# "Transcendental weight in supersymmetric field theories"

Verbeek, Bram

## ABSTRACT

To accommodate the ever-growing need for accurate theoretical predictions of scattering events, recent years have seen a tremendous amount of effort towards understanding the mathematical structure of scattering amplitudes. As a testing ground for developing new computational techniques, theorists have turned to idealised setups such as N = 4 SYM, the maximally supersymmetric Yang- Mills theory in four dimensions. In this theory, scattering amplitudes display an enormous amount of interesting structure. One of the most remarkable traits is the property of maximal transcendental weight, which implies that all polylogarithmic functions appearing in an L-loop amplitude have weight 2L. As of yet for generic amplitudes this property remains an observation. Furthermore, N = 4 SYM is the only gauge theory known to exhibit this feature. In this thesis, we aim to investigate this property. First, we identify a special kinematic regime called the multi-Regge or high-energy limit. Turning to pl...

UCLouvain
Institut de recherche
en mathématique et physique

# Transcendental Weight in Supersymmetric Field Theories

Doctoral dissertation presented by

BRAM VERBEEK

in fulfilment of the requirement for the degree of Doctor in Sciences.

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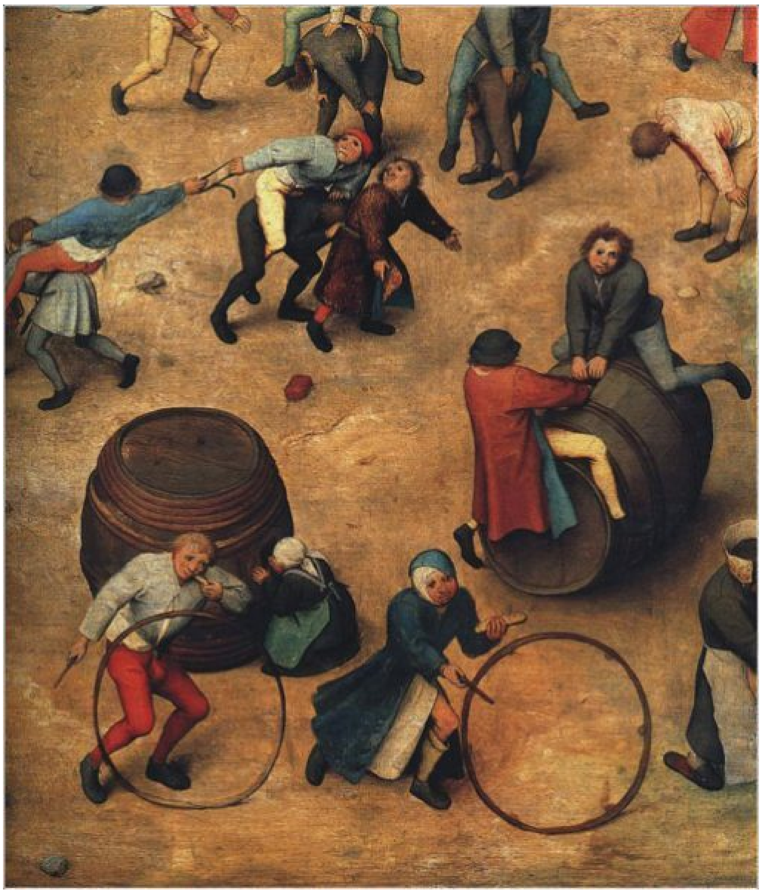
2019

*You may never get to touch the Master,*
*but you can tickle his creatures.*

Thomas Pynchon



A detail from Pieter Breugel the Elder's *Children's Games* (1560).

# Abstract

To accommodate the ever-growing need for accurate theoretical predictions of scattering events, recent years have seen a tremendous amount of effort towards understanding the mathematical structure of scattering amplitudes. As a testing ground for developing new computational techniques, theorists have turned to idealised setups such as $\mathcal{N} = 4$ SYM, the maximally supersymmetric Yang-Mills theory in four dimensions. In this theory, scattering amplitudes display an enormous amount of interesting structure. One of the most remarkable traits is the property of maximal transcendental weight, which implies that all polylogarithmic functions appearing in an $\ell$-loop amplitude have weight $2\ell$. As of yet for generic amplitudes this property remains an observation. Furthermore, $\mathcal{N} = 4$ SYM is the only gauge theory known to exhibit this feature.

In this thesis, we aim to investigate this property. First, we identify a special kinematic regime called the multi-Regge or high-energy limit. Turning to planar $\mathcal{N} = 4$ SYM, we develop a computational technique in this regime which is recursive in the perturbative order, allowing the maximal transcendentality of the high-energy limit of planar $\mathcal{N} = 4$ SYM to be shown to all orders. Building on this, we study part of the parton-parton cross section in the high-energy limit for generic gauge theories, aiming to find other theories which may display maximal transcendentality. We find simple constraints in terms of the matter content which imply maximal transcendental weight for this object. For theories with matter in the adjoint and fundamental representation of the gauge group, these constraints are satisfied by supersymmetric theories with a vanishing one-loop beta function, hinting towards a connection between maximal transcendentality and superconformal symmetry. Finally, we study a two-loop amplitude in general kinematics in $\mathcal{N} = 2$ superconformal quantum chromodynamics, one of the theories found to display maximal transcendentality in our computation in the high energy limit. Our aim is to see whether this maximal transcendentality continues to hold beyond the high-energy limit. We find that maximal transcendentality is broken, but the lower-weight terms at leading-colour are given by a single multiple zeta value, confirming published results. Beyond leading-colour, we observe an interplay between the lower-weight terms and Catani infrared subtraction.

# Acknowledgements

Though only one name is printed on the cover of this thesis, there are many more who deserve credit for the work it contains. This page constitutes an effort to voice my gratitude to those who have been invaluable to me for the past four years.

First and foremost I would like to thank Claude for offering me the chance to start this adventure, for his guidance, support and encouragement and for sharing a small sliver of his encyclopaedic knowledge of physics and mathematics with me.

At CP3 I would like to thank Robin, for our fruitful collaboration and discussions these past four years, and for the shared joy and misery throughout. Liam, for his curiosity and integrity. Martin, Matteo, Samuel, Zhengwen and many others at the Cyclotron for their help and for all the chats, coffee breaks and lunches. On the CERN side I would like to thank the other half of the team, Brenda, Lorenzo and Sophia for their remote help during many a weekly group meeting and for the enjoyable local interactions.

Thanks to all my external collaborators, Falko, Georgios, Gregor, Gustav, Henrik, James, Stefan and Vittorio, from whom I have learned a great deal and with whom it was always a pleasure to work. Furthermore I would like to thank my thesis committee for taking the time to read this manuscript and for their comments which greatly improved its quality.

In tegenstelling tot het universum bestaat het leven niet uit fysica alleen en ook buiten het instituut lopen er velen rond die zich verantwoordelijk voor dit boekje moeten voelen.

Mijn vrienden in Brussel wil ik bedanken voor alle fijne momenten in de mooiste stad ter wereld. De Kempenaren in mijn leven wil ik bedanken voor de regelmatige dosis nuchterheid en gezelligheid, de hoogste waarden van onze voedingsbodem. Alex, Arnaud, Jonas en Ruben wil ik bedanken voor het onderhouden van onze lange-afstandsvriendschappen de voorbije jaren.

Het gezin Torfs-Boving zou ik graag bedanken om mij te herbergen tijdens menig weekend, voor bed, bord en vooral babbel. Aan mijn grote zus Babs heb ik mijn eerste lessen wiskunde - het startschot van dit werk - en een luisterend oor te danken. Mijn ouders zou ik willen bedanken voor een kwart eeuw zorg en liefde, een geschenk waarop geen waarde geplakt kan worden.

Ten slotte rest er één persoon zonder wie de voorbije jaren nog niet half zo leuk waren geweest. Alies, het is tijd dat we koers inzetten naar ons volgende avontuur. Bedankt om er te zijn.

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# Introduction

## Fundamental Physics and Quantum Field Theory

Understanding the fundamental constituents of matter and their interactions has been a major theme throughout the history of (first) philosophy and (later) physics. As early as 500 B.C., the pre-Socratic philosophers in ancient Greece postulated various theories on the fundamental nature of matter. Chief among these was the *atomism* of Leukippos and Demokritos. In this theory, macroscopic matter was thought to be built up of building blocks called *atoms* — from the Greek "$\alpha\tau o\mu o\nu$" meaning "indivisible". Though the view that matter has a discontinuous character at the smallest scales remains present in modern physics, we have come a long way from the original atomistic theory. The development of experimental techniques led to a confirmation of the existence of atoms – which correspond to a certain chemical element, as had been hypothesised in the early development of chemistry – in the early 20th century. Shortly after, the existence of the negatively-charged *electron* was shown and it was understood that the (no longer aptly-named) atom had a substructure. This substructure consisted of a nucleus made up of positively charged *protons* and *neutrons* with zero charge, surrounded by a cloud of orbiting electrons. The atomic substructure was a welcome discovery, as it unified atomic chemistry by showing that all chemical elements were to be understood as different compositions of the same three building blocks, restoring the spirit of reductionism from which the atomic ideology stemmed.

Probing further into the heart of matter requires studying its consituents at ever higher energies. Thus we are dealing with physical objects at subatomic scales travelling at high speeds. The physical theory which studies such objects is called *quantum field theory* (QFT), in which elementary particles are considered to be excitations of quantised fields permeating space-time. There are two such types of fields, *fermionic* fields which correspond to matter particles such as the electron and *bosonic* fields which describe the elementary forces such as electromagnetism, mediated by its force-carrying particle the *photon*. A theory is then defined by a list of these fields and their interactions.

In the past decades, studying particle collisions has allowed us to discover many more elementary particles, grouped in a set called *the Standard Model*. For one, it was found that the proton and neutron are not elementary but rather are made up of two types of *quarks* called *up* and *down*, held together by *gluons* which mediate the strong nuclear force that binds them. The theory governing these particles is called *quantum chromodynamics (QCD)*. A neutral matter particle called the *neutrino* was theorised from radioactive decay and later experimentally found to exist. Said radioactive decay called for the need to introduce another elementary force, called the weak nuclear force, mediated by particles called the *W and Z-bosons*. This force was then unified with the electromagnetic force into *electroweak theory*. Furthermore, it was found that there are three *generations* of matter particles, by which we mean that each matter particle comes in three versions, only differentiated by mass. The most recent discovery and final puzzle piece of the Standard Model is the *Higgs boson* which was discovered at the Large Hadron Collider (LHC). This particle is required for the internal mathematical consistency of the Standard Model. The particle content of the Standard Model is summarised in Figure 1.

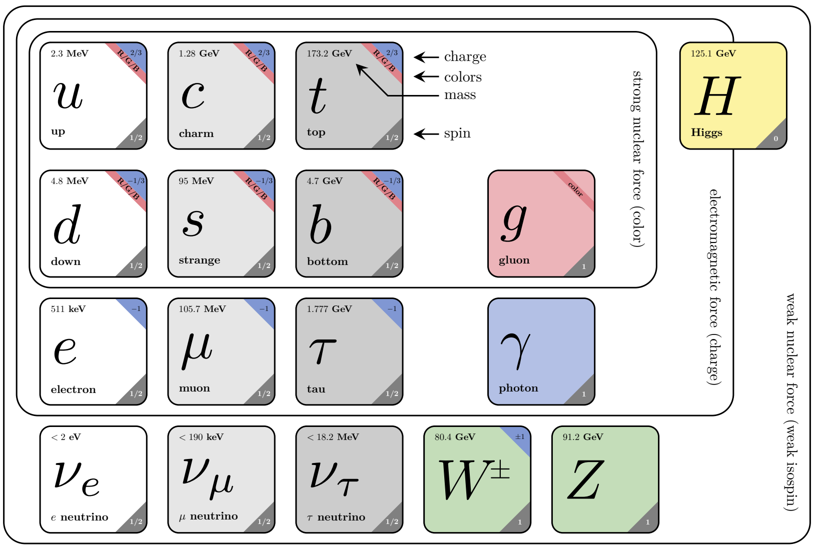



**Figure 1:** The particle content of the Standard Model of particle physics. Coloured fields represent bosons, whereas the grey fields correspond to fermions, the three generations being distinguished by shade. (Modified version of graphic due to C. Burgard and D. Galbraith.)

Though the Standard Model describes all known fundamental phenomena (barring gravity) to striking accuracy, there are many experimental observations and theoretical motivations which lead us to believe that it should be expanded upon. To detect such physics beyond the Standard Model, we need accurate theoretical predictions which can be compared to experimental results. Let us now turn to the canonical way to compute such predictions in QFT.

## Scattering Processes and Feynman Diagrams

Particle collisions are described by a quantity called the *scattering cross section*, which represents the probability for a given process to occur. This probability can be computed through the absolute value squared of a *scattering amplitude*, the central objects of study in this thesis. Consider a scattering process of two particles scattering, resulting in $(N-2)$ particles being emitted from the collision as represented in Figure 2.

