$ with $|\mathfrak{N}| = k$ for the N^{k-2}MHV superamplitude. Therefore, we are always able to evaluate any superamplitude using the formula (4.38) with fixing $\{3, n\} \subseteq \mathfrak{P}$ as well as $\{1, 2\} \subseteq \mathfrak{N}$. The all component (e.g. gluon) amplitudes can be extracted from the η-expansion of the superamplitude according to the on-shell superfield. For the case of $\{3, n\} \not\subseteq \mathfrak{P}$, we have also performed a numerical analysis and found the solutions t_a have slightly different behaviour in QMRK. Here we do not show it here explicitly and we suggest interested readers to appendix B of paper [138] for details.

Finally, let us make a comment at this point. Equation (7.26) implies that the variables σ_a scale in every helicity sector in the same way as in the MHV sector, cf. eq. (7.22). The scaling of the variables t_a in eq. (7.26), however, depends on the "helicity configuration". This is natural in the sense that, even away from Regge kinematics the solutions σ_a are independent of the ways of dividing n labels into two subsets $\mathfrak{N}$ and $\mathfrak{P}$, $|\mathfrak{N}| = k$, $2 \leq k \leq n-2$, and these solutions form a subset of the $(n-3)!$ solutions to the D-dimensional scattering equations. Very differently, the variables t_a are specific to the four-dimensional scattering equations and take different values in different configurations for given k.

7.4 Discussions

These conjectures are among the main results of this work. Although we do currently not have any proof of our conjectures, we find it remarkable that in all cases we have investigated the solutions to the scattering equations display this universal asymptotic behaviour, independently of the helicity sector. In the next two chapters, we explore the consequences of our conjectures. In particular, we will show that these conjectures imply the expected factorization of tree-level amplitudes in quasi-multi-Regge limits when combined with the contour integration representation of tree-level amplitudes. This result is important for two reasons: First, it gives support to our conjectures, because it shows that they allow us to recover the expected behaviour of tree-level amplitudes in quasi-multi-Regge limits. Second, it reveals the origin of Regge factorization from the angle of the scattering equations.

8 Multi-Regge factorization from the scattering equations

In the previous chapter, we observed that in the QMRK the solutions to the scattering equations admit the same hierarchy as the rapidity ordering, and we conjectured that this behaviour holds independently of the number of external legs. In this chapter, we apply the conjecture to the case of multi-Regge kinematics. In MRK, any amplitude in gauge and gravity theories takes a simple factorized form which is determined by MHV-type building blocks, as shown in detail in the first section of Chapter 7. We show that our conjecture implies that in MRK the four-dimensional scattering equations have a unique solution. The amplitude in MRK is then determined uniquely by this solution in both Yang-Mills and Einstein gravity. This is very similar to MHV amplitudes, where the CHY-type formula has support only on a single solution of the scattering equations.

8.1 An exact solution of the scattering equations

Let us begin with the study of the four-dimensional scattering equations (4.5) in multi-Regge kinematics.

We have shown that the equations (4.5) can be nicely rewritten in terms of lightcone momenta, as shown in eq. (7.17) in the previous chapter. According to our conjecture, in MRK any solution to the scattering equations (7.17) satisfies

$$\frac{1}{\sigma_a - \sigma_b} \simeq \frac{1}{\sigma_a} \text{ when } a < b, \quad \tau_a = \mathcal{O}(k_a^\perp). \tag{8.1}$$

Expanding the scattering equations (7.17) to leading power in MRK according to (8.1), they get significantly simplified

$$
\begin{aligned}
\mathcal{S}_i^1 &= 1 + \tau_i \left(1 + \sum_{I \in \overline{\mathfrak{N}}_{<i}} \zeta_I \right) = 0, \\
\mathcal{S}_i^2 &= 1 + \zeta_i \left(1 - \sum_{I \in \overline{\mathfrak{N}}_{>i}} \tau_I \right) = 0, \\
\bar{\mathcal{S}}_I^{\dot{1}} &= k_I^\perp + \tau_I \sum_{i \in \mathfrak{P}_{<I}} \zeta_i \, k_i^\perp = 0, \\
\bar{\mathcal{S}}_I^{\dot{2}} &= (k_I^\perp)^* - \zeta_I \sum_{i \in \mathfrak{P}_{>I}} \tau_i (k_i^\perp)^* = 0,
\end{aligned}
\tag{8.2}
$$

where $M_{<a}$ ($M_{>a}$) denotes the subset of M of elements that are less (greater) than a, and we have defined the rescaled variables

$$
\zeta_a \equiv \frac{k_a^+}{k_a^\perp} \frac{\tau_a}{\sigma_a}, \quad 3 \le a \le n \,. \tag{8.3}
$$

Let us rewrite the scattering equations (8.2) as:

$$
\begin{aligned}
\mathcal{S}_i^1 &= 1 + a_i \, \tau_i = 0 \,, \qquad & \bar{\mathcal{S}}_I^{\dot{2}} &= (k_I^\perp)^* + b_I \, \zeta_I = 0 \,, \\
\mathcal{S}_i^2 &= 1 + c_i \, \zeta_i = 0 \,, \qquad & \bar{\mathcal{S}}_I^{\dot{1}} &= k_I^\perp + d_I \, \tau_I = 0 \,,
\end{aligned}
\tag{8.4}
$$

with short-handed notations

$$
\begin{aligned}
a_i &\equiv 1 + \sum_{I \in \overline{\mathfrak{N}}_{<i}} \zeta_I \,, \qquad & b_I &\equiv - \sum_{i \in \mathfrak{P}_{>I}} \tau_i \, (k_i^\perp)^*, \\
c_i &\equiv 1 - \sum_{I \in \overline{\mathfrak{N}}_{>i}} \tau_I \,, \qquad & d_I &\equiv \sum_{i \in \mathfrak{P}_{<I}} \zeta_i \, k_i^\perp .
\end{aligned}
\tag{8.5}
$$

We observe that in MRK the equations $(\mathcal{S}_i^1, \bar{\mathcal{S}}_I^{\dot{2}}) = 0$ only depend on the variables τ_i and ζ_I, while the equations $(\mathcal{S}_i^2, \bar{\mathcal{S}}_I^{\dot{1}}) = 0$ only depend on τ_I and ζ_i. We also see that each equation in (8.4) is linear in one of the variables. We can use this very special structure to find an explicit analytic solution of the scattering equations in MRK.

To start, we can use the fact that the equations $\mathcal{S}_i^\alpha = 0$ are linear in τ_i and ζ_i to give

$$\tau_i \;=\; -\frac{1}{a_i}, \qquad \zeta_i \;=\; -\frac{1}{c_i}. \tag{8.6}$$

Inserting eq. (8.6) into the expressions for b_I and d_I in eq. (8.5), we find

$$b_I \;=\; \sum_{i\in\mathfrak{P}_{>I}} \frac{(k_i^\perp)^*}{a_i} \;=\; \sum_{i\in\mathfrak{P}_{>I}} (k_i^\perp)^* \left(1+\sum_{J\in\overline{\mathfrak{N}}_{<i}} \zeta_J\right)^{-1}, \tag{8.7}$$

$$d_I \;=\; -\sum_{i\in\mathfrak{P}_{<I}} \frac{k_i^\perp}{c_i} \;=\; -\sum_{i\in\mathfrak{P}_{<I}} k_i^\perp \left(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}} \tau_J\right)^{-1}. \tag{8.8}$$

The b_I are solely determined by the variables ζ_I, and similarly the d_I are determined by the variables τ_I. In order to proceed, we write $\overline{\mathfrak{N}} = \{I_\ell\}_{1\le\ell\le m}$ ($m \equiv k-2$) with $I_\ell < I_{\ell+1}$. The quantities b_I and d_I satisfy the recursions, for $1 \le r \le m$,

$$b_{I_r} \;=\; b_{I_{r+1}} + \left(1+\sum_{l=1}^{r} \zeta_{I_l}\right)^{-1} (q_{I_{r+1}}^\perp - q_{I_r+1}^\perp)^*, \tag{8.9}$$

$$d_{I_r} \;=\; d_{I_{r-1}} - \left(1-\sum_{l=r}^{m} \tau_{I_l}\right)^{-1} (q_{I_r}^\perp - q_{I_{r-1}+1}^\perp). \tag{8.10}$$

The recursion starts with $b_{I_{m+1}} = d_{I_0} = q_{I_{m+1}} = q_{I_0+1} = 0$. Indeed, we have

$$\begin{aligned} b_{I_r} &= \sum_{i\in\mathfrak{P}_{>I_r}} (k_i^\perp)^* \Bigg(1 + \sum_{J\in\overline{\mathfrak{N}}_{<i}} \zeta_J\Bigg)^{-1} \\ &= \Bigg(\sum_{i\in\mathfrak{P}_{>I_{r+1}}} + \sum_{I_r<i<I_{r+1}}\Bigg)(k_i^\perp)^* \Bigg(1 + \sum_{J\in\overline{\mathfrak{N}}_{<i}} \zeta_J\Bigg)^{-1} \\ &= b_{I_{r+1}} + \Bigg(1+\sum_{l=1}^{r}\zeta_{I_l}\Bigg)^{-1} \sum_{I_r<i<I_{r+1}} (k_i^\perp)^* \\ &= b_{I_{r+1}} + \Bigg(1+\sum_{l=1}^{r}\zeta_{I_l}\Bigg)^{-1} (q_{I_{r+1}}^\perp - q_{I_r+1}^\perp)^*. \end{aligned} \tag{8.11}$$

The proof of the recursion for d_{I_r} is similar. We can combine the recursions for b_I and d_I with the scattering equations and turn them into recursions for τ_I and ζ_I. We illustrate this procedure explicitly on d_I and τ_I. The procedure for b_I and ζ_I is similar. We start from the equation $\bar{\mathcal{S}}^{\dot{1}}_{I_r} = 0$ and insert the recursion in eq. (8.9). For $r = 1$, we find

$$\begin{aligned} 0 = \bar{\mathcal{S}}^{\dot{1}}_{I_1} &= k_{I_1}^\perp + d_{I_1}\tau_{I_1} \\ &= -\Bigg(1-\sum_{l=1}^{m}\tau_{I_l}\Bigg)^{-1}\Bigg[-k_{I_1}^\perp\Bigg(1-\sum_{l=2}^{m}\tau_{I_l}\Bigg) + \tau_{I_1} q_{I_1+1}^\perp\Bigg]. \end{aligned} \tag{8.12}$$

This immediately leads to

$$\tau_{I_1} = \frac{k_{I_1}^\perp}{q_{I_1+1}^\perp}\Bigg(1-\sum_{l=2}^{m}\tau_{I_l}\Bigg). \tag{8.13}$$

Similarly, using eq. (8.13) we obtain for $r = 2$,

$$0 = \bar{\mathcal{S}}^{\mathrm{i}}_{I_2} = k^{\perp}_{I_2} + d_{I_2}\tau_{I_2} \tag{8.14}$$

$$= k^{\perp}_{I_2} - \tau_{I_2}\left[\frac{k^{\perp}_{I_1}}{\tau_{I_1}} + \left(1 - \sum_{l=2}^{m}\tau_{I_l}\right)^{-1}(q^{\perp}_{I_2} - q^{\perp}_{I_1+1})\right] \tag{8.15}$$

$$= \left(1 - \sum_{l=2}^{m}\tau_{I_l}\right)^{-1}\left[k^{\perp}_{I_2}\left(1 - \sum_{l=3}^{m}\tau_{I_l}\right) - \tau_{I_2}q^{\perp}_{I_2+1}\right], \tag{8.16}$$

and so we have

$$\tau_{I_2} = \frac{k^{\perp}_{I_2}}{q^{\perp}_{I_2+1}}\left(1 - \sum_{l=3}^{m}\tau_{I_l}\right). \tag{8.17}$$

A similar formula holds for general r, and the τ_{I_r} satisfy the recursion,

$$\tau_{I_r} = \frac{k^{\perp}_{I_r}}{q^{\perp}_{I_r+1}}\left(1 - \sum_{l=r+1}^{m}\tau_{I_l}\right). \tag{8.18}$$

The recursion starts with

$$\tau_{I_m} = \frac{k^{\perp}_{I_m}}{q^{\perp}_{I_m+1}}. \tag{8.19}$$

It is easy to show by induction that the recursion admits the explicit solution

$$\tau_{I_r} = \frac{k^{\perp}_{I_r}}{q^{\perp}_{I_r+1}}\prod_{l=r+1}^{m}\frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}}, \qquad 1 \leq r < m. \tag{8.20}$$

The explicit solution for the variables ζ_{I_r} can be obtained in the same way. More precisely, one can show that there is a recursion

$$\zeta_{I_r} = \left(\frac{k^{\perp}_{I_r}}{q^{\perp}_{I_r}}\right)^{*}\left(1 + \prod_{l=1}^{r-1}\zeta_{I_l}\right), \qquad 1 < r \leq m, \tag{8.21}$$

with

$$\zeta_{I_1} = \left(\frac{k_{I_1}^{\perp}}{q_{I_1}^{\perp}}\right)^*, \tag{8.22}$$

and the recursion admits the explicit solution

$$\zeta_{I_r} = \left(\frac{k_{I_r}^{\perp}}{q_{I_r}^{\perp}}\right)^* \left(\prod_{l=1}^{r-1} \frac{q_{I_l+1}^{\perp}}{q_{I_l}^{\perp}}\right)^*, \qquad 1 < r \leq m\,. \tag{8.23}$$

Together with eqs. (8.6), (8.20) and (8.23) provide the explicit solution of the scattering equations in MRK. We see that there is indeed a unique (independent) solution to the four-dimensional scattering equations in MRK, which can be traced back to the fact that at every step we only needed to solve linear equations. We can easily obtain explicit solutions for the coefficients c_i and a_i that appear in eq. (8.6). For example, using eq. (8.18) and (8.20), we find

$$c_i = 1 - \sum_{I \in \overline{\mathfrak{N}}_{>i}} \tau_I = \prod_{I \in \overline{\mathfrak{N}}_{>i}} \frac{q_I^{\perp}}{q_{I+1}^{\perp}}\,. \tag{8.24}$$

We recall that we use the convention that products over empty ranges are unity. Similarly for a_i, we have

$$a_i = 1 + \sum_{I \in \overline{\mathfrak{N}}_{<i}} \zeta_I = \left(\prod_{I \in \overline{\mathfrak{N}}_{<i}} \frac{q_{I+1}^{\perp}}{q_I^{\perp}}\right)^*. \tag{8.25}$$

Hence, we find

$$\tau_i = -\left(\prod_{I \in \overline{\mathfrak{N}}_{<i}} \frac{q_I^{\perp}}{q_{I+1}^{\perp}}\right)^* \quad \text{and} \quad \zeta_i = -\prod_{I \in \overline{\mathfrak{N}}_{>i}} \frac{q_{I+1}^{\perp}}{q_I^{\perp}}\,. \tag{8.26}$$

Finally, we can also write this solution in the original variables (t_a, σ_a) using eqs. (7.15) and (8.3).

$$t_a = \sqrt{k_a^+} \times \begin{cases} \dfrac{-1}{\sqrt{\chi}} \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{<a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right)^*, & a \in \mathfrak{P}, \\[2ex] \dfrac{\sqrt{\chi}}{q_{a+1}^\perp} \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{>a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right), & a \in \overline{\mathfrak{N}}, \end{cases} \tag{8.27}$$

$$\sigma_a = \frac{k_a^+}{k_a^\perp} \times \begin{cases} \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{<a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right)^* \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{>a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right), & a \in \mathfrak{P} \\[2ex] \dfrac{k_a^\perp}{q_{a+1}^\perp} \left(\dfrac{q_a^\perp}{k_a^\perp} \right)^* \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{<a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right)^* \left(\displaystyle\prod_{I \in \overline{\mathfrak{N}}_{>a}} \frac{q_I^\perp}{q_{I+1}^\perp} \right), & a \in \overline{\mathfrak{N}} \end{cases} \tag{8.28}$$

Here we interpret a product over an empty range as 1. Let us make some comments about this solution (referred to as the *MRK solution*). First, we immediately see that the solution in eqs. (8.27) and (8.28) has the same hierarchy as the rapidities of the produced particles, in agreement with Conjecture 4d.

Second, we find it remarkable that in MRK we can explicitly solve the scattering equations for arbitrary multiplicity and arbitrary helicity configuration. For each configuration $\mathfrak{N}$ of particle labels in sector k, we find a solution of the four-dimensional scattering equations (4.5). As shown in Chapter 4, the solutions σ_a of the four-dimensional scattering equations also solve the D-dimensional ones. It is very rare that one can solve the scattering equations for arbitrary multiplicities, and so far this has only been done for the MHV sector. We can recover the MHV solution of eq. (7.22) from eqs. (8.27) and (8.28) by taking $\mathfrak{P} = \{3, \cdots, n\}$. Indeed, in that case we have $\overline{\mathfrak{N}}_{<a} = \overline{\mathfrak{N}}_{>a} = \emptyset$, and so the products in eqs. (8.27) and (8.28) do not contribute, and we immediately recover eq. (7.22).

Finally, in the following sections we will show that we can obtain the multi-Regge factorization of amplitudes in both gauge and gravity theories by localizing the CHY-type integrals to the MRK solution.

8.2 Gluonic scattering in multi-Regge kinematics

In the previous section, we have shown that our conjecture implies that in MRK the four-dimensional scattering equations have a unique solution. In this section, we show that the conjecture also implies that any n-gluon amplitude in MRK can be written in the factorized form in eq. (7.4). Since eqs. (8.27) and (8.28) hold for arbitrary helicity configurations and multiplicities, this shows that the MRK factorization holds independently of the helicity configuration and the number of legs. This nicely complements the results of ref. [48], where factorization was shown to hold for the simplest (MHV) helicity configurations. We emphasise, however, that we cannot present at this point a rigorous proof that multi-Regge factorization holds for arbitrary helicity configurations because our derivation relies on the conjecture presented in the previous chapter. Finding a rigorous mathematical proof of our conjecture would thus eventually imply an elegant proof of the factorization of tree-level gluon amplitudes in MRK, for arbitrary multiplicities and helicity configurations.

We start by considering n-gluon scattering amplitudes in the case where the two incoming gluons (labelled by 1 and 2 respectively) carry the same helicity, and we comment on the case where they have different helicities at the end of this section. The gluon amplitudes can be easily extracted from the superamplitude (4.38). When we use $\mathfrak{N}$ and $\mathfrak{P}$ to collect the labels of gluons with negative and positive helicities respectively, the formula for pure gluon amplitudes does not receive an additional contribution from the Grassmann delta-functions in (4.32). As shown earlier, the scattering equations (4.5) can be rewritten in terms of lightcone momenta by resacling t_a variables acooring to (7.15) and fixing the $\mathrm{GL}(2,\mathbb{C})$ redundancy according to eq. (7.14). Similarly, using the new equations (7.17), a n gluon amplitude can be written as

$$\begin{aligned}\mathcal{A}_n(1^-,2^-,\dots,n) = -s\int\prod_{a=3}^{n}\frac{d\sigma_a d\tau_a}{\tau_a}\,\frac{1}{\sigma_{34}\cdots\sigma_{n-1,n}\sigma_n} \\ \times\left(\prod_{i\in\mathfrak{P},I\in\overline{\mathfrak{N}}}\frac{k_I^\perp}{k_i^\perp}\right)\prod_{I\in\overline{\mathfrak{N}}}\delta^2(\bar{\mathcal{S}}_I^{\dot\alpha})\prod_{i\in\mathfrak{P}}\delta^2(\mathcal{S}_i^\alpha),\end{aligned} \tag{8.29}$$

where we have eliminated four of the scattering equations and identify them with the momentum conservation delta-functions (see Appendix B), i.e.,

$$\delta^2\big(\bar{\mathcal{S}}_1^{\dot{\alpha}}\big)\delta^2\big(\bar{\mathcal{S}}_2^{\dot{\alpha}}\big) \;=\; \delta^4\left(\sum_{a=1}^{n} k_a^\mu\right). \tag{8.30}$$

We would like to emphasise that eq. (8.29) is exact and no Regge limit has been applied to it.

From Conjecture 4d, it follows that in MRK the solutions to the scattering equations satisfy (8.1). Inserting it into eq. (8.29) and passing to the rescaled variables defined in eqs. (7.15) and (8.3), we see that the formula for the amplitude in MRK can be reduced to

$$\begin{aligned}&\mathcal{A}_n(1^-,2^-,\ldots,n) \\ &\quad\simeq -s\left(\int\prod_{a=3}^{n}\frac{d\tau_a d\zeta_a}{\zeta_a\tau_a}\right)\left(\prod_{i\in\mathfrak{P}}\frac{1}{k_i^\perp}\,\delta^2\big(S_i^\alpha\big)\right)\left(\prod_{I\in\overline{\mathfrak{N}}}k_I^\perp\,\delta^2\big(\bar{\mathcal{S}}_I^{\dot{\alpha}}\big)\right),\end{aligned} \tag{8.31}$$

where the arguments of the δ-functions are given in eq. (8.2). We know that these equations have a unique solution given by eqs. (8.27) and (8.28). We now show that when the integral in eq. (8.31) is localized on this solution, then we recover the factorized form of a tree-level amplitude in MRK. We do this by a procedure very similar to the one used in the previous section to find the solution. The proof of eqs. (8.27) and (8.28) relies heavily on the fact that in MRK the scattering equations decouple into a set of linear equations that can be solved one-by-one. Here we use this fact to solve the δ-functions in the integrand of eq. (8.31) one at the time. All the manipulations are identical to those performed in the previous section, so we will be brief on the derivation and only highlight some aspects related to solving the δ-functions.

We start by performing the integrations over the τ_i and ζ_i. From eq. (8.2) we see that these integrations are independent from each other. We can localize them completely using $\delta\big(\mathcal{S}_i^1\big)$ and $\delta\big(\mathcal{S}_i^2\big)$, and we find

$$\int \frac{d\tau_i}{\tau_i}\,\delta\big(\mathcal{S}_i^1\big) \;=\; \int \frac{d\tau_i}{\tau_i}\,\delta\big(1+a_i\tau_i\big) \;=\; -1\,, \tag{8.32}$$

$$\int \frac{d\zeta_i}{\zeta_i}\,\delta\big(\mathcal{S}_i^2\big) \;=\; \int \frac{d\zeta_i}{\zeta_i}\,\delta\big(1+c_i\zeta_i\big) \;=\; -1\,. \tag{8.33}$$

After this step all the variables τ_i and ζ_i have been integrated out.

Using the same reasoning as in the previous section, we see that the integrations over τ_I and ζ_I are now independent of each other, and so we can discuss the integrations over the τ_I variables independently from the integrations over the ζ_I variables. Let us start with the τ_I-integrations. We use the δ-functions $\delta\big(\bar{\mathcal{S}}_I^{\dot{1}}\big)$ combined with the recursive procedure of the previous section to localize all the τ_I variables. In particular, we localize the τ_I variables in increasing order from I_1 to I_m. Let us look at the integration over τ_{I_1}. We start from eq. (8.12) to obtain

$$\begin{aligned}\int \frac{d\tau_{I_1}}{\tau_{I_1}}\,\delta\big(\bar{\mathcal{S}}_{I_1}^{\dot{1}}\big) \;&=\; \oint_{\mathcal{C}} \frac{d\tau_{I_1}}{\tau_{I_1}}\,\frac{1}{\bar{\mathcal{S}}_{I_1}^{\dot{1}}}\\ &=\; \oint_{\mathcal{C}} \frac{d\tau_{I_1}}{\tau_{I_1}}\,\frac{1-\sum_{l=1}^m \tau_{I_l}}{k_{I_1}^\perp\big(1-\sum_{l=2}^m \tau_{I_l}\big)-\tau_{I_1}q_{I_1+1}^\perp}\\ &=\; -\frac{1}{k_{I_1}^\perp}\,\frac{q_{I_1}^\perp}{q_{I_1+1}^\perp}\,,\end{aligned} \tag{8.34}$$

where we have interpreted the integral over the δ-function as a contour integral over a contour $\mathcal{C}$ encircling the zero of the argument of the δ-function. The remaining integrals over the τ_I variables can be performed in the same way in increasing order, with the simple result

$$\int \frac{d\tau_I}{\tau_I}\,\delta\big(\bar{\mathcal{S}}_I^{\dot{1}}\big) \;=\; -\frac{1}{k_I^\perp}\,\frac{q_I^\perp}{q_{I+1}^\perp}\,. \tag{8.35}$$

Finally, we are left with the integrals over the ζ_I variables to perform. The procedure is the same as for the τ_I variables, combined with the recursive procedure in the previous section. We find

$$\int \frac{d\zeta_I}{\zeta_I}\,\delta(\bar{\mathcal{S}}_I^{\dot{2}}) \;=\; -\left(\frac{1}{k_I^\perp}\,\frac{q_{I+1}^\perp}{q_I^\perp}\right)^*. \tag{8.36}$$

We see that we can perform all integrations one after the other. As a result, the amplitude takes a completely factorized form, which has its origin in the fact that in MRK all the integrations in the CHY-type formula decouple and can be performed one-by-one. The final result takes the very simple factorized form in eq. (7.4) and rewrite here

$$\begin{aligned}&\mathcal{A}_n(1,\ldots,n)\\ &\quad\simeq s\,\mathcal{C}(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^\perp|^2}\mathcal{V}(q_{n-1};n{-}1;q_n)\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1;n).\end{aligned}$$

Our conjecture thus implies the factorized form of the tree-level gluon amplitude in MRK. Unlike previous derivations, we stress that our derivation is completely independent of the helicities of the produced particles, giving strong support to the idea that MRK-factorization of tree-level amplitudes holds for any multiplicity and for any helicity assignment.

Finally, we end discussions in this section with some comments on other helicity configurations and color orderings.

Let us first see what happens in the case where the incoming gluons 1 and 2 have opposite helicities, for example, when gluon 1 has positive helicity. Since helicity is conserved by the impact factor $\mathcal{C}(1;n)$, we obtain a non-zero result in MRK only when gluon n has negative helicity. According to the formula for superamplitudes in (4.38), amplitudes $\mathcal{A}_n(1^-,2^-,\ldots,n^+)$ and $\mathcal{A}_n(1^+,2^-,\ldots,n^-)$ can be computed using the same formula (8.29), up to adding an additional factor $(1n)^{-4}$ to the integrand. If the $\mathrm{GL}(2,\mathbb{C})$ redundancy is fixed according to eq. (7.14), then this additional factor only depends on the variables t_n and σ_n. More precisely, we have,

$$\frac{1}{(1\,n)} \;=\; \frac{t_n}{\sigma_n} \;=\; \sqrt{\frac{k_n^+}{\chi}\,\frac{k_n^\perp}{k_n^+}}\,\zeta_n\,. \tag{8.37}$$

In MRK ζ_n is uniquely fixed by the equation $\mathcal{S}_n^2 = 1 + \zeta_n = 0$. Thus we have

$$\frac{1}{(1n)^4}\bigg|_{\zeta_n=-1} = \left(\frac{k_n^\perp}{(k_n^\perp)^*}\right)^2 . \tag{8.38}$$

This factor combines with the impact factor $\mathcal{C}(1^-; n^+)$ to give

$$\mathcal{C}(1^-; n^+)\left(\frac{k_n^\perp}{(k_n^\perp)^*}\right)^2 = \frac{k_n^\perp}{(k_n^\perp)^*} = \mathcal{C}(1^+; n^-)\,. \tag{8.39}$$

We see that in the case where $h_1 = -h_2 = +1$, only the impact factor $\mathcal{C}(1; n)$ changes, in agreement with the factorization of tree-level amplitudes in MRK.