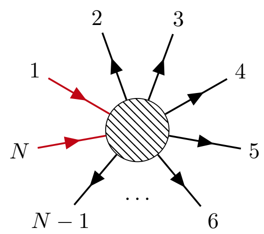



**Figure 2:** A pictorial representation of a generic scattering process

We can model this process as an operation mapping an initial state $|\text{IN}\rangle$ to a final state $|\text{OUT}\rangle$. The operator which defines this evolution is called the *S-matrix* and the probability for the process to occur is given by $|\langle\text{OUT}|\, S\, |\text{IN}\rangle|^2$. Separating out the trivial part of the $S$-matrix for which no scattering occurs as $S = 1 + i\mathcal{T}$, the scattering amplitude $\mathcal{A}$ is schematically given by equation

$$\langle\text{OUT}|\, \mathcal{T}\, |\text{IN}\rangle \sim \delta^{(4)}\left(p_1 + p_N - \sum_{i=2}^{N-1} p_i\right)\mathcal{A}\,, \tag{1}$$

where the delta function imposes momentum conservation. The canonical way to compute such objects is by expanding them in terms of the *coupling constant*, a scalar variable which characterises the strength of the force exerted in an interaction. When this number is small, we can get a good estimate of the final result by computing the first few terms in this expansion, called the *perturbative expansion*. These terms can be computed by making use of the technique of *Feynman diagrams*. In this technique, the interactions of particles are translated into vertices of graphs, which can then be arranged into diagrams schematically representing the scattering event in an infinite number of ways. Such diagrams can then be translated to contributions to the scattering amplitude using the *Feynman rules*, which assign to each diagram a function of the external states. The topology of the diagram is related to the perturbative order at which it contributes. The leading terms are characteristically given by *tree-level* diagrams, which contain no closed cycles. Higher-order contributions are referred to as *loop diagrams*, with the next-to-leading terms having one closed cycle (*one-loop*), next-to-next-to-leading terms having two closed cycles (*two-loop*), and so on. Examples of such diagrams are given in Figure 3. For this reason, the perturbative orders are also referred to as *loop orders*.

Loop calculations are significantly more involved than computing at tree-level, since the contribution associated to a loop diagram involves an integral over the momentum running through the loop. Performing these integrations is no easy feat, as there is no algorithmic way of tackling them at arbitrary orders, and little knowledge on the special functions to which they may evaluate in general. Furthermore, the number of diagrams which need to be computed increases dramatically with the number of external particles, also called *legs* or *points* of a diagram. For this reason, the state-of-the-art for computing scattering amplitudes can be referred to as the *loop-leg frontier*. In QCD, where state-of-the-art results are typically four-particle scattering events at two loops, higher-loop and -leg results are incredibly challenging to obtain. To push this loop-leg frontier further, there has been a lot of effort the last few decades towards understanding the mathematical structure of scattering amplitudes, with the aim of making these computations more feasible.

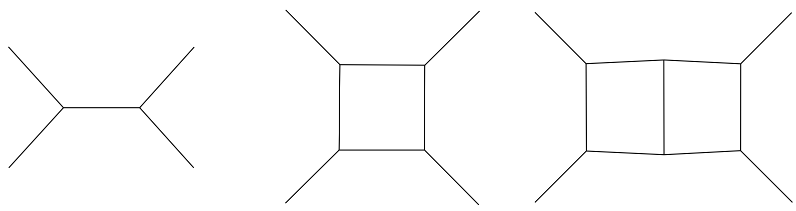


**Figure 3:** An example of a tree-level, one-loop and two-loop diagram, respectively.

In order to study the mathematical structures arising from the computation of scattering amplitudes, physicists have turned to theories which exhibit a high degree of symmetry as mathematical laboratories, hoping that the techniques developed in such idealised theories can be extended in some sense to theories which represent physical nature. Maximally supersymmetric $\mathrm{SU}(N_c)$ gauge theory has proven to be particularly fertile ground for studying structures arising in perturbative calculations. In this theory, loop integrals can often be expressed by a class of special functions called polylogarithms, which display a tremendously rich amount of algebraic structure.

It is natural to associate a certain number to each polylogarithm called the weight, which is given by the number of integrations involved in the definition of the function. An incredibly peculiar property of the maximally supersymmetric $\mathrm{SU}(N_c)$ gauge theory, is that for all known results in this theory, all polylogarithmic functions appearing in an $\ell$-loop amplitude have weight $2\ell$. This property is referred to as the property of maximal transcendental weight and maximally supersymmetric $\mathrm{SU}(N_c)$ gauge theory is the only known theory which obeys this structure. There is as of yet no general proof that this property holds for any result to all orders, nor an understanding of which property of the maximally supersymmetric theory lies at its root. In this thesis, we undergo a study of transcendental weight in various contexts. For the remainder of this chapter, we give an overview of what is to come.

## Outline of this Thesis

The work presented in this thesis aims to study the transcendental weight of special functions appearing in scattering amplitudes in idealised theories with a high degree of supersymmetry. It is organised as follows.

In Chapter 1, we define *supersymmetry* and its implications for quantum field theories. In particular, we will review the supersymmetric version of the Poincaré algebra and the field theories we can construct from it. After these general considerations, we will turn to the theories in which this work situated: maximally supersymmetric Yang-Mills theory and supersymmetrised versions of quantum chromodynamics.

In Chapter 2, a short overview of modern techniques for computing scattering amplitudes is presented. We will first review some methods for separating the various types of functional dependences which arise in these computations, before turning our attention to the computation of loop integrals and the special functions which arise from such computations. We introduce the notion of *transcendental weight*, a scalar quantity associated to such special functions and whose possible range for a scattering amplitude in supersymmetric field theories is the central topic of this thesis. Finally, we will comment on the planar limit of maximally supersymmetric Yang-Mills theory, a setting in which the scattering amplitudes display remarkable simplicity.

The original work starts in Chapter 3, where we study the high-energy limit of the planar maximally supersymmetric Yang-Mills theory [1–6]. After defining precisely what this limit means for the kinematics, we review a conjectural representation for the amplitude to all orders in terms of a certain integral transform called the Fourier-Mellin transform. This conjectural representation is obtained by resumming the perturbative expansion in terms of large logarithms which appear in this limit. We will see that the geometry of the configuration space restricts the special functions which may arise, and this in turn allows us to simplify the computations dramatically. We present a method for computing amplitudes in this theory recursively in loop order, and eventually comment on the consequences this recursive structure has on the analytic structure of scattering amplitudes, in particular with regard to the transcendental weight.

Having studied the high-energy limit in a specific theory, we turn to a more general setup in Chapter 4, where we study a part of the resummed cross-section for parton-parton scattering [7]. This object can easily be defined for a gauge theory with generic matter content. Using quantum chromodynamics as a testing case, we compute the various contributions to this object at next-to-leading order in the large logarithm to high orders in the coupling constant and consequently study their analytic properties and transcendental weight. Subsequently, turning to generic matter content allows us to translate demands on the transcendental weight of the result into constraints on the matter content. In particular, we classify the theories with adjoint and fundamental matter for which this object has uniform transcendental weight.

Finally, in Chapter 5 we turn to half-maximally supersymmetric quantum chromodynamics, one of the theories we found to have uniform transcendental weight in the high-energy limit. We aim to see whether uniform transcendentality continues to hold beyond the high-energy limit. We review the computation of the four-gluon two-loop amplitude in this theory [8] and study its mathematical structure.

This work is based on the following publications [1–3, 6–8]:

- V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou & B. Verbeek, *Multi-Regge Kinematics and the Moduli Space of Riemann Spheres with Marked Points*, JHEP 1608 152.
- V. Del Duca, C. Duhr, R. Marzucca & B. Verbeek, *The Analytic Structure and the Transcendental Weight of the BFKL Ladder at NLL Accuracy*, JHEP 1710 001.
- V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou & B. Verbeek, *The Seven-gluon Amplitude in Multi-Regge Kinematics beyond Leading Logarithmic Accuracy*, JHEP 1806 116.
- R. Marzucca & B. Verbeek, *The Multi-Regge Limit of the Eight-particle Amplitude beyond Leading Logarithmic Accuracy*, JHEP 1907 039.
- C. Duhr, H. Johansson, G. Kälin, G. Mogull & B. Verbeek, *The Full-Color Two-Loop Four-Gluon Amplitude in $\mathcal{N} = 2$ Super-QCD*, [1904.05299], (Submitted to PRL).
- V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou & B. Verbeek, *Amplitudes in Multi-Regge Kinematics to All Orders*, (in preparation).

And on conference proceedings [4, 5]:

- V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou & B. Verbeek, *Amplitudes in the Multi-Regge Limit of $\mathcal{N} = 4$ SYM* , [1811.10588], (Accepted by Acta Pol Supp B).
- V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou & B. Verbeek, *Multi-Loop Amplitudes in the High-Energy Limit in $\mathcal{N} = 4$ SYM* , PoS LL2018 (2018) 026.

# 1 | Supersymmetric Field Theories

Symmetry has been a central principle in physics for the past decades. On the one hand it operates as a guiding light for simplicity, as the most symmetric description of a physical process will be the most elegant, often facilitating further generalisation. On the other hand, beyond being an important ideal, symmetry can be a useful tool for theorists. Imposing a symmetry which is not present in nature allows one to study the mathematical structure of an idealised setup before unleashing the developed machinery onto less symmetric theories which model physical reality. The work described in this thesis was done entirely in such mathematical laboratories, namely extended supersymmetric field theories.

In this chapter we introduce the supersymmetry algebra and its effect on field theories, before introducing the main protagonists of our story: $\mathcal{N} = 4$ super Yang-Mills theory and supersymmetric quantum chromodynamics.

# 1.1. Supersymmetry Algebra and Multiplets

## 1.1.1. Embedding the Poincaré Algebra

The fundamental space-time symmetries of a quantum field theory in four dimensions are described by the *Poincaré algebra*, defined by the set of generators $\{P_\mu, M_{\mu\nu}\}$ where $P_\mu$ generates *translations* and $M_{\mu\nu}$ generates *Lorentz transformations.* Their commutation relations are given by

$$\begin{aligned} [P_\mu, P_\nu] &= 0\,, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} + i\eta_{\mu\sigma}M_{\nu\rho}\,, \\ [M_{\mu\nu}, P_\rho] &= -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu\,, \end{aligned} \tag{1.1}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ denotes the Minkowski metric in mostly-minus convention. The generators $J_i$ of rotations and $K_i$ of Lorentz boosts are given by

$$J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}\,, \qquad K_i = M_{0i}\,, \tag{1.2}$$

where $\varepsilon_{ijk}$ is the four-dimensional Levi-Civita tensor. Note that the algebra of $M_{\mu\nu}$ is called the *Lorentz algebra* and its subalgebra which is connected to the identity $\mathrm{SO}^+(3,1)$ has universal covering group $\mathrm{SL}(2,\mathbb{C})$. The generators of $\mathrm{SL}(2,\mathbb{C})$ are given by the matrices

$$\sigma_{\mu\nu} = \frac{i}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)\,, \tag{1.3}$$

where $\sigma_\mu$ denotes a vector of Pauli matrices

$$\sigma_\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{1.4}$$

and we will refer to the elements $\psi_\alpha$ of the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ as *left-handed Weyl spinors*, while the elements $\bar{\chi}_{\dot{\alpha}}$ of the conjugate representation will be called *right-handed Weyl spinors.*

It is a natural question whether this symmetry algebra can be extended in a meaningful way. Can one define quantum field theories with a larger symmetry algebra into which the Poincaré algebra is embedded non-trivially? In a seminal paper published in 1967 [9], Coleman and Mandula showed that any such bosonic extension of space-time symmetries would render the S-matrix trivial. Since we are interested in interacting field theories this tells us that — at least as far as bosonic generators go — the Poincaré algebra is the full story.

It is however possible to add fermionic operators if we consider a *graded extension* of the Poincaré algebra, which contains both even ($E$) and odd ($O$) operators such that

$$\{O, O\} = E, \qquad [E, E] = E, \qquad [E, O] = O, \tag{1.5}$$

where $\{.,.\}$ denotes the anticommutator. Such extensions, dubbed *supersymmetry* (SUSY) were considered in the early 70s both in the context of field theory [10,11] and in the early development of string theory [12–14]. It is natural to wonder whether a no-go result similar to the Coleman-Mandula theorem could be formulated in the graded case, as a priori there was no bound on the properties of the odd operators. A paper from 1975 by Haag, Łopuszański and Sohnius [15] showed that if one allows odd generators, they must have spin 1/2. In what follows, we will study the consequences of adding such objects to the Poincaré algebra. For a more complete treatment of supersymmetry see [16].

### 1.1.2. Superalgebra

We add spin-1/2 generators $\{Q^i_\alpha \mid i = 1, \ldots, \mathcal{N}\}$ with conjugates[1] $\bar{Q}^i_{\dot{\alpha}} = (Q^i_\alpha)^\dagger$ to the Poincaré algebra and call the resulting set the generators of the *super-Poincaré algebra*. The odd generators $Q^i_\alpha$ and their conjugates will henceforth be refered to as *supercharges*.

We wish to determine the structure relations of these supercharges with the generators of the Poincaré algebra. Their interaction with the Lorentz group can be inferred from their spinorial character. Since supercharges are Weyl spinors, the Lorentz group must act on them via the $\mathrm{SL}(2, \mathbb{C})$ generators,

$$Q^i_\alpha \to Q^{i\,\prime}_\alpha = \exp\left(-\frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q^i_\beta \,. \tag{1.6}$$

On the other hand, the supercharges are operators, which implies that Lorentz transformations $U = \exp\left(-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right)$ act on them as

$$Q^i_\alpha \to Q^{i\,\prime}_\alpha = U^\dagger Q^i_\alpha U \,. \tag{1.7}$$

Comparing (1.6) and (1.7) to first order in $\omega^{\mu\nu}$, one can see that

$$[Q^i_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha^{\ \beta} Q^i_\beta \,. \tag{1.8}$$

The commutator of the supercharges with the generator of translations can be inferred from Jacobi relations. The index structure implies that it must be of the form

$$[Q^i_\alpha, P^\mu] = \mathrm{c}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{Q}^{i\dot{\alpha}} \,. \tag{1.9}$$

[1] Throughout this thesis, we will denote the conjugate of an object $a$ by $\bar{a}$.

From the Jacobi identity

$$\big[P^\mu, [P^\nu, Q^i_\alpha]\big] + \big[P^\nu, [Q^i_\alpha, P^\mu]\big] = 0\,, \tag{1.10}$$

we can see that

$$\begin{aligned} 0 &= \mathrm{c}\,(\sigma^\mu)_{\alpha\dot\alpha}[P^\nu, \bar Q^{i\dot\alpha}] - \mathrm{c}\,(\sigma^\nu)_{\alpha\dot\alpha}[P^\mu, \bar Q^{i\dot\alpha}]\,,\\ &= |\mathrm{c}|^2(\sigma^\nu\bar\sigma^\mu - \sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta} Q_\beta\,, \end{aligned} \tag{1.11}$$

which implies $\mathrm{c} = 0$. From this we get

$$[Q^i_\alpha, P^\mu] = 0\,. \tag{1.12}$$

The only structure relations left to be determined are those which contain only supercharges. The anticommutator $\{Q^i_\alpha, \bar Q^j_{\dot\beta}\}$ must be of the form

$$\{Q^i_\alpha, \bar Q^j_{\dot\beta}\} = 2\mathrm{C}^{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,. \tag{1.13}$$

The matrix $C$ is necessarily Hermitian, so we can diagonalise it, as there exists some unitary matrix U such that $\mathrm{UCU}^{-1} = \mathrm{diag}(\bar{\mathrm{c}})$ and by transforming

$$Q^i_\alpha \to \mathrm{U}^i_l Q^l_\alpha\,, \qquad \bar Q^j_{\dot\beta} \to (\mathrm{U}^{-1})^j_k \bar Q^k_{\dot\beta}\,, \tag{1.14}$$

we get

$$\{Q^l_\alpha, \bar Q^k_{\dot\beta}\} = 2(\bar{\mathrm{c}})_l\ \delta^{lk}(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,. \tag{1.15}$$

We require $(\bar{\mathrm{c}})_l > 0$, as the anticommutator is intimately tied to the energy of the states in our system, so we may rescale the supercharges to obtain

$$\{Q^l_\alpha, \bar Q^k_{\dot\beta}\} = 2\ \delta^{lk}(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,. \tag{1.16}$$

Finally, we are left to determine the anticommutator $\{Q^i_\alpha, Q^j_\beta\}$. The most general term we can write down which matches the index structure is

$$\{Q^i_\alpha, Q^j_\beta\} = \varepsilon_{\alpha\beta}Z^{ij} + \mathrm{C}^{ij}\ (\sigma^{\mu\nu})_{\alpha\beta}M_{\mu\nu}\,. \tag{1.17}$$

However, the left hand side commutes with $P_\mu$ due to (1.12) and the Jacobi relation, whereas the second term in the right hand side does not. Thus we are left with

$$\{Q^i_\alpha, Q^j_\beta\} = \varepsilon_{\alpha\beta}Z^{ij}\,, \tag{1.18}$$

where we have introduced new objects called *central charges* $Z^{ij} = -Z^{ji}$ which commute with all generators. These central charges form an algebra of internal symmetries and are a unique feature of massive theories with *extended supersymmetry*, by which we mean $\mathcal{N} > 1$.

To summarize, the structure relations involving the supercharges are of the form

$$\begin{aligned} [Q^i_\alpha, P^\mu] &= 0\,, \\ [Q^i_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^\beta Q^i_\beta\,, \\ \{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} &= 2\,\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta} P_\mu\,, \\ \{Q^i_\alpha, Q^j_\beta\} &= \varepsilon_{\alpha\beta} Z^{ij}\,. \end{aligned} \tag{1.19}$$

Combined with the commutation relations found in (1.1), this fixes our entire algebra. In the following section we will discuss the representations of the super-Poincaré algebra for various degrees of supersymmetry.

### 1.1.3. Supermultiplets

In what follows, we will refer to the representations of the super-Poincaré algebra as *supermultiplets*. To start, let us discuss a few important facts regarding the action of supercharges on states in quantum field theory. First of all, since the supercharges are spinorial objects, if we construct a non-zero state $|\omega'\rangle = Q|\omega\rangle$ by acting with the supercharges on an arbitrary state, $|\omega\rangle$ and $|\omega'\rangle$ must have opposite statistics. The supercharges map bosons to fermions and vice versa. Furthermore, two such states must be degenerate in mass, since $[Q^i_\alpha, P^\mu] = 0$ and so also $[Q^i_\alpha, P^2] = 0$. A third consequence is that any representation of the supersymmetric algebra will contain an equal amount of bosonic and fermionic degrees of freedom. To see this, define the *fermion number operator* $(-)^F$ by its action

$$(-)^F|F\rangle = -|F\rangle\,, \qquad (-)^F|B\rangle = |B\rangle\,, \tag{1.20}$$

where $|F\rangle, |B\rangle$ denote fermionic and bosonic states respectively. Clearly this object anticommutes with the supercharges. Now compute the trace

$$\mathrm{Tr}\left((-)^F\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\}\right) = \mathrm{Tr}\left((-)^F Q^i_\alpha \bar{Q}^j_{\dot\beta} - (-)^F \bar{Q}^j_{\dot\beta} Q^i_\alpha\}\right) = 0\,. \tag{1.21}$$

On the other hand, we know that

$$\begin{aligned} \mathrm{Tr}\left((-)^F\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\}\right) &= \mathrm{Tr}\left((-)^F 2\,\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta} P_\mu\right)\,, \\ &= 2\,\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta}\, p_\mu\, \mathrm{Tr}\left((-)^F\right)\,, \end{aligned} \tag{1.22}$$

where $p_\mu$ denotes the eigenvalues of $P^\mu$ and hence

$$0 = \mathrm{Tr}\left((-)^F\right) = \sum \langle B|(-)^F|B\rangle + \sum \langle F|(-)^F|F\rangle = N_B - N_F\,, \tag{1.23}$$

where $N_B$ and $N_F$ are the total number of bosons and fermions respectively.

Having covered some general properties of supermultiplets, we will now construct the massless multiplet explicitly. If we restrict ourselves to massless particles, we have $P^2 = 0$. In this case, we may write the generator of translations in canonical form as

$$P^\mu = (E, 0, 0, E)\,. \tag{1.24}$$

This implies that

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\beta}}\} = 4E\,\delta^{ij} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}}\,. \tag{1.25}$$

From this we can see that $Q^i_2$ must vanish. Note that this implies that the central charges $Z^{ij} = \varepsilon_{\alpha\beta}\{Q^i_\alpha, Q^j_\beta\}$ must also vanish, hence we observe that they are features of massive theories only. If we define operators

$$\alpha^i = \frac{1}{2\sqrt{E}} Q^i_1\,, \qquad \alpha^{i\dagger} = \frac{1}{2\sqrt{E}} \bar{Q}^i_{\dot{1}}\,, \tag{1.26}$$

these trivialise the anticommutation relations $\{\alpha^i, \alpha^{j\dagger}\} = \delta^{ij}$ and can be interpreted as annihilation and creation operators. Now, start from a state $|\omega\rangle$ such that

$$\alpha^i |\omega\rangle = 0\,, \tag{1.27}$$

for all $\alpha^i$ and suppose this state has helicity $\lambda$ such that

$$M_{12}|\omega\rangle = \lambda|\omega\rangle\,. \tag{1.28}$$

We can construct $\mathcal{N}$ new states from $|\omega\rangle$ by acting with $\alpha^{i\dagger}$. The helicities of these states can be determined from the structure relation (1.8)

$$M_{12}\alpha^{i\dagger}|\omega\rangle = \left(\alpha^{i\dagger} M_{12} + \frac{1}{2}\alpha^{i\dagger}\right)|\omega\rangle = \left(\lambda + \frac{1}{2}\right)\alpha^{i\dagger}|\omega\rangle\,. \tag{1.29}$$

We can continue constructing states in this manner, taking into account that $(\alpha^{i\dagger})^2 = 0$. Starting from the state $|\omega\rangle$ which is annihilated by all the operators $\alpha^i$, we obtain the set of states listed in Table 1.1. Note that the total number of states of a supermultiplet is given by

$$1 + \sum_{k=1}^{\mathcal{N}} \frac{1}{k!} \prod_{l=0}^{k-1} (\mathcal{N} - l) = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}\,. \tag{1.30}$$

In what follows, we will define the supersymmetric theories which will be the main objects of study in this thesis.

| state | helicity | #(states) |
|---|---|---|
| $\|\omega\rangle$ | $\lambda$ | 1 |
| $\alpha^{i\dagger}\|\omega\rangle$ | $\lambda+\frac{1}{2}$ | $\mathcal{N}$ |
| $\alpha^{j\dagger}\alpha^{i\dagger}\|\omega\rangle$ | $\lambda+1$ | $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ |
| $\alpha^{k\dagger}\alpha^{j\dagger}\alpha^{i\dagger}\|\omega\rangle$ | $\lambda+\frac{3}{2}$ | $\frac{1}{6}\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\alpha^{\mathcal{N}\dagger}\alpha^{\mathcal{N}-1\dagger}\dots\alpha^{i\dagger}\|\omega\rangle$ | $\lambda+\frac{\mathcal{N}}{2}$ | 1 |

**Table 1.1:** The structure of a generic massless supermultiplet of a $\mathcal{N}$-supersymmetric theory.

## 1.2. Maximally Supersymmetric Yang-Mills Theory

A priori there is no reason to believe that there should be an upper bound on the number of supersymmetries we can put in a consistent gauge theory. Looking at Table 1.1 however, we notice that arbitrarily high amounts of SUSY will introduce arbitrarily high helicities to our supermultiplets. Put otherwise, a bound on the possible helicities in a consistent quantum field theory would imply a maximal degree of supersymmetry.

A heuristic argument for such a bound (which can be found in [17] in more detail) is that massless particles with helicity $> 1/2$ must couple to conserved quantities and these are limited by the symmetries of our system. Particles of helicity $\pm 1$ couple to internal symmetry generators, particles of helicity $\pm 3/2$ couple to the supercharges and particles of helicity $\pm 2$ couple to the momentum 4-vector. There are no conserved quantities for higher spin fields to couple to and hence we state that the maximal/minimal helicity for particles in a consistent quantum field theory is $\pm 2$. This argument also implies that any massless spin-2 field must be interpreted as a *graviton*, and any massless spin-3/2 field through its coupling to *local supersymmetry*[2] must be interpreted as the *gravitino*, the graviton's superpartner.

If we restrict ourselves to global supersymmetry, the maximal helicity $\lambda_{\text{max}} = +1$ and the minimal helicity is $\lambda_{\text{min}} = -1$. A quick look at Table 1.1 will teach us that the maximal degree of SUSY $\mathcal{N}_{\text{max}}$ is given by

$$\lambda_{\text{max}} - \lambda_{\text{min}} = \frac{\mathcal{N}_{\text{max}}}{2}\,. \tag{1.31}$$

[2]By turning supersymmetry into a local symmetry, one must automatically gauge the Poincaré algebra as they are joined in the same algebra. Any theory of local supersymmetry is a theory of supersymmetric gravity or *supergravity (SUGRA)*. For a pedagogical introduction see [18].

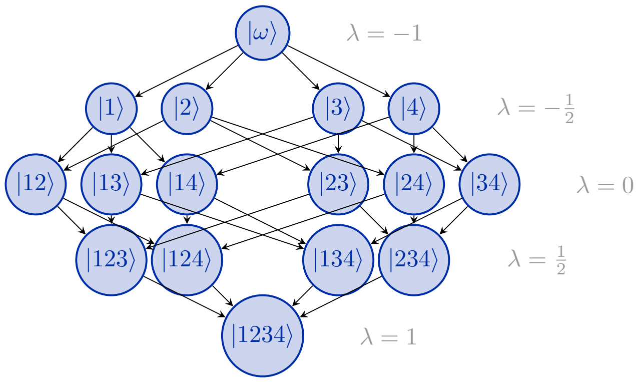



**Figure 1.1:** The $\mathcal{N} = 4$ supermultiplet. The shorthand $\alpha^{a_1\dagger}\alpha^{a_2\dagger}\ldots\alpha^{a_n\dagger}|\omega\rangle = |a_1 a_2 \ldots a_n\rangle$ was used for compactness and $\lambda$ represents the helicities of the states defined at the respective level of recursion.

We conclude that the maximal degree of global supersymmetry allowed in a four-dimensional quantum field theory is $\mathcal{N} = 4$. In this case, the multiplet is entirely fixed by the high degree of symmetry.