Second, we discuss the case where the cyclic color ordering in partial gluon amplitude is not aligned with the rapidity ordering. Previously, we have shown how the MRK factorization of tree-level amplitudes where the cyclic color-ordering is aligned with the rapidity ordering follows from the CHY-type representation and Conjecture 4d. Here we show that all other cyclic color orderings are suppressed in MRK. This nicely complements the results of ref. [48], where this result had been shown to hold only for the MHV sector.

We have seen that in MRK the scattering equations have a unique solution. As a consequence, we can obtain a very simple relation between amplitudes in MRK with different cyclic orderings,

$$\mathcal{A}_n^{\text{MRK}}(1^-, 2^-, \mu_3, \ldots, \mu_n) = \mathcal{A}_n^{\text{MRK}}(1^-, 2^-, 3, \ldots, n)\,\mathcal{R}[\mu]\,, \tag{8.40}$$

where $\mathcal{A}_n^{\text{MRK}}$ is the n-point amplitude in MRK and $(\mu_3, \ldots, \mu_n)$ is a permutation of $(3, \ldots, n)$ and we defined,

$$\begin{aligned}\mathcal{R}[\mu] &= \frac{\sigma_3\sigma_4\cdots\sigma_n}{\sigma_{\mu_3\mu_4}\sigma_{\mu_4\mu_5}\cdots\sigma_{\mu_{n-1}\mu_n}\sigma_{\mu_n}}\bigg|_{\text{MRK solution (8.28)}} \\ &= \prod_{i=3}^{n-1}\mathcal{R}_{\mu_i}\bigg|_{\text{MRK solution (8.28)}}, \qquad \mathcal{R}_{\mu_i} \equiv \frac{\sigma_{\mu_i}}{\sigma_{\mu_i\mu_{i+1}}}.\end{aligned} \tag{8.41}$$

Our conjecture implies that

$$\mathcal{R}_{\mu_i} \simeq \begin{cases} 1, & \mu_i < \mu_{i+1} \\ -\frac{\sigma_{\mu_i}}{\sigma_{\mu_{i+1}}}, & \mu_i > \mu_{i+1} \end{cases} = \begin{cases} 1, & \mu_i < \mu_{i+1}, \\ \mathcal{O}\big(k^+_{\mu_i}/k^+_{\mu_{i+1}}\big), & \mu_i > \mu_{i+1}. \end{cases} \tag{8.42}$$

It is then easy to see that $\mathcal{R}[\mu]$ is suppressed in MRK kinematics unless the cyclic color-ordering is aligned with the rapidity ordering, in agreement with known results for MHV amplitudes [48].

8.3 Gravitational scattering in multi-Regge kinematics

In this section, we extend the analysis in the previous section from Yang-Mills theory to Einstein gravity. While the framework presented in Sections 8.1 and 8.2 is applicable to graviton amplitudes because of the universality of the scattering equations, this is still highly non-trivial since graviton amplitudes have a rather complicated structure even in the MHV sector.

Using a similar way with Yang-Mills, the formula for tree-level N^{k-2}MHV graviton amplitude can be obtained from the superamplitude (4.40). Using lightcone variables, it reads

$$\mathcal{M}_n = -s^2 \left(\int \prod_{a=3}^{n} \frac{d\tau_a d\sigma_a}{\tau_a^3} \right) \left(\prod_{I\in\overline{\mathfrak{N}}} \frac{(k_I^\perp)^3}{k_I^+} \delta^2(\bar{\mathcal{S}}_I^{\dot\alpha}) \right) \tag{8.43}$$
$$\times \left(\prod_{i\in\mathfrak{P}} \frac{1}{k_i^+ k_i^\perp} \delta^2(\mathcal{S}_i^\alpha) \right) \det{}' \overline{\mathsf{H}} \det{}' \mathsf{H}.$$

Here we employ again the convention where particles 1 and 2 carry negative helicity and use $\mathfrak{N}$ and $\mathfrak{P}$ to collect the babels of negative and positive helicity gravitons. For convenience, we have rescaled the two matrices defined in (4.41) according to

$$\mathsf{H}_{ab} := \chi^{-1}\mathbb{H}_{ab} \ \text{ for } \ a,b\in\mathfrak{N}; \quad \overline{\mathsf{H}}_{ab} := \chi\,\overline{\mathbb{H}}_{ab} \ \text{ for } \ a,b\in\mathfrak{P}. \tag{8.44}$$

More precisely, in MRK the entries of H are given by

$$\mathsf{H}_{12} = -1, \quad \mathsf{H}_{1I} = -\frac{\tau_I}{\sigma_I}\frac{k_I^+}{k_I^\perp}, \quad \mathsf{H}_{2I} = \tau_I, \tag{8.45}$$

$$\mathsf{H}_{IJ} = \frac{\tau_I \tau_J}{\sigma_I - \sigma_J}\left(\frac{k_I^+}{k_I^\perp} - \frac{k_J^+}{k_J^\perp}\right) \quad \text{for } I \neq J, \tag{8.46}$$

$$\mathsf{H}_{aa} = -\sum_{b\in\mathfrak{N}, b\neq a} \mathsf{H}_{ab}, \quad a \in \mathfrak{N}, \tag{8.47}$$

and for $\overline{\mathsf{H}}$ we have

$$\overline{\mathsf{H}}_{ij} = \frac{\tau_i \tau_j}{\sigma_i - \sigma_j}\left(k_i^+ k_j^{\perp *} - k_j^+ k_i^{\perp *}\right) \text{ for } i \neq j, \tag{8.48}$$

$$\overline{\mathsf{H}}_{ii} = -\sum_{j\in\mathfrak{P}, j\neq i} \overline{\mathsf{H}}_{ij}. \tag{8.49}$$

We emphasize that the formula (8.43) evaluate graviton amplitudes for arbitrary kinematics.

Similarly, according to our conjecture, $\sigma_a - \sigma_b \simeq \sigma_a$ for $a < b$ in MRK, expanding the formula (8.43) to leading power, it significantly simplified as

$$\mathcal{M}_n \simeq -s^2 \left(\int \prod_{a=3}^{n} \frac{d\tau_a d\zeta_a}{\tau_a^2 \zeta_a^2}\right)\left(\prod_{I\in\overline{\mathfrak{N}}} (k_I^\perp)^2\, \delta^2(\bar{\mathcal{S}}_I^{\dot\alpha})\right) \times \left(\prod_{i\in\mathfrak{P}} \frac{1}{(k_i^\perp)^2}\delta^2(\mathcal{S}_i^\alpha)\right) \det{}'\overline{\mathsf{H}} \det{}'\mathsf{H}, \tag{8.50}$$

where $\bar{\mathcal{S}}_I$ and $\mathcal{S}_i$ are given in eq. (8.2). The unique solution of the scattering equations is given by eqs. (8.20), (8.23) and (8.26), or by eq. (8.27) and (8.28) in (σ_a, t_a) variables. Like in Yang-Mills case, we fix $\{3, n\} \subseteq \mathfrak{P}$ as a convention, and $\zeta_a \sim \tau_a/\sigma_a$ is defined in (8.3). The matrices H and $\overline{\mathsf{H}}$ of course also get simplified and we will give their explicit expressions later.

In the following, we show that we can obtain the multi-Regge factorization of graviton amplitudes, shown in eq. (7.7), by localizing integrals in (8.50) to the

MRK solution. Let us begin by recalling the results for the gluon amplitude

$$\left(\int\prod_{a=3}^{n}\frac{d\tau_a d\zeta_a}{\zeta_a\tau_a}\right)\left(\prod_{i\in\mathfrak{P}}\frac{1}{k_i^{\perp}}\,\delta^2(S_i^{\alpha})\right)\left(\prod_{I\in\overline{\mathfrak{N}}}k_I^{\perp}\,\delta^2(\bar{\mathcal{S}}_I^{\dot\alpha})\right)$$
$$=(-1)^n\left(\prod_{I\in\overline{\mathfrak{N}}}\frac{1}{k_I^{\perp *}}\frac{q_I^{\perp}q_{I+1}^{\perp *}}{q_I^{\perp *}q_{I+1}^{\perp}}\right)\left(\prod_{i\in\mathfrak{P}}\frac{1}{k_i^{\perp}}\right). \tag{8.51}$$

Form it we can conclude that for any function $\mathcal{F}(\tau_a,\zeta_a)$ of τ_a and ζ_a we have

$$\int\prod_{a=3}^{n}\frac{d\zeta_a d\tau_a}{\zeta_a\tau_a}\prod_{I\in\overline{\mathfrak{N}}}\delta^2(\bar{\mathcal{S}}_I^{\dot\alpha})\prod_{i\in\mathfrak{P}}\delta^2(\mathcal{S}_i^{\alpha})\mathcal{F}(\zeta_a,\tau_a)$$
$$=(-1)^n\left(\prod_{I\in\overline{\mathfrak{N}}}\frac{1}{|k_I^{\perp}|^2}\frac{q_I^{\perp}q_{I+1}^{\perp *}}{q_I^{\perp *}q_{I+1}^{\perp}}\right)\mathcal{F}(\zeta_a,\tau_a)\Big|_{\text{MRK solution}}. \tag{8.52}$$

For gravitational scattering amplitudes, from (8.50) we can easily read that $\mathcal{F}$ function takes

$$\mathcal{F}^{(\text{GR})}=-s^2\left(\prod_{i\in\mathfrak{P}}\frac{1}{(k_i^{\perp})^2}\frac{1}{\zeta_i\tau_i}\right)\det{}'\,\overline{\mathsf{H}}\left(\prod_{I\in\overline{\mathfrak{N}}}\frac{(k_I^{\perp})^2}{\zeta_I\tau_I}\right)\det{}'\,\mathsf{H}. \tag{8.53}$$

The main task of the rest part of this section is to calculate this quantity on the support of the unique solution of the four-dimensional scattering equations in MRK.

MHV sector

Let us first consider the $\mathcal{F}$ function defined in (8.53) in the MHV sector. The experience from the MHV sector will be useful for evaluating the determinants in other N^{k}MHV sectors.

In this case, $\mathfrak{P} = \{3, \ldots, n\}$, the solution in (8.26) becomes $\zeta_i = \tau_i = -1$, and we have

$$\overline{\mathsf{H}}_{ij} = (k_j^{\perp}\zeta_j)\,(k_i^{\perp *}\tau_i) = (k_i^{\perp}k_j^{\perp})\,x_i \ \text{ when } \ i > j, \tag{8.54}$$

$$\overline{\mathsf{H}}_{ij} = (k_i^{\perp}k_j^{\perp})\,x_j \ \text{ when } \ i < j, \tag{8.55}$$

$$\overline{\mathsf{H}}_{ii} = -\sum_{j\in\mathfrak{P}, j\neq i} \overline{\mathsf{H}}_{ij} = (k_i^{\perp})^2\,\big(x_i + v_i\big), \tag{8.56}$$

where we define

$$x_a \equiv \frac{k_a^{\perp *}}{k_a^{\perp}}, \qquad v_a \equiv \frac{k_a^{\perp}q_a^{\perp *} - q_a^{\perp}k_a^{\perp *}}{(k_a^{\perp})^2}. \tag{8.57}$$

Let us choose to delete the first column and row corresponding to the particle label "3" from the matrix $\overline{\mathsf{H}}$. Then the reduced determinant can be written as

$$\det{}'\,\overline{\mathsf{H}} = \frac{1}{(k_3^{\perp})^2}\left(\prod_{i\in\mathfrak{P}}(k_i^{\perp})^2\right)\det\bar{\phi}, \tag{8.58}$$

with

$$\bar{\phi} = \begin{pmatrix} x_4{+}v_4 & x_5 & \cdots & x_{n-1} & x_n \\ x_5 & x_5{+}v_5 & \cdots & x_{n-1} & x_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_{n-1} & \cdots & x_{n-1}{+}v_{n-1} & x_n \\ x_n & x_n & \cdots & x_n & x_n \end{pmatrix}. \tag{8.59}$$

This matrix is nothing but the leading order approximation of Hodges' matrix[1] [148, 149] in MRK. We can observe a lot of nice properties. In particular, a conspicuous feature is that the entries $\bar{\phi}_{ij}$ are equal when $j < i$ for each i-th row. As we now show in the following, this implies further simplification.

[1]For arbitrary external kinematics, in the MHV sector, the four-dimensional scattering equations have only one set of independent of solutions and the formula (8.43) is simply reduced to Hodges' formula where the MHV amplitude of gravitons is given by the determinant of a symmetric matrix.

By performing some elementary row/column transformations of the matrix, we have

$$\det \bar{\phi} \;=\; \det \bar{\phi}', \tag{8.60}$$

with

$$\bar{\phi}' \;=\; \begin{pmatrix} v_4 & x_5{-}x_4{-}v_4 & \cdots & x_{n-1}{-}x_4{-}v_4 & x_n{-}x_4 \\ 0 & v_5 & \cdots & x_{n-1}{-}x_5 & x_n{-}x_5 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{n-1} & x_n{-}x_{n-1} \\ x_n & 0 & \cdots & 0 & x_n \end{pmatrix}. \tag{8.61}$$

This is almost an upper triangular matrix. We find it remarkable that we can nicely compute its determinant by employing the so-called the *matrix determinant lemma*. In order to proceed, let first us present this theorem in brief as follows:

Matrix determinant lemma (MDL) Let $\mathbf{A}$ be an invertible matrix, $\mathbf{u}$ and $\mathbf{v}$ be two vectors. Then the matrix determinant lemma states that [227, 228]

$$\det\left(\mathbf{A} + \mathbf{u}^{\mathrm{T}}\mathbf{v}\right) \;=\; \left(1 + \mathbf{v}\mathbf{A}^{-1}\mathbf{u}^{\mathrm{T}}\right)\,\det(\mathbf{A}). \tag{8.62}$$

Proof: Let us first see the special case of the identity matrix, i.e. $\mathbf{A} = \mathbb{1}$. Noting the following identity [228]

$$\begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{v} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} + \mathbf{u}^{\mathrm{T}}\mathbf{v} & \mathbf{u}^{\mathrm{T}} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ -\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbf{u}^{\mathrm{T}} \\ \mathbf{0} & 1 + \mathbf{v}\mathbf{u}^{\mathrm{T}} \end{pmatrix}, \tag{8.63}$$

calculating the determinant of both sides ends the proof for the case of $\mathbf{A} = \mathbb{1}$. Then from

$$\mathbf{A} + \mathbf{u}^{\mathrm{T}}\mathbf{v} \;=\; \mathbf{A}\left(\mathbb{1} + \mathbf{A}^{-1}\mathbf{u}^{\mathrm{T}}\mathbf{v}\right), \tag{8.64}$$

we can prove the lemme, i.e.,

$$\begin{aligned}\det\left(\mathbf{A}+\mathbf{u}^{\mathrm{T}}\mathbf{v}\right) &= \det\left(\mathbf{A}\right)\det\left(\mathbb{1}+\mathbf{A}^{-1}\mathbf{u}^{\mathrm{T}}\mathbf{v}\right)\\ &= \det(\mathbf{A})\left(1+\mathbf{v}\mathbf{A}^{-1}\mathbf{u}^{\mathrm{T}}\right).\end{aligned} \tag{8.65}$$

We first simply decompose the matrix $\bar{\phi}'$ defined in (8.61) into an upper triangular part and a matrix that has only non-zero element x_n in the lower left corner. To be more precise, we write

$$\bar{\phi}' = \varphi+\mu^{\mathrm{T}}\nu \quad \text{with} \quad \mu=(0,\ldots,0,1),\ \nu=(x_n,0,\ldots,0), \tag{8.66}$$

where φ is nothing but the matrix $\bar{\phi}'$ with replacing the first element of the last row x_n by zero. Then by making use of the matrix determinant lemma, we have

$$\begin{aligned}\det\bar{\phi}' &= \left(\det\varphi\right)\left(1+\nu\varphi^{-1}\mu^{\mathrm{T}}\right)\\ &= v_4\,v_5\cdots v_{n-1}\,x_n\left(1+x_n\left(\varphi^{-1}\right)_{1,n-3}\right).\end{aligned} \tag{8.67}$$

In general, it seems difficult to exactly find the inverse of the matrix φ. Fortunately, it is not hard to obtain the entry $(\varphi^{-1})_{1,n-3}$ by induction (see Appendix D for the detail of derivation),

$$(\varphi^{-1})_{1,n-3} = -\frac{k_3^{\perp}+k_n^{\perp}}{k_n^{\perp *}}. \tag{8.68}$$

Plugging it into (8.67) immediately gives

$$\det\bar{\phi}' = -\frac{k_3^{\perp}}{k_n^{\perp}}\,v_4\,v_5\cdots v_{n-1}\,x_n. \tag{8.69}$$

Noting

$$\mathcal{V}_g(q_i,i^+,q_{i+1}) = q_i^{\perp *}\,v_i\,q_{i+1}^{\perp}, \qquad \mathcal{V}_g(1^-;n^+) = x_n^2, \tag{8.70}$$

and equations (8.69), (8.58), (8.53) and (8.52), we obtain

$$\begin{aligned}&\mathcal{M}_n(1^-,2^-) \qquad (8.71)\\ &\quad= -s^2\,\mathcal{C}_g(2^-;3^+)\,\frac{-1}{|q_4^\perp|^2}\,\mathcal{V}_g(q_4,4^+,q_5)\,\cdots\,\frac{-1}{|q_n^\perp|^2}\,\mathcal{C}_g(1^-;n^+).\end{aligned}$$

In summary, we derived the correct multi-Regge factorization of any MHV amplitude of gravitons. In order to extend the analysis in the MHV sector to other helicity sectors, at this point let us summarize some key technical ideas that have been used above. First, we can transform the matrix into a near upper triangular form by some elementary row and column operations since the matrix has a particular structure in MRK. Then the matrix determinant lemma can be used to compute its determinant. We show in the next section that this technique is useful for other helicity configurations.

All helicity configurations

Let us first consider the positive helicity part $\mathfrak{P}$. In MRK, the entries of $(n-k)\times(n-k)$ matrix $\overline{\mathsf{H}}$ take

$$\overline{\mathsf{H}}_{ij} = (k_j^\perp\zeta_j)\,(k_i^{\perp *}\tau_i) = (k_i^\perp\tau_i)\,(k_j^\perp\zeta_j)\,x_i, \quad \text{for } i>j \qquad (8.72)$$

$$\overline{\mathsf{H}}_{ij} = (k_j^\perp\tau_j)(k_i^\perp\zeta_i)\,x_j = (k_i^\perp\tau_i)(k_j^\perp\zeta_j)\,c_{ij},\ c_{ij} = \frac{\zeta_i\tau_j\,x_j}{\zeta_j\tau_i},\ \text{for } i<j.$$

For diagonal elements, we have

$$\begin{aligned}\overline{\mathsf{H}}_{ii} &= -\sum_{j\in\mathfrak{P}_{<i}}\overline{\mathsf{H}}_{ij} - \sum_{j\in\mathfrak{P}_{>i}}\overline{\mathsf{H}}_{ij}\\ &= -(\tau_i k_i^{\perp *})\sum_{j\in\mathfrak{P},j<i}\zeta_j\,k_j^\perp - (\zeta_i k_i^\perp)\sum_{j\in\mathfrak{P},j>i}\tau_j\,k_j^{\perp *}\\ &= -\tau_i k_i^{\perp *}\sum_{j\in\mathfrak{P},j<i}\zeta_j\,k_j^\perp + \zeta_i k_i^\perp\sum_{j\in\mathfrak{P},j\le i}\tau_j\,k_j^{\perp *}\\ &= (k_i^\perp\zeta_i)\,(k_i^\perp\tau_i)\,\big(x_i+u_i\big), \qquad (8.73)\end{aligned}$$

where one has used the momentum conservation (8.30) in the third line, and u_i is defined as

$$u_i \equiv \sum_{j\in\mathfrak{P}_{<i}} \left(\frac{\tau_j\, k_j^\perp}{\tau_i\, k_i^\perp} x_j - \frac{\zeta_j\, k_j^\perp}{\zeta_i\, k_i^\perp} x_i \right) . \tag{8.74}$$

Using the scattering equations (8.2) and their solution, one can further obtain

$$u_i = v_i = \frac{k_i^\perp q_i^{\perp *} - q_i^\perp k_i^{\perp *}}{(k_i^\perp)^2} . \tag{8.75}$$

Let us delete the first column and row corresponding to particle label "3", then the reduced determinant becomes

$$\det{}' \overline{\mathsf{H}} = \left(\prod_{i\in\mathfrak{P}, i\neq 3} (k_i^\perp)^2 \zeta_i\, \tau_i \right) \det \overline{\mathsf{H}}', \tag{8.76}$$

where

$$\overline{\mathsf{H}}' = \begin{pmatrix} v_{i_1}+x_{i_1} & c_{i_1 i_2} & c_{i_1 i_3} & \cdots & c_{i_1 n} \\ x_{i_2} & v_{i_2}+x_{i_2} & c_{i_2 i_3} & \cdots & c_{i_2 n} \\ x_{i_3} & x_{i_3} & v_{i_3}+x_{i_3} & \cdots & c_{i_3 n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & x_n & \cdots & x_n \end{pmatrix} . \tag{8.77}$$

Here labels satisfy $3 < i_1 < i_2 < \cdots < n$. In the case of the MHV sector, since $c_{ij} = x_j$ $(i < j)$, this matrix is identical to the matrix $\bar{\phi}$ in (8.59). More remarkably, they have a similar structure and share some useful properties. As a consequence, we can use the same technique to calculate the determinant as in the MHV sector. We summarize the result as follows (see Appendix D for a detailed derivation):

$$\det{}' \overline{\mathsf{H}} = \left(\prod_{i\in\mathfrak{P}} (k_i^\perp)^2\, \zeta_i\, \tau_i \right) \frac{x_n}{q_4^\perp q_n^\perp} \prod_{\substack{i\in\mathfrak{P} \\ i\neq 3,n}} v_i\, . \tag{8.78}$$

Let us now discuss the $k \times k$ matrix H. It is reasonable to expect that the similar structure appears in this matrix such that we can compute its determinant using the matrix determinant lemma. Let us first compute its entries as follows:

$$\mathsf{H}_{12} = -1, \qquad \mathsf{H}_{1I} = -\zeta_I, \tag{8.79}$$

$$\mathsf{H}_{I2} = (x_I \tau_I) x_I^*, \qquad \mathsf{H}_{2I} = c_{2I}\, \zeta_I, \quad c_{2I} \equiv \frac{\tau_I}{\zeta_I}, \tag{8.80}$$

$$\mathsf{H}_{IJ} = (x_I \tau_I \zeta_J) x_I^*, \quad I > J \tag{8.81}$$

$$\mathsf{H}_{IJ} = c_{IJ} (x_I \tau_I \zeta_J) x_I^*, \quad c_{IJ} \equiv \frac{\zeta_I \tau_J}{\tau_I \zeta_J}, \quad I < J \tag{8.82}$$

$$\mathsf{H}_{22} = -\mathsf{H}_{12} - \sum_{I \in \overline{\mathfrak{N}}} \mathsf{H}_{2I}, \tag{8.83}$$

$$\mathsf{H}_{II} = -\mathsf{H}_{1I} - \mathsf{H}_{2I} - \sum_{J \in \overline{\mathfrak{N}}_{<I}} \mathsf{H}_{IJ} - \sum_{J \in \overline{\mathfrak{N}}_{>I}} \mathsf{H}_{IJ}. \tag{8.84}$$

By using the scattering equations (8.2) and their unique solution, it is easy to obtain

$$\mathsf{H}_{22} = 1 - \sum_{I \in \overline{\mathfrak{N}}} \tau_I = \prod_{I \in \overline{\mathfrak{N}}} \frac{q_I^\perp}{q_{I+1}^\perp}, \tag{8.85}$$

$$\mathsf{H}_{II} = \zeta_I \Bigg(1 - \sum_{J \in \overline{\mathfrak{N}}_{>I}} \tau_J \Bigg) - \tau_I \Bigg(1 + \sum_{J \in \overline{\mathfrak{N}}_{<I}} \zeta_J \Bigg) = x_I \zeta_I \tau_I \, (v_I^* + x_I^*).$$

Then we have

$$\det{}' \mathsf{H} = \Bigg(\prod_{I \in \overline{\mathfrak{N}}} x_I \zeta_I \tau_I \Bigg) \det \mathsf{H}', \tag{8.86}$$

where one choose to remove the first column and row corresponding to particle label "1", and H' is defined as

$$\mathsf{H}' = \begin{pmatrix} \mathsf{H}_{22} & c_{2I_1} & c_{2I_2} & \cdots & c_{2I_m} \\ x^*_{I_1} & v^*_{I_1}+x^*_{I_1} & c_{I_1I_2}x^*_{I_1} & \cdots & c_{I_1I_m}x^*_{I_1} \\ x^*_{I_2} & x^*_{I_2} & v^*_{I_2}+x^*_{I_2} & \cdots & c_{I_2I_m}x^*_{I_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^*_{I_m} & x^*_{I_m} & x^*_{I_m} & \cdots & v^*_{I_m}+x^*_{I_m} \end{pmatrix}, \tag{8.87}$$

where $I_1 < \cdots < I_m \in \overline{\mathfrak{N}}$, $m = k{-}2$. This matrix again displays a similar structure with $\bar{\phi}$ in (8.59). Hence we can calculate its determinant by a similar technique based on the matrix determinant lemma. The final result is

$$\det{}' \mathsf{H} = \left(\prod_{I\in\overline{\mathfrak{N}}} x_I \zeta_I \tau_I\right) \prod_{I\in\overline{\mathfrak{N}}} v^*_I. \tag{8.88}$$

The details of the derivation can be found in Appendix D.