We give a graphical representation of the multiplet in Figure 1.1. In total, the multiplet has 16 degrees of freedom which are grouped as one vector field $\mathcal{A}_\mu$, four Weyl fermions $\psi^i_\alpha$ and six real scalars $\Phi^i$. We can construct a quantum field theory by associating the vector field to a $\mathrm{SU}(N_c)$ gauge boson. Such a supersymmetrized version of $\mathrm{SU}(N_c)$ gauge theory is referred to as *super Yang-Mills theory (SYM)*, in our case $\mathcal{N} = 4$ super Yang-Mills theory. Since all particles are grouped into one multiplet, all fields are in the adjoint representation of the gauge group. The Lagrangian of the theory can be obtained by dimensional reduction of ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory on $T^6$ [19] and is given by

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4} = \mathrm{Tr}\bigg( &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D_\mu\Phi^i)^2 + \frac{\alpha^2}{4}[\Phi^i,\Phi^j]^2 \\ &+ \frac{i}{2}\bar{\Psi}\not{D}\Psi + \frac{\alpha}{2}\bar{\Psi}\Gamma^i[\Phi_i,\Psi]\bigg), \end{aligned} \tag{1.32}$$

where the $\Gamma^i$ are gamma-matrices of the ten-dimensional Clifford algebra and the fermionic degrees of freedom were captured by ten-dimensional Weyl spinors $\Psi$. Note that we consider the theory at the origin of its moduli space where there are no scalar vacuum expectation values.

Due its high degree of symmetry, $\mathcal{N} = 4$ SYM is a remarkable gauge theory with many interesting properties. One these is that it is conformal [20], meaning that there is no inherent mass scale in the theory. Any theory with only massless fields and marginal couplings will be conformal at the classical level, but $\mathcal{N} = 4$ SYM stays conformal even at the quantum level. In particular its beta function is zero to all orders in perturbation theory. This can be readily observed at one-loop order by plugging the particle content into the expression

$$\beta_0 = -\frac{\alpha^2}{2\pi^2}\left(\frac{11}{3}N_c - \frac{1}{6}\sum_i C_i^S - \frac{2}{3}\sum_j C_j^F\right), \tag{1.33}$$

which is valid for $\mathrm{SU}(N_c)$ gauge theories with coupling $\alpha$ and where $C_i^{S/F}$ denotes the quadratic Casimirs of the real scalars and Weyl fermions in the theory respectively. As all particles are in the adjoint representation their quadratic Casmirs are $N_c$ and one readily obtains $\beta_0 = 0$. To observe the all-order vanishing, we may use the NSVZ beta function [21] which is an explicit form for the all-order beta function for super Yang-Mills theories with added matter multiplets. For pure super Yang-Mills theories it is given by

$$\beta_{\mathrm{NSVZ}}^{\mathrm{SYM}} = -\frac{\alpha^2}{2\pi^2}N_c\big(4-\mathcal{N}\big)\left[1 - \frac{\alpha N_c(2-\mathcal{N})}{2\pi}\right]^{-1}, \tag{1.34}$$

which clearly vanishes when $\mathcal{N} = 4$. In the following chapters, we will discuss many more properties of $\mathcal{N} = 4$ super Yang-Mills theory, in particular in the context of perturbative computations.

## 1.3. Supersymmetric Quantum Chromodynamics

A second class of supersymmetric theories we will consider in this thesis are supersymmetrized versions of *quantum chromodynamics (QCD)*, by which we mean $\mathrm{SU}(N_c)$ Yang-Mills theories with $N_f$ additional fermions in the fundamental representation of the gauge group. In the next subsections we will introduce both the minimal and extended supersymmetric versions of this theory.

### 1.3.1. $\mathcal{N} = 1$ Supersymmetric QCD

$\mathcal{N} = 1$ supersymmetric QCD (SQCD) is given by $\mathcal{N} = 1$ super Yang-Mills theory, which is made up of the $\mathcal{N} = 1$ vector multiplet and its conjugate in the

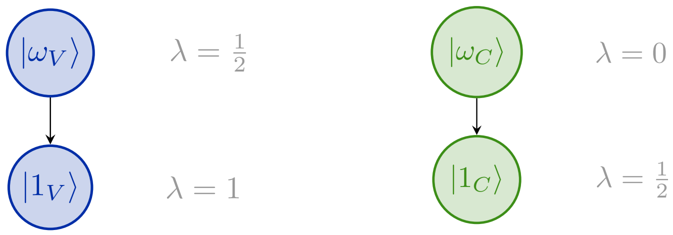



**Figure 1.2:** The $\mathcal{N} = 1$ vector- and chiral multiplet respectively. The shorthand $\alpha^{i\dagger}|\omega_I\rangle = |i_I\rangle$ was used for compactness and $\lambda$ represents the helicities of the states defined at the respective level of recursion.

adjoint representation of $\mathrm{SU}(N_c)$, with $N_f$ added $\mathcal{N} = 1$ chiral multiplets in the fundamental and antifundamental representation and their conjugate partners. The relevant super-multiplets can be found in Figure 1.2. The total particle content consists of $N_c$ adjoint vectors and Weyl fermions, $N_f$ (anti)fundamental Weyl fermions and $2N_f$ (anti)fundamental real scalars. Plugging this into (1.33), we obtain

$$\beta_0^{\mathcal{N}=1\ \mathrm{SQCD}} = -\frac{\alpha^2}{2\pi^2}\left(3N_c - N_f\right) . \tag{1.35}$$

The one-loop beta function vanishes when $N_f = 3N_c$. The all-order beta function is given by NSVZ as follows

$$\beta_{\mathrm{NSVZ}}^{\mathcal{N}=1\ \mathrm{SQCD}} = -\frac{\alpha^2}{2\pi^2}\Big(3N_c + N_f(\gamma(\alpha) - 1)\Big)\left(1 - N_c\frac{\alpha}{2\pi}\right)^{-1} , \tag{1.36}$$

where $\gamma(\alpha)$ denotes the anomalous dimension of the matter multiplets. In [22], it was shown that within the region $3N_c/2 < N_f < 3N_c$, we can tune the parameters of the theory such that the all-order beta function vanishes. We will henceforth refer to this region as the *conformal window* of $\mathcal{N} = 1$ SQCD.

### 1.3.2. $\mathcal{N} = 2$ Supersymmetric QCD

$\mathcal{N} = 2$ supersymmetric QCD (SQCD) is given by $\mathcal{N} = 2$ super Yang-Mills theory, which is made up of the $\mathcal{N} = 2$ vector multiplet and its conjugate in the adjoint representation of $\mathrm{SU}(N_c)$, with $N_f$ added $\mathcal{N} = 2$ hypermultiplets in the fundamental and antifundamental representation. The relevant supermultiplets are given in Figure 1.3. This implies a matter content of $N_c$ adjoint vectors, $2N_c$ adjoint Weyl fermions, $2N_c$ adjoint real scalars, $N_f$ (anti)fundamental Weyl fermions and $2N_f$ (anti)fundamental real scalars.

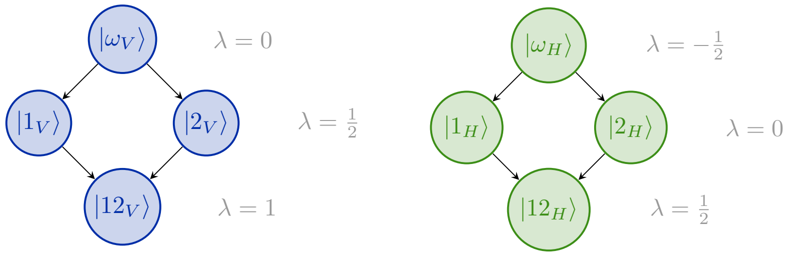



**Figure 1.3:** The $\mathcal{N} = 2$ vector- and hypermultiplet respectively. The shorthand $\alpha^{i\dagger}\alpha^{j\dagger}|\omega_I\rangle = |ij_I\rangle$ was used for compactness and $\lambda$ represents the helicities of the states defined at the respective level of recursion.

For the one-loop beta function (1.33), we obtain

$$\beta_0^{\mathcal{N}=2 \text{ SQCD}} = -\frac{\alpha^2}{2\pi^2}\left(2N_c - N_f\right) . \tag{1.37}$$

This tells us that the one-loop beta function vanishes when $N_f = 2N_c$. However, $\mathcal{N} = 2$ super Yang-Mills theories with additional matter multiplets are one-loop exact. By this we mean that if the beta function vanishes at first order in perturbation theory, it will vanish to all orders and the theory is conformal. We can observe this in our case from the NSVZ beta function [23]

$$\begin{aligned} \beta_{\text{NSVZ}}^{\mathcal{N}=2 \text{ SQCD}} &= -\frac{\alpha^2}{2\pi^2}\left(2N_c - N_f\right) , \\ &= \beta_0^{\mathcal{N}=2 \text{ SQCD}} , \end{aligned} \tag{1.38}$$

which exactly equals the one-loop in (1.37). At the value $N_f = 2N_c$, which we will refer to as the *conformal point*, the theory is refered to as $\mathcal{N} = 2$ *superconformal QCD (SCQCD)*.

# 2 | Scattering Amplitudes

Our current most important experimental probes of fundamental physics are scattering experiments. The theoretical predictions with which these experiments aim to compare can be deduced from scattering amplitudes. These scattering amplitudes are traditionally determined by the calculation of Feynman diagrams. This technique, however, has its limitations. For one, the number of diagrams which need to be computed grows dramatically with the multiplicity of the process. Furthermore, results at higher orders in perturbation theory involve the computation of arbitrarily complicated loop integrals for which no straightforward algorithm exists.

To push the loop-leg frontier, many techniques have been developed in the past decades to facilitate and compartmentalise these computations. In the context of idealised setups such as $\mathcal{N} = 4$ super Yang-Mills theory in particular, this has led to a field of study which aims to reformulate the entire computational scheme. In what follows, we will give a brief review of some techniques and results from this field of study which we will invoke in later chapters.

## 2.1. Quantum Number Management

Scattering amplitudes in gauge theory are functions of the initial and final states of the scattering process. More precisely, this means they depend on the types of external particles as well as their momenta, helicities and colour charges. In order to facilitate computation, several techniques have been developed throughout the past decades to separate these different types of functional dependence, which we review here. For a pedagogical introduction see [24].

### 2.1.1. Supersymmetric Ward Identities

As we saw in the previous chapter, different particles in supersymmetric field theories are related by action of the supercharges. Since amplitudes are given by wedging the non-trivial part of the $S$-matrix with separated particle states at infinity, there are relations amongst different amplitudes for which the external states are in the same supermultiplet implied by the action of the supercharges. Such relations are called *supersymmetric Ward identities* [25].

To see how they arise, rewrite the definition of a generic amplitude (1) such that it is given by field operators acting on the vacuum. Consider an amplitude with all particles outgoing and label the field operators as $\mathcal{O}_i = \mathcal{O}_i(p_i, c_i, \lambda_i)$, we then get

$$\mathcal{A}_n = \langle 0|\mathcal{O}_1 \dots \mathcal{O}_N|0\rangle \,, \tag{2.1}$$

where the operators act to the left. Now, if the vacuum is supersymmetric, it is necessarily invariant under the action of the supercharges

$$Q^i|0\rangle = 0 = \bar{Q}^i|0\rangle \,. \tag{2.2}$$

From this we may deduce that

$$\begin{aligned} 0 &= \langle 0|[\bar{Q}^i, \mathcal{O}_1 \dots \mathcal{O}_N]|0\rangle \,, \\ &= \sum_{i=1}^{n} (-1)^{\sum_{j<i} F(\mathcal{O}_j)} \langle 0|\mathcal{O}_1 \dots [\bar{Q}^i, \mathcal{O}_i] \dots \mathcal{O}_N|0\rangle \,, \end{aligned} \tag{2.3}$$

where $F(\mathcal{O}_j) = 1$ if $\mathcal{O}_j$ corresponds to a fermionic field and is 0 otherwise. Taking into account the discussion in Section 1.1.3 where we studied the effect of the supercharges on particle states, we see that (2.3) describes a linear relation of amplitudes with different helicities and matter content. Solving the resulting set of equations allows us to find a set of amplitudes with particular matter content and helicities which span all other external states in the same supermultiplet, see for example [26]. In what follows, we will restrict ourselves to gluon amplitudes.

### 2.1.2. Helicity Amplitudes

An $N$-gluon amplitude depends on helicities, colour charges and momenta. Amplitudes for which the helicities are specified are called *helicity amplitudes*. SUSY Ward identities link amplitudes with different values for the helicities of the external particles and constrain the set of helicity amplitudes. For example take $\mathcal{O}(g_\pm)$ and $\mathcal{O}(\lambda_\pm)$ the field operators of the $\pm1$ helicity gluon and $\pm1/2$ helicity gluino[1] respectively, we then see from (2.3) and (2.2) that

$$\langle 0|[\bar{Q}^i, \mathcal{O}_1(\lambda_+)\mathcal{O}_2(g_+)\ldots\mathcal{O}_N(g_+)]|0\rangle = 0 \;\rightarrow\; \mathcal{A}\,[1^+2^+\ldots N^+] = 0\,, \qquad (2.4)$$

where $\mathcal{A}\,[1^+2^+\ldots N^+]$ is the $N$-gluon amplitude with all helicities positive. A similar argument can be used to show that the gluon amplitude where one gluon has negative helicity vanishes

$$\mathcal{A}\,[1^+\ldots i^-\ldots N^+] = 0\,, \qquad 1 \le i \le N\,. \qquad (2.5)$$

Note that while these constraints hold to all orders for supersymmetric field theories, this implies they must also hold for non-supersymmetric theories at tree level. The non-zero gluon amplitude with the most uniform set of helicities is referred to as *maximally helicity violating* (MHV) and is given by

$$\mathcal{A}\,[1^+\ldots i^-\ldots j^-\ldots N^+]\,, \qquad 1 \le i \le j \le N\,. \qquad (2.6)$$

Flipping more helicities amounts to *(next-to)$^k$-maximally helicity violating* (N$^k$MHV) contributions

$$\mathcal{A}\,[1^+\ldots i_1^-\ldots i_2^-\ldots i_k^-\ldots N^+]\,, \qquad 1 \le i_1 \le \cdots \le i_k \le N\,. \qquad (2.7)$$

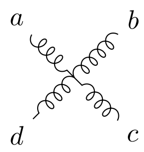
Note that helicity configurations can also be related by parity conjugation. For example, the parity conjugate of the MHV $\mathcal{A}\,[1^-\ldots i^+\ldots j^+\ldots N^-]$ is referred to anti-MHV.

### 2.1.3. Colour Ordering

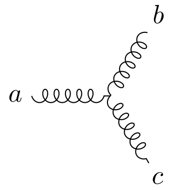
Now we consider how to factor out the colour dependence [27–30] of gluon scattering amplitudes. We consider pure SU($N_c$) Yang-Mills theories. Then, colour factors in amplitudes arise from gluon-gluon interactions through the

[1] The fermionic particle in the SU($N_c$)-gauged vector multiplet.

following vertices




$$\sim \sum_e f^{abe} f^{ecd}\,,$$




$$\sim f^{abc}\,, \tag{2.8}$$

where $f^{abc}$ and are the structure constants of the $\mathrm{SU}(N_c)$ algebra. To gain insight in the different types of colour factors which can appear in this context it is useful to express the structure constants explicitly

$$f^{abc} \sim \mathrm{Tr}\left(T^a[T^b, T^c]\right)\,, \tag{2.9}$$

to write everything in terms of generators. The colour factors in the amplitude are (products of) traces of generators of the form $\mathrm{Tr}(\dots T^a \dots)\,\mathrm{Tr}(\dots T^a \dots)$. We can use the completeness relation

$$(T^a)_i^{\,j}(T^a)_k^{\,l} = \delta_i^l \delta_k^j - \frac{1}{N_c}\delta_i^j \delta_k^l\,, \tag{2.10}$$

to simplify such factors. For a gluon amplitude $\mathcal{A}_N^{(\ell)}$ with $\ell$ loops and $N$ legs, this leads to a decomposition in powers of $N_c$

$$\begin{aligned} \mathcal{A}_N^{(\ell)} = \sum_{\sigma \in S_N/Z_N} & g^{N-2+2\ell}\, N_c^\ell\, \mathrm{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(N)}})\, A_{N;1}^{(\ell)}(\sigma(1), \dots, \sigma(N)) \\ & + \mathcal{O}(N_c^{\ell-1})\,, \end{aligned} \tag{2.11}$$

where $g$ denotes the coupling strength, $S_N/Z_N$ are all non-cyclic permutations of the $N$-tuple and we call $A_{N;1}^{(\ell)}$ the *colour-ordered* amplitude. Colour-ordered amplitudes are functions of the kinematics only and as their name suggests, have a particle-ordering which is fixed by the colour structure. The subleading terms, which appear only at loop level, carry a multi-trace colour structure and their kinematically dependent parts are referred to as *partial* amplitudes.

Now that we have discussed methods to strip off the helicity and colour information of the amplitude, we will discuss the computation of the kinematical dependence in the next section.

## 2.2. Special Functions and Transcendentality

### 2.2.1. Feynman Integrals

To compute the kinematical dependence of a scattering amplitude, one must evaluate a multiple tensor integral over the loop momenta. The canonical first approach to this problem is to rewrite the tensor integral as a leading tensorial factor multiplying a sum of scalar integrals referred to as *Feynman integrals*. The generic Feynman integral for massless theories can be written as

$$I(\{p_i\},\{D_i\},\{a_i\}) = \int \prod_{i=0}^{\ell} \frac{\mathrm{d}^D l_i}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{(D_j)^{a_j}}\,, \tag{2.12}$$

where the denominators $D_i$ are given by squares of generic linear combinations of loop and external momenta

$$D_i = \left( \sum_{j=0}^{n} \alpha_{ij} p_j + \sum_{j=0}^{n} \beta_{ij} l_j \right)^2 \,. \tag{2.13}$$

We may represent these integrals pictorially by the graph which reproduces the integral upon application of the Feynman rules. From a certain scattering process one obtains Feynman integrals with the same set of denominators $\{D_i\}$ but different exponents $\{a_i\}$, referred to as an *integral family*.

Said Feynman integral family can be reduced to a minimal set using so-called *integration-by-parts* (IBP) identities [31, 32] which rely on the fact that for any well-behaved function $f(k)$ in $D$ dimension the integral is shift-independent, meaning for an arbitrary $h \in \mathbb{R}$ and momentum $k$ one gets

$$\int \mathrm{d}^D p\, f(p) = \int \mathrm{d}^D p\, f(p + hk) \;\rightarrow\; \int \mathrm{d}^D p\, \frac{\partial f}{\partial k^\mu}(p) = 0\,. \tag{2.14}$$

Applying such a relation to an integral as defined in (2.12) with a single vector numerator $v^\mu(p_i, l_i)$ given by one of the external or loop momenta results in

$$\int \prod_{i=0}^{\ell} \frac{\mathrm{d}^D l_i}{(2\pi)^D} \frac{\partial}{\partial l_a^\mu} \prod_{j=1}^{n} \frac{v^\mu}{(D_j)^{a_j}} = 0\,, \tag{2.15}$$

where we decide to differentiate in one of the loop momenta $l_a$. Now note that

$$\frac{\partial}{\partial l_a^\mu} \prod_{j=1}^{n} \frac{v^\mu}{(D_j)^{a_j}} = \frac{\partial v^\mu}{\partial l_a^\mu} \prod_{j=1}^{n} \frac{1}{(D_j)^{a_j}} + \sum_{b=1}^{n} \frac{n_b}{D_b} \frac{\partial D_b}{\partial l_a^\mu} \prod_{j=1}^{n} \frac{v^\mu}{(D_j)^{a_j}}\,. \tag{2.16}$$

Clearly $\partial v^\mu / \partial l_a^\mu$ will be 1 if $v^\mu = l_a^\mu$ and 0 if $v^\mu$ equals any other momentum. In the second term, $v^\mu(\partial D_b / \partial l_a^\mu)$ can be expressed as a linear combination of scalar products of momenta, which in its turn can be rewritten as a polynomial of the denominators. From this, one can see that (2.15) defines a linear relation of Feynman integrals in the same family. These IBP identities can be used to reduce the set of scalar integrals in a certain family to a minimal set called *master integrals*. An example of this reduction from tensor integrals to a set of masters will be considered in Section 5.1.2.

Note that the Feynman integrals in (2.12) are defined for a generic number of dimensions $D$. This is done because the integral typically diverges in four dimensions due to the unboundedness of the loop momenta over which we integrate on the one hand, or due to internal loop momenta becoming *collinear* with external momenta or external momenta becoming *soft*, meaning having low energy on the other. Such divergences are called *ultraviolet (UV) divergences* and *infrared (IR) divergences* respectively and can be regulated by setting $D = 4 - 2\epsilon$, subsequently expanding in $\epsilon$ and thus making the appearance of the divergence for $\epsilon \to 0$ manifest. This procedure is called *dimensional regularisation*. These divergences are then taken care of by the process of *renormalisation*.

The evaluation of master integrals — both analytically and numerically — is an active field of study with interest from both physics and pure mathematics. In particular, the study of special functions which may arise from such integrations and the properties of these function spaces has proven to be a very rich research topic and the understanding of these structures have led to very powerful computational techniques. In the next section we will explore one such space of functions which we will encounter throughout this thesis, namely the space of polygarithmic functions.

### 2.2.2. Multiple Polylogarithms

Polylogarithmic functions arise naturally from the computation of Feynman integrals. The largest class of such functions is called *multiple polylogarithms* (MPLs) [33, 34]. It is defined by an iterated integral [35] over rational kernels

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_n} G(a_1, \ldots, a_{n-1}; t)\,, \tag{2.17}$$

where $G(;z) = 1$, $a_i \in \mathbb{C}$ is a set of constants which define the function and $z$ is a complex variable. When all the $a_i$ are zero we define the function to be

$$G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z)\,. \tag{2.18}$$

Many well-known functions can be expressed as MPLs, such as arbitrary powers of logarithms through (2.18), the *classical polylogarithms*

$$\mathrm{Li}_n(z) = -G(\vec{0}_{n-1}, 1; z)\,, \tag{2.19}$$

as well as the *harmonic polylogarithms* (HPLs) [36]

$$H(\vec{a}; z) = (-1)^p G(\vec{a}; z)\,, \tag{2.20}$$

where $a_i \in \{-1, 0, +1\}$ and $p$ is the number of elements in $\vec{a}$ which are equal to $+1$. For a polylogarithm $G(a_1, \ldots, a_n; z)$ the number of indices $n$, which corresponds to the number of integrations involved in its definition, is referred to as its *weight*. We refer to linear combinations of MPLs with rational numbers as coefficients as *pure functions*.

The vector space of polylogarithmic functions $\mathcal{H}_{\mathrm{MPL}}$ forms multiple mathematical structures. The first we will discuss is the *shuffle algebra* [37]. This structure implies that products of polylogarithmic functions which depend on the same variable can always be rewritten as a sum of re-shuffled polylogarithms of higher weight. Take for example the product of two weight-one polylogarithms

$$G(a; z)\, G(b; z) = G(a, b; z) + G(b, a; z)\,, \tag{2.21}$$

this can be easily shown by using Fubini's theorem and splitting the resulting integration domain into two triangles. For a generic product we write

$$G(\vec{a}_1; z)\, G(\vec{a}_2; z) = \sum_{\vec{a} \in \vec{a}_1 \sqcup\!\sqcup\, \vec{a}_2} G(\vec{a}; z)\,, \tag{2.22}$$

where $\vec{a}_1 \sqcup\!\sqcup\, \vec{a}_2$ represents the set of shuffles of the two vectors of indices, meaning all permutations of their union which retain the internal ordering of the vectors $\vec{a}_1$ and $\vec{a}_2$. Note that if we define the weight of a product of two polylogarithms to be the sum of the weights of its terms, these relations preserve the weight. A few more examples of shuffle relations are given below

$$G(a; z)\, G(b, c; z) = G(a, b, c; z) + G(b, a, c; z) + G(b, c, a; z)\,, \tag{2.23}$$

$$\begin{aligned} G(a, b; z)\, G(c, d; z) = {} & G(a, b, c, d; z) + G(a, c, b, d; z) + G(a, c, d, b; z) \\ & + G(c, a, b, d; z) + G(c, a, d, b; z) + G(c, d, a, b; z)\,. \end{aligned} \tag{2.24}$$

The polylogarithms also form a structure known as a *Hopf algebra*. Such a structure is defined by a vector space $\mathcal{H}$ with multiplication $\mu$ and *coproduct* $\Delta$

$$\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}\,, \qquad \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}\,, \tag{2.25}$$

such that $\Delta(\mu(a,b)) = \mu(\Delta(a),\Delta(b))$ and $(\mathbb{I}\otimes\Delta)\Delta = (\Delta\otimes\mathbb{I})\Delta$, and a linear map $S$ called the *antipode* which is defined by the relation

$$\mu(S\otimes\mathbb{I})\Delta(x) = 0\,, \qquad \forall x \notin \mathbb{Q}\,. \tag{2.26}$$

The antipode is an involution, $S^2 = \mathbb{I}$ with $\mathbb{I}$ the identity, and it preserves the product and the coproduct,

$$S(\mu(a,b)) = \mu(S(b),S(a)) \;\text{ and }\; \Delta S = (S\otimes S)\tau\Delta\,, \tag{2.27}$$

with $\tau(a\otimes b) = b\otimes a$. This structure can be endowed upon the vector space $\tilde{\mathcal{H}}_{\text{MPL}} \equiv (\mathcal{H}_{\text{MPL}} \bmod \pi)$. In this context it is convenient to use an alternative notation for polylogarithms, where we allow for more general lower integration limits. Following [34], we define

$$I(a_0;a_n,\ldots,a_1;z) = \int_{a_0}^{z} \frac{\mathrm{d}t}{t-a_n}\, I(a_0;a_{n-1},\ldots,a_1;t)\,. \tag{2.28}$$

It is easy to see that $G(a_1,\ldots,a_n;z) = I(0;a_n,\ldots,a_1;z)$. As a shorthand, consider the word $\vec{a} \equiv a_n\ldots a_1$ and write $I(0;\vec{a};z) = I(0;a_n,\ldots,a_1;z)$. If $\vec{b}$ and $\vec{c}$ are sub-words of $\vec{a}$, then we denote by $I_{\vec{b}}(0;\vec{c};z)$ the iterated integral with the same integrand as $I(0;\vec{c};z)$, but whose integration contour encircles the points $z = b_i$ with $b_i \in \vec{b}$ in the order defined by $\vec{b}$. We can always express $I_{\vec{b}}(0;\vec{c};z)$ in terms of polylogarithms:

1. If $\vec{b} = \emptyset$ is the empty word, then $I_{\emptyset}(0;\vec{c};z) = I(0;\vec{c};z)$ is just a polylogarithm.
2. $I_{\vec{b}}(0;\vec{c};z) = 0$, unless $\vec{b}$ is a sub-word of $\vec{c}$, because otherwise we take residues at points where there are no singularities in the integrand.
3. $I_{\vec{b}}(0;\vec{b};z) = (2\pi i)^{|\vec{b}|}$, where $|\vec{b}|$ denotes the length of the word $\vec{b}$.
4. If $\vec{b} = c_{i_1}\ldots c_{i_m}$ is a proper sub-word of $\vec{c} = c_1\ldots c_k$, we have

$$I_{\vec{b}}(0;\vec{c};z) = (2\pi i)^{|\vec{b}|} I(0;c_1,\ldots,c_{i_1-1};c_{i_1})\ldots I(c_{i_m};c_{i_m+1},\ldots,c_k;z)\,.$$

Using this notation, the coproduct on polylogarithms is given by [34],

$$\Delta(I(a_0;\vec{a};z)) = \sum_{\emptyset\subseteq\vec{b}\subseteq\vec{a}} I(a_0;\vec{b};z)\otimes\left[(2\pi i)^{-|\vec{b}|}\, I_{\vec{b}}(a_0;\vec{a};z)\right]\,, \tag{2.29}$$

where the sum runs over all subwords of $\vec{a}$. The coproduct of the logarithm and the classical polylogarithms are given by

$$\Delta(\log^n(z)) = \sum_{k=0}^{n} \binom{n}{k} \log^k(z)\otimes\log^{n-k}(z)\,, \tag{2.30}$$

and

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k(z)}{k!} \,. \tag{2.31}$$

Note that the sum of the weights of the tensor product components one obtains from the coproduct always equals the weight of the original function. The coproduct can be iterated in a unique way to decompose the polylogarithm into multiple tensor products of lower weight components.