Putting everything together, we obtain the fully-localized form of any N^{k}MHV amplitude of gravitons

$$\begin{aligned} \mathcal{M}_n \simeq\; & -s^2\, \mathcal{C}_g(2;3) \frac{-1}{|q_4^\perp|^2} \mathcal{V}_g(q_4;4;q_5) \cdots \\ & \times \cdots \frac{-1}{|q_{n-1}^\perp|^2} \mathcal{V}_g(q_{n-1};n{-}1;q_n) \frac{-1}{|q_n^\perp|^2} \mathcal{C}_g(1;n), \end{aligned}$$

with

$$\mathcal{V}_g(q_i, i^+, q_{i+1}) = q_i^{\perp *} v_i\, q_{i+1}^\perp = \frac{q_i^{\perp *}\left(k_i^\perp q_i^{\perp *} - k_i^{\perp *} q_i^\perp\right) q_{i+1}^\perp}{(k_i^\perp)^2}, \tag{8.89}$$

$$\mathcal{V}_g(q_I, I^-, q_{I+1}) = q_I^\perp v_I^*\, q_{I+1}^{\perp *} = \frac{q_I^\perp\left(k_I^{\perp *} q_I^\perp - k_I^\perp q_I^{\perp *}\right) q_{I+1}^{\perp *}}{(k_I^{\perp *})^2}, \tag{8.90}$$

in agreement with the Lipatov formula (7.7). Finally, we provide an elegant derivation of the tree-level multi-Regge factorization of gravitational scattering amplitudes.

Let us conclude this section by making some comments. First, we have assumed in previous sections that the two incoming particles 1 and 2 carry the same helicity. Here we show that all conclusions hold for the case where the gravitons 1 and 2 have opposite helicities. For example, let us consider amplitude $\mathcal{M}_n$ with helicity configuration $(1^+, 2^-, n^-, \ldots)$. In this case, in MRK, we have

$$\mathcal{M}_n(1^+, 2^-, n^-, \ldots) \;=\; \mathcal{M}_n(1^-, 2^-, n^+, \ldots) \left(\frac{1}{(1\,n)^8} \bigg|_{\text{MRK}} \right). \tag{8.91}$$

The factor $(1\,n)^{-8}$ combines with the impact factor $\mathcal{C}_g(1^+; n^-)$ to give

$$\left(\frac{1}{(1\,n)^8} \bigg|_{\text{MRK}} \right) \mathcal{C}_g(1^-; n^+) \;=\; \mathcal{C}_g(1^-; n^+)^* \;=\; \mathcal{C}_g(1^+; n^-). \tag{8.92}$$

Second, we would like to show that the helicity is conserved in gravitational impact factors, like gauge theory. As an example, we consider amplitude $\mathcal{M}_n(1^-, 2^-, 3^-, 4^+, \ldots)$. In the multi-Regge limit, we have

$$\begin{aligned} &\mathcal{M}_n(1^-, 2^-, 3^-, 4^+, \ldots) \\ &\quad = \mathcal{M}_n(1^-, 2^-, 3^+, 4^-, \ldots) \left(\frac{1}{(3\,4)^8} \bigg|_{\text{MRK}} \right) \\ &\quad = \mathcal{M}_n(1^-, 2^-, 3^+, 4^+, \ldots) \left(-\frac{(k_4^\perp)^2\, q_4^\perp q_5^{\perp *}}{(k_4^{\perp *})^2\, q_4^{\perp *} q_5^\perp} \right) \left(\frac{1}{(3\,4)^8} \bigg|_{\text{MRK}} \right). \end{aligned} \tag{8.93}$$

This shows that the amplitude $\mathcal{M}_n(1^-, 2^-, 3^-, 4^+, \ldots)$ is suppressed in MRK since

$$\left(\frac{1}{(3\,4)^8} \bigg|_{\text{MRK}} \right) \simeq \mathcal{O}\Big(\big(k_4^+ / k_3^+ \big)^4 \Big), \tag{8.94}$$

which is implied by our conjecture in the solutions of the scattering equations in MRK.

Finally, let us see what happens in the case where particles with other spins in supergravity multiplet are involved. For example, in case of one pair of gravitinos, i.e. $(1^-_{\tilde{G}}, n^+_{\tilde{G}})$

$$\mathcal{M}_n(1^-_{\tilde{G}}, n^+_{\tilde{G}}, \ldots) = \mathcal{M}_n(1^-, n^+, \ldots) \left(\frac{-1}{(1\,n)} \bigg|_{\text{MRK}} \right). \tag{8.95}$$

Similarly, a combination of the factor $(1\,n)^{-1}$ and the graviton impact factor $\mathcal{C}_g(1^+; n^-)$ gives

$$\left(\frac{-1}{(1\,n)} \bigg|_{\text{MRK}} \right) \mathcal{C}_g(1^-; n^+) = \left(\frac{k_n^{\perp *}}{k_n^{\perp}} \right)^{3/2} = \mathcal{C}_g(1^-_{\tilde{G}}; n^+_{\tilde{G}}), \tag{8.96}$$

which exactly agrees with the result in [221, 222]. It is also easy to obtain the result for other cases where particles with other spins in the supergravity multiplet are involved.

8.4 Conclusions

We have shown that our conjecture implies that the scattering equations, as well as the CHY-type formulas for amplitudes in gauge theory and gravity, simplify vastly in MRK. We have explicitly determined the unique solution (up to $\mathrm{GL}(2, \mathbb{C})$ redundancy) of the four-dimensional scattering equations in MRK for any multiplicity and any helicity configuration, and furthermore, derived the expected multi-Regge factorization of the amplitudes by inserting the MRK solution into the CHY-type formulas in both Yang-Mills and Einstein gravity in this chapter. These give very strong support to the validity of our conjecture.

We will extend our investigation to the quasi-multi-Regge kinematics in the following chapter.

9 Quasi-multi-Regge factorization from the scattering equations

In the previous chapter, we have shown that one can derive the factorization of gluon and graviton amplitudes from the scattering equations baed on the conjecture presented in Chapter 7. One feature of the derivation is independent of the number n of external legs as well as their helicities. In this chapter, we extend the analysis to quasi-multi-Regge regimes. More precisely, we investigate quasi-multi-Regge limits of gluon amplitudes. We will show that, in agreement with expectations from Regge theory, in each of these two cases our conjecture implies that the amplitude factorizes into a set of universal building blocks which are multi-particle generalisations of the impact factors and Lipatov vertices in the case of MRK, independently of the multiplicity and the helicities of the produced particles. As a byproduct, we obtain CHY-type representations for these building blocks.

9.1 The first type of quasi-multi-Regge limits

The goal of this section is to show that our conjecture implies that in the quasi-multi-Regge limit where

$$y_3 \simeq \cdots \simeq y_r \gg \cdots \gg y_n, \tag{9.1}$$

any tree-level gluon amplitude factorizes as

$$\begin{aligned}\mathcal{A}_n(1,\ldots,n) \simeq s\,\mathcal{C}(2;3,\ldots,r)\,\frac{-1}{|q_{r+1}^{\perp}|^2}\,\mathcal{V}(q_{r+1};r+1;q_{r+2})\cdots \\ \times\cdots\times\frac{-1}{|q_{n-1}^{\perp}|^2}\,\mathcal{V}(q_{n-1};n-1;q_n)\,\frac{-1}{|q_n^{\perp}|^2}\,\mathcal{C}(1;n),\end{aligned} \tag{9.2}$$

where $\mathcal{C}(2;3,\ldots,r)$ is a generalised impact factor that only depends on the subset of momenta $(k_2,\ldots,k_r)$ and corresponding helicities. We will now show that the generalised impact factors are universal, i.e., they do not depend on the quantum numbers of the other particles involved in the scattering.

Let us start by analysing the case $r = n-1$. We assume without loss of generality that particles 1 and n have negative and positive helicities respectively. For simplicity, we also assume that $h_2 = -1$. The case $h_2 = +1$ can be recovered by using the same argument described in the previous chapter. The general logic will be similar to the MRK case, so we will be brief and we will not describe all the steps in detail here. We start by fixing the $\mathrm{GL}(2,\mathbb{C})$ redundancy as in eq. (7.14), and we apply Conjecture 4d to expand the scattering equations to leading order in the limit. We observe that there is a subset of scattering equations that become linear in τ_n and ζ_n,

$$\mathcal{S}_n^1 = 1+\tau_n\left(1+\sum_{I\in\overline{\mathfrak{N}}}\zeta_I\right) = 0\,, \tag{9.3}$$

$$\mathcal{S}_n^2 = 1+\zeta_n = 0\,, \tag{9.4}$$

where we use the rescaled variables defined in eqs. (7.15) and (8.3). Note that here we keep only the leading-power terms in the limit (9.1) with $r = n-1$. We can easily evaluate the corresponding residues,

$$\frac{(q_n^\perp)^*}{k_n^\perp}\int\frac{d\tau_n}{\tau_n}\int\frac{d\zeta_n}{\zeta_n}\,\delta\big(S_n^1\big)\delta\big(S_n^2\big) = \frac{(k_n^\perp)^*}{k_n^\perp} = \mathcal{C}(1^-;n^+)\,, \tag{9.5}$$

where we have identified it with the impact factor $\mathcal{C}(1;n)$. The amplitude then takes the expected factorized form

$$\mathcal{A}_n(1^-,2^-,\ldots,n^+) \simeq s\,\mathcal{C}(2^-;3,\ldots,n-1)\,\frac{-1}{|q_n^\perp|^2}\,\mathcal{C}(1^-;n^+)\,, \tag{9.6}$$

where $\mathcal{C}(1^-; n^+)$ is the same impact factor as in MRK, cf. eq. (7.5). The generalised impact factor $\mathcal{C}(2^-; 3, \ldots, n{-}1)$ admits a CHY-type representation,

$$\begin{aligned}
&\mathcal{C}\big(2^-; 3, \ldots, n{-}1\big) \qquad (9.7)\\
&= q_n^\perp \int \prod_{a=3}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \frac{1}{\sigma_{34}\cdots\sigma_{n-2,n-1}\sigma_{n-1}} \left(\prod_{i\in\mathfrak{P}, I\in\overline{\mathfrak{N}}} \frac{k_I^\perp}{k_i^\perp} \right)\\
&\quad \times \prod_{I\in\overline{\mathfrak{N}}} \delta\left(k_I^\perp - \sum_{i\in\mathfrak{P}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} k_i^+ \right)\\
&\quad \times \prod_{I\in\overline{\mathfrak{N}}} \delta\left((k_I^\perp)^* - \frac{k_I^+}{k_I^\perp} \sum_{i\in\mathfrak{P}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} (k_i^\perp)^* - \zeta_I \frac{(q_n^\perp)^*}{1+\sum_{J\in\overline{\mathfrak{N}}}\zeta_J} \right)\\
&\quad \times \prod_{i\in\mathfrak{P}} \delta\left(1 + \tau_i - \sum_{I\in\overline{\mathfrak{N}}} \frac{\tau_i\tau_I}{\sigma_i{-}\sigma_I} \frac{k_I^+}{k_I^\perp} \right) \delta\left(1 + \zeta_i - \frac{k_i^+}{k_i^\perp} \sum_{I\in\overline{\mathfrak{N}}} \frac{\tau_i\tau_I}{\sigma_i{-}\sigma_I} \right),
\end{aligned}$$

where $\overline{\mathfrak{N}} = \mathfrak{N}\backslash\{2\}$ and $q_n = \sum_{a=2}^{n-1} k_a$ is the total momentum exchanged in the t-channel. A similar formula can be derived when $h_2 = +1$. We have checked that our formula correctly reproduces various known results in the literature [71, 225, 229]. In particular, eq. (9.7) correctly reproduces known results for the MHV-type impact factors $\mathcal{C}\big(2^-; 3^+, \ldots, (n{-}1)^+\big)$ for arbitrary multiplicities. All results obtained are collected in Appendix E. Moreover, we have checked that this formula reproduces the correct results for up to $n = 7$ numerically.

In Appendix F, we also show the formula in eq. (9.7) has the factorization properties expected from a gluon amplitude at tree level. In particular, if the gluon i becomes soft, $k_i \to 0$, the impact factor factorizes into an impact factor with one particle less, times the usual eikonal factor [79]. For example, if the soft gluon carries positive helicity, we have

$$\mathcal{C}\big(2; 3, \ldots, a, i, b, \ldots, n{-}1\big) \simeq \mathcal{C}\big(2; 3, \ldots, a, b, \ldots, n{-}1\big) \frac{\langle a\, b\rangle}{\langle a\, i\rangle \langle i\, b\rangle}, \qquad (9.8)$$

where $(a, b) = (i{-}1, i{+}1)$ are the gluons adjacent to i for the chosen color-ordering. Similarly, if two produced gluons, say i and $i{+}1$, become collinear,

we have

$$\begin{aligned}
&\mathcal{C}\big(2;3,\ldots,n{-}1\big)\\
&\qquad\simeq \mathcal{C}\big(2;3,\ldots,i{-}1,p^h,i{+}2,\ldots,n{-}1\big)\,\texttt{Split}_{-h}\big(i,i{+}1\big)\,,
\end{aligned}\tag{9.9}$$

where $p = k_i + k_{i+1}$ denotes the momentum of the parent gluon before the splitting, and Split denotes the usual tree-level splitting function (c.f. eq. (2.18) for explicit forms). Finally, in the case where particle $n{-}1$ has much smaller rapidity as the other produced gluons, we have

$$\mathcal{C}\big(2;3,\ldots,n{-}1\big) \simeq \mathcal{C}\big(2;3,\ldots,n{-}2\big)\,\frac{-1}{|q_{n-1}^{\perp}|^2}\,\mathcal{V}(q_{n-1};n{-}1;q_n), \tag{9.10}$$

with $q_n = q_{n-1} + k_{n-1} = k_2 + k_3 + \cdots + k_{n-1}$.

The previous considerations also imply that the amplitude has the expected factorization behaviour in the quasi-multi-Regge limit where $y_3 \simeq \cdots \simeq y_r \gg \cdots \gg y_n$. Indeed, we can simply apply eq. (9.10) iteratively to add one large rapidity gap at the time, and we immediately see that the amplitude takes the factorized form given in eq. (9.2). We note that this factorization is a direct consequence of Conjecture 4d and the CHY-type formulas of gluon amplitudes, and it does not rely on any other assumptions. In particular, we see that the factorized form holds for arbitrary multiplicities and helicity configurations. Moreover, we see that the impact factors are universal, in the sense they do not depend on the quantum numbers of the other produced particles, in agreement with general expectations.

9.2 The second type of quasi-multi-Regge limits

In this section, we study another type of quasi-multi-Regge limits

$$y_3 \gg \cdots \gg y_r \simeq \cdots \simeq y_s \gg \cdots \gg y_n, \tag{9.11}$$

where the amplitude is expected to factorize as follows:

$$\mathcal{A}_n(1,\ldots,n) \simeq s\,\mathcal{C}(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}(q_4;4;q_5)\cdots\frac{-1}{|q_r^\perp|^2}\mathcal{V}(q_r;r,\ldots,s;q_{s+1})\cdots \\ \times\cdots\frac{-1}{|q_{n-1}^\perp|^2}\mathcal{V}(q_{n-1};n{-}1;q_n)\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1;n). \tag{9.12}$$

We start by analysing the case $(r,s)=(4,n{-}1)$, and we assume that $(h_1,h_2)=(-1,-1)$. Helicity conservation implies that we obtain a non-zero result only if $(h_3,h_n)=(+1,+1)$, which we assume from now on. We proceed in the usual way, and we use Conjecture 4d to expand the scattering equations to leading order in the limit. We then observe that the equations $\mathcal{S}_3^\alpha=\mathcal{S}_n^\alpha=0$ are linear in ζ_i and τ_i, $i\in\{3,n\}$. We can thus localize the integrals over these variables on the residues obtained by solving these linear equations. We arrive at the following factorized form for the amplitude:

$$\mathcal{A}_n(1^-,2^-,3,\ldots,n) \\ \simeq s\,\mathcal{C}(2^-;3)\frac{-1}{|q_4^\perp|^2}\,\mathcal{V}(q_4;4,\ldots,n{-}1;q_n)\,\frac{-1}{|q_n^\perp|^2}\,\mathcal{C}(1^-;n), \tag{9.13}$$

where the generalised Lipatov vertices admit the following CHY-type representation,

$$\mathcal{V}\big(q_4;4,\ldots,n{-}1;q_n\big) \\ = (q_4^\perp)^* q_n^\perp \int\prod_{a=4}^{n-1}\frac{d\sigma_a dt_a}{t_a}\,\frac{1}{\sigma_{45}\cdots\sigma_{n-2,n-1}\sigma_{n-1}}\left(\prod_{i\in\mathfrak{P},I\in\mathfrak{N}}\frac{k_I^\perp}{k_i^\perp}\right) \\ \times\prod_{I\in\mathfrak{N}}\delta\left(k_I^\perp-\sum_{i\in\mathfrak{P}}\frac{t_i t_I}{\sigma_I-\sigma_i}k_i^+ + \frac{t_I}{1-\sum_{J\in\mathfrak{N}}t_J}\,q_4^\perp\right) \\ \times\prod_{I\in\mathfrak{N}}\delta\left(k_I^{\perp *}-\frac{k_I^+}{k_I^\perp}\sum_{i\in\mathfrak{P}}\frac{t_i t_I}{\sigma_I-\sigma_i}k_i^{\perp *}-\frac{\zeta_I}{1+\sum_{J\in\mathfrak{N}}\zeta_J}\,q_n^{\perp *}\right) \\ \times\prod_{i\in\mathfrak{P}}\delta\left(1-\sum_{I\in\mathfrak{N}}\frac{t_i t_I}{\sigma_i-\sigma_I}\frac{k_I^+}{k_I^\perp}+t_i\right)\delta\left(1-\frac{k_i^+}{k_i^\perp}\sum_{I\in\mathfrak{N}}\frac{t_i t_I}{\sigma_i-\sigma_I}+\zeta_i\right). \tag{9.14}$$

Here $\mathfrak{N}$ and $\mathfrak{P}$ denote the subsets of $\{4,\ldots,n-1\}$ of particles with negative and positive helicity respectively. In previous analyses, we always remove the four delta-functions $\delta^2(\bar{\mathcal{S}}_1^{\dot\alpha})\delta^2(\bar{\mathcal{S}}_2^{\dot\alpha})$ and identify them with the momentum conservation constraint, cf. eq. (8.30). Alternatively, we can also use the equations $\bar{\mathcal{S}}_1^{\dot\alpha}=\bar{\mathcal{S}}_2^{\dot\alpha}=0$ to localize the integrals over $\{\sigma_3,\tau_3,\sigma_n,\tau_n\}$ and identify $\delta^2(\mathcal{S}_3^{\alpha})\delta^2(\mathcal{S}_n^{\alpha})$ with the δ-function expressing momentum conservation:

$$\delta^2\big(\mathcal{S}_3^{\alpha}\big)\,\delta^2\big(\mathcal{S}_n^{\alpha}\big) \;=\; s\,k_3^{\perp}k_n^{\perp^*}\,\delta^4\left(\sum_{a=1}^{n}k_a^{\mu}\right). \tag{9.15}$$

Consequently, this leads to the following alternatively equivalent formula for the generalized Lipatov vertex:

$$\begin{aligned}
&\mathcal{V}\big(q_4;4,\ldots,n-1;q_n\big)\\
&\;=\;(q_4^{\perp^*}q_n^{\perp})\int\prod_{a=4}^{n-1}\frac{d\sigma_a d\tau_a}{\tau_a}\,\frac{\mathcal{J}_{1,2;3,n}}{\sigma_{45}\cdots\sigma_{n-2,n-1}\sigma_{n-1}}\left(\prod_{i\in\mathfrak{P},I\in\mathfrak{N}}\frac{k_I^{\perp}}{k_i^{\perp}}\right)\\
&\quad\times\prod_{I\in\mathfrak{N}}\delta\left(k_I^{\perp}-\sum_{i\in\mathfrak{P}}\left(\frac{\tau_I\tau_i}{\sigma_I-\sigma_i}+\frac{\tau_I\tau_i}{\sigma_i}\right)k_i^{+}-\tau_I q_n^{\perp}\right)\\
&\quad\times\prod_{I\in\mathfrak{N}}\delta\left(k_I^{\perp^*}-\sum_{i\in\mathfrak{P}}\left(\frac{k_I^{+}}{k_I^{\perp}}\frac{\tau_I\tau_i}{\sigma_I-\sigma_i}-\zeta_I\tau_i\right)k_i^{\perp^*}-\zeta_I q_4^{\perp^*}\right)\\
&\quad\times\prod_{i\in\mathfrak{P}}\delta\left(1-\sum_{I\in\mathfrak{N}}\frac{\tau_i\tau_I}{\sigma_i-\sigma_I}\frac{k_I^{+}}{k_I^{\perp}}+\tau_i\right)\delta\left(1-\frac{k_i^{+}}{k_i^{\perp}}\sum_{I\in\mathfrak{N}}\frac{\tau_i\tau_I}{\sigma_i-\sigma_I}+\zeta_i\right),
\end{aligned} \tag{9.16}$$

where

$$\mathcal{J}_{1,2;3,n}=q_4^{\perp}q_n^{\perp^*}\left(q_n^{\perp}+\sum_{i\in\mathfrak{P}}\zeta_i k_i^{\perp}\right)^{-1}\left(q_4^{\perp^*}-\sum_{i\in\mathfrak{P}}\tau_i k_i^{\perp^*}\right)^{-1}. \tag{9.17}$$

We have checked that the two CHY-type formulas are consistent with known results in the literature for generalised Lipatov vertices for the production up to three particles, as well as for NMHV-type Lipatov vertices for the production of up to four particles, see Appendix E for analytic results. In addition, we show in Appendix F that eq. (9.14) has the correct factorization properties. In particular, if y_4 (y_{n-1}) is much greater (smaller) than the rapidities of all the other emitted particles, then the Liptatov vertex itself factorizes. More precisely, in the limit $y_4 \gg y_i$ with $5 \leq i \leq n-1$

$$\mathcal{V}\big(q_4; 4, \ldots, n-1; q_n\big) \;\simeq\; \mathcal{V}\big(q_4; 4; q_5\big)\, \frac{-1}{|q_5^\perp|^2}\, \mathcal{V}\big(q_5; 5, \ldots, n-1; q_n\big), \quad (9.18)$$

and in the limit $y_{n-1} \ll y_i$ with $4 \leq i \leq n-2$ we have

$$\begin{aligned} &\mathcal{V}\big(q_4; 4, \ldots, n-1; q_n\big) \\ &\quad \simeq\; \mathcal{V}\big(q_4; 4, \ldots, n-2; q_{n-1}\big)\, \frac{-1}{|q_{n-1}^\perp|^2}\, \mathcal{V}\big(q_{n-1}; n-1; q_n\big). \end{aligned} \quad (9.19)$$

Using these properties we can iterate the factorization and gradually approach the quasi-multi-Regge limit $y_3 \gg \cdots \gg y_r \simeq \cdots \simeq y_s \gg \cdots \gg y_n$, and we see that the amplitude takes the factorized form in eq. (9.12). We emphasise again the factorization is a direct consequence of Conjecture 4d and the CHY-type formulas for gluon amplitudes. In particular, we find that this factorization holds for arbitrary helicity configurations and that the ensuing generalised Lipatov vertices are universal and do not depend on the quantum numbers of the other particles involved in the scattering.

9.3 Comments

In the previous sections we have studied two families of quasi-multi-Regge limits, and we have shown that in those cases our conjecture implies that the amplitude has the expected factorization into universal building blocks. In principle, one could also consider more general quasi-multi-Regge limits, like for example

$$y_3 \simeq \ldots \simeq y_{r-1} \gg y_r \simeq \ldots \simeq y_s \gg y_{s+1} \simeq \ldots \simeq y_n\,. \quad (9.20)$$

In this limit the amplitude is expected to factorize as

$$\mathcal{A}_n(1,\ldots,n) \simeq s\,\mathcal{C}(2;3,\ldots,r-1)\frac{-1}{|q_r^\perp|^2}\mathcal{V}(q_r;r,\ldots,s;q_{s+1}) \times \frac{-1}{|q_{s+1}^\perp|^2}\mathcal{C}(1;s{+}1,\ldots,n). \tag{9.21}$$

Unfortunately, we are currently not able to derive this factorization from Conjecture 4d and the CHY-type formula of the amplitude. The main obstacle is that for these more general quasi-multi-Regge limits we cannot identify a set of variables that enter the scattering equations linearly in the limit. This property, however, was the cornerstone in previous sections to prove factorization in (quasi-)multi-Regge limits. We stress that our inability to derive factorizations for more general QMRKs from Conjecture 4d and the CHY-type formula of the amplitude does by no means imply that no such factorization exists. Indeed, if we consider for example eq. (9.21) and we insert the CHY-type formulas for the generalised impact factors and Lipatov vertices in eqs. (9.7) and (9.14) we obtain a representation for the amplitude in this limit that is consistent with the known factorizations of the amplitude in all soft, collinear and multi-Regge limits. We then find it hard to imagine that the amplitude could take any other form in this limit than the one given in eq. (9.21).

A second comment is about amplitudes involving quarks. There are various quasi-multi-Regge limits which involve one or more quark pairs in the final state. For example, in the QMRK $y_3 \gg y_4 \simeq y_5 \gg y_6$, we have

$$\mathcal{A}_6(1,2,3,4_q,5_{\bar{q}},6) \simeq s\,\mathcal{C}(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}(q_4;4_q,5_{\bar{q}};q_6)\frac{-1}{|q_6^\perp|^2}\mathcal{C}(1;6). \tag{9.22}$$

Since there are CHY-type formulas for tree-level amplitudes involving massless quarks [133], and since the scattering equations are universal and do not depend on the details of the theory, we can immediately extend our analysis to these amplitudes and use our conjecture to derive their factorization in various quasi-multi-Regge limits. Similarly, we can also obtain CHY-type representations for the corresponding generalised impact factors and Lipatov vertices, and we have checked that in this case we are able to reproduce the known analytic expressions from the literature [48,71,225,230]. For example, we list the analytic result for $\mathcal{V}(q_2;4_{\bar{q}}^+,5_q^-;q_1)$ in Appendix E.

9.4 Conclusions and discussions

Together with the previous two chapters, we have initiated the study of Regge kinematics through the lens of the scattering equations. Based on numerical studies, we have formulated a precise conjecture about the behaviour of the solutions to the scattering equations in the Regge limit, both in their D and four-dimensional versions in Chapter 7. While we currently have no proof of our conjecture, we have tested its validity by showing that we can derive the expected factorization of the amplitudes when we combine the conjecture with the four-dimensional CHY-type formulas. This is a highly non-trivial prediction of our conjecture, which gives us confidence that it describes the correct asymptotic behaviour of the solutions of the scattering equations in QMRK.