The maximal iteration which results in a tensor product of weight one functions is referred to as the *symbol* $\mathcal{S}$ of the polylogarithm. In this maximal iteration, since all factors are logarithms, the explicit 'log' forms are often dropped and it is written purely as a tensor product of the arguments of the logarithms. For example,

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{n-1 \text{ times}} \,. \tag{2.32}$$

The possible entries of the symbol of a function is commonly referred to as its *alphabet*. The symbol has proved to be an invaluable tool to the physics community for simplifying the complicated functions which arise from loop computations [38, 39].

A particularly interesting property of the coproduct is its interaction with the analytic properties of functions. In particular, it can be shown that for the derivative $\partial_z$ and the discontinuity Disc across some branch cut, one gets [40]

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathbb{I})\Delta \,, \tag{2.33}$$

$$\Delta \partial_z = (\mathbb{I} \otimes \partial_z)\Delta \,. \tag{2.34}$$

The antipode is determined recursively in the weight through the condition

$$\mu(S \otimes \mathbb{I})\Delta(G(\vec{a}; z))) = \mu(\mathbb{I} \otimes S)\Delta(G(\vec{a}; z))) = 0 \,, \text{if } |\vec{a}| \geq 1 \,. \tag{2.35}$$

For example, if $|\vec{a}| = 1$, (2.35) takes the form

$$S(G(a; z)) + G(a; z) = 0 \,, \tag{2.36}$$

and so the antipode of polylogarithms of weight one is uniquely determined. Similarly, for polylogarithms of weight two (2.35) reduces to

$$\begin{aligned} 0 &= S(G(a, b; z)) + S(G(a; z))\, G(b; a) \\ &\quad + S(G(b; z))\, [G(a; z) - G(a; b)] + G(a, b; z) \\ &= S(G(a, b; z)) - G(a; z)\, G(b; a) \\ &\quad - G(b; z)\, [G(a; z) - G(a; b)] + G(a, b; z) \,, \end{aligned} \tag{2.37}$$

where the last step follows upon inserting the corresponding weight-one result. We see that $S(G(a,b;z))$ is uniquely determined by (2.37). The action of the antipode can be made more manifest when considering the way it acts on the symbol. By commuting the antipode with the symbol map, we can naturally define its action on symbols. Upon doing this, we see that

$$S(a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes \cdots \otimes a_1 \,, \tag{2.38}$$

and so the antipode represents an inversion of the symbol, with an overall minus sign depending on its length.

Much of the mathematical structure of multiple polylogarithms has been implemented in the `Mathematica` package `PolyLogTools` [41], which was used in the work described throughout this thesis.

We conclude by mentioning that MPLs are certainly not the end of the story. Starting from two-loop integrals, functions appear which can not be classified in this set. Though progress has been made in understanding the most generic iterated integral which may arise from a Feynman integral [42, 43], there is as of yet no full understanding of the largest possible function space and its properties. A larger set of functions which has been getting a lot of attention recently are iterated integrals which include elliptic integration kernels. These objects are referred to as *elliptic multiple polylogarithms* [44] and much of their mathematical structure is currently being uncovered [45–49]. However, the work in this thesis is purely polylogarithmic, by which we mean that it will always be possible to write the results of our computations in terms of MPLs.

### 2.2.3. Transcendental Weight

In the previous section we introduced the space of multiple polylogarithms and the rich mathematical structure they describe. We mentioned the existence of a scalar quantity called weight which is associated to each MPL and which was preserved by both the shuffle relations and the action of the coproduct. This implies that the space lends itself to be written as a direct sum of subspaces made up of polylogarithms with a fixed weight

$$\begin{gathered}\mathcal{H}_{\text{MPL}} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\text{MPL}} \,, \qquad \mathcal{H}^{(a)}_{\text{MPL}} \cdot \mathcal{H}^{(b)}_{\text{MPL}} \subset \mathcal{H}^{(a+b)}_{\text{MPL}} \,, \\ \Delta \tilde{\mathcal{H}}^{(a)}_{\text{MPL}} \subset \bigoplus_{i=0}^{a} \tilde{\mathcal{H}}^{(a-i)}_{\text{MPL}} \otimes \tilde{\mathcal{H}}^{(i)}_{\text{MPL}} \,,\end{gathered} \tag{2.39}$$

in this case the algebra is called *graded*. The weight is also referred to as *transcendental weight* or *transcendentality*, which refers to the fact that MPLs are conjectured to be *transcendental functions*. Transcendental functions are defined as functions which are not *algebraic*, where algebraic functions are given by the root of a polynomial with coefficients that are rational functions in the variables. It is also conjectured that there are no relations between MPLs of different weights, thus separating the subspaces of fixed weight completely.

The space of MPLs implicitly defines another graded algebra, the set of *multiple zeta values* (MZVs)

$$\zeta(a_1, \ldots, a_k) = \sum_{n_1 > \cdots > n_k} \prod_{i=1}^{k} \frac{1}{n_i^{a_i}} , \tag{2.40}$$

which are conjectured to be *transcendental numbers*, meaning numbers which aren't roots of any polynomial with rational coefficients. The MZVs are related to MPLs via the following equality

$$\zeta(a_1, \ldots, a_k) = (-1)^k G(\underbrace{0, \ldots, 0}_{a_k - 1}, 1, \underbrace{0 \ldots, 0}_{a_{k-1} - 1}, 1, \ldots, \underbrace{0, \ldots, 0}_{a_1 - 1}, 1; 1) , \tag{2.41}$$

and so we can naturally associate the weight $\sum_{i=1}^{k} a_i$ to any MZV $\zeta(a_1, \ldots, a_k)$. It is worth mentioning that for even $n$ the zeta function $\zeta(n)$ is proportional to an even power of $\pi$ via

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n}}{2(2n)!} (2\pi)^{2n} , \tag{2.42}$$

where $B_{2n}$ denotes rational numbers called the *Bernoulli numbers*. Thus we naturally associate weight one to $\pi$, which is a transcendental number through the Hermite-Lindemann theorem. The shuffle product on polylogarithms naturally implies a shuffle product of MZVs and so these numbers also define a graded algebra. Furthermore, analogously to the MPL case, it is conjectured that there are no relations amongst MZVs which mix the weights.

Coming back to physics, transcendental weight is of interest for the study of scattering amplitudes. Since amplitudes can be described as sums of Feynman integrals, when these integrals can be expressed in terms of MPLs (and MZVs) we can study what the weights are of the functions that appear. In $D = 4 - 2\epsilon$ dimensions, coefficients of $\epsilon^k$ of an $\ell$-loop amplitude are expected to contain terms of weight at most $2\ell + k$. However, there is typically no lower bound on the transcendentality of these coefficients.

There is one gauge theory for which there seems to be a lower bound, namely $\mathcal{N} = 4$ super Yang-Mills theory. For $\mathcal{N} = 4$ SYM, all amplitudes which have been computed satisfy the upper bound exactly, see e.g. [38, 50–72]. This property is referred to as the *maximal transcendentality* of $\mathcal{N}=4$ SYM. The maximal transcendental weight property of $\mathcal{N}=4$ SYM was not only observed for scattering amplitudes, but it was also established for certain anomalous dimensions [73–80], form factors [81–85] and correlation functions [86–89]. As of yet it is not understood why $\mathcal{N}=4$ SYM exhibits this property, and it has only been shown for the MHV amplitude in a special limit [90]. Note that beyond gauge theories, there is growing evidence that amplitudes in $\mathcal{N}=8$ SUGRA have the same uniform weight as in $\mathcal{N}=4$ SYM [91–95].

The central topic to the work described in this thesis is to study this property further, and to see whether it can be proven in special setups. Furthermore, we investigate whether this constraint also manifests itself for gauge theories which are similar to $\mathcal{N}=4$ SYM to understand better on which property of the theory it hinges exactly.

## 2.3. The Planar Limit and Dual Conformal Symmetry

In this thesis we study various $\mathrm{SU}(N_c)$ gauge theories. It is natural to wonder what happens when we take $N_c$ to be very large, as this could offer a simplified setup. Define the limit

$$N_c \to \infty \qquad \text{where} \qquad g^2 N_c \sim \mathcal{O}(1)\,, \tag{2.43}$$

to be the *'t Hooft limit*. It is clear from (2.11) that in this case only the leading-colour term contributes. However, there are even diagrammatic consequences to taking this limit. In [96], 't Hooft introduced a double-line notation for amplitudes in $\mathrm{SU}(N_c)$ gauge theories which allows one to determine the proportionality to $N_c$ immediately from the diagram itself. From this it can be concluded that in the 't Hooft limit, only diagrams which can be represented entirely in two dimensions or *planar* diagrams contribute. For this reason the limit is also referred to as the *planar limit*.

The planar limit of $\mathcal{N}=4$ super Yang-Mills theory (or *planar* $\mathcal{N}=4$ *SYM*) is a particularly interesting setup to study scattering amplitudes, due to the extra symmetries which arise. A leading-colour $N$-gluon helicity amplitude is a function of $N$ colour-ordered momenta $\{p_i\}$ which we may choose to be outgoing. Due to momentum conservation, these momenta can be arranged in a closed $N$-gon. This polygon is equivalently described by specifying the vertices

$$x_i - x_{i-1} = p_i\,, \tag{2.44}$$

which we henceforth refer to as *dual coordinates* [97].

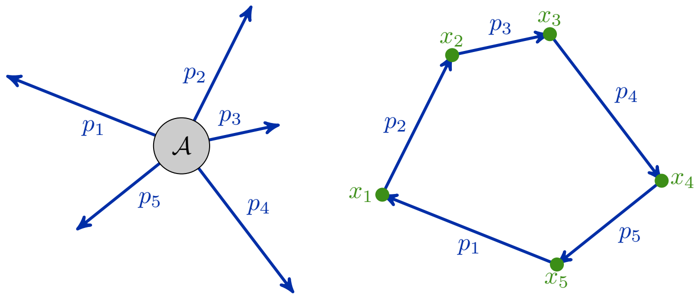



**Figure 2.1:** The relation between momenta $p_i$ and dual coordinates $x_i$ for five particles scattering with outgoing momenta.

The dual coordinates give an alternative functional dependence for scattering amplitudes. Upon expressing scattering amplitudes in these dual variables, it turns out the integrand is also invariant under conformal transformations of the $x_i$. This is referred to as *dual conformal symmetry* [98–103]. It is an entire second symmetry group which is independent of the original superconformal symmetry. The ordinary and dual (super)conformal symmetry groups close together into an infinite symmetry group called the *Yangian* [104], which hints towards the integrability of planar $\mathcal{N} = 4$ SYM [105].

Dual conformal symmetry can be made manifest by writing the amplitudes in terms of variables which are covariant under the action of the symmetry generators [106]. First, note that any null vector $p_i$ can be written in terms of a negative $\lambda^\alpha$ and positive chirality $\tilde{\lambda}^{\dot{\alpha}}$ spinor [28] such that

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \, \tilde{\lambda}_i^{\dot{\alpha}} \,. \tag{2.45}$$

This set of variables is referred to as *spinor helicity variables*. They naturally define scalar invariants through the two-dimensional Levi-Civita tensor

$$\langle \lambda_1 \lambda_2 \rangle = \varepsilon_{\alpha\beta} \lambda_1^\alpha \lambda_2^\beta, \quad \text{and} \quad [\tilde{\lambda}_1 \tilde{\lambda}_2] = \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_1^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} \,. \tag{2.46}$$

The dual coordinates can also be described in terms of spinorial quantities. Writing

$$\mu_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} \lambda_{i\,\alpha} \,, \tag{2.47}$$

allows us to define twistor variables called *momentum twistors* [107]

$$Z_i^I = (\lambda_{i+1}^\alpha, \mu_{i+1}^{\dot{\alpha}}) \,. \tag{2.48}$$

Since the spinor helicity variables allow for arbitrary rescalings of the form

$$\lambda_i^\alpha \to t\, \lambda_i^\alpha \,, \qquad \tilde{\lambda}_i^{\dot{\alpha}} \to t^{-1}\, \tilde{\lambda}_i^{\dot{\alpha}} \,, \tag{2.49}$$

and (2.47) is homogeneous, the momentum twistors $Z_i^I$ are elements of complex projective space[2] $\mathbb{CP}^3$. To form invariants out of these momentum twistors, we contract the $SU(2,2)$ index $I$ with the four-dimensional Levi-Civita tensor to get the *four-bracket*

$$\langle i, j, k, l \rangle \equiv \varepsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \,. \tag{2.50}$$

Differences of dual coordinates $x_{ij} = x_i - x_j$ can be expressed in terms of these four-brackets upon choosing an arbitrary *infinity twistor* $I$ which fixes the null cone at infinity in the coordinate patch given by the dual coordinates. For cross ratios of such differences referred to as *dual conformal cross ratios*, this choice cancels out and we obtain

$$U_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} = \frac{\langle i-1, i, j-1, j\rangle \langle k-1, k, l-1, l\rangle}{\langle i-1, i, k-1, k\rangle \langle j-1, j, l-1, l\rangle} \,, \tag{2.51}$$

where indices are cyclically identified. These dual conformal cross ratios make dual conformal symmetry manifest and are the preferred variables for expressing amplitudes in planar $\mathcal{N} = 4$ Super Yang-Mills. Since the kinematics is determined by $N$ momentum twistors, the configuration space is given by a *Grassmanian*[3] [90]

$$\mathrm{Conf}_N(\mathbb{CP}^3) \cong \mathbb{G}(4, N)/(\mathbb{C}^*)^{N-1} \,. \tag{2.52}$$

Investigating the consequences of this Grassmanian geometry has been a major theme in the study of amplitudes in planar $\mathcal{N} = 4$ Super Yang-Mills and has led to a much deeper understanding of their mathematical structure. One such structure associated to Grassmanian spaces are *cluster algebras* [108, 109], which are commutative rings equipped with a distinguished set of generators called *cluster variables* which can be constructed from an *initial cluster* through a process called *mutation.* In [110], it was observed that the symbol alphabet of the six- and seven-particle MHV amplitude at two loops is described entirely by a special set of variables defined on the cluster algebra of the corresponding Grassmanian, refered to as *cluster Poisson variables* or $\mathcal{X}$-coordinates. Representing the initial cluster for $\mathbb{G}(4, N)$ in a quiver of four-brackets in Figure 2.2, we can readily determine these $\mathcal{X}$-coordinates. The nodes in boxes are called *frozen nodes* and the others are called *unfrozen.* For each unfrozen node one can form $\mathcal{X}$-coordinates by taking the product of all nodes connected by incoming arrows and dividing by the product of all nodes

[2] Complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines through the origin of $\mathbb{C}^{n+1}$

[3] The Grassmanian $Gr(k, n)$ over a field $\mathbb{F}$ is defined as the set of all $k$-dimensional subspaces of $\mathbb{F}^n$. We denote the complex Grassmanian by $\mathbb{G}(k, n)$.

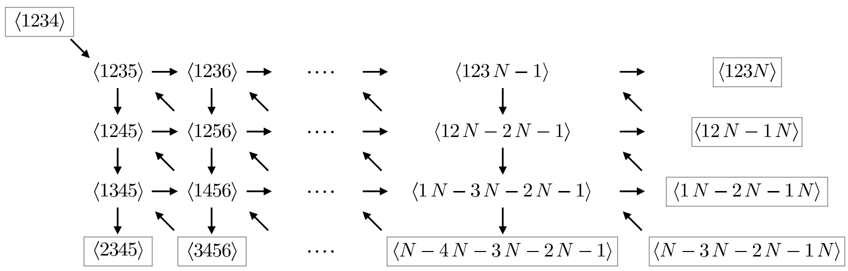



**Figure 2.2:** The initial quiver for $\mathbb{G}(4, N)$ with frozen nodes in boxes.

connected by outgoing ones. We label the $\mathcal{X}$-coordinates as $\mathcal{X}_{ij}$ for $i = 1, 2, 3$ and $j = 1, \ldots, N-5$ following the obvious structure of Fig. 2.2. Explicitly, they are given by

$$\begin{aligned}
\mathcal{X}_{1j} &= \frac{\langle 1\,2\,3\,j+3\rangle\langle 1\,2\,j+4\,j+5\rangle}{\langle 1\,2\,3\,j+5\rangle\langle 1\,2\,j+3\,j+4\rangle}\,,\\
\mathcal{X}_{2j} &= \frac{\langle 1\,2\,3\,j+4\rangle\langle 1\,2\,j+2\,j+3\rangle\langle 1\,j+3\,j+4\,j+5\rangle}{\langle 1\,2\,3\,j+3\rangle\langle 1\,2\,j+4\,j+5\rangle\langle 1\,j+2\,j+3\,j+4\rangle}\,,\\
\mathcal{X}_{3j} &= \frac{\langle 1\,2\,j+3\,j+4\rangle\langle 1\,j+1\,j+2\,j+3\rangle\langle j+2\,j+3\,j+4\,j+5\rangle}{\langle 1\,2\,j+2\,j+3\rangle\langle 1\,j+3\,j+4\,j+5\rangle\langle j+1\,j+2\,j+3\,j+4\rangle}\,.
\end{aligned} \tag{2.53}$$

We may represent these coordinates in a quiver once again, as is shown in Figure 2.3. The $\mathcal{X}$-coordinates are conjectured to form a complete set of coordinates for the kinematical dependence of the scattering amplitude. Thus, the cluster algebra attached to the configuration space describing the kinematics of an amplitude is related its singularities.

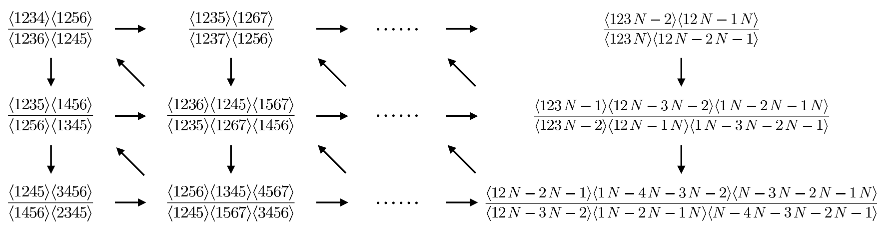



**Figure 2.3:** The $\mathcal{X}$-coordinates of the initial quiver for $\mathbb{G}(4, N)$.

Dual conformal symmetry was observed for the integrand, but is broken upon integration through the introduction of infrared divergences. The MHV amplitude can be factorised into an all-order Ansatz which captures the IR divergences, the so-called *Bern-Dixon-Smirnov (BDS) Ansatz* [51], and a *remainder function* which is IR finite

$$\mathcal{A}_N^{\mathrm{MHV}} = \mathcal{A}_N^{\mathrm{BDS}} \exp\left(R_N\right)\,. \tag{2.54}$$

The remainder function $R_N$ is dual conformal invariant [111, 112], and so it can be written as a function of dual conformal cross ratios (2.51). Of these cross ratios, only $3N - 15$ are algebraically independent in four dimensions. This can be seen straightforwardly, as there are $4N$ coordinates associated to the $x_i$, constrained by $N$ relations $x^2_{i,i+1} = 0$ and the dual conformal group in four dimensions, which has 15 generators. Hence we see that for the four- and five-gluon scattering amplitudes, the only dual-conformal invariant functions are constants, and because of this fact the BDS Ansatz is exact and the remainder function vanishes to all loop orders, $R_4 = R_5 = 0$. This tells us that for up to five external particles, the scattering amplitude is completely fixed by symmetry considerations which is an incredibly strong statement. Starting from six particles, there is a non-trivial finite remainder [112–114]. Beyond MHV the amplitude is described by the *ratio function*

$$\mathcal{P}_N^{\mathrm{N}^k\mathrm{MHV}} = \frac{\mathcal{A}_N^{\mathrm{N}^k\mathrm{MHV}}}{\mathcal{A}_N^{\mathrm{MHV}}}\,, \tag{2.55}$$

which is also conjectured to be a dual conformal invariant object [97].

These remainder and ratio functions have been the focus of many computational efforts. Planar $\mathcal{N} = 4$ SYM is the theory in which the loop-leg frontier can be pushed the furthest. For example, the six-particle remainder function is known up to seven loops in both the MHV and the NMHV configuration [38, 52–58, 60, 62, 64]. For seven particles the MHV amplitude is known analytically at two loops [115] and there are symbol level results at four loops in the MHV case and at three loops in the NMHV case [59, 61]. Though these achievements are immense indeed, there is much to be discovered in the context of the perturbative expansion of planar $\mathcal{N} = 4$ SYM. For one, there is as of yet no full understanding of its function space and in particular, the range of transcendental weights which can appear in a generic amplitude is not understood – though it is expected to be maximally transcendental. To gain insight in these questions, in the following chapter we will consider a special kinematic limit of planar $\mathcal{N} = 4$ SYM. In this simplified setup, we will see that understanding the geometry of the configuration space of amplitudes allows us to deduce their analytic properties.

# 3 | The High-Energy Limit of Planar $\mathcal{N} = 4$ SYM

In this chapter we explore the high energy limit of planar $\mathcal{N} = 4$ Super Yang-Mills theory. The high energy limit was originally studied in the early days of QCD, when it was observed that four-particle scattering[1] in the *Regge limit* $s \gg |t|$ where the scattering process is dominated by gluon exchange in the $t$-channel, exhibits a rich analytic structure. This study led to the BFKL equation in QCD, which resums the corrections in $\alpha_S \log(s/|t|) \sim \mathcal{O}(1)$ to parton-parton scattering at leading logarithmic accuracy (LLA) [116–118] and next-to-LLA (NLLA) [119–121]. The building blocks which appear in this resummation at LLA are gluon amplitudes in the high energy or *multi-Regge limit*, where the outgoing gluons are strongly ordered in rapidity. In the BFKL formalism, which we will turn to in Chapter 4, the gluon rapidities are integrated out, and the BFKL equation is reduced to a two-dimensional problem in terms of purely transverse degrees of freedom.

In planar $\mathcal{N} = 4$ SYM in the Euclidean region where all Mandelstam invariants are negative, scattering amplitudes in the multi-Regge limit factorise into building blocks given by the effective resummed propagator in the $t$-channel and a vertex describing the emission of gluons along said resummed propagator. These building blocks can be determined to all orders from four- and five-particle scattering and hence scattering amplitudes in the multi-Regge limit are trivial in the Euclidean region [122–126]. Starting from six external particles, scattering amplitudes have Regge cuts that are not captured by this factorised form. As a consequence, amplitudes in the multi-Regge limit are no longer trivial if this limit is taken after analytic continuation to a Mandelstam region, already at leading logarithmic accuracy [122, 123]. The discontinuity across the cut is described to all orders by a dispersion relation, and the integrand of the dispersion integral factorises in Fourier-Mellin space. The building blocks

[1] With *Mandelstam variables* $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_3)^2$.

describing the factorisation are closely related to the energy spectrum and the $S$-matrix of the flux-tube excitations in the collinear OPE approach [127, 128].

A lot of effort has gone into determining scattering amplitudes in planar $\mathcal{N} = 4$ SYM in the multi-Regge limit, both at strong [129, 130] and at weak coupling [54, 58, 127, 128, 131–137]. In particular it was observed in [134] that the six-point amplitude in the multi-Regge limit can be expressed perturbatively in terms of single-valued polylogarithmic functions [138].

In this chapter, we will show that in the multi-Regge limit, one can completely describe the geometry underlying the scattering for any number of external particles, and hence classify all the iterated integrals that appear in the amplitude. We will start by introducing the high energy limit, followed by the study of its impact on the form of amplitudes in this theory. Then, we will review a formalism introduced in [1] and further developed in [2, 3, 6] which simplifies the computation of these amplitudes greatly, and which allows us to study the function space and transcendentality of the theory to all orders in the coupling constant [1, 6].

## 3.1. The Geometry of High-Energy Scattering

### 3.1.1. Multi-Regge Kinematics

The focus of this chapter are planar colour-ordered scattering amplitudes in $\mathcal{N}=4$ SYM in a special region of the phase space of $N$-gluon scattering, the so-called *multi-Regge kinematics (MRK)* [139].

To define this limit, we consider $1+2\to 3+\cdots+N$ scattering with all particles outgoing. We will work in lightcone coordinates

$$p_k^\pm \equiv p_k^0 \pm p_k^z, \qquad \mathbf{p}_k \equiv p_{k\perp} = p_k^x + i p_k^y\,, \tag{3.1}$$

where the scalar product of two vectors $p$ and $q$ is given by

$$2\,p\cdot q = p^+q^- + p^-q^+ - \mathbf{p}\bar{\mathbf{q}} - \bar{\mathbf{p}}\mathbf{q}\,. \tag{3.2}$$

Without loss of generality, we choose the reference frame in which the momenta of the initial-state gluons lie on the $z$-axis with $p_2^0 = p_2^z$, implying $p_1^+ = p_2^- = \mathbf{p}_1 = \mathbf{p}_2 = 0$. In this case the multi-Regge limit corresponds to the limit where

$$p_3^+ \gg p_4^+ \gg \cdots \gg p_{N-1}^+ \gg p_N^+, \qquad |\mathbf{p}_3| \simeq \cdots \simeq |\mathbf{p}_N|\,. \tag{3.3}$$

The on-shell conditions $p_i^2 = p_i^+ p_i^- - |\mathbf{p}_i|^2 = 0$ then imply

$$p_3^- \ll p_4^- \ll \cdots \ll p_{N-1}^- \ll p_N^-\,. \tag{3.4}$$

We are considering planar $\mathcal{N}=4$ SYM, which exhibits dual conformal invariance. This implies that the kinematical dependence can be expressed in terms of conformal cross-ratios (2.51). Following [140, 141], from the set of all the $U_{ijkl}$ one can pick a particular algebraically independent set of $3N-15$ cross ratios as (see Figure 3.1),

$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2}\,,\quad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2}\,,\quad u_{3i} = \frac{x_{1\,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2}\,. \tag{3.5}$$

All other cross ratios can be expressed as algebraic functions in these $3N-15$ independent cross ratios.