Our conjecture is not only of formal interest, but it has concrete applications to tree-level scattering amplitudes. In particular, we have applied our conjecture to show that in MRK the four-dimensional scattering equations have a unique solution (up to $\mathrm{GL}(2,\mathbb{C})$ redundancy), independently of the multiplicity, and we have explicitly determined this MRK solution in Chapter 8. We find it remarkable that in MRK we may find the exact solution of the four-dimensional scattering equations for any sector k for arbitrary multiplicities. Indeed, so far this has only been achieved for the MHV sector. In QMRK we cannot obtain exact solutions to the scattering equations anymore. Instead, we have derived CHY-type formulas for the generalised impact factors and Lipatov vertices valid for an arbitrary number of particles, and we have checked that these formulas reproduce known analytic results from the literature for low multiplicities in Chapter 9.

We see two possible directions for future research. First, it would be interesting to find proof of our conjecture. This would not only clarify some new mathematical property of the scattering equations, but it would have direct implications for tree-level scattering amplitudes. Indeed, so far the factorization of color-ordered helicity amplitudes has only been rigorously proven for arbitrary multiplicities for the simplest helicity configurations. In this work, we have shown that our conjecture implies the expected factorization for arbitrary helicity configurations. A rigorous mathematical proof of our conjecture would thus immediately lead to an elegant proof of the Regge factorization of all tree-level amplitudes in Yang-Mills and gravity.

Second, while in this work we have focused exclusively on gluon and graviton amplitudes, the scattering equations and the CHY-type formula are valid for much larger classes of massless quantum field theories, and thus our conjecture applies also to theories other than gauge theory and gravity. In particular, since we have shown that our conjecture implies the factorization of gauge theory amplitudes in various quasi-multi-Regge limits, it would be interesting to investigate if similar factorizations can be derived for other massless quantum field theories in QMRK. The study of Regge kinematics has so far mostly focused on Yang-Mills and gravity, and it would be interesting to study the implications for other theories.

10 Solving the scattering equations by homotopy continuation

We present an efficient method to solve the scattering equations numerically in this chapter. The content in this chapter is based on the paper [231].

10.1 Homotopy continuation

Let us first give a brief introduction to homotopy continuation [232, 233], which is the primary method in numerical algebraic geometry that we will use throughout this chapter. In order to solve a system of equations $p(z) = 0$ with $p(z) = (p_1(z), \ldots, p_N(z))$ and $z = (z_1, \ldots, z_N)$, the fundamental idea is to introduce a continuous deformation (*homotopy*)

$$p(z) \to p(z,t), \quad t \in [0,1], \tag{10.1}$$

which connects the *target system* $p(z,1) = p(z)$ with a *start system* $p(z,0) = 0$ whose solutions $z(0)$ are known. Then the solutions $z(1)$ of the target system can be obtained from $z(0)$ via smooth paths as the continuation parameter t varies from 0 to 1. To be explicit, constructing a differentiable homotopy $p(z,t)$ and differentiating it with respect to t lead to a system of *ordinary differential equations (ODEs)* on $z = z(t)$ as follows:

$$\frac{\mathrm{d}p_i(z,t)}{\mathrm{d}t} = \sum_{j=1}^{N} \frac{\partial p_i(z,t)}{\partial z_j} \frac{\mathrm{d}z_j(t)}{\mathrm{d}t} + \frac{\partial p_i(z,t)}{\partial t} = 0. \tag{10.2}$$

Viewing this as a system of linear equations on $\mathrm{d}z_i/\mathrm{d}t$, it can be transformed into the following standard form:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right) = -\left(\frac{\partial p(z,t)}{\partial z}\right)^{-1}\left(\frac{\partial p(z,t)}{\partial t}\right), \tag{10.3}$$

where terms in parentheses should be understood as matrices. Providing the initial condition $z(0)$, the desired solutions $z(1)$ of the target system can be obtained by integrating the system of the ODEs (10.3). Usually, numerical algorithms for initial value problems [234] are applied to obtain an estimate for $z(1)$. This approximated solution serves as the initial guess of the true solution, and are fed to the Newton method to improve its precision further [232].

10.2 The homotopy continuation of the scattering equations

The homotopy continuation method described above has been well-studied, in particular for polynomial systems, during the past decades. Therefore, it is natural to straightforwardly apply this technique to the polynomial form of the scattering equations [51], as shown in (3.17). A frequently used homotopy in the mathematics literature is

$$h_m(\sigma,t) = t\,h_m(\sigma) + (1-t)(\sigma_{m+2}^{m}-1), \quad i \in \{1,\ldots,n-3\}. \tag{10.4}$$

The advantage of such construction is that the start system has $(n-3)!$ known solutions and the number of solutions remains unchanged for any regular t. Although such a homotopy can be used to solve the scattering equations in principle, with some experimentations, we found that it is highly inefficient. One reason is on the technical side, saying that the complexity of evaluating ODEs (10.2) corresponding to the polynomial system is too high. Another reason is that the initial system is significantly different from the target system; this implies that a huge number of steps are spent to reach the target system.

In this work, we extend the homotopy continuation method to solve the fractional scattering equations (3.5) by establishing an appropriate homotopy.

Instead of constructing the homotopy continuation for scattering equations directly, we propose the *physical* homotopy in the kinematic space $\mathcal{K}_n$ (the space spanned by linearly independent Lorentz invariants $s_{ij} = 2k_i \cdot k_j$ [235]), i.e.,

$$\mathcal{S} \to \mathcal{S}_t, \tag{10.5}$$

where $\mathcal{S} \in \mathcal{K}_n$ denotes a set of kinematical variables. The momentum conservation and on-shell conditions hold for $\mathcal{S}_t$ at any t. More explicitly, a simple construction is:

$$s_{ij}(t) = (1-t)\,\bar{s}_{ij} + t\,s_{ij}, \tag{10.6}$$

where $\bar{s}_{ij}$ and s_{ij} are two sets of Mandelstam variables belonging to the physical region of interest in $\mathcal{K}_n$. Clearly, as long as on-shell conditions and momentum conservation are satisfied for $\bar{s}_{ij}$ and s_{ij}, they are satisfied for $s_{ij}(t)$. By abuse of terminology, we define the *kinematic homotopy* as a one-parameter smooth path in the kinematic space $\mathcal{K}_n$, shown by (10.5). The physical kinematic homotopy connects different points in $\mathcal{K}_n$, and this may be used to establish the connection between the physics quantities evaluated at different points.

The kinematic homotopy $\mathcal{S}_t$ naturally induces a homotopy for the scattering equations

$$f_a(t) = \sum_{b\neq a} \frac{s_{ab}(t)}{\sigma_a(t) - \sigma_b(t)} = 0. \tag{10.7}$$

Since the physical homotopy preserves on-shellness and momentum conservation, the system has exact $(n-3)!$ solutions for any regular t. To proceed, let us use the $\mathrm{SL}(2,\mathbb{C})$ redundancy to fix three punctures, for example $(\sigma_1, \sigma_2, \sigma_n) \to (0, 1, \infty)$. The last equation $f_n = 0$ is then trivially satisfied [23]. Differentiating other equations with respect to t gives the following system of ODEs:

$$\sum_{j=3}^{n-1} \Phi_{ij}\dot{\sigma}_j + f_i' = 0, \quad i \in \{1, 2, \ldots, n-1\} \tag{10.8}$$

with

$$\dot{\sigma}_i \equiv \frac{\mathrm{d}\sigma_i(t)}{\mathrm{d}t}, \qquad \Phi_{ij} \equiv \frac{\partial f_i(\sigma, t)}{\partial \sigma_j}, \qquad f_i' \equiv \frac{\partial f_i(\sigma, t)}{\partial t}. \tag{10.9}$$

An important property is that the matrix $\Phi(t)$ has exactly rank $n-3$ at any t [23]. This ensures that there is no singularity in our algorithm. To improve numerical stability, we retain all $(n-1)$ equations except $f_n = 0$ which is satisfied trivially, and employ matrix decomposition methods [234] to generate the standard form, like eq. (10.3). Therefore, once the solutions of the scattering equations for $\bar{s}_{ij}$ is known, the solutions for s_{ij} can be obtained by numerically integrating the ODEs in eq. (10.8).

10.3 Algorithm

However, so far the *initial solutions* (the solutions of the scattering equations for kinematical invariants $s_{ij}(0) = \bar{s}_{ij}$) are not readily available yet. We would like to emphasize that it is highly non-trivial to obtain the initial solutions, in particular when the multiplicity n is large. In order to initiate our program, we develop an algorithm based on the properties of the scattering equations in some special kinematical regions as well as the homotopy continuation technique. This algorithm will be described in detail in the following.

We employ the homotopy (10.6) again, i.e. $s_{ij}(t) = (1-t)\,\hat{s}_{ij} + t\bar{s}_{ij}$. Here the kinematical invariants $\hat{s}_{ij}$ satisfy

$$\hat{s}_{1i} > 0, \quad \hat{s}_{2i} > 0, \quad \hat{s}_{ij} > 0, \quad i, j \in \{3, \ldots, n-1\}, \tag{10.10}$$

which are referred to as the *positive region* denoted by $\mathcal{K}_n^+$ in [236]. A remarkable property is that all $(n-3)!$ solutions of the scattering equations in $\mathcal{K}_n^+$ are real if we fix the $\mathrm{SL}(2,\mathbb{C})$ according to $(\sigma_1, \sigma_2, \sigma_n) \to (0, 1, \infty)$ [236]. More interestingly, all variables $(\sigma_3, \ldots, \sigma_{n-1})$ live inside the interval $(0, 1)$ and distinct from each other for each solution. It is clear that due to this feature, the scattering equations in $\mathcal{K}_n^+$ can be solved much more quickly, compared to generic kinematic regions. As will be detailed below, all $(n-3)!$ real solutions can be obtained using the homotopy continuation technique again.[1] Once these solutions are readily available, they will serve as initial solutions, and we can use the homotopy (10.6) and integrate the system of the ODEs (10.8) to generate the solutions for general kinematics $\bar{s}_{ij}$. It is also worth stressing that

[1] In [236], for the kinematics in the positive region, one kind of algorithms were proposed based on interpreting the scattering equations as the equilibrium equations for a stable system of $n-3$ particles on the real interval $(0, 1)$.

we may encounter singularities if we still adopt the real contour for t from 0 to 1, since the starting and target points live in unphysical and physical regions of $\mathcal{K}_n$ respectively. A solution to avoiding singularities is to employ a complex contour for t. In our program, we choose a simple contour consisting of two line segments in the complex t plane: $0 \to 0.5 + 0.5\,i \to 1$.

Now the final task is to obtain all solutions to the scattering equations for one point in $\mathcal{K}_n^+$. Inspired by the soft limit of the scattering equations, we propose the following homotopy contnuation[2]

$$\begin{aligned} \hat{s}_{1i}(t) &= \hat{s}_{1i}, \quad \hat{s}_{2i}(t) = \hat{s}_{2i}, \quad 3 \le i \le n-2 \\ \hat{s}_{ij}(t) &= \hat{s}_{ij}, \quad 3 \le i < j \le n-2 \\ \hat{s}_{a,n-1}(t) &= t\,\hat{s}_{a,n-1}, \quad 1 \le a \le n-2. \end{aligned} \tag{10.11}$$

We refer to it as the *inverse soft homotopy*. All the remaining kinematic invariants can be easily obtained via on-shell conditions and momentum conservation. Clearly, this homotopy preserves the "positivity" of the kinematic region $\mathcal{K}_n^+$. Another remarkable property is that in the limit $t \to 0$ which defines the soft limit $k_{n-1} \to 0$, the kinematic space of n particles is reduced to $(n-1)$-particle one which is still in the positive region. As shown in Section 3.3, in this limit, $f_{n-1}(t)$ is invariant up to a factor t, i.e.,

$$f_{n-1}(t) = t\,\tilde{f}_{n-1}(t), \quad \tilde{f}_{n-1}(t) = \sum_{a=1}^{n-2} \frac{\hat{s}_{a,n-1}}{\sigma_{n-1}-\sigma_a}, \tag{10.12}$$

while other equations become nothing but the system of scattering equations associated with $(n-1)$ particles without the soft momentum in $\mathcal{K}_{n-1}^+$.

In order to solve the scattering equations in $\mathcal{K}_{n-1}^+$, we can use the *inverse soft homotopy* (10.11) recursively until the four-particle case, whose unique solution is known, i.e. $\sigma_3 = -s_{12}/s_{13}$ with gauge fixing $(\sigma_1, \sigma_2, \sigma_4) \to (0, 1, \infty)$. The equation corresponding to the soft particle $\tilde{f}_{n-1} = 0$ (referred to as the *soft equation*) is equivalent to a polynomial equation of degree $n-3$ in σ_{n-1}. For each solution of the scattering equations for the $(n-1)$-point system without the soft particle, the $n-3$ zeroes of the soft equation $\tilde{f}_{n-1}(\sigma_{n-1}) = 0$ are

[2]Based on the soft limit, one alternative algorithm was constructed and implemented in `Mathematica` in [23].

distributed in the $n-3$ sub-intervals of $(0,1)$, separated by σ_3, σ_4, $\cdots$, σ_{n-2}. Thus simple numerical techniques such as the bisection method can be applied to obtain all $n-3$ roots. For using the inverse soft homotopy (10.11) each time, a similar method can be used to solve the soft equation. Here it should be noted that the $f_{n-1}(t)$ is always replaced by $\tilde{f}_{n-1}(t)$ when we employ the inverse soft homotopy (10.11). Finally, we will obtain all $(n-3)!$ solutions to the scattering equations for any point in $\mathcal{K}_n^+$.

With the initial solutions from solving the scattering equations in $\mathcal{K}_n^+$, by integrating the corresponding differential equations given in (10.8), we can obtain the solutions to the scattering equations for one point in $\mathcal{K}_n$.

To summarise, we have proposed a homotopy continuation method to solve the scattering equations and given a workable framework in detail. As shown schematically below (superscript (s) stands for the soft limit), our method consists of two main steps.

$$\underbrace{\mathcal{K}_5^{+(\mathrm{s})} \xrightarrow{(10.11)} \cdots \xrightarrow{(10.11)} \mathcal{K}_n^{+(\mathrm{s})} \xrightarrow{(10.11)} \mathcal{K}_n^{+} \xrightarrow{(10.6)} \mathcal{K}_n}_{\text{Step I}} \underbrace{\xrightarrow{(10.6)} \mathcal{K}_n}_{\text{Step II}} \tag{10.13}$$

The first step is to obtain the initial solutions, which consists of two substeps: First, solve the scattering equations in $\mathcal{K}_n^+$ by using the inverse soft homotopy (10.11) recursively. Then, with these solutions as initial solutions, we can use the homotopy (10.6) to solve the scattering equations for one point in the realistic target region. As the next step, once we have all $(n-3)!$ solutions to the scattering equations for one physically realistic point in the kinematic space, we can track these solutions to any point in the kinematic space using the homotopy (10.6). In the second step, the solutions of the start system can be continued to the target system much more easily, since they both live in the same physically realistic region.

The method presented above has been implemented into a C++ program. For the numerical integration of differential equations, we adopt the Runge-Kutta-Fehlberg method [237] provided by ODEINT [238], and for the numerical solution of linear equation system, we employ the Householder QR decomposition with column pivoting provided by EIGEN [239]. In obtaining the initial solutions, the local accuracy is set to be 10^{-15}, while in the second step, the local accuracy is set to be 10^{-7}. In both steps, the Newton method is adopted to increase the precision to 10^{-15}. The code is publicly available at

$$\texttt{https://github.com/zxrlha/sehomo}.$$

In order to validate our algorithm, we consider the randomly selected non-exceptional points in the phase space corresponding to $2 \to n-2$ scattering up to $n = 13$. All tests were performed on a Macintosh laptop with a 2.7 GHz processor. The results of the computation times are summarized in Table 10.1. In the table, t_n are the computation times for obtaining all $\sharp(n) = (n-3)!$ solutions, and $\bar{t}_n \equiv t_n/(n-3)!$ represents the average time for each solution, for solving the scattering equations with a set of prepared initial solutions in the physically realistic region of $\mathcal{K}_n$. That is to say, they correspond to Step II showed in (10.13). Here we would also like to note that our algorithm for obtaining the initial solutions (i.e. the Step I in (10.13)) works well. More precisely, in this step, the time cost is dominated by tracking solutions from unphysical positive region to physically realistic region in $\mathcal{K}_n$, while solving the scattering equations in the positive region recursively is very fast. For example, it costs less than 30 minutes for $n = 11$ case.

As a consequence of the Newton method, all solutions can be obtained with an accuracy of 10^{-15}. We have also checked that all solutions are distinct from each other; thus we can verify that no solution is missed.

We observed that the total time to obtain all solutions increases significantly as n increases, mainly due to a factorial increase in the number of solutions. On the other hand, the average time of obtaining one solution increases much more slowly, and it is still at $\mathcal{O}(\mathrm{ms})$ level even for $n = 13$. It is noteworthy that obtaining different solutions is completely independent, thus can be done in parallel.

We also found that the time costs are dominated by solving the differential equations. Therefore if higher precision on solutions is requested, only the last step, i.e. the Newton iterations should be performed within higher precision, which has an only small impact on the total time cost.

n	$\sharp(n)$	t_n	$\bar{t}_n$ (ms)
5	2	1.3 ms	0.7
6	6	5.0 ms	0.8
7	24	35 ms	1.5
8	120	0.22 s	1.8
9	720	1.3 s	1.8
10	5040	13 s	2.5
11	40 320	2.3 min	3.2
12	362 880	30 min	4.9
13	3 628 800	5.6 h	5.5

Table 10.1: The total time costs t_n of solving n-point scattering equations are shown. The number of solutions as well as the averaged time per solution $\bar{t}_n \equiv t_n/(n-3)!$ are also shown.

In addition, due to the property of the algorithm, for two neighbouring points in the phase space, clearly it will be much easier to obtain the solutions of the scattering equations from each other. Therefore, one could speed up the calculation through a book-keeping method: first the initial solutions are prepared at several typical kinematic points rather than only one point, and the closest point are adopted as the initial point when doing the actual calculation.

Lastly, we extend our method to solve the four-dimensional scattering equations. Here we consider $2 \to (n-2)$ scattering. It is a good idea to work with the equations in light-cone variables (7.17) since they are fully equivalent to the four-dimensional scattering equations (4.5). Working in the center-of-momentum frame (c.f. Section 7.1 for detail), we propose a homotopy continuation for momenta:

$$
\begin{aligned}
k_i^{\perp}(t) &= (1-t)\,p_i^{\perp} + t\,p'^{\perp}_i, \qquad i \in \{4,\ldots,n\} \\
k_i^{+}(t) &= (1-t)\,p_i^{+} + t\,p'^{+}_i, \qquad i \in \{3,\ldots,n\} \\
k_3^{\perp}(t) &= -\sum_{i=4}^{n}(1-t)\,p_i^{\perp} + t\,p'^{\perp}_i, \\
k_2^{+}(t) &= -\sum_{i=3}^{n}(1-t)\,p_i^{+} + t\,p'^{+}_i,
\end{aligned}
\tag{10.14}
$$

where p and p' are two sets of momenta. The last two equations follow from the momentum conservation. $k_i^-(t)$ can be obtained using on-shell and momentum conservation conditions and have a more complicated dependence on the continuation parameter t. Fortunately, all k_i^- components are not needed since they do not explicitly appear in (7.17). Substituting (10.14) into the equations (7.17), it is straightforward to establish a system of ODEs. The initial conditions can be obtained using a similar algorithm with D-dimensional case, as shown previously.

We will not provide more details, but make a comparison with ref. [66], where a method was introduced to solve the four-dimensional scattering equations (4.5) and implemented in `Mathematica`. Overall, our algorithm is much faster than the one in [66]. Here we identify some significant differences as follows. As already pointed out, obtaining solutions is utterly independent of each other in our method. In contrast, in [66] the solutions are obtained sequentially, and as more solutions obtained, finding the next solution becomes increasingly difficult. Consequently, we can easily obtain all solutions for high points (e.g. $n = 13$), while even for $n = 10$ it is quite challenging to solve the equations for all helicity sectors by the method developed in [66].

10.4 Conclusion and discussions

We have proposed the kinematic homotopy continuation, which connects different points in kinematic space. Such a deformation naturally induces a homotopy continuation of the scattering equations. As a result, the solutions of the scattering equations with different points in $\mathcal{K}_n$ can be related to each other.

We have developed an efficient algorithm to generate all numerical solutions of the scattering equations. This opens a new window of opportunity for further explorations in various prospectives.

First of all, this powerful method allows us to solve the scattering equations with high accuracy and high efficiency in different contexts. It is interesting to investigate the properties of the scattering equations and the CHY formulas in various kinematical regions, such as collinear and multi-Regge limits. While the discussion above is limed at the tree level, our method can be directly generalized to solve the scattering equations at the loop level, which have been derived from ambitwistor strings.

In practical terms, it allows one to develop a new framework to compute scattering amplitudes at tree and loop level. Once one obtains all solutions to the scattering equations, as a next step, it is straightforward to generate tree amplitudes or loop integrands by summing up the contributions from these solutions. For instance, since the scheme to extend the CHY formalism to loop level has been developed at least for gauge and gravity theories, this may make possible to compute the amplitudes in these theories up to the two-loop order.

More interestingly, the kinematic homotopy developed in this work has further significance beyond solving the scattering equations. Intriguingly, the kinematic homotopy may provide an avenue to study various physical quantities, such as scattering amplitudes and scattering forms [235, 240, 241], in the kinematic space directly.

11 Conclusions

Throughout this work, we have explored many aspects of the scattering equations. In this concluding chapter, we summarize the main results achieved and make some comments on further avenues of research.

Our results have led to a deeper understanding of the mathematical structures underlying the scattering equations, which are universal for all massless theories. First, we clarified the equivalence between two types of the four-dimensional scattering equations and worked out how to exactly transform the two corresponding formulations of S-matrices one another in Chapter 4. Second, we have initiated the study of the asymptotic behavior of the scattering equations in the high-energy limit in Chapters 7, 8 and 9. We observed that the solutions to the scattering equations display the same hierarchy as the rapidity ordering in various quasi-Regge regimes, and we conjectured that this behaviour holds for any multiplicity. In particular, we explicitly solved the four-dimensional scattering equations for any helicity configuration for any multiplicity and derived the correct factorized form of tree amplitudes in Yang-Mills and Einstein gravity in the multi-Regge kinematics. Our conjecture also implies the expected factorization of the amplitude in Yang-Mills theory in quasi-multi-Regge kinematics. Finally, a good understanding of the scattering equations has also led us to develop efficient methods of solving the scattering equations in Chapter 10. We introduced the *physical* homotopy continuation of the scattering equations and designed an ingenious algorithm to generate all numerical solutions of the scattering equations with high accuracy and high efficiency based on the properties of the scattering equations in various kinematic regimes.

Our results also have broadened the scope of the applications of the scattering equation formalism. Based on the four-dimensional scattering equations, we have obtained the new representations for the complete tree-level S-matrix in maximally supersymmetric DBI-VA theory as well as in the $U(N)$ NLSM and a special Galileon theory in Chapter 5. These new formulas also have shown the power of producing various double soft theorems. In Chapter 6, we have also extended the scattering equation formalism designed for on-shell amplitudes to form factors where a composite operator carries off-shell momentum. With the help of a set of modified scattering equations, we constructed new compact formulas for form factors with the chiral stress-tensor multiplet operator and a specific family of scalar operators inserted in $\mathcal{N}=4$ SYM. Our results strongly support the availability and universality of the scattering-equation method for form factors and also have initiated to study off-shell quantities in this context. We expect that our results can be generalized to form factors for general operators and even purely off-shell correlation functions in $\mathcal{N}=4$ SYM.

While our research has focused primarily on the formal aspects of the scattering equations, we hope our results shed new light on particle phenomenology. So far, only partial results for amplitudes with gluons and a few quark pairs or/and massive bosons exist (see e.g. [133,242–244]). Once one has a contour-integral representation of scattering amplitudes in the full Standard Model, it may be combined with the numerical algorithm presented in this work to develop a faster and more reliable numerical evaluation of the tree-level S-matrix of the SM than traditional results obtained by summing a prohibitively large number of Feynman diagrams.

Another longstanding open question is how to generalize the scattering equations to loop-level amplitudes. Even though the impressive progress has been made regarding constructing compact representations for loop amplitudes (integrands), in particular in supersymmetric theories, from the scattering equations based on ambitwistor strings [31–37], it is still highly non-trivial and desirable to obtain new representations for loop amplitudes in realistic theories such as QCD. This is especially crucial to apply the scattering equations to "*simplify predictions and analyses of LHC experiments*" [245].

A The spinor-helicity formalism

We provide a brief introduction to the spinor-helicity formalism in this appendix. We massively follow refs. [8, 246, 247] in the following.

Let us begin by relating a four-vector k^μ to the following matrix

$$k_{\alpha\dot{\alpha}} = k_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} k^0 + k^3 & k^1 - ik^2 \\ k^1 + ik^2 & k^0 - k^3 \end{pmatrix}, \tag{A.1}$$

with $k_\mu = (k^0, \vec{k})$ and $\sigma^\mu = (\mathbf{1}_{2\times 2}, \vec{\sigma})$, where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli matrices. If k is light-like, the matrix $k_{\alpha\dot{\alpha}}$ is rank deficient since $\det(k_{\alpha\dot{\alpha}}) = k^2 = 0$. Hence it can be represented as

$$k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{A.2}$$

where the two-component objects λ and $\tilde{\lambda}$ are known as spinors carrying different helicities respectively, as explained later. It is clear that for any given null momentum k, the λ, $\tilde{\lambda}$ are unique up to a *little group transformation*

$$(\lambda,\, \tilde{\lambda}) \longrightarrow (t\lambda,\, t^{-1}\tilde{\lambda}), \quad t \in \mathbb{C}^*. \tag{A.3}$$

An explicit representation used in this work is

$$\lambda_a = \frac{e^{i\theta}}{\sqrt{k_a^+}} \begin{pmatrix} k_a^+ \\ k_a^\perp \end{pmatrix}, \qquad \tilde{\lambda}_a = \frac{e^{-i\theta}}{\sqrt{k_a^+}} \begin{pmatrix} k_a^+ \\ k_a^{\perp *} \end{pmatrix}. \tag{A.4}$$

The λ and $\tilde{\lambda}$ transform in the representations $(1/2, 0)$ and $(0, 1/2)$ of Lorentz group respectively[1]. There is an invariant antisymmetric tensor

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{A.6}$$

to raise and lower spinor indices. More precisely,

$$\lambda^{\alpha} = \epsilon^{\alpha\beta}\lambda_{\beta}, \quad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}, \quad \lambda_{\alpha} = \epsilon_{\alpha\beta}\lambda^{\beta}, \quad \tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}, \tag{A.7}$$

$$k^{\dot{\alpha}\alpha} \equiv \tilde{\lambda}^{\dot{\alpha}}\lambda^{\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}}k_{\mu} = \frac{1}{2}\bar{\sigma}^{\nu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_{\nu}\sigma^{\mu}_{\beta\dot{\beta}}k_{\mu} = \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}k^{\mu}, \tag{A.8}$$

where $\bar{\sigma}^{\mu} \equiv (\mathbf{1}_{2\times 2}, -\vec{\sigma})$ and one has used the following identities (c.f. [248])

$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_{\mu} \quad \text{and} \quad \mathrm{Tr}\left(\sigma^{\mu}\bar{\sigma}^{\nu}\right) = \mathrm{Tr}\left(\bar{\sigma}^{\mu}\sigma^{\nu}\right) = 2\eta^{\mu\nu}. \tag{A.9}$$

The kinematic invariants also be written in terms of spinors. Let us define spinor products as follows:

$$\langle i\, j\rangle \equiv \epsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta} = \lambda_{i\alpha}\lambda^{\alpha}_{j} \;\text{ and }\; [i\, j] \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_{j\dot{\beta}} = \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{j}. \tag{A.10}$$

From the definition, it is easy to see $\langle i\, j\rangle = -\langle j\, i\rangle$ and $[i\, j] = -[j\, i]$. For two massless momenta $k_i = \lambda_i\tilde{\lambda}_i$ and $k_j = \lambda_j\tilde{\lambda}_j$, the Lorentz product can be expressed as

$$s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j = \langle ij\rangle\, [ij]. \tag{A.11}$$

It has also become clear that the representation of a null momentum k^{μ} as a product of two spinors makes manifest the on-shell condition $k^2 = 0$ since $\langle i\, i\rangle = 0 = [i\, i]$. By viewing spinors as two-component vectors in $\mathbb{C}^2$, we can decompose any spinor λ_i as

$$\lambda_i = \frac{\langle ik\rangle}{\langle jk\rangle}\lambda_j + \frac{\langle ij\rangle}{\langle kj\rangle}\lambda_k \;\Rightarrow\; \langle ij\rangle\langle kl\rangle + \langle jk\rangle\langle il\rangle + \langle ki\rangle\langle jl\rangle = 0, \tag{A.12}$$

[1] In four dimensions, the complexified Lorentz group is locally isomorphic to

$$SO(1,3,\mathbb{C}) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C}), \tag{A.5}$$

thus finite dimensional representations are classified as (i, j) with integers or half-integers i, j.

which is known as the famous *Schouten identity*.

Here we give the physical interpretation of spinors λ and $\tilde{\lambda}$. It is evident that they satisfy the Weyl equations, i.e.,

$$k^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \lambda_\alpha = 0 \quad \text{and} \quad k_\mu \sigma^\mu_{\alpha\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} = 0, \tag{A.13}$$

They clearly show that the spinor λ (or $\tilde{\lambda}$) is the wavefunction of a massless particle of helicity $-1/2$ (or $+1/2$) in momentum space.

It is also natural to find an analogous description of wavefunctions for massless vector bosons. For positive helicity and negative helicity vectors, one has [249–254]

$$\epsilon^+_{\alpha\dot{\alpha}}(\lambda, \tilde{\lambda}; \eta) = \frac{\eta_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \eta\, \lambda \rangle}, \quad \epsilon^-_{\alpha\dot{\alpha}}(\lambda, \tilde{\lambda}; \tilde{\eta}) = \frac{\lambda_\alpha \tilde{\eta}_{\dot{\alpha}}}{[\tilde{\lambda}\, \tilde{\eta}]}, \tag{A.14}$$

where η and $\tilde{\eta}$ are two arbitrary chosen reference spinors corresponding to gauge freedom, often judiciously chosen to simplify expressions. This representation was sometimes referred to "*Chinese Magic*".

The wave functions of (massless) particles with other spins also can be built in terms of spinor variables. In particular, a polarization tensor of graviton can be expressed in terms of the symmetric-traceless tensor product of two polarization vectors, $\epsilon^{\mu\nu} = \epsilon^\mu \epsilon^\nu$. Therefore the graviton polarization tensor can be written in terms of spinor variables[2]:

$$\begin{aligned} \epsilon_i^{+,\alpha\dot{\alpha}\beta\dot{\beta}}(\lambda_i, \tilde{\lambda}_i; \lambda_x, \lambda_y) &= \frac{\lambda_x^\alpha \tilde{\lambda}_i^{\dot{\alpha}}}{\langle x\, i \rangle} \frac{\lambda_y^\beta \tilde{\lambda}_i^{\dot{\beta}}}{\langle y\, i \rangle} + (x \leftrightarrow y), \\ \epsilon_i^{-,\alpha\dot{\alpha}\beta\dot{\beta}}(\lambda_i, \tilde{\lambda}_i; \tilde{\lambda}_x, \tilde{\lambda}_y) &= \frac{\lambda_i^\alpha \tilde{\lambda}_x^{\dot{\alpha}}}{[i\, x]} \frac{\lambda_i^\beta \tilde{\lambda}_y^{\dot{\beta}}}{[i\, y]} + (x \leftrightarrow y), \end{aligned} \tag{A.15}$$

where x and y are arbitrary reference spinors.

Now we have constructed the wave functions $\psi(\lambda, \tilde{\lambda}, h)$ in terms of the spinors for massless particles with various spins, an interesting and important obser-

[2]This representation for graviton polarization tensors satisfies $\epsilon_i^+ \cdot \epsilon_i^- \equiv \epsilon_i^{+,\mu\nu} \epsilon^-_{i,\mu\nu} = 1$.

vation is

$$\left(\lambda^{\alpha}\frac{\partial}{\partial\lambda^{\alpha}}-\tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}}\right)\psi(\lambda,\tilde{\lambda},h)\;=\;-2h\psi(\lambda,\tilde{\lambda},h). \tag{A.16}$$

Having established a unified description for the wave functions of massless particles with various spins, we turn to see scattering amplitudes. From the Feynman rules, it is easy to see that the amplitude must be a multi-linear function of the wave function ψ_i corresponding to the external leg "i", and also depends on the momenta of external legs because of the propagators. Therefore, any scattering amplitude can be regarded as a function of the spinors λ_a and $\tilde{\lambda}_a$ as well as helicities h_a, i.e.,

$$\mathcal{A}_n\big(\{\lambda_i,\tilde{\lambda}_i,h_i\}\big). \tag{A.17}$$

This is also the origin of the name of the *helicity amplitude*. Since the amplitude is linear in each external wave function ψ_i, eq. (A.16) immediately implies

$$\left(\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}}-\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}\right)\mathcal{A}_n\big(\{\lambda_j,\tilde{\lambda}_j,h_j\}\big)\;=\;-2h_i\mathcal{A}_n\big(\{\lambda_j,\tilde{\lambda}_j,h_j\}\big), \tag{A.18}$$

for each particle i with helicity h_i. These conditions give strong constraints on helicity amplitudes, and are very useful in *bootstrapping* amplitudes [255] (see also [256]).

B Conservation of momentum

We show the four-dimensional scattering equations, eqs. (4.5), imply momentum conservation in detail in this appendix.

As shown in the previous appendix, for any spinor λ_a^α we can decompose it along two nonparallel spinors, e.g. λ_1^α and λ_2^α

$$\lambda_a^\alpha = \frac{\langle 2\, a\rangle}{\langle 2\, 1\rangle}\lambda_1^\alpha + \frac{\langle 1\, a\rangle}{\langle 1\, 2\rangle}\lambda_2^\alpha. \tag{B.1}$$

Then momentum conservation delta function can be represented as

$$\begin{aligned}\delta^4\left(\sum_{a=1}^n \lambda_a^\alpha \tilde\lambda_a^{\dot\alpha}\right) &= \delta^4\left(\lambda_1^\alpha\left[\tilde\lambda_1^{\dot\alpha} + \sum_{a=3}^n \frac{\langle 2\, a\rangle}{\langle 2\, 1\rangle}\tilde\lambda_a^{\dot\alpha}\right] + \lambda_2^\alpha\left[\tilde\lambda_2^{\dot\alpha} + \sum_{a=3}^n \frac{\langle 1\, a\rangle}{\langle 1\, 2\rangle}\tilde\lambda_a^{\dot\alpha}\right]\right) \\ &= \frac{1}{\langle 1\, 2\rangle^2}\,\delta^2\left(\tilde\lambda_1^{\dot\alpha} + \sum_{a=3}^n \frac{\langle 2\, a\rangle}{\langle 2\, 1\rangle}\tilde\lambda_a^{\dot\alpha}\right)\delta^2\left(\tilde\lambda_2^{\dot\alpha} + \sum_{a=3}^n \frac{\langle 1\, a\rangle}{\langle 1\, 2\rangle}\tilde\lambda_a^{\dot\alpha}\right).\end{aligned} \tag{B.2}$$

In the following we prove

$$\delta^2\left(\tilde\lambda_I^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{Ii}\,\tilde\lambda_i^{\dot\alpha}\right)\delta^2\left(\tilde\lambda_J^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{Ji}\,\tilde\lambda_i^{\dot\alpha}\right) = \langle I J\rangle^2\,\delta^4\left(\sum_{a=1}^n \lambda_a^\alpha\tilde\lambda_a^{\dot\alpha}\right), \tag{B.3}$$

for arbitrary $\{I, J\} \subseteq \mathfrak{N}$, where $c_{Ii} = 1/(I\, i)$. Without loss generality, we consider the case of $I = 1$ and $J = 2$. By using the following equations

$$\lambda_i^\alpha + \sum_{I\in\overline{\mathfrak{N}}} c_{Ii}\lambda_I^\alpha + c_{1i}\lambda_1^\alpha + c_{2i}\lambda_2^\alpha = 0, \quad \overline{\mathfrak{N}} \equiv \mathfrak{N}\backslash\{1,2\} \tag{B.4}$$

we have

$$c_{1i} \;=\; \frac{1}{\langle 1\,2\rangle}\Bigg(\langle 2\,i\rangle + \sum_{I\in\overline{\mathfrak{N}}} c_{Ii}\,\langle 2\,I\rangle\Bigg), \tag{B.5}$$

$$c_{2i} \;=\; \frac{1}{\langle 2\,1\rangle}\Bigg(\langle 1\,i\rangle + \sum_{I\in\overline{\mathfrak{N}}} c_{Ii}\,\langle 1\,I\rangle\Bigg). \tag{B.6}$$

Plugging them back into the delta functions on the left-hand side of (B.3) gives

$$\begin{aligned}\delta^2\Bigg(\tilde\lambda_1^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{1i}\,\tilde\lambda_i^{\dot\alpha}\Bigg) \;&=\; \delta^2\Bigg(\tilde\lambda_1^{\dot\alpha} - \sum_{i\in\mathfrak{P}}\frac{\langle 2\,i\rangle}{\langle 1\,2\rangle}\tilde\lambda_i^{\dot\alpha} - \sum_{I\in\overline{\mathfrak{N}}}\frac{\langle 2\,I\rangle}{\langle 1\,2\rangle}\sum_{i\in\mathfrak{P}} c_{Ii}\tilde\lambda_i^{\dot\alpha}\Bigg)\\ &=\; \delta^2\Bigg(\tilde\lambda_1^{\dot\alpha} + \sum_{a=3}\frac{\langle 2\,a\rangle}{\langle 2\,1\rangle}\tilde\lambda_a^{\dot\alpha}\Bigg),\end{aligned} \tag{B.7}$$

where we used the scattering equations

$$\sum_{i\in\mathfrak{P}} c_{Ii}\tilde\lambda_i^{\dot\alpha} \;=\; \tilde\lambda_I^{\dot\alpha}. \tag{B.8}$$

Similarly we have

$$\delta^2\Bigg(\tilde\lambda_2^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{2i}\,\tilde\lambda_i^{\dot\alpha}\Bigg) \;=\; \delta^2\Bigg(\tilde\lambda_2^{\dot\alpha} + \sum_{a=3}\frac{\langle 1\,a\rangle}{\langle 1\,2\rangle}\tilde\lambda_a^{\dot\alpha}\Bigg). \tag{B.9}$$

A combination of eqs. (B.7), (B.9) and (B.2) yields

$$\delta^2\Bigg(\tilde\lambda_1^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{1i}\,\tilde\lambda_i^{\dot\alpha}\Bigg)\,\delta^2\Bigg(\tilde\lambda_2^{\dot\alpha} - \sum_{i\in\mathfrak{P}} c_{2i}\,\tilde\lambda_i^{\dot\alpha}\Bigg) = \langle 1\,2\rangle^2\,\delta^4\Bigg(\sum_{a=1}^{n}\lambda_a^{\alpha}\tilde\lambda_a^{\dot\alpha}\Bigg). \tag{B.10}$$

C Lipatov formula for the Graviton Reggeization

This appendix provides a review of the multi-Regge factorization of tree-level gravitational scattering amplitudes.

In the multi-Regge kinematical regime, the compact formula for tree-level amplitudes with any number of external gravitons was obtained from t-channel unitarity methods by Lev Nikolaevich Lipatov more than three decades ago [221, 222]. The formula takes a surprisingly simple factorized form: a product of universal building blocks connected by t-channel propagators, as shown in Figure 7.1 in Chapter 7. To be clear, here we rewrite it explicitly:

$$\mathcal{M}_n \qquad \text{(C.1)}$$
$$\simeq -s^2\mathcal{C}_g(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}_g(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^\perp|^2}\mathcal{V}_g(q_{n-1};n-1;q_n)\frac{-1}{|q_n^\perp|^2}\mathcal{C}_g(1;n).$$

There are two types of universal building blocks, $\mathcal{C}_g$ and $\mathcal{V}_g$ (referred to as gravitational impact factor and gravitational Lipatov vertex respectively). We define them explicitly below.

The effective graviton-graviton-(Reggeized graviton) vertex reads

$$\Gamma_{\mu\nu,\alpha\beta} \equiv \Gamma_{\mu\alpha}\Gamma_{\nu\beta} + \Gamma_{\mu\beta}\Gamma_{\nu\alpha}, \qquad \text{(C.2)}$$

which is manifestly a double copy of the gluon-gluon-(Reggeized gluon[1]) vertex that is defined as

$$\Gamma_{2;3}^{\mu\alpha} = -\eta^{\mu\alpha} + \frac{2(k_1^\mu k_2^\alpha - k_3^\mu k_1^\alpha)}{s} + s_{23}\,\frac{2k_1^\alpha k_1^\mu}{s^2}, \tag{C.3}$$

$$\Gamma_{1;n}^{\mu\alpha} = -\eta^{\mu\alpha} + \frac{2(k_2^\mu k_1^\alpha - k_n^\mu k_2^\alpha)}{s} + s_{1n}\,\frac{2k_2^\alpha k_2^\mu}{s^2}. \tag{C.4}$$

Similarly, the effective (Reggeized graviton)-(Reggeized graviton)-graviton vertex can also be obtained as the double copy of gauge theory vertices:

$$\Gamma_i^{\mu\nu}(q_i, q_{i+1}) \equiv 2\big(C_i^\mu C_i^\nu - N_i^\mu N_i^\nu\big), \tag{C.5}$$

where the C^μ is the famous Lipatov vertex of (Reggeized gluon)-(Reggeized gluon)-gluon in QCD [218]

$$\begin{aligned} &C_i^\mu(q_i, q_{i+1}) \\ &\quad= -(q_i^\perp)^\mu - (q_{i+1}^\perp)^\mu + \left(\frac{2q_{i+1}^2}{s_{1i}} - \frac{s_{2i}}{s}\right) k_1^\mu - \left(\frac{2q_i^2}{s_{2i}} + \frac{s_{1i}}{s}\right) k_2^\mu, \end{aligned} \tag{C.6}$$

with $(q^\perp)^\mu \equiv (0, 0; q^\perp)$, while N^μ is the so-called QED Bremsstrahlung vertex:

$$N_i^\mu(q_i, q_{i+1}) = \sqrt{q_i^2 q_{i+1}^2}\,\left(\frac{k_1^\mu}{s_{1i}} - \frac{k_2^\mu}{s_{2i}}\right). \tag{C.7}$$

The contractions between vertices and the polarization tensors of external gravitons give the gravitational impact factor and gravitational Lipatov vertex appearing in formula (7.7), i.e.,

$$\begin{aligned} \mathcal{C}_g(2;3) &= \Gamma_{\mu\nu,\alpha\beta}(k_2, k_3)\,\epsilon_2^{\mu\nu}(k_2)\epsilon_3^{\alpha\beta}(k_3), \\ \mathcal{C}_g(1;n) &= \Gamma_{\mu\nu,\alpha\beta}(k_1, k_n)\,\epsilon_1^{\mu\nu}(k_1)\epsilon_n^{\alpha\beta}(k_n), \\ \mathcal{V}_g(q_i, i, q_{i+1}) &= \Gamma_{i,\mu\nu}(q_i, q_{i+1})\,\epsilon_i^{\mu\nu}(k_i), \ \ i \in \{3, \ldots, n\}. \end{aligned} \tag{C.8}$$

Here we translate these building blocks to the modern language, say spinor-helicity variables. In four dimensions, in addition to momenta, the wavefunc-

[1] Reggeized gluon is often referred to as “Reggeon” in the literature.

tions of massless particles (including graviton polarization tensors) may be written in terms of two-component Weyl spinors, see Appendix A. Using the spinor-helicity variables defined in eqs. (7.13) and (A.15), it is easy to compute the gravitational impact factors and Lipatov vertices defined in (C.8). A straightforward calculation gives

$$\begin{aligned}
&\mathcal{C}_g(2^+;3^+) = \mathcal{C}_g(2^-;3^-) = \mathcal{C}_g(1^+;n^+) = \mathcal{C}_g(1^-;n^-) = 0,\\
&\mathcal{C}_g(2^-;3^+) = \mathcal{C}_g(2^+;3^-) = 1,\\
&\mathcal{C}_g(1^-;n^+) = \mathcal{C}_g(1^+;n^-)^* = \left(\frac{k_n^{\perp *}}{k_n^\perp}\right)^2,
\end{aligned} \tag{C.9}$$

and

$$\begin{aligned}
\mathcal{V}_g\big(q_i;i^+;q_{i+1}\big) &= \mathcal{V}_g\big(q_i;i^-;q_{i+1}\big)^* \\
&= \frac{q_i^{\perp *}\big(q_i^{\perp *}q_{i+1}^\perp - q_i^\perp q_{i+1}^{\perp *}\big)q_{i+1}^\perp}{(k_i^\perp)^2} \qquad \text{(C.10)}\\
&= \frac{q_i^{\perp *}\big(k_i^\perp q_i^{\perp *} - k_i^{\perp *} q_i^\perp\big)q_{i+1}^\perp}{(k_i^\perp)^2}. \qquad \text{(C.11)}
\end{aligned}$$

We see from the first line in (C.9) that helicity is conserved by the impact factors, like in gauge theory.

Finally, we would like to make some comments on the effective vertices (C.2) and (C.5) in gravity in MRK. These effective vertices have also been derived from an effective action (see e.g. [219, 257, 258]). It is extremely remarkable that the double copy relation between gravity and gauge theories was uncovered for the first time in multi-Regge kinematics. In general kinematics, Kawai, Lewellen and Tye (KLT) found that a closed string amplitude can be expressed in terms of sums of products of two open string amplitudes [91]. In the field theory limit, the KLT relation naturally implies the double copy relation between amplitudes in gravity and Yang-Mills. More recently, Bern, Carrasco and Johansson discovered that the amplitudes in gauge theory and gravity can be related via the so-called “color-kinematics duality” [74, 103].

D Proof of the three identities in Chapter 8

In this appendix, we prove three identities used to derive the multi-Regge factorization of the graviton amplitudes in Chapter 8. They are eq. (8.68), eq. (8.78) and eq. (8.88).

Identity in (8.68)

Here we focus on the following triangular matrix:

$$\varphi = \begin{pmatrix} v_4 & x_5-x_4-v_4 & \cdots & x_{n-1}-x_4-v_4 & x_n-x_4 \\ 0 & v_5 & \cdots & x_{n-1}-x_5 & x_n-x_5 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{n-1} & x_n-x_{n-1} \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix}. \tag{D.1}$$

Our goal is to find $(\varphi^{-1})_{1,n-3}$. Let us use $\alpha = (\alpha_4, \alpha_5, \ldots, \alpha_n)^{\mathrm{T}}$ to denote the last column of the inverse of this matrix, then it satisfies the following equation

$$\varphi\,\alpha = (0, \ldots, 0, 1)^{\mathrm{T}}. \tag{D.2}$$

In the following we show that its solution is

$$\alpha_i = \frac{k_i^{\perp}}{k_n^{\perp *}}, \quad i = n, n-1, \ldots, 5 \tag{D.3}$$

$$\alpha_4 = -\frac{k_3^{\perp} + k_n^{\perp}}{k_n^{\perp *}}. \tag{D.4}$$

First, we can easily get from $x_n \alpha_n = 1$

$$\alpha_n = \frac{1}{x_n} = \frac{k_n^\perp}{k_n^{\perp *}}. \tag{D.5}$$

Then we assume all α_j with $j > i$ are given by (D.3), and let us solve α_i from the following equation:

$$v_i \alpha_i + \sum_{j=i+1}^{n} (x_j - x_i)\alpha_j = 0. \tag{D.6}$$

Plugging the values of α_j ($j > i$) given in (D.3) into this equation gives

$$\alpha_i = -\frac{1}{v_i} \sum_{j=i+1}^{n} (x_j - x_i)\alpha_j = \frac{k_i^\perp}{k_n^{\perp *}}, \tag{D.7}$$

which agrees with eq. (D.3). Finally, solving the last equation gives

$$\begin{aligned}(\varphi^{-1})_{1,n-3} \equiv \alpha_4 &= -\frac{1}{v_4}\left(\sum_{j=5}^{n}(x_j - x_4)\alpha_j - v_4 \sum_{j=5}^{n-1} \alpha_j\right) \\ &= -\frac{k_3^\perp + k_n^\perp}{k_n^{\perp *}}. \end{aligned} \tag{D.8}$$

Identity in (8.78)

In the following, we consider the reduced determinant of the $\overline{\mathsf{H}}_{k\times k}$ whose indices take values from the set $\mathfrak{P}$.

First fo all, it is useful to introduce a new notation related to particle labels: $I_{\ell_i} \in \mathfrak{N}$ denotes the smallest number that satisfies $I_{\ell_i} > i \in \mathfrak{P}$. For example, $\ell_3 = 1$ because of $3 < I_1 \in \mathfrak{N}$. Then, by abuse of multiple subscripts, we can rewrite ζ_i and τ_i in terms of ζ_I and τ_I as follows:

$$\tau_i = \begin{cases} -\dfrac{k_{I_{\ell_i}}^{\perp *}}{q_{I_{\ell_i}}^{\perp *}} \zeta_{I_{\ell_i}}^{-1}, & i < I_m, \\ -\displaystyle\prod_{1\le l\le m} \frac{q_{I_l}^{\perp *}}{q_{I_l+1}^{\perp *}}, & i > I_m, \end{cases} \qquad \zeta_i = \begin{cases} -\dfrac{k_{I_{\ell_i}}^{\perp}}{q_{I_{\ell_i}}^{\perp}} \tau_{I_{\ell_i}}^{-1}, & i < I_m, \\ -1, & i > I_m. \end{cases} \tag{D.9}$$

Let us also rewrite the matrix (8.77):

$$\overline{\mathsf{H}}' = \begin{pmatrix} v_{i_1}+x_{i_1} & c_{i_1 i_2} & \cdots & c_{i_1 i_p} & c_{i_1 n} \\ x_{i_2} & v_{i_2}+x_{i_2} & \cdots & c_{i_2 i_p} & c_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i_p} & x_{i_p} & \cdots & v_{i_p}+x_{i_p} & c_{i_p n} \\ x_n & x_n & \cdots & x_n & x_n \end{pmatrix}, \tag{D.10}$$

where particle labels in $\mathfrak{P}$ have been reordered as $3 < i_1 < \cdots < i_p < n$ with $p = (n-k)-2$. By performing some elementary row and column transformations, we have

$$\det \overline{\mathsf{H}}' = \det \overline{\mathsf{H}}'', \tag{D.11}$$

with

$$\overline{\mathsf{H}}'' = \begin{pmatrix} v_{i_1} & c_{i_1 i_2}-x_{i_1}-v_{i_1} & \cdots & c_{i_1 i_p}-x_{i_1}-v_{i_1} & c_{i_1 n}-x_{i_1} \\ 0 & v_{i_2} & \cdots & c_{i_2 i_p}-x_{i_2} & c_{i_2 n}-x_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{i_p} & c_{i_p n}-x_{i_p} \\ x_n & 0 & \cdots & 0 & x_n \end{pmatrix}. \tag{D.12}$$

Using the matrix determinant lemma, we have

$$\det \overline{\mathsf{H}}' = x_n \left(\prod_{i \in \mathfrak{P}, i \neq 3, n} v_i \right) \left(1 + x_n \left(\bar{\Phi}^{-1}\right)_{1,p+1}\right), \tag{D.13}$$

where $\bar{\Phi}$ is nothing but $\overline{\mathsf{H}}''$ with replacing the first element of the last row x_n by zero. Now our task becomes to calculate the entry in the upper right corner of the inverse of the matrix $\bar{\Phi}$. Let us denote the last column of the inverse of $\bar{\Phi}$ as:

$$(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p}, \alpha_n)^{\mathrm{T}}. \tag{D.14}$$

Clearly, $\alpha_{i_1} = (\bar{\Phi}^{-1})_{1,p+1}$. Then it can be determined by the following linear equations:

$$\bar{\Phi} \cdot (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_n)^{\mathrm{T}} = (0, 0, \ldots, 1)^{\mathrm{T}}. \tag{D.15}$$

The solution is

$$\alpha_j = -\frac{k_j^{\perp}}{k_n^{\perp *}} \zeta_j, \quad i_1 < j \leq n, \tag{D.16}$$

$$\alpha_{i_1} = x_n^* \left(\frac{k_3^{\perp}}{k_n^{\perp}} \zeta_3 - 1 \right). \tag{D.17}$$

Plugging (D.16) into (D.13) immediately leads to

$$\det \overline{\mathsf{H}}' = -x_n \left(\prod_{i \in \mathfrak{P}, i \neq 3, n} v_i \right) \frac{k_3^{\perp}}{k_n^{\perp}} \zeta_3 \, \tau_3 \,. \tag{D.18}$$

We immediately obtain (8.78) by inserting $\det \overline{\mathsf{H}}'$ into (8.76).

We show how to obtain (D.16) and (D.17) by induction in the following. First, it is very easy to obtain α_n from equation (D.15),

$$\alpha_n = x_n^* = -\frac{k_n^{\perp}}{k_n^{\perp *}} \zeta_n, \tag{D.19}$$

where one uses the solution of the MRK scattering equations, $\zeta_n = -1$. As a next step, we assume all α_j's ($j > i$) are given by (D.16), and then let us solve α_i which satisfies the following equation:

$$v_i \alpha_i + \sum_{j \in \mathfrak{P}, j > i} (c_{ij} - x_i) \alpha_j = 0, \qquad i > i_1. \tag{D.20}$$

Using the definition of c_{ij} and α_j given by (D.16), we have

$$\sum_{j\in\mathfrak{P},j>i}(c_{ij}-x_i)\alpha_j = \sum_{j\in\mathfrak{P},j>i}\left(-\frac{\zeta_i\tau_j}{\tau_i}\frac{k_j^{\perp *}}{k_n^{\perp *}}+x_i\frac{k_j^{\perp}}{k_n^{\perp *}}\zeta_j\right)$$
$$= \frac{\zeta_i}{k_n^{\perp *}}\sum_{j\in\mathfrak{P},j>i}\left(-\frac{\tau_j k_j^{\perp *}}{\tau_i}+x_i\frac{\zeta_j k_j^{\perp}}{\zeta_i}\right)$$
$$= \frac{\zeta_i}{k_n^{\perp *}}\left(-\frac{\mathcal{T}_i}{\tau_i}+\frac{x_i\mathcal{Z}_i}{\zeta_i}\right), \tag{D.21}$$

where we denote

$$\mathcal{T}_i = \sum_{j\in\mathfrak{P},j>i}\tau_j k_j^{\perp *}, \qquad \mathcal{Z}_i = \sum_{j\in\mathfrak{P},j>i}\zeta_j k_j^{\perp}. \tag{D.22}$$

Next, we calculate these tow terms for two cases respectively: the label i is bigger than the label of any negative-helicity particle or not.

- We first consider the case of the label i is less than the largest label carried by negative-helicity particles, i.e. $i < I_m$. In this case, we have

$$\mathcal{T}_i = \sum_{j\in\mathfrak{P},i<j<I_{\ell_i}}\tau_j k_j^{\perp *} + \sum_{j\in\mathfrak{P},j>I_{\ell_i}}\tau_j k_j^{\perp *} \tag{D.23}$$
$$= \tau_i\sum_{i<j<I_{\ell_i}}k_j^{\perp *} + \frac{k_{I_{\ell_i}}^{\perp *}}{\zeta_{I_{\ell_i}}} \tag{D.24}$$
$$= \tau_i\sum_{i<j<I_{\ell_i}}k_j^{\perp *} - \tau_i\, q_{I_{\ell_i}}^{\perp *} \tag{D.25}$$
$$= -\tau_i\, q_{i+1}^{\perp *}. \tag{D.26}$$

Here we used the scattering equations $\bar{\mathcal{S}}_I^{\dot{2}} = 0$, (8.2), in the second line (D.24), and the solution (D.9) in the third line (D.25). Similarly, for $\mathcal{Z}_i$

we have

$$\mathcal{Z}_i = -\sum_{j\in\mathfrak{P}, j\leq i} \zeta_j k_j^\perp \tag{D.27}$$

$$= -\sum_{j\in\mathfrak{P}, j<I_{\ell_i}} \zeta_j k_j^\perp + \sum_{j\in\mathfrak{P}, i<j<I_{\ell_i}} \zeta_j k_j^\perp \tag{D.28}$$

$$= \frac{k_{I_{\ell_i}}^\perp}{\tau_{I_{\ell_i}}} + \zeta_i \sum_{j\in\mathfrak{P}, i<j<I_{\ell_i}} k_j^\perp \tag{D.29}$$

$$= -\zeta_i\, q_{I_{\ell_i}}^\perp + \zeta_i \sum_{j\in\mathfrak{P}, i<j<I_{\ell_i}} k_j^\perp \tag{D.30}$$

$$= -\zeta_i\, q_{i+1}^\perp \tag{D.31}$$

- In the other case, i.e. $i > I_m$, it is easy to obtain

$$\mathcal{T}_i = \tau_i \sum_{i<j\leq n} k_j^{\perp *} = -\tau_i\, q_{i+1}^{\perp *}, \tag{D.32}$$

$$\mathcal{Z}_i = -\zeta_i \sum_{i<j\leq n} k_j^\perp = -\zeta_i\, q_{i+1}^\perp. \tag{D.33}$$

In both cases, as expected, we obtain the same results for $\mathcal{T}_i$ and $\mathcal{Z}_i$. By inserting them into (D.21), we find

$$\sum_{j\in\mathfrak{P}, j>i} (c_{ij} - x_i)\alpha_j = \frac{\zeta_i}{k_n^{\perp *}} \left(-\frac{\mathcal{T}_i}{\tau_i} + \frac{x_i \mathcal{Z}_i}{\zeta_i} \right) = \frac{k_i^\perp}{k_n^{\perp *}}\, v_i\, \zeta_i\,. \tag{D.34}$$

Finally, equation (D.20) can be solved exactly by

$$\alpha_i = -\frac{k_i^\perp}{k_n^{\perp *}}\, \zeta_i, \tag{D.35}$$

which proves (D.16).

As a final step, we prove (D.17) via finding α_{i_1} which satisfies

$$v_{i_1}\alpha_{i_1} + \sum_{j\in\mathfrak{P}, j>i_1} \big(c_{i_1 j} - x_{i_1} - v_{i_1}\big)\alpha_j + v_{i_1}\alpha_n \;=\; 0. \tag{D.36}$$

Noting that

$$\begin{aligned}\sum_{j\in\mathfrak{P}, j>i_1} \big(c_{i_1 j} - x_{i_1} - v_{i_1}\big)\alpha_j \;&=\; \frac{\zeta_{i_1}}{k_n^{\perp *}}\left(-\frac{\mathcal{T}_{i_1}}{\tau_{i_1}} + \big(x_{i_1} + v_{i_1}\big)\frac{\mathcal{Z}_{i_1}}{\zeta_{i_1}}\right) \\ &=\; -\frac{q_{i_1}^{\perp}}{k_n^{\perp *}}\, v_{i_1}\,\zeta_{i_1},\end{aligned} \tag{D.37}$$

we have

$$\alpha_{i_1} \;=\; \frac{q_{i_1}^{\perp}}{k_n^{\perp *}}\,\zeta_{i_1} - x_n^* \;=\; -x_n^*\left(\frac{k_3^{\perp}}{k_n^{\perp}}\,\zeta_3\,\tau_3 + 1\right), \tag{D.38}$$

which ends the proof.

Identity in (8.88)

Let us now discuss another part corresponding to the set $\mathfrak{N}$. Our goal is to evaluate the determinant of the following matrix:

$$\mathsf{H}' \;=\; \begin{pmatrix} \mathsf{H}_{22} & c_{2I_1} & c_{2I_2} & \cdots & c_{2I_m} \\ x_{I_1}^* & v_{I_1}^* + x_{I_1}^* & c_{I_1 I_2} x_{I_1}^* & \cdots & c_{I_1 I_m} x_{I_1}^* \\ x_{I_2}^* & x_{I_2}^* & v_{I_2}^* + x_{I_2}^* & \cdots & c_{I_2 I_m} x_{I_2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{I_m}^* & x_{I_m}^* & x_{I_m}^* & \cdots & v_{I_m}^* + x_{I_m}^* \end{pmatrix} \tag{D.39}$$

with

$$\mathsf{H}_{22} = \prod_{I\in\overline{\mathfrak{N}}} \frac{q_I^{\perp}}{q_{I+1}^{\perp}}, \quad c_{2I} = \frac{\tau_I}{\zeta_I}, \quad c_{IJ} = \frac{\zeta_I \tau_J}{\tau_I \zeta_J} \;\text{ for }\; I < J. \tag{D.40}$$

Using a little linear algebra, $\det \mathsf{H}'$ becomes

$$\begin{vmatrix} \mathsf{H}_{22} & c_{2I_1}-\mathsf{H}_{22} & c_{2I_2}-\mathsf{H}_{22} & \cdots & c_{2I_m}-\mathsf{H}_{22} \\ 0 & v^*_{I_1} & (c_{I_1I_2}-1)x^*_{I_1} & \cdots & (c_{I_1I_m}-1)x^*_{I_1} - x^*_{I_1}x_{I_m}v^*_{I_m} \\ 0 & 0 & v^*_{I_2} & \cdots & (c_{I_2I_m}-1)x^*_{I_2} - x^*_{I_2}x_{I_m}v^*_{I_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^*_{I_m} & 0 & 0 & \cdots & v^*_{I_m} \end{vmatrix}. \tag{D.41}$$

By the matrix determinant lemma, we have

$$\det \mathsf{H}' = \mathsf{H}_{22}\left(\prod_{I\in\overline{\mathfrak{N}}} v^*_I\right)\left(1 + x^*_{I_m}\left(\Phi^{-1}\right)_{1,m+1}\right), \tag{D.42}$$

where

$$\Phi = \begin{pmatrix} \mathsf{H}_{22} & c_{2I_1}-\mathsf{H}_{22} & c_{2I_2}-\mathsf{H}_{22} & \cdots & c_{2I_m}-\mathsf{H}_{22} \\ 0 & v^*_{I_1} & (c_{I_1I_2}-1)x^*_{I_1} & \cdots & (c_{I_1I_m}-1)x^*_{I_1} - x^*_{I_1}x_{I_m}v^*_{I_m} \\ 0 & 0 & v^*_{I_2} & \cdots & (c_{I_2I_m}-1)x^*_{I_2} - x^*_{I_2}x_{I_m}v^*_{I_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v^*_{I_m} \end{pmatrix}. \tag{D.43}$$

Thus now our task is to calculate the last entry in the first row of the inverse of this matrix. As we will see in the following, a similar technique used in previous sections also works in this case.

Let us denote the first row of the inverse of the Φ as:

$$(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_m) \tag{D.44}$$

They can be determined by the following linear equations:

$$(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_m)\,\Phi \;=\; (1, 0, \ldots, 0) \tag{D.45}$$

The solution of this equation is

$$\bar{\alpha}_0 \;=\; \frac{1}{\mathsf{H}_{22}} \;=\; \prod_{I \in \bar{\mathfrak{N}}} \frac{q^{\perp}_{I+1}}{q^{\perp}_{I}}, \tag{D.46}$$

$$\bar{\alpha}_a \;=\; \bar{\alpha}_0\, x_{I_a} \tau_{I_a} \quad \text{for } 1 \le a < m, \tag{D.47}$$

$$\bar{\alpha}_m \;=\; \big(\bar{\alpha}_0 - 1\big) x_{I_m}. \tag{D.48}$$

In the following, we prove them by induction. First of all, it is straightforward to obtain $\bar{\alpha}_0$ and $\bar{\alpha}_1$ by solving the first two equations in (D.45). In next step, we assume all $\bar{\alpha}_b$'s ($b < a$) are given by (D.46) and (D.47), then let us solve $\bar{\alpha}_a$ which satisfies the following equation:

$$v^{*}_{I_a} \bar{\alpha}_a + \big(c_{2I_a} - \mathsf{H}_{22}\big)\bar{\alpha}_0 + \sum_{r=1}^{a-1} \big(c_{I_r I_a} - 1\big) x^{*}_{I_r} \bar{\alpha}_r \;=\; 0, \tag{D.49}$$

for $1 < a < m$. Here we first consider the second term on the left-hand side of the equation. By observing the MRK, given by eqs. (8.26), (8.20) and (8.23), we find that H_{22} and c_{2I} can be written as

$$\mathsf{H}_{22} \;=\; \prod_{I \in \bar{\mathfrak{N}}} \frac{q^{\perp}_{I}}{q^{\perp}_{I+1}} \;=\; \frac{q^{\perp}_{I_a}}{k^{\perp}_{I_a}} \left(\prod_{l=1}^{a-1} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right) \tau_{I_a}, \tag{D.50}$$

$$c_{2I_a} \;=\; \frac{\tau_{I_a}}{\zeta_{I_a}} \;=\; \frac{q^{\perp *}_{I_a}}{k^{\perp *}_{I_a}} \left(\prod_{l=1}^{a-1} \frac{q^{\perp *}_{I_l}}{q^{\perp *}_{I_l+1}} \right) \tau_{I_a}. \tag{D.51}$$

Therefore we have

$$\big(c_{2I_a} - \mathsf{H}_{22}\big)\bar{\alpha}_0 \;=\; \big(h_a - h'_a\big)\bar{\alpha}_0\, x_{I_a} \tau_{I_a}, \tag{D.52}$$

where

$$h_a \equiv \frac{k^{\perp}_{I_a} q^{\perp *}_{I_a}}{\left(k^{\perp *}_{I_a}\right)^2} \left(\prod_{l=1}^{a-1} \frac{q^{\perp *}_{I_l}}{q^{\perp *}_{I_l+1}} \right), \qquad h'_a \equiv \frac{q^{\perp}_{I_a}}{k^{\perp *}_{I_a}} \left(\prod_{l=1}^{a-1} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right). \tag{D.53}$$

Let us now turn to the last term on the left-hand side of (D.49). Comparing to the solution of the MRK scattering equations in eqs. (8.26), (8.20) and (8.23), we have

$$\tau_{I_r} = \frac{k^{\perp}_{I_r} q^{\perp}_{I_a+1}}{k^{\perp}_{I_a} q^{\perp}_{I_r+1}} \left(\prod_{l=r+1}^{a} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right) \tau_{I_a}, \quad r < a. \tag{D.54}$$

This leads to

$$\begin{aligned}
\left(c_{I_r I_a} - 1\right) x^*_{I_r} \bar{\alpha}_r &= \left(c_{I_r I_a} - 1\right) x^*_{I_r} \bar{\alpha}_0 \, x_{I_r} \frac{k^{\perp}_{I_r} q^{\perp}_{I_a+1}}{k^{\perp}_{I_a} q^{\perp}_{I_r+1}} \left(\prod_{l=r+1}^{a} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right) \tau_{I_a} \\
&= \left(f_{ra} - g_{ra}\right) \left(\bar{\alpha}_0 \, x_{I_a} \tau_{I_a}\right),
\end{aligned} \tag{D.55}$$

where we introduce some short-handed notations:

$$g_{ra} = \frac{k^{\perp}_{I_r} q^{\perp}_{I_a+1}}{k^{\perp *}_{I_a} q^{\perp}_{I_r+1}} \left(\prod_{l=r+1}^{a} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right), \tag{D.56}$$

$$f_{ra} = \frac{k^{\perp}_{I_r} q^{\perp}_{I_a+1}}{k^{\perp *}_{I_a} q^{\perp}_{I_r+1}} \left(\prod_{l=r+1}^{a} \frac{q^{\perp}_{I_l}}{q^{\perp}_{I_l+1}} \right) c_{I_r I_a} = \frac{k^{\perp}_{I_a} k^{\perp *}_{I_r} q^{\perp *}_{I_a}}{\left(k^{\perp *}_{I_a}\right)^2 q^{\perp *}_{I_r}} \left(\prod_{l=r}^{a-1} \frac{q^{\perp *}_{I_l}}{q^{\perp *}_{I_l+1}} \right).$$

Then equation (D.49) becomes

$$v^*_{I_a} \bar{\alpha}_a + \left[\left(h_a - h'_a\right) + \sum_{r=1}^{a-1} \left(f_{ra} - g_{ra}\right) \right] \left(\bar{\alpha}_0 \, x_{I_a} \tau_{I_a}\right) = 0 \tag{D.57}$$

for $1 < a < m$. Then by performing a lot of straightforward calculations, we obtain

$$h_a + \sum_{r=1}^{a-1} f_{ra} \;=\; \frac{k_{I_a}^{\perp} q_{I_a}^{\perp\,*}}{\left(k_{I_a}^{\perp\,*}\right)^2}, \tag{D.58}$$

$$h'_a + \sum_{r=1}^{a-1} g_{ra} \;=\; \frac{q_{I_a}^{\perp}}{k_{I_a}^{\perp\,*}}. \tag{D.59}$$

By plugging them into (D.57) gives immediately

$$\bar{\alpha}_a \;=\; \bar{\alpha}_0\, x_{I_a} \tau_{I_a}. \tag{D.60}$$

As a final step, we can obtain $\bar{\alpha}_m$ by solving the following equation:

$$v_{I_m}^* \bar{\alpha}_m + \left(c_{2I_m} - \mathsf{H}_{22}\right)\bar{\alpha}_0 + \sum_{r=1}^{m-1}\left(c_{I_r I_m} - 1\right) x_{I_r}^* \bar{\alpha}_r - \sum_{r=1}^{m-1} x_{I_r}^* x_{I_m} v_{I_m}^* \bar{\alpha}_r = 0. \tag{D.61}$$

For the second and the third terms, we have

$$\left(c_{2I_m} - \mathsf{H}_{22}\right)\bar{\alpha}_0 + \sum_{r=1}^{m-1}\left(c_{I_r I_m} - 1\right) x_{I_r}^* \bar{\alpha}_r \;=\; -v_{I_m}^* \bar{\alpha}_0\, x_{I_m} \tau_{I_m}. \tag{D.62}$$

For the last term, we have

$$\begin{aligned}\sum_{r=1}^{m-1} x_{I_r}^* x_{I_m} v_{I_m}^* \bar{\alpha}_r \;&=\; \sum_{r=1}^{m-1} x_{I_r}^* x_{I_m} v_{I_m}^* \bar{\alpha}_0\, x_{I_r} \tau_{I_r} \\ &=\; v_{I_m}^* \bar{\alpha}_0\, x_{I_m} \sum_{r=1}^{m-1} \tau_{I_r}.\end{aligned} \tag{D.63}$$

Using the scattering equations (8.2) and their unique solution in eqs. (8.26), (8.20) and (8.23), we have

$$\sum_{r=1}^{m-1} x_{I_r}^* x_{I_m} v_{I_m}^* \bar{\alpha}_r \;=\; v_{I_m}^* \bar{\alpha}_0\, x_{I_m}\Big(1 - \mathsf{H}_{22} - \tau_{I_m}\Big). \tag{D.64}$$

Finally, we find

$$\bar{\alpha}_m \;=\; \bar{\alpha}_0\, x_{I_m}\big(1 - \mathsf{H}_{22}\big) \;=\; x_{I_m}\big(\bar{\alpha}_0 - 1\big). \tag{D.65}$$

Plugging it into (D.42) gives

$$\det \mathsf{H}' \;=\; \prod_{I \in \overline{\mathfrak{N}}} v_I^* \,. \tag{D.66}$$

E Explicit results for Lipatov vertices and impact factors

As both consistency checks as well as applications of the CHY-type formulas for the generalized Lipatov vertex and impact factor and obtained in Chapter 9, we analytically evaluate them and compare with known results for some helicity configurations in this appendix. These include MHV and anti-MHV sectors for any number of legs, as well as some more complicated examples beyond the MHV sector.

MHV

First, we consider the MHV sector in which the four-dimensional scattering equations have only one independent solution, as shown in previous chapters. In this case, since the scattering equations have unique MHV solution, it is easy to obtain analytical results. Using the MHV solution we have computed the MHV amplitudes in Yang-Mills and MHV form factors for some certain operators. Here we exactly evaluate our formulas in the MHV sector.

Let us first see the MHV Lipatov factor where $\mathfrak{P} = \{3, \ldots, n\}$ and $\mathfrak{N} = \emptyset$. For this object, the Jacobian (9.17) in formula (9.16) is just one, and the two formulas for the Lipatov vertex, i.e. eqs. (9.14) and (9.16), become identical in MHV sector. Simply substituting the MHV solution into the formula gives

$$\mathcal{V}_{n-4}\big(q_2; 4^+, \ldots, (n{-}1)^+; q_1\big) = \frac{{q_2^\perp}^* q_1^\perp}{k_4^\perp} \sqrt{\frac{k_4^+}{k_{n-1}^+}} \frac{1}{\langle 4\, 5\rangle \cdots \langle n{-}2\; n{-}1\rangle}. \tag{E.1}$$

Using formula (9.7) we can also obtain the MHV impact factor:

$$\mathcal{C}_{n-3}\big(2^-;3^+,\ldots,(n-1)^+;q_1\big) = -\frac{q_1^\perp}{k_3^\perp}\sqrt{\frac{k_3^+}{k_{n-1}^+}}\frac{1}{\langle 3\,4\rangle\cdots\langle n-2\;n-1\rangle}. \quad \text{(E.2)}$$

Very similarly, it is also easy to evaluate our formulas for Lipatov vertices and impact factors in the $\overline{\text{MHV}}$ sector.

NMHV and NNMHV

In the following, we evaluate analytically several more complicated examples by following the similar procedure used in [140,211]. We illustrate this method in the example below.

Let us first consider a NMHV Lipatov vertex for $g^*g^* \to g^+g^-$ for which our formula (9.16) gives

$$\begin{aligned}\mathcal{V}\big(q_2;4^+,5^-;q_1\big) \;&=\; \frac{|q_2^\perp|^2|q_1^\perp|^2k_5^\perp}{k_4^\perp}\int\frac{d\sigma_4 d\tau_4 d\sigma_5 d\tau_5}{\tau_4\tau_5\,\sigma_{45}\sigma_5}\\ &\quad\times\frac{\delta^2(\mathcal{S}_4^\alpha)\delta^2(\bar{\mathcal{S}}_5^{\dot\alpha})}{\big(q_1^\perp-\zeta_4k_4^\perp\big)\big(q_2^{\perp *}+\tau_4k_4^{\perp *}\big)}\end{aligned} \quad \text{(E.3)}$$

where $q_1 = k_1 + k_6$ and $q_2 = -k_2 - k_3$ and the scattering equations are give by

$$\mathcal{S}_4^1 \;=\; 1-\frac{\tau_4\tau_5}{\sigma_{45}}\frac{k_5^+}{k_5^\perp}+\tau_4, \quad \text{(E.4)}$$

$$\mathcal{S}_4^2 \;=\; 1-\frac{k_4^+}{k_4^\perp}\frac{\tau_4\tau_5}{\sigma_{45}}+\zeta_4, \quad \text{(E.5)}$$

$$\bar{\mathcal{S}}_5^{\dot 1} \;=\; k_5^\perp+\left(\frac{\tau_5\tau_4k_4^+}{\sigma_{45}}-\tau_5\zeta_4k_4^\perp\right)+\tau_5q_1^\perp, \quad \text{(E.6)}$$

$$\bar{\mathcal{S}}_5^{\dot 2} \;=\; k_5^{\perp *}+\left(\frac{k_5^+}{k_5^\perp}\frac{\tau_5\tau_4}{\sigma_{45}}+\zeta_5\tau_4\right)k_4^{\perp *}+\zeta_5q_2^{\perp *}. \quad \text{(E.7)}$$

We first introduce a new variable

$$z = \frac{\tau_4 \tau_5}{\sigma_{45}}. \tag{E.8}$$

Then we can observe that the three equations $\mathcal{S}_4^1$, $\mathcal{S}_4^2$ and $\bar{\mathcal{S}}_5^{\dot{1}}$ are linear in variables σ_5, τ_5 and z. After fixing these three variables, the formula becomes a one-dimensional contour integral over τ_4, where the contour is determined by the zeros of $\bar{\mathcal{S}}_5^{\dot{2}} = 0$, and we can evaluate it by residue theorem. The final result is

$$\begin{aligned}
&\mathcal{V}_2\big(q_2; 4^+, 5^-; q_1\big) \\
&= +\frac{x_5^2\, q_1^\perp q_2^\perp \big(k_4^{\perp *}\big)^3}{s_{561} x_4 \big(\sqrt{x_5}\,[4\,5] + \sqrt{x_4}\, q_1^{\perp *}\big)\big(\sqrt{x_5}\,\langle 4\,5\rangle\, k_4^{\perp *} - \sqrt{x_4}\, k_5^\perp q_2^{\perp *}\big)} \\
&- \frac{x_4^2\, q_1^{\perp *} |q_2^\perp|^2 \big(k_5^\perp\big)^4}{\sqrt{x_5}\,\langle 45\rangle\, k_4^\perp \big(k_4^\perp + k_5^\perp\big)\big(\sqrt{x_5}\,\langle 45\rangle\, k_4^{\perp *} + \sqrt{x_4}\, k_5^\perp k_3^{\perp *}\big)\big(x_5 |k_4^\perp|^2 + x_4 |k_5^\perp|^2\big)} \\
&+ \frac{x_4^2 |q_1^\perp|^2 q_2^{\perp *}}{\sqrt{x_5}\big(x_4 + x_5\big)[4\,5]\big(k_4^\perp + k_5^\perp\big)\big(\sqrt{x_5}[4\,5] + \sqrt{x_4} q_1^{\perp *}\big)},
\end{aligned} \tag{E.9}$$

where $x_i = k_i^+/(k_4^+ + k_5^+)$ for $i = 4, 5$. Our result exactly agrees with previously known results, including

$$\begin{aligned}
&\mathcal{V}_2\big(q_2; 4^+, 5^-; q_1\big) \\
&\quad = -\frac{x_4^{3/2} q_2^{\perp *} \big(k_5^\perp + q_1^\perp\big)^3}{k_4^\perp \big(\sqrt{x_5}\,\langle 4\,5\rangle + \sqrt{x_4}\, q_1^\perp\big)\big(x_4 |k_4^\perp - q_2^\perp|^2 + x_5 |k_4^\perp|^2\big)} \\
&\qquad + \frac{x_4^{3/2} |q_2^\perp|^2 \big(k_5^\perp\big)^3}{\sqrt{x_5}\,\langle 4\,5\rangle\, k_4^\perp \big(k_4^\perp + k_5^\perp\big)\big(x_5 |k_4^\perp|^2 + x_4 |k_5^\perp|^2\big)} \\
&\qquad + \frac{\sqrt{x_4}\, x_5^{3/2}\, q_1^\perp q_2^\perp}{\langle 4\,5\rangle\, \big(x_4 (k_4^\perp - q_2^\perp) + x_5 k_4^\perp\big)} \\
&\qquad + \frac{x_4^{3/2}\, q_1^\perp q_2^{\perp *}}{\sqrt{x_5}\,[4\,5]\, \big(k_4^\perp + k_5^\perp\big)}
\end{aligned} \tag{E.10}$$

which was computed by the CSW rules in [225], and another one obtained earlier from the Feynman diagrams [71, 220, 229, 259]

$$\mathcal{V}_2\big(q_2; 4^+, 5^-; q_1\big) \tag{E.11}$$
$$= \frac{k_4^{\perp *}}{k_4^{\perp}} \Bigg\{ -\frac{1}{s_{45}} \left(\frac{\big(k_5^{\perp}\big)^2 |q_2^{\perp}|^2}{\big(k_4^- + k_5^-\big) k_5^+} + \frac{\big(k_4^{\perp}\big)^2 |q_1^{\perp}|^2}{\big(k_4^+ + k_5^+\big) k_4^-} + \frac{s_{561} k_4^{\perp} k_5^{\perp}}{k_4^- k_5^+} \right)$$
$$+ \frac{\big(q_1^{\perp} + k_5^{\perp}\big)^2}{s_{561}} - \frac{q_1^{\perp} + k_5^{\perp}}{s_{45}} \left(\frac{\big(k_4^- + k_5^-\big) k_4^{\perp}}{k_4^-} - \frac{\big(k_4^+ + k_5^+\big) k_5^{\perp}}{k_5^+} \right) \Bigg\}.$$

In similar way, evaluating our formula for the Lipatov vertex of $g^* g^* \to q\bar{q}$ gives

$$\mathcal{V}_2\big(q_2; 4_{\bar{q}}^+, 5_q^-; q_1\big)$$
$$= \frac{x_4^2\, q_1^{\perp *} |q_2^{\perp}|^2 \big(k_5^{\perp}\big)^3}{\langle 45\rangle\, \big(k_4^{\perp} + k_5^{\perp}\big) \big(x_5 |k_4^{\perp}|^2 + x_4 |k_5^{\perp}|^2\big) \big(x_5 |k_4^{\perp}|^2 + x_4 k_5^{\perp} (k_5^{\perp *} + q_1^{\perp *})\big)}$$
$$- \frac{x_4^2\, |q_1^{\perp}|^2 |q_2^{\perp}|^2}{[4\,5]\, k_3^{\perp} \big(k_4^{\perp} + k_5^{\perp}\big) \big(x_4 (k_4^{\perp *} - q_2^{\perp *}) + x_5 k_4^{\perp *}\big)}$$
$$- \frac{\sqrt{x_4}\, x_5^{3/2}\, q_1^{\perp} q_2^{\perp} \big(k_4^{\perp *}\big)^2 \big(k_5^{\perp *} + q_1^{\perp *}\big)}{s_{561} \big(x_5 k_4^{\perp *} - x_4 (k_5^{\perp *} + q_1^{\perp *})\big) \big(x_5 |k_4^{\perp}|^2 + x_4 k_5^{\perp} (k_5^{\perp *} + q_1^{\perp *})\big)},$$

which also agrees with the known result [71, 220, 229, 259]:

$$\mathcal{V}_2\big(q_2; 4_{\bar{q}}^+, 5_q^-; q_1\big) \tag{E.12}$$
$$= \sqrt{\frac{k_4^+}{k_5^+}} \Bigg\{ \frac{k_5^- k_5^{\perp} |q_2^{\perp}|^2}{k_4^{\perp} \big(k_4^- + k_5^-\big) s_{45}} + \frac{k_5^+ |q_1^{\perp}|^2}{\big(k_4^+ + k_5^+\big) s_{45}} + \frac{k_4^{\perp *} k_5^+ \big(k_5^{\perp} + q_1^{\perp}\big)}{k_4^+ s_{156}}$$
$$- \frac{|k_5^{\perp}|^2}{s_{45}} + \frac{\big(k_5^{\perp} + q_1^{\perp}\big) \big(-k_5^{\perp} (k_5^{\perp *} + q_1^{\perp *}) + k_4^- k_5^+ - k_4^{\perp *} k_5^{\perp}\big)}{k_4^{\perp} s_{45}} \Bigg\}.$$

Let us turn to study the next-to-next-to-leading impact factor which describes $g^*g \to ggg$. By using a similar technique, our formula (9.7) also can obtain this impact factor with various helicity configurations analytically. The results are listed below. In these cases, we define $x_i = k_i^+/(k_3^+ + k_4^+ + k_5^+)$, $i = 3, 4, 5$.

$$\begin{aligned}
&\mathcal{C}_3\big(2^-; 3^-, 4^+, 5^+; q_1\big) \\
&\qquad = -\frac{x_4^2\, q_1^\perp}{\sqrt{x_3 x_4 x_5}\,[3\,4]\big(k_3^\perp + k_4^\perp\big)\big(\sqrt{x_3}\,\langle 3\,5\rangle + \sqrt{x_4}\,\langle 4\,5\rangle\,\big)} \\
&\qquad\quad - \frac{\sqrt{x_4}\,(x_4{+}x_5)^3 q_1^\perp}{\sqrt{x_5}\,\langle 4\,5\rangle\, k_3^{\perp\,*}\big(x_4 k_3^\perp + (x_4{+}x_5)k_4^\perp\big)} \\
&\qquad\quad + \frac{\big(\sqrt{x_4}\,\langle 3\,4\rangle + \sqrt{x_5}\,\langle 3\,5\rangle\,\big)^3}{s_{345}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\big(\sqrt{x_3}\,\langle 3\,5\rangle + \sqrt{x_4}\,\langle 4\,5\rangle\,\big)} \\
&\qquad\quad - \frac{x_4^{1/2} x_5^2\,(k_3^\perp)^3}{x_3^{3/2} s_{561}\,\langle 3\,4\rangle\,\big(k_3^\perp + k_4^\perp\big)\big(x_4 k_3^\perp + (x_4{+}x_5)k_4^\perp\big)}
\end{aligned} \tag{E.13}$$

$$\begin{aligned}
&\mathcal{C}_3\big(2^-; 3^+, 4^-, 5^+; q_1\big) \\
&= -\frac{x_3^{1/2} x_5^{3/2}\, q_1^\perp}{\sqrt{x_4}\,(x_4{+}x_5)\,[4\,5]\,k_3^\perp\big(\sqrt{x_4}\,\langle 3\,4\rangle + \sqrt{x_5}\,\langle 3\,5\rangle\,\big)} \\
&\quad - \frac{x_3^2\, q_1^\perp}{\sqrt{x_3 x_4 x_5}\,[3\,4]\,\big(k_3^\perp + k_4^\perp\big)\big(\sqrt{x_3}\,\langle 35\rangle + \sqrt{x_4}\,\langle 45\rangle\,\big)} \\
&\quad - \frac{x_3^2\, x_4^{5/2}\, q_1^\perp}{\sqrt{x_5}(x_4{+}x_5)\,\langle 45\rangle\, k_3^{\perp\,*}\big(x_4 k_3^\perp + (x_4{+}x_5)k_4^\perp\big)} \\
&\quad - \frac{\sqrt{x_3}\, x_5^2\,\big(k_4^\perp\big)^4}{x_4^{3/2} s_{561}\,\langle 34\rangle\, k_3^\perp\big(k_3^\perp + k_4^\perp\big)\big(x_4 k_3^\perp + (x_4{+}x_5)k_4^\perp\big)} \\
&\quad + \frac{\big(\sqrt{x_3}\,\langle 34\rangle - \sqrt{x_5}\,\langle 45\rangle\,\big)^4}{s_{345}\,\langle 34\rangle\,\langle 45\rangle\,\big(\sqrt{x_4}\,\langle 34\rangle + \sqrt{x_5}\,\langle 35\rangle\,\big)\big(\sqrt{x_3}\,\langle 35\rangle + \sqrt{x_4}\,\langle 45\rangle\,\big)}
\end{aligned} \tag{E.14}$$

$$
\begin{aligned}
&\mathcal{C}_3\big(2^-;3^+,4^+,5^-;q_1\big)\\
&=\frac{(x_3{+}x_4)^4|q_1^\perp|^2}{\sqrt{x_3x_5}\,\langle 3\,4\rangle\,\big({k_3^\perp}^*{+}{k_4^\perp}^*\big)\big(\sqrt{x_3}\,[3\,5]+\sqrt{x_4}\,[4\,5]\big)\big(\sqrt{x_3}\,\langle 3\,4\rangle-\sqrt{x_4}\,q_1^\perp\big)}\\
&\quad-\frac{\sqrt{x_4x_5}\,[3\,4]^3q_1^\perp}{s_{345}\,[4\,5]\big(\sqrt{x_3}\,[3\,5]+\sqrt{x_4}\,[4\,5]\big)\big(\sqrt{x_5}\,[3\,4]\,k_4^\perp+\sqrt{x_4}\,[3\,5]\,k_5^\perp\big)}\\
&\quad-\frac{x_4^2\,|q_1^\perp|^2}{\sqrt{x_5}\,(x_4{+}x_5)[4\,5]\,k_3^\perp\big(k_4^\perp+k_5^\perp\big)\big(\sqrt{x_5}\,[4\,5]+\sqrt{x_4}\,{q_1^\perp}^*\big)}\\
&\quad-\frac{1}{x_5\,[3\,4]\,k_4^\perp+\sqrt{x_4x_5}\,[3\,5]\,k_5^\perp+\sqrt{x_3x_4}\,k_5^\perp\,{q_1^\perp}^*}\\
&\quad\times\left\{\frac{\sqrt{x_4}\,x_5^2\,[3\,4]^3q_1^\perp\big((x_3{+}x_5){k_3^\perp}^*{+}\,x_3{k_4^\perp}^*\big)}{s_{561}{k_3^\perp}^*\big({k_5^\perp}^*{+}\,{q_1^\perp}^*\big)\big(\sqrt{x_5}\,[4\,5]+\sqrt{x_4}\,{q_1^\perp}^*\big)}\right.\\
&\left.-\frac{x_3^2\,x_4\,{q_1^\perp}^*\big(k_5^\perp\big)^4}{x_5\,\langle 45\rangle\,\big(k_4^\perp{+}k_5^\perp\big)\big(\sqrt{x_3}\,\langle 34\rangle-\sqrt{x_4}\,q_1^\perp\big)\big(\sqrt{x_5}\,[34]\,k_4^\perp+\sqrt{x_4}\,[35]\,k_5^\perp\big)}\right\}.
\end{aligned}
\tag{E.15}
$$

All these results agree with known results obtained using the CSW rule [225] and helicity amplitudes [71] respectively. Moreover, we discuss the Regge limit of these impact factors. Let us first see the last two objects given by (E.14) and (E.15) respectively. In the limit $y_3 \gg y_4 \simeq y_5$, they behave as

$$
\mathcal{C}_3(2^-;3^+,4^\pm,5^\mp;q_1) \simeq \mathcal{C}(2^-;3^+)\,\frac{1}{|q_2^\perp|^2}\,\mathcal{V}_2(q_2;4^\pm,5^\mp;q_1), \tag{E.16}
$$

where the leading impact factor $\mathcal{C}(2^-;3^+) = 1$, as shown exactly in eq. (7.5), while next-to-leading Lipatov factors are given by

$$
\begin{aligned}
&\mathcal{V}_2(q_2;4^+,5^-;q_1) \\
&\quad = \frac{x_4^2\,|q_1^\perp|^2\,q_2^{\perp\,*}}{[4\,5]\big(k_4^\perp+k_5^\perp\big)\big(x_5\,[4\,5]+\sqrt{x_4x_5}\,q_1^{\perp\,*}\big)} \\
&\qquad + \frac{1}{x_5|k_4^\perp|^2+x_4k_5^\perp\big(k_5^{\perp\,*}+q_1^{\perp\,*}\big)} \\
&\qquad \times\left(\frac{x_4^{5/2}\,q_1^{\perp\,*}\,|q_2^\perp|^2(k_5^\perp)^4}{\sqrt{x_5}\,\langle 4\,5\rangle\,k_4^\perp\big(k_4^\perp+k_5^\perp\big)\big(x_5|k_4^\perp|^2+x_4|k_5^\perp|^2\big)}\right. \\
&\qquad\quad \left. + \frac{x_5^2\,q_1^\perp\,q_2^\perp\,(k_4^{\perp\,*})^3\big(k_4^{\perp\,*}-q_2^{\perp\,*}\big)}{\sqrt{x_4}\,s_{561}\big(k_5^{\perp\,*}+q_1^{\perp\,*}\big)\big(\sqrt{x_5}\,[4\,5]+\sqrt{x_4}\,q_1^{\perp\,*}\big)}\right),
\end{aligned}
\tag{E.17}
$$

$$
\begin{aligned}
\mathcal{V}_2(q_2;4^-,5^+;q_1) \;=\;& \frac{x_5^{3/2}\,|q_2^\perp|^2\,(k_4^\perp)^3}{\sqrt{x_4}\,\langle 4\,5\rangle\,k_5^\perp\big(k_4^\perp+k_5^\perp\big)\big(x_5|k_4^\perp|^2+x_4|k_5^\perp|^2\big)} \\
&+ \frac{x_4^{5/2}q_1^\perp q_2^\perp}{\sqrt{x_5}\,\langle 4\,5\rangle\,\big(k_4^\perp-x_4q_2^\perp\big)} \\
&+ \frac{x_5^{3/2}\,q_1^\perp q_2^{\perp\,*}}{\sqrt{x_4}\,[4\,5]\,\big(k_4^\perp+k_5^\perp\big)} \\
&+ \frac{q_1^\perp|q_2^\perp|^2}{k_4^{\perp\,*}k_5^\perp\big(q_2^\perp-k_4^\perp\big)} \\
&+ \frac{x_5^2\,q_2^{\perp\,*}(k_4^\perp)^3}{x_4s_{561}\big(k_4^\perp-q_2^\perp\big)\big(k_4^\perp-x_4q_2^\perp\big)},
\end{aligned}
\tag{E.18}
$$

which are complex conjugates of each other, and we have checked that they are equivalent to formulas (E.9), (E.10) or (E.11). For the helicity impact factor

$\mathcal{C}_3(2^-;3^-,4^+,5^+;q_1)$, in limit $y_3 \gg y_4 \simeq y_5$, it behaves as

$$\begin{aligned}
\mathcal{C}_3\big(2^-;3^-,4^+,5^+;q_1\big) \;=\; & \frac{\big(k_4^+ + k_5^+\big)^2}{\big(k_3^+\big)^2}\Bigg(\frac{\sqrt{x_4}\,q_1^\perp}{\sqrt{x_5}\,q_2^{\perp\,*}\,\langle 4\,5\rangle\,\big(k_4^\perp - x_4 q_2^\perp\big)} \\
& + \frac{x_4^{3/2}x_5^{3/2}\,\big(k_4^\perp + k_5^\perp\big)^3}{\langle 4\,5\rangle\,k_4^\perp k_5^\perp\big(x_5|k_4^\perp|^2 + x_4|k_5^\perp|^2\big)} \\
& + \frac{x_4 x_5^2\,(q_2^\perp)^3}{k_4^\perp\,s_{561}\,\big(k_4^\perp - q_2^\perp\big)\big(k_4^\perp - x_4 q_2^\perp\big)} \\
& + \frac{x_4^2\,q_1^\perp}{k_4^{\perp\,*}k_5^\perp\big(q_2^\perp - k_4^\perp\big)}\Bigg),
\end{aligned} \tag{E.19}$$

which is suppressed by the prefactor $\mathcal{O}\big((k_4^+ + k_5^+/k_3^+)^2\big)$. This is consistent with the fact that the leading impact factor $\mathcal{C}(2^-;3^-)$ is vanishing.

F Factorizations of impact factors and Lipatov vertices

In this appendix, we show that the generalised impact factors and Lipatov vertices that appear in QMRK have the expected factorizations in soft, collinear and Regge limits.

Soft limits

Let us start by discussing the soft limits of the impact factors and the Lipatov vertices defined through the CHY-type formulas in eqs. (9.7) and (9.14). The argument is the same in both cases and follows the analysis of the soft limit of the full amplitude in refs. [24, 25] closely. In the following we only discuss the case of the generalised Lipatov vertex $\mathcal{V}(q_4; 4, \ldots, n{-}1; q_n)$.

Without loss of generality we assume that $k_5 \to 0$, and we assume that the soft gluon has positive helicity. The argument in other cases is identical. Keeping only the leading behaviour as $k_5 \to 0$ in eq. (9.14), we can write

$$\mathcal{V}\big(q_4; 4, \ldots, n{-}1; q_n\big) \simeq \frac{1}{k_5^{\perp}} \int \prod_{a=4}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \left(\frac{1}{\sigma_{45}} + \frac{1}{\sigma_{56}}\right) \delta^2\big(\mathcal{S}_5^{\alpha}\big)\, \mathcal{I}_{n-5}\,, \tag{F.1}$$

where $\mathcal{I}_{n-5}$ collects the terms in the integrand of eq. (9.14) independent of σ_5 and τ_5,

$$\mathcal{I}_{n-5} = \frac{(q_4^\perp)^* q_n^\perp}{\sigma_{46}\sigma_{67}\cdots\sigma_{n-2,n-1}\sigma_{n-1}} \left(\prod_{i\in\mathfrak{P}\setminus\{5\}, I\in\mathfrak{N}} \frac{k_I^\perp}{k_i^\perp} \right) \tag{F.2}$$
$$\times \prod_{I\in\mathfrak{N}} \delta\left(k_I^\perp - \sum_{i\in\mathfrak{P}\setminus\{5\}} \frac{\tau_i\tau_I}{\sigma_I - \sigma_i} k_i^+ + \frac{\tau_I}{1-\sum_{J\in\mathfrak{N}} t_J} q_4^\perp \right)$$
$$\times \prod_{I\in\mathfrak{N}} \delta\left(k_I^{\perp *} - \frac{k_I^+}{k_I^\perp} \sum_{i\in\mathfrak{P}\setminus\{5\}} \frac{\tau_i\tau_I}{\sigma_I - \sigma_i} k_i^{\perp *} - \frac{\zeta_I}{1+\sum_{J\in\mathfrak{N}} \zeta_J} q_n^{\perp *} \right)$$
$$\times \prod_{i\in\mathfrak{P}\setminus\{5\}} \delta\left(1 - \sum_{I\in\mathfrak{N}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I} \frac{k_I^+}{k_I^\perp} + \tau_i \right) \delta\left(1 - \frac{k_i^+}{k_i^\perp} \sum_{I\in\mathfrak{N}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I} + \zeta_i \right).$$

Let us write down $\mathcal{S}_5^\alpha$ explicitly:

$$\mathcal{S}_5^1 = 1 + \tau_5 \left(1 - \sum_{I\in\mathfrak{N}} \frac{\tau_I}{\sigma_5 - \sigma_I} \frac{k_I^+}{k_I^\perp} \right), \tag{F.3}$$

$$\mathcal{S}_5^2 = 1 + \tau_5 \frac{k_5^+}{k_5^\perp} \left(\frac{1}{\sigma_5} - \sum_{I\in\mathfrak{N}} \frac{\tau_I}{\sigma_5 - \sigma_I} \right). \tag{F.4}$$

As a first step, let us use the delta function $\delta(\mathcal{S}_5^1)$ in eq. (F.3) to fix τ_5. We then obtain

$$\mathcal{V}(q_4; 4, \ldots, n-1; q_n) \tag{F.5}$$
$$\simeq \frac{1}{k_5^\perp} \int \prod_{a=4, a\neq 5}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \mathcal{I}_{n-5} \oint_{\mathcal{C}} \frac{d\sigma_5}{\mathcal{S}_5^2} \left(\frac{1}{\sigma_{45}} + \frac{1}{\sigma_{56}} \right),$$

where the contour $\mathcal{C}$ is defined to encircle the zeros of $\mathcal{S}_5^2$, which is given by

$$\mathcal{S}_5^2\big|_{\mathcal{S}_5^1=0} = 1 - \frac{k_5^+}{k_5^\perp} \left(\frac{1}{\sigma_5} - \sum_{I\in\mathfrak{N}} \frac{\tau_I}{\sigma_5 - \sigma_I} \right) \left(1 - \sum_{I\in\mathfrak{N}} \frac{\tau_I}{\sigma_5 - \sigma_I} \frac{k_I^+}{k_I^\perp} \right)^{-1}. \tag{F.6}$$

We apply the global residue theorem, which allows us to express the residue at $\mathcal{S}_5^2 = 0$ in terms of the residues at $\sigma_5 = \sigma_4$ and $\sigma_5 = \sigma_6$. Even though the computation depends on the helicity configuration of the two adjacent legs, the final result does not, and a straightforward calculation gives

$$\mathcal{V}\big(q_4; 4, 5, \ldots, n{-}1; q_n\big) \simeq \frac{\langle 4\,6\rangle}{\langle 4\,5\rangle\,\langle 5\,6\rangle}\,\mathcal{V}\big(q_4; 4, 6, \ldots, n{-}1; q_n\big). \tag{F.7}$$

Collinear limit

We now study the generalised impact factors and Lipatov vertices in the limit where a pair of produced particles become collinear. We again restrict ourselves to the study of Lipatov vertices, and we assume that the momenta k_4 and k_5 are collinear. We take the following parametrization for the two collinear massless momenta [260],

$$\begin{aligned}
k_4^+ &= zk_x^+ - 2\epsilon\sqrt{z(1{-}z)}\big(k_x^+ k_y^+\big)^{1/2} + \epsilon^2(1{-}z)k_y^+\,,\\
k_4^- &= zk_x^- - \epsilon\sqrt{z(1{-}z)}\big((k_x^\perp)^* k_y^\perp + k_x^\perp (k_y^\perp)^*\big)\big(k_x^+ k_y^+\big)^{-1/2} + \epsilon^2(1{-}z)k_y^-\,,\\
k_4^\perp &= zk_x^\perp - \epsilon\sqrt{z(1{-}z)}\big(k_x^\perp k_y^+ + k_x^+ k_y^\perp\big)\big(k_x^+ k_y^+\big)^{-1/2} + \epsilon^2(1{-}z)k_y^\perp\,,\\
k_5^+ &= (1{-}z)k_x^+ + 2\epsilon\sqrt{z(1{-}z)}\big(k_x^+ k_y^+\big)^{1/2} + \epsilon^2 z k_y^+\,,\\
k_5^- &= (1{-}z)k_x^- + \epsilon\sqrt{z(1{-}z)}\big((k_x^\perp)^* k_y^\perp + k_x^\perp (k_y^\perp)^*\big)\big(k_x^+ k_y^+\big)^{-1/2} + \epsilon^2 z k_y^-\,,\\
k_5^\perp &= (1{-}z)k_x^\perp + \epsilon\sqrt{z(1{-}z)}\big(k_x^\perp k_y^+ + k_x^+ k_y^\perp\big)\big(k_x^+ k_y^+\big)^{-1/2} + \epsilon^2 z k_y^\perp\,.
\end{aligned} \tag{F.8}$$

The collinear limit is realised as $\epsilon \to 0$ and the collinear direction is $K = k_4 + k_5 = k_x + \mathcal{O}(\epsilon^2)$.

In ref. [47] it was shown that in the limit where two gluons become collinear only those solutions contribute where σ_4 and σ_5 coincide in the limit. Like the case of the double soft limit (see Chapter 5), it is then useful to perform the change of variables from (σ_4, σ_5) to (ρ, ξ):

$$\sigma_4 \,=\, \rho + \frac{1}{2}\epsilon\xi + \mathcal{O}(\epsilon^2)\,, \qquad \sigma_5 \,=\, \rho - \frac{1}{2}\epsilon\xi + \mathcal{O}(\epsilon^2)\,, \tag{F.9}$$

such that that $\sigma_4-\sigma_5 = \epsilon\xi + \mathcal{O}(\epsilon^2)$ and $d\sigma_4 d\sigma_5 = \epsilon d\xi d\rho$. In these new variables eq. (9.14) takes the form:

$$\mathcal{V}(q_4;4,5,\ldots,n{-}1;q_n) = (q_4^\perp)^* q_n^\perp \left(\prod_{i\in\mathfrak{P},I\in\mathfrak{N}} \frac{k_I^\perp}{k_i^\perp}\right) \tag{F.10}$$
$$\times \int \prod_{a=6}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \int \frac{d\xi d\rho d\tau_4 d\tau_5}{\xi\tau_4\tau_5} \frac{\prod_{I\in\mathfrak{N}} \delta^2\big(\mathcal{S}_I^{\dot\alpha}\big) \prod_{i\in\mathfrak{P}} \delta^2\big(\mathcal{S}_i^{\alpha}\big)}{(\rho-\sigma_6)\sigma_{67}\cdots\sigma_{n-2,n-1}\sigma_{n-1}}.$$

We now show that after integrating out ξ and some linear combination of τ_4 and τ_5 we can recover the expected factorized form of the Lipatov vertex in the collinear limit. In order to proceed, we need to analyse separately the cases where the collinear particles have either the same or opposite helicities.

- $(h_4, h_5) = (+,+)$

Let us first consider the case where the two particles have positive helicity. It is convenient to perform the change of variables from (τ_4, τ_5) to (τ_x, τ_y) defined by,

$$\tau_x = z\,\tau_4 + (1{-}z)\,\tau_5 \quad \text{and} \quad \tau_y = \sqrt{z(1{-}z)}\,\big(\tau_5 - \tau_4\big), \tag{F.11}$$

such that

$$d\tau_4 d\tau_5 = \frac{1}{\sqrt{z(1{-}z)}} d\tau_x d\tau_y. \tag{F.12}$$

Equation (F.10) then becomes,

$$\mathcal{V}(q_4;4,5,\ldots,n{-}1;q_n) = \frac{(q_4^\perp)^* q_n^\perp}{(k_x^\perp)^2} \frac{1}{(z(1{-}z))^{3/2}} \left(\prod_{i\in\mathfrak{P}\backslash\{4,5\},I\in\mathfrak{N}} \frac{k_I^\perp}{k_i^\perp}\right)$$
$$\times \int \prod_{a=6}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \int \frac{d\rho d\tau_x}{\tau_x} \int \frac{d\xi d\tau_y}{\xi} \frac{\tau_x}{\tau_4\tau_5} \frac{\prod_{I\in\mathfrak{N}} \delta^2\big(\mathcal{S}_I^{\dot\alpha}\big) \prod_{i\in\mathfrak{P}} \delta^2\big(\mathcal{S}_i^{\alpha}\big)}{(\rho-\sigma_6)\sigma_{67}\cdots\sigma_{n-2,n-1}\sigma_{n-1}},$$

where the scattering equations are given below in detail. First, $\bar{\mathcal{S}}_I^{\dot{\alpha}}$ become

$$\bar{\mathcal{S}}_I^{\dot{1}} = k_I^\perp - \sum_{i\in\mathfrak{P}\backslash\{4,5\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i}k_i^+ - \frac{\tau_I\tau_x}{\sigma_I{-}\rho}k_x^+ + \frac{\tau_I\, q_4^\perp}{1-\sum_{J\in\mathfrak{N}} t_J} + \mathcal{O}(\epsilon),$$

$$\begin{aligned}\bar{\mathcal{S}}_I^{\dot{2}} = {k_I^\perp}^* - \frac{k_I^+}{k_I^\perp} \sum_{i\in\mathfrak{P}\backslash\{4,5\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i}{k_i^\perp}^* \\ - \frac{k_I^+}{k_I^\perp}\frac{\tau_I\tau_x}{\sigma_I{-}\rho}(k_x^\perp)^* - \frac{\zeta_I}{1+\sum_{J\in\mathfrak{N}}\zeta_J}\,(q_n^\perp)^* + \mathcal{O}(\epsilon),\end{aligned}$$

up to leading order in the collinear limit. While $\mathcal{S}_i^\alpha$ with $i\neq 4,5$ remain unchanged, $\mathcal{S}_4^\alpha$ and $\mathcal{S}_5^\alpha$ become

$$\mathcal{S}_4^1 = 1+\tau_4\left[1-\sum_{I\in\mathfrak{N}}\left(\frac{\tau_I}{\rho{-}\sigma_I} - \frac{\epsilon\xi}{2}\frac{\tau_I}{(\rho{-}\sigma_I)^2}\right)\frac{k_I^+}{k_I^\perp}\right] + \mathcal{O}(\epsilon^2), \tag{F.13}$$

$$\mathcal{S}_5^1 = 1+\tau_5\left[1-\sum_{I\in\mathfrak{N}}\left(\frac{\tau_I}{\rho{-}\sigma_I} + \frac{\epsilon\xi}{2}\frac{\tau_I}{(\rho{-}\sigma_I)^2}\right)\frac{k_I^+}{k_I^\perp}\right] + \mathcal{O}(\epsilon^2), \tag{F.14}$$

$$\begin{aligned}\mathcal{S}_4^2 = 1+\tau_4\frac{k_x^+}{k_x^\perp}\Bigg\{\frac{1}{\rho} - \sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho{-}\sigma_I} + \Bigg[\sqrt{\frac{1{-}z}{z}}\frac{\langle xy\rangle}{k_x^\perp}\left(\frac{1}{\rho}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho{-}\sigma_I}\right) \\ - \frac{\xi}{2}\left(\frac{1}{\rho^2}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{(\rho{-}\sigma_I)^2}\right)\Bigg]\epsilon\Bigg\} + \mathcal{O}(\epsilon^2),\end{aligned} \tag{F.15}$$

$$\begin{aligned}\mathcal{S}_5^2 = 1+\tau_5\frac{k_x^+}{k_x^\perp}\Bigg\{\frac{1}{\rho} - \sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho{-}\sigma_I} - \Bigg[\sqrt{\frac{z}{1{-}z}}\frac{\langle xy\rangle}{k_x^\perp}\left(\frac{1}{\rho}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho{-}\sigma_I}\right) \\ - \frac{\xi}{2}\left(\frac{1}{\rho^2}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{(\rho{-}\sigma_I)^2}\right)\Bigg]\epsilon\Bigg\} + \mathcal{O}(\epsilon^2).\end{aligned} \tag{F.16}$$

Next, we consider the following linear combinations,

$$\mathcal{S}_x^\alpha \;\equiv\; z\,\mathcal{S}_4^\alpha + (1{-}z)\,\mathcal{S}_5^\alpha, \tag{F.17}$$

$$\mathcal{S}_y^\alpha \;\equiv\; \sqrt{z(1{-}z)}\,\big(-\mathcal{S}_4^\alpha + \mathcal{S}_5^\alpha\big). \tag{F.18}$$

such that $\delta^2\big(\mathcal{S}_4^\alpha\big)\delta^2\big(\mathcal{S}_5^\alpha\big) = z(1-z)\,\delta^2\big(\mathcal{S}_x^\alpha\big)\delta^2\big(\mathcal{S}_y^\alpha\big)$. At leading order eq. (F.17) reduces to

$$\begin{aligned}
\mathcal{S}_x^1 &= 1+\tau_x\left(1-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho-\sigma_I}\frac{k_I^+}{k_I^\perp}\right)+\mathcal{O}(\epsilon),\\
\mathcal{S}_x^2 &= 1+\tau_x\left(\frac{1}{\rho}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho-\sigma_I}\right)\frac{k_x^+}{k_x^\perp}+\mathcal{O}(\epsilon).
\end{aligned} \tag{F.19}$$

The interpretation of these equations is as follows: the collinear momenta have been replaced by a new parent particle with momentum k_x and positive helicity, and to this particle we associate the variables (ρ,τ_x). The remaining step is then to integrate out the variables (ξ,τ_y) associated to the collinear splitting using the equations $\mathcal{S}_y^\alpha = 0$ in eq. (F.18).

To leading order in ϵ the equations $\mathcal{S}_y^\alpha = 0$ are independent of ξ, so we need to expand them to next-to-leading order,

$$\mathcal{S}_y^1 = \tau_y\Lambda_1 - \frac{1}{2}\epsilon\,\xi\Big(2\sqrt{z(1-z)}\tau_x + (2z-1)\tau_y\Big)\Lambda_2 + \mathcal{O}(\epsilon^2), \tag{F.20}$$

$$\begin{aligned}
\mathcal{S}_y^2 = \frac{k_x^+}{k_x^\perp}\Bigg\{&\Lambda_3\tau_y - \left(\tau_x - \frac{1-2z}{\sqrt{z(1-z)}}\tau_y\right)\frac{\langle x\,y\rangle}{k_x^\perp}\Lambda_3\epsilon\\
&+\frac{\xi}{2}\Big(2\sqrt{z(1-z)}\tau_x + (2z-1)\tau_y\Big)\Lambda_4\epsilon\Bigg\} + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{F.21}$$

where we introduced the shorthands,

$$\begin{aligned}
\Lambda_1 &= 1-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho-\sigma_I}\frac{k_I^+}{k_I^\perp}, &\qquad \Lambda_2 &= \sum_{I\in\mathfrak{N}}\frac{\tau_I}{(\rho-\sigma_I)^2}\frac{k_I^+}{k_I^\perp},\\
\Lambda_3 &= \frac{1}{\rho}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{\rho-\sigma_I}, &\qquad \Lambda_4 &= \frac{1}{\rho^2}-\sum_{I\in\mathfrak{N}}\frac{\tau_I}{(\rho-\sigma_I)^2}.
\end{aligned} \tag{F.22}$$

We use eq. (F.20) to fix τ_y,

$$\tau_y = \frac{\epsilon\xi\sqrt{z(1-z)}\,\Lambda_2\,\tau_x}{\Lambda_1-\frac{1}{2}\epsilon\xi(2z-1)\Lambda_2}. \tag{F.23}$$

Finaly, we use the global residue theorem to express the residue at $\mathcal{S}_y^2$ in terms of the residues at $\xi = 0$ and $\xi = \infty$. The residue at $\xi = \infty$ vanishes, and we obtain

$$\begin{aligned}
&\mathcal{V}\big(q_4; 4^+, 5^+, \ldots, n{-}1; q_n\big) \\
&\quad \simeq \frac{1}{\epsilon\,\langle x\,y\rangle}\,\frac{1}{\sqrt{z(1{-}z)}}\,\mathcal{V}\big(q_4; K^+, 6, \ldots, n{-}1; q_n\big) \\
&\quad = \frac{1}{\epsilon}\,\texttt{Split_}(4^+, 5^+)\,\mathcal{V}\big(q_4; K^+, 6, \ldots, n{-}1; q_n\big)\,. \qquad \text{(F.24)}
\end{aligned}$$

- $(h_4, h_5) = (+, -)$

In the case where the collinear particles have opposite helicities, the general philosophy is similar to the previous case, though some of the steps are technically more involved. We only highlight the main steps here. We perform the following change of variables,

$$\tau_x \;=\; z\,\tau_4 + \tau_5 \;\text{ and }\; \tau_y \;=\; \sqrt{\frac{z}{1{-}z}}\,\big(\tau_5 - (1{-}z)\tau_4\big)\,, \qquad \text{(F.25)}$$

such that $d\tau_4 d\tau_5 \;=\; \sqrt{\frac{1-z}{z}}\,d\tau_x d\tau_y$, and eq. (F.10) reduces to

$$\begin{aligned}
\mathcal{V}\big(q_4; 4, 5, \ldots, n{-}1; q_n\big) \;\simeq\; &\big(q_4^{\perp\,*}\,q_n^{\perp}\big)\left(\prod_{\substack{i\in\mathfrak{P}\setminus\{4\}\\ I\in\mathfrak{N}\setminus\{5\}}}\frac{k_I^{\perp}}{k_i^{\perp}}\right)\left(\frac{1{-}z}{z}\right)^{3/2} \\
\times \int\prod_{a=6}^{n-1}\frac{d\sigma_a d\tau_a}{\tau_a}\int d\rho\, d\tau_x\int &\frac{d\xi\, d\tau_y}{\xi\tau_4\tau_5}\,\frac{\prod_{I\in\mathfrak{N}}\delta^2\big(\mathcal{S}_I^{\dot\alpha}\big)\prod_{i\in\mathfrak{P}}\delta^2\big(\mathcal{S}_i^{\alpha}\big)}{(\rho{-}\sigma_6)\sigma_{67}\cdots\sigma_{n-2,n-1}\sigma_{n-1}}\,.
\end{aligned}$$

(F.26)

The scattering equations entering eq. (F.26) can be cast in the form:

$$\mathcal{S}_i^1 = 1 + \tau_i - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I}\frac{k_I^+}{k_I^\perp} - \frac{\tau_i\tau_5}{\sigma_i - \rho}\frac{k_x^+}{k_x^\perp} + \mathcal{O}(\epsilon), \tag{F.27}$$

$$\mathcal{S}_i^2 = 1 + \frac{k_i^+}{k_i^\perp}\left(\frac{\tau_i}{\sigma_i} - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_i\tau_I}{\sigma_i - \sigma_I} - \frac{\tau_i\tau_5}{\sigma_i - \rho}\right) + \mathcal{O}(\epsilon), \tag{F.28}$$

$$\bar{\mathcal{S}}_I^{\dot 1} = k_I^\perp - \sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_I\tau_i}{\sigma_I - \sigma_i}k_i^+ - \frac{\tau_I\tau_4}{\sigma_I - \rho}\, z k_x^+ + a\tau_I + \mathcal{O}(\epsilon), \tag{F.29}$$

$$\bar{\mathcal{S}}_I^{\dot 2} = k_I^{\perp *} - \frac{k_I^+}{k_I^\perp}\left(\sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_I\tau_i}{\sigma_I - \sigma_i}k_i^{\perp *} + \frac{\tau_I\tau_4}{\sigma_I - \rho} z k_x^{\perp *} + b_0\frac{\tau_I}{\sigma_I}\right) + \mathcal{O}(\epsilon), \tag{F.30}$$

$$\mathcal{S}_4^1 = 1 + \tau_4 - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_4\tau_I}{\rho - \sigma_I + \frac{1}{2}\epsilon\xi}\frac{k_I^+}{k_I^\perp} - \frac{\tau_4\tau_5}{\epsilon\,\xi}\frac{k_5^+}{k_5^\perp}, \tag{F.31}$$

$$\mathcal{S}_4^2 = 1 + \frac{k_4^+}{k_4^\perp}\left(\frac{\tau_4}{\rho + \epsilon\,\xi/2} - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_4\tau_I}{\rho - \sigma_I + \frac{1}{2}\epsilon\xi} - \frac{\tau_4\tau_5}{\epsilon\,\xi}\right), \tag{F.32}$$

$$\bar{\mathcal{S}}_5^{\dot 1} = k_5^\perp - \sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_5\tau_i}{\rho - \sigma_i - \frac{1}{2}\epsilon\xi}k_i^+ + \frac{\tau_5\tau_4}{\epsilon\,\xi}k_4^+ + a\tau_5, \tag{F.33}$$

$$\bar{\mathcal{S}}_5^{\dot 2} = k_5^{\perp *} - \frac{k_5^+}{k_5^\perp}\left(\sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_5\tau_i}{\rho - \sigma_i - \frac{1}{2}\epsilon\xi}k_i^{\perp *} - \frac{\tau_5\tau_4}{\epsilon\,\xi}(k_4^\perp)^* + b\frac{\tau_5}{\rho - \frac{1}{2}\epsilon\xi}\right), \tag{F.34}$$

with

$$a \equiv q_4^\perp\left(1 - \sum_{J\in\mathfrak{N}} t_J\right)^{-1}, \quad b \equiv q_n^{\perp *}\left(1 + \sum_{J\in\mathfrak{N}} \frac{t_J}{\sigma_J}\frac{k_J^+}{k_J^\perp}\right)^{-1}, \tag{F.35}$$

and b_0 in eq. (F.30) is the leading order approximation of b in ϵ.

In the first step, we use $\delta(\mathcal{S}_4^1)$ to fix τ_y. Since $\mathcal{S}_4^1 = 0$ is a quadratic equation in τ_y, it has two solutions as follows:

$$\tau_y^{\pm} = \frac{\left((2z{-}1)k_5^+\tau_x + \epsilon\xi\Xi k_5^{\perp}\right) \pm \sqrt{-4z\epsilon\xi k_5^{\perp}k_5^+ + \left(k_5^+\tau_x - \epsilon\xi\Xi k_5^{\perp}\right)^2}}{2k_5^+\sqrt{z(1{-}z)}}, \tag{F.36}$$

with

$$\Xi \equiv 1 - \sum_{I\in\mathfrak{N}\backslash\{5\}} \frac{\tau_I}{\rho{-}\sigma_I{+}\frac{1}{2}\epsilon\xi}\frac{k_I^+}{k_I^{\perp}}. \tag{F.37}$$

Equation (F.26) then reduces to

$$\mathcal{V}\big(q_4;4,5,\ldots,n{-}1;q_n\big) \simeq (q_4^{\perp})^* q_n^{\perp}\left(\prod_{\substack{i\in\mathfrak{P}\backslash\{4\}\\ I\in\mathfrak{N}\backslash\{5\}}}\frac{k_I^{\perp}}{k_i^{\perp}}\right)(1-z)^{1/2}z^{-3/2}$$
$$\times\int\prod_{a=6}^{n-1}\frac{d\sigma_a d\tau_a}{\tau_a}\int\frac{d\rho d\tau_x}{\tau_x}\int\frac{d\xi}{\xi}\left(\Delta\big|_{\tau_y=t_y^+} - \Delta\big|_{\tau_y=t_y^-}\right)\frac{\tau_x}{\tau_y^+ - \tau_y^-}, \tag{F.38}$$

where

$$\Delta \equiv \frac{\delta\big(\mathcal{S}_4^2\big)\delta^2\big(\bar{\mathcal{S}}_5^{\dot\alpha}\big)}{(1+\tau_4\Xi)}\,\frac{\prod_{i\in\mathfrak{P}\backslash\{4\}}\delta^2\big(\mathcal{S}_i^{\alpha}\big)\prod_{I\in\mathfrak{N}\backslash\{5\}}\delta^2\big(\bar{\mathcal{S}}_I^{\dot\alpha}\big)}{(\rho{-}\sigma_6)\sigma_{67}\cdots\sigma_{n-2,n-1}\sigma_{n-1}}. \tag{F.39}$$

Finally, we again use the global residue theorem and integrate out ξ by summing the residues at $\xi = 0$ and $\xi = \infty$. The residue at $\xi = \infty$ again vanishes, and we find

$$\mathcal{V}(q_4;4,5,\ldots,n{-}1;q_n) = -q_4^{\perp^*} q_n^\perp \left(\prod_{i\in\mathfrak{P}\setminus\{4\}, I\in\mathfrak{N}\setminus\{5\}} \frac{k_I^\perp}{k_i^\perp} \right) \left(\frac{1{-}z}{z} \right)$$
$$\times \int \prod_{a=6}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \int \frac{d\rho d\tau_x}{\tau_x} \Big(\Delta|_{(\xi,\tau_y)=(0,\tau_y^+)} - \Delta|_{(\xi,\tau_y)=(0,\tau_y^-)} \Big). \tag{F.40}$$

Let us analyse the contributions from the two terms in eq. (F.40) separately. We start by considering the term containing $\Delta|_{(\xi,\tau_y)=(0,\tau_y^+)}$. We have

$$\lim_{\xi\to 0} \tau_y^+ = \tau_x \sqrt{\frac{z}{1{-}z}}, \tag{F.41}$$

which implies $\tau_4 = 0$ and $\tau_5 = \tau_x$. Inserting this into the scattering equations, we obtain

$$\mathcal{S}_i^1 = 1 + \tau_i - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_i \tau_I}{\sigma_i{-}\sigma_I} \frac{k_I^+}{k_I^\perp} - \frac{\tau_i\tau_x}{\sigma_i{-}\rho} \frac{k_x^+}{k_x^\perp} + \mathcal{O}(\epsilon), \tag{F.42}$$

$$\mathcal{S}_i^2 = 1 + \frac{k_i^+}{k_i^\perp} \left(\frac{\tau_i}{\sigma_i} - \sum_{I\in\mathfrak{N}\setminus\{5\}} \frac{\tau_i\tau_I}{\sigma_i{-}\sigma_I} - \frac{\tau_i\tau_x}{\sigma_i{-}\rho} \right) + \mathcal{O}(\epsilon), \tag{F.43}$$

$$\bar{\mathcal{S}}_I^{\dot{1}} = k_I^\perp - \sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} k_i^+ + a'\tau_I + \mathcal{O}(\epsilon), \tag{F.44}$$

$$\bar{\mathcal{S}}_I^{\dot{2}} = (k_I^\perp)^* - \frac{k_I^+}{k_I^\perp} \left(\sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} (k_i^\perp)^* + b' \frac{\tau_I}{\sigma_I} \right) + \mathcal{O}(\epsilon), \tag{F.45}$$

$$\bar{\mathcal{S}}_5^{\dot{1}} = k_x^\perp - \sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_x\tau_i}{\rho{-}\sigma_i} k_i^+ + a'\tau_x + \mathcal{O}(\epsilon) \tag{F.46}$$

$$\bar{\mathcal{S}}_5^{\dot{2}} = (k_x^\perp)^* - \frac{k_x^+}{k_x^\perp} \left(\sum_{i\in\mathfrak{P}\setminus\{4\}} \frac{\tau_x\tau_i}{\rho{-}\sigma_i} (k_i^\perp)^* + b' \frac{\tau_x}{\rho} \right) + \mathcal{O}(\epsilon), \tag{F.47}$$

with $\zeta_x \equiv \frac{\tau_x}{\rho}\frac{k_x^+}{k_x^\perp}$ and

$$\begin{aligned} a' &= q_2^\perp\Big(1-\tau_x-\sum_{J\in\mathfrak{N}\backslash\{5\}} t_J\Big)^{-1}, \\ b' &= q_1^{\perp *}\Big(1+\zeta_x+\sum_{J\in\mathfrak{N}\backslash\{5\}} \zeta_J\Big)^{-1}. \end{aligned} \tag{F.48}$$

In addition, in this case we have

$$1+\tau_4\Xi \;=\; 1, \qquad \mathcal{S}_4^2 \;=\; -\frac{1}{\sqrt{z(1-z)}}\,\frac{\langle x\,y\rangle}{k_x^\perp}\epsilon+\mathcal{O}(\epsilon^2). \tag{F.49}$$

We see that $\bar{\mathcal{S}}_5^{\dot\alpha}$ only depends on k_x, and not on k_4 and k_5 separately. Thus, if we let $\bar{\mathcal{S}}_x^{\dot\alpha} = \bar{\mathcal{S}}_5^{\dot\alpha}$, we obtain the scattering equations associated with a single parent particle with momentum k_x and negative helicity, and we have

$$-\frac{1}{\epsilon\,\langle x\,y\rangle}\,\frac{(1-z)^2}{\sqrt{z(1-z)}}\,\mathcal{V}\big(q_4;K^-,6,\ldots,n-1;q_n\big) \tag{F.50}$$

$$=\;\frac{1}{\epsilon}\,\mathtt{Split}_+(4^+,5^-)\,\mathcal{V}\big(q_4;K^-,6,\ldots,n-1;q_n\big). \tag{F.51}$$

We can perform a similar analysis for the term $\Delta|_{(\xi,\tau_y)=(0,\tau_y^-)}$, and letting $\mathcal{S}_x^1 = \bar{\mathcal{S}}_5^{\dot 1}/k_x^\perp$ and $S_x^2 = z\,\mathcal{S}_4^2$ we obtain scattering equations for a parent parton with momentum k_x and positive helicity. This gives a contribution

$$\frac{1}{\epsilon\,[x\,y]}\,\frac{z^2}{\sqrt{z(1-z)}}\,\mathcal{V}\big(q_4;K^+,6,\ldots,n-1;q_n\big) \tag{F.52}$$

$$=\;\frac{1}{\epsilon}\,\mathtt{Split}_-(4^+,5^-)\,\mathcal{V}\big(q_4;K^+,6,\ldots,n-1;q_n\big). \tag{F.53}$$

Quasi multi-Regge limits

In this section, we prove that our CHY-type formulas for the impact factors and the Lipatov vertices have the expected factorization properties if the last particle, with momentum k_{n-1}, has much smaller rapidity as the other particles. A similar analysis can be performed in the case where the first particle has much greater rapidity than the other.

Here we only discuss the impact factor $\mathcal{C}(2;3,\ldots,n{-}1)$ and derive the factorization in the limit $y_3 \simeq \cdots \simeq y_{n-2} \gg y_{n-1}$. The argument follows the same lines as the derivation of the factorization of the amplitude in QMRK in Section 9.1, so we will be brief. More precisely, we use the Conjecture 1 and two δ-functions to localize the integrals over τ_{n-1} and ζ_{n-1}. The derivation depends on the helicity of the particle $n{-}1$, and we now discuss each case in turn.

Let us first study the case where the gluon with momentum k_{n-1} has positive helicity. The two equations $\mathcal{S}^{\alpha}_{n-1} = 0$ are linear in ζ_{n-1} and τ_{n-1}. More precisely, we have

$$\mathcal{S}^1_{n-1} = 1 + \tau_{n-1}\left(1 + \sum_{I\in\overline{\mathfrak{N}}} \zeta_I\right) = 0, \quad \mathcal{S}^2_{n-1} = 1 + \zeta_{n-1} = 0. \tag{F.54}$$

We can use these equations to fix ζ_{n-1} and τ_{n-1}. We obtain,

$$\zeta_{n-1} \;=\; -1 \;\text{ and }\; \tau_{n-1} \;=\; -\frac{1}{1+\sum_{I\in\overline{\mathfrak{N}}}\zeta_I}. \tag{F.55}$$

We can insert eq. (F.55) into the remaining scattering equations $\bar{S}^{\dot\alpha}_I$, and we get

$$\bar{S}^1_I = k_I^\perp - \sum_{i\in\mathfrak{P}\setminus\{n-1\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} k_i^+, \tag{F.56}$$

$$\bar{S}^2_I = k_I^{\perp\,*} - \frac{k_I^+}{k_I^\perp} \sum_{i\in\mathfrak{P}\setminus\{n-1\}} \frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i} k_i^{\perp\,*} - \zeta_I \frac{(q_n^\perp - k_{n-1}^\perp)^*}{1+\sum_{J\in\overline{\mathfrak{N}}}\zeta_J}. \tag{F.57}$$

After ζ_{n-1} and τ_{n-1} have been integrated out, we immediately find

$$\begin{aligned} &\mathcal{C}\big(2^-,3,\ldots,n{-}1\big) \\ &\qquad \simeq \mathcal{C}\big(2^-,3,\ldots,n{-}2\big)\,\frac{-1}{|q_{n-1}^\perp|^2}\,\mathcal{V}\big(q_{n-1};n{-}1;q_n\big), \end{aligned} \tag{F.58}$$

where $q_{n-1} = q_n - k_{n-1}$, and the impact factor $\mathcal{C}\big(2^-,3,\ldots,n{-}2\big)$ in the right-hand side is given again by the CHY-type formula in eq. (9.7).

Similarly, in another case where the gluon with momentum k_{n-1} has negative helicity, we first use the equation

$$\bar{\mathcal{S}}^{\dot{2}}_{n-1} \;=\; k^{\perp *}_{n-1} - \zeta_{n-1}\,\frac{q^{\perp *}_{n}}{1+\sum_{J\in\overline{\mathfrak{N}}}\zeta_J} \;=\; 0 \tag{F.59}$$

to localize the integral over ζ_{n-1}, and the formula for the impact factor becomes

$$q_n^{\perp}\times\frac{-k^{\perp}_{n-1}\,(q^{\perp}_n)^*}{(k^{\perp}_{n-1})^*(q^{\perp}_{n-1})^*}\int\prod_{a=3}^{n-2}\frac{d\sigma_a d\tau_a}{\tau_a}\,\frac{1}{\sigma_{34}\cdots\sigma_{n-2}}\left(\prod_{i\in\mathfrak{P},I\in\overline{\mathfrak{N}}\backslash\{n-1\}}\frac{k^{\perp}_I}{k^{\perp}_i}\right)$$
$$\times\prod_{I\in\overline{\mathfrak{N}}\backslash\{n-1\}}\delta\left(k^{\perp}_I-\sum_{i\in\mathfrak{P}}\frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i}k^{+}_i\right)$$
$$\times\prod_{I\in\overline{\mathfrak{N}}\backslash\{n-1\}}\delta\left((k^{\perp}_I)^*-\frac{k^{+}_I}{k^{\perp}_I}\sum_{i\in\mathfrak{P}}\frac{\tau_I\tau_i}{\sigma_I{-}\sigma_i}(k^{\perp}_i)^*-\zeta_I\,\frac{(q^{\perp}_n-k^{\perp}_{n-1})^*}{1+\sum\limits_{J\in\overline{\mathfrak{N}}\backslash\{n-1\}}\zeta_J}\right)$$
$$\times\int\frac{d\tau_{n-1}}{\tau_{n-1}}\,\delta\left(k^{\perp}_{n-1}+\tau_{n-1}\sum_{i\in\mathfrak{P}}\zeta_i k^{\perp}_i\right)$$
$$\times\prod_{i\in\mathfrak{P}}\delta\left(1+\tau_i-\sum_{I\in\overline{\mathfrak{N}}}\frac{\tau_i\tau_I}{\sigma_i{-}\sigma_I}\,\frac{k^{+}_I}{k^{\perp}_I}\right)\delta\left(1+\zeta_i-\frac{k^{+}_i}{k^{\perp}_i}\sum_{I\in\overline{\mathfrak{N}}}\frac{\tau_i\tau_I}{\sigma_i{-}\sigma_I}-\zeta_i\tau_{n-1}\right).$$

Finally, we use the residue theorem to localize the integral over τ_{n-1} on the residues at $\tau_{n-1} = 0$ and $\tau_{n-1} = \infty$. The residue at $\tau_{n-1} = \infty$ vanishes, and the residue at $\tau_{n-1} = 0$ immediately reproduces the desired factorization formula.

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