In the multi-Regge limit, scattering is described by an effective $t$-channel exchange (see Figure 3.2). To describe the kinematics accordingly, we will define momenta $q_i$ which determine the $t$-channel invariants

$$q_i = -\sum_{k=2}^{i+3} p_k\,, \qquad t_{i+1} = q_i^2\,. \tag{3.6}$$

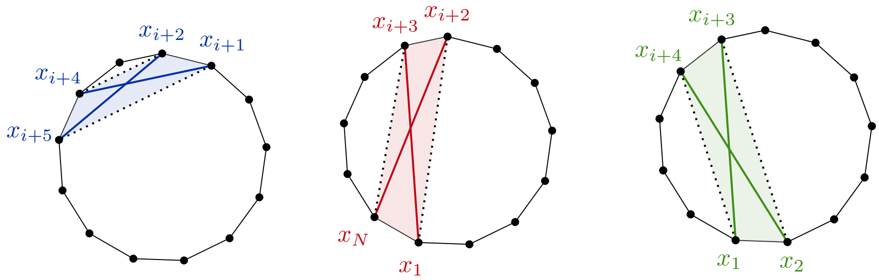



**Figure 3.1:** The algebraically independent set of cross-ratios $\{u_{1i}, u_{2i}, u_{3i}\}$.

We will also relabel the momenta of the gluons emitted along the $t$-channel propagator

$$k_i \equiv p_{i+3}\,, \qquad 1 \leq i \leq N-4\,. \tag{3.7}$$

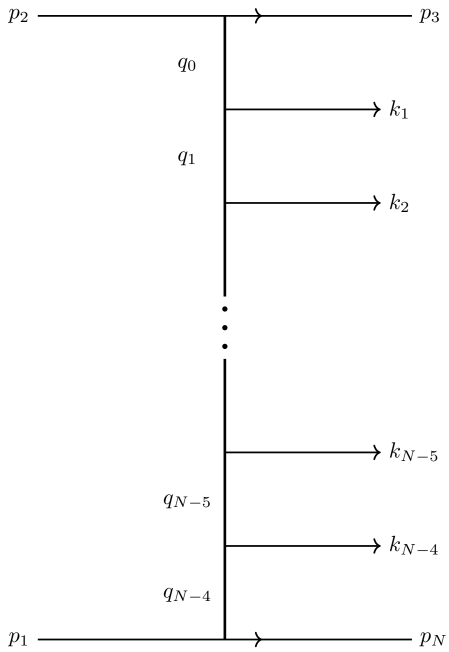



**Figure 3.2:** A diagram representing $t$-channel exchange, illustrating the definitions of momenta $k_i$ and $q_i$.

The conformally invariant cross ratios $(u_{1i}, u_{2i}, u_{3i})$ of (3.5) can be rewritten in terms of these variables [140, 141]. In MRK they take the form

$$\begin{aligned} u_{1i} &= 1 - \delta_i \, \frac{|\mathbf{k}_i + \mathbf{k}_{i+1}|^2}{|\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i^2)\,, \\ u_{2i} &= \delta_i \, \frac{|\mathbf{q}_{i-1}|^2}{|\mathbf{q}_i|^2} + \mathcal{O}(\delta_i^2)\,, \\ u_{3i} &= \delta_i \, \frac{|\mathbf{q}_{i+1}|^2 \, |\mathbf{k}_i|^2}{|\mathbf{q}_i|^2 \, |\mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i^2)\,, \end{aligned} \tag{3.8}$$

where we define the ratio $\delta_i \equiv k_{i+1}^+/k_i^+$. From (3.3) it is clear that in the multi-Regge limit $\delta_i \to 0$, and so we see that all the $u_{1i}$ tend to 1 at the same speed as the $u_{2i}$ and $u_{3i}$ vanish. It is convenient to define the reduced cross ratios [140, 141],

$$\begin{aligned} \tilde{u}_{2i} = \frac{u_{2i}}{1 - u_{1i}} &= \frac{|\mathbf{q}_{i-1}|^2 \, |\mathbf{k}_{i+1}|^2}{|\mathbf{q}_i|^2 \, |\mathbf{k}_i + \mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i)\,, \\ \tilde{u}_{3i} = \frac{u_{3i}}{1 - u_{1i}} &= \frac{|\mathbf{q}_{i+1}|^2 \, |\mathbf{k}_i|^2}{|\mathbf{q}_i|^2 \, |\mathbf{k}_i + \mathbf{k}_{i+1}|^2} + \mathcal{O}(\delta_i)\,. \end{aligned} \tag{3.9}$$

If we introduce dual coordinates in the transverse space $\mathbb{CP}^1$

$$\mathbf{q}_i = \mathbf{x}_{i+2} - \mathbf{x}_1\,, \qquad \mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}\,, \tag{3.10}$$

the reduced cross ratios $\tilde{u}_{2i}$ and $\tilde{u}_{3i}$ can be written as (squares of) cross ratios in $\mathbb{CP}^1$,

$$\tilde{u}_{2i} \simeq |\xi_{2i}|^2\,, \qquad \tilde{u}_{3i} \simeq |\xi_{3i}|^2\,, \tag{3.11}$$

with

$$\xi_{2i} = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+3} - \mathbf{x}_{i+2})}{(\mathbf{x}_1 - \mathbf{x}_{i+2})\,(\mathbf{x}_{i+3} - \mathbf{x}_{i+1})}\,, \quad \xi_{3i} = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+2})\,(\mathbf{x}_{i+3} - \mathbf{x}_{i+1})}\,. \tag{3.12}$$

These objects are not independent, $\xi_{2i} = 1 - \xi_{3i}$, and we introduce the transverse cross ratios which we will henceforth refer to as *Fourier-Mellin coordinates*

$$z_i \equiv 1 - \frac{1}{\xi_{2i}} = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1}\,\mathbf{k}_i}{\mathbf{q}_{i-1}\,\mathbf{k}_{i+1}}\,. \tag{3.13}$$

Note that these cross ratios encode the full kinematical information of the process. In the next section, we will take a closer look of the geometry of the configuration space of scattering in the multi-Regge limit.

### 3.1.2. The Moduli Space of Riemann Spheres with Marked Points

In the previous section we argued that in the multi-Regge limit, the configuration space is described entirely by $N-2$ *transverse dual coordinates* $\mathbf{x}_i$. Since planar $\mathcal{N}=4$ SYM is dual conformal invariant, these transverse dual coordinates can be interpreted as points on the complex projective line $\mathbb{CP}^1$. Such a configuration space is equivalent to the moduli space of Riemann spheres with $N-2$ marked points

$$\mathrm{Conf}_{N-2}(\mathbb{CP}^1) \equiv \mathfrak{M}_{0,N-2}\,. \tag{3.14}$$

To demonstrate this limit behaviour explicitly, one can make use of the cluster algebra structure discussed in the previous chapter. More precisely, one can show that the cluster algebra associated to $\mathrm{Conf}_N(\mathbb{CP}^3)$ in full kinematics reduces to the cluster algebra of $\mathfrak{M}_{0,N-2}$ [1]. To show this, it is useful to note that the dual conformal invariance of planar $\mathcal{N}=4$ SYM implies that the multi-Regge limit defined in (3.3) is equivalent to the strongly-ordered multi-soft limit where the momenta $p_i$, $3\leq i\leq N-3$, are soft, with $p_i$ softer than $p_{i+1}$. The soft limit of $p_{i+1}$ corresponds to the limit where the momentum twistors $Z_{i-1}$, $Z_i$ and $Z_{i+1}$ are aligned. In general, let us consider a limit where the twistor $Z_i$ approaches the line between $Z_j$ and $Z_k$. One may parametrise this situation as follows

$$\begin{aligned} Z_i \to \hat{Z}_i =& Z_j + \alpha_i \tfrac{\langle j\, j-1\, k+1\, k+2\rangle}{\langle k\, j-1\, k+1\, k+2\rangle} Z_k \\ &+ \epsilon_i \tfrac{\langle j\, k\, k+1\, k+2\rangle}{\langle j-1\, k\, k+1\, k+2\rangle} Z_{j-1} - \epsilon_i\beta_i \tfrac{\langle j\, j-1\, k\, k+2\rangle}{\langle k+1\, j-1\, k\, k+2\rangle} Z_{k+1}\,, \end{aligned} \tag{3.15}$$

where the limit where $Z_i$, $Z_j$ and $Z_k$ are aligned corresponds to the limit $\epsilon_i \to 0$ and $\alpha_i$ and $\beta_i$ are scalar parameters which characterise this multi-soft limit. One can show using this parameterisation that the multi-soft limit where we sequentially take the momenta $p_i$, $3\leq i\leq N-3$ to be soft[2] corresponds to the multi-Regge limit. This allows us to easily compute the multi-Regge limit of momentum twistors, and thus also of the $\mathcal{X}$-coordinates.

Taking said limit, one sees that all $\mathcal{X}$-coordinates as given in (2.53) of the form $\mathcal{X}_{2j}$ vanish, while the others reduce to either holomorphic or anti-holomorphic cross ratios in $\mathbb{CP}^1$,

$$\mathcal{X}_{1j} = \frac{(\overline{\mathbf{x}}_2-\overline{\mathbf{x}}_{j+2})(\overline{\mathbf{x}}_{j+3}-\overline{\mathbf{x}}_{j+4})}{(\overline{\mathbf{x}}_2-\overline{\mathbf{x}}_{j+4})(\overline{\mathbf{x}}_{j+2}-\overline{\mathbf{x}}_{j+3})}\,,\ \mathcal{X}_{3j} = \frac{(\mathbf{x}_1-\mathbf{x}_{j+1})(\mathbf{x}_{j+2}-\mathbf{x}_{j+3})}{(\mathbf{x}_1-\mathbf{x}_{j+3})(\mathbf{x}_{j+1}-\mathbf{x}_{j+2})}\,. \tag{3.16}$$

These MRK $\mathcal{X}$-coordinates are singular when two points $\mathbf{x}_i$ coincide, which is precisely the singularity structure of the moduli space $\mathfrak{M}_{0,N-2}$. However,

[2] This corresponds to taking twistor $Z_2$ to the line $(Z_1Z_3)$, then $Z_3$ to the line $(Z_1Z_4)$ and so on.

we have obtained two copies of coordinate sets, a holomorphic and an anti-holomorphic one. This can be understood from the cluster algebra in Figure 2.3, since in the multi-Regge limit the middle line in the quiver vanishes, and so the cluster algebra splits into two disconnected parts. Each of these two parts is isomorphic to the cluster algebra $A_{N-5}$, which is the cluster algebra that describes the singularity structure of $\text{Conf}_{N-2}(\mathbb{CP}^1) \simeq \mathfrak{M}_{0,N-2}$. Hence, we conclude that in MRK the cluster algebra of $\text{Conf}_N(\mathbb{CP}^3)$ reduces to the cluster algebra $A_{N-5} \times A_{N-5}$, where the two copies of $A_{N-5}$ are complex conjugate to each other.

As a consequence, we expect that planar scattering amplitudes in $\mathcal{N} = 4$ SYM in MRK can be expressed through iterated integrals with singularities precisely when the $\mathcal{X}$-coordinates in (3.16) are singular, i.e., iterated integrals made out of the one-forms $d\log(\mathbf{x}_i - \mathbf{x}_j)$ (and their complex conjugates). Note that the cluster algebra in MRK is of finite type, independently of the number of external particles. The cluster algebras associated to the six and seven-point amplitudes in full kinematics are of finite type, but starting from eight-particle scattering, the cluster algebra is infinite [110, 142]. Remarkably, we conclude that the cluster algebra in full kinematics must always reduce to a cluster algebra of finite type in the multi-Regge limit.

The space $\mathfrak{M}_{0,N-2}$ exhibits a $SL(2,\mathbb{C})$ symmetry, related to the action of the conformal group on the transverse plane. Since $\dim_{\mathbb{C}} SL(2,\mathbb{C}) = 3$ we see

$$\dim_{\mathbb{C}} \mathfrak{M}_{0,N-2} = N - 5\,. \tag{3.17}$$

This implies that the kinematics is equally well parameterised by $N - 5$ coordinates upon exploiting the symmetry to fix three coordinates to certain values. Such an example is given by the *simplicial coordinates*, obtained by fixing three of the $N - 2$ points to 0, 1, and $\infty$. A particularly useful set of simplicial coordinates for our purposes is named *simplicial MRK coordinates*

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{N-2}) \to (1, 0, \rho_1, \ldots, \rho_{N-5}, \infty)\,. \tag{3.18}$$

They are related to the Fourier-Mellin coordinates by

$$z_i = \frac{(1-\rho_{i+1})(\rho_i - \rho_{i-1})}{(1-\rho_{i-1})(\rho_i - \rho_{i+1})}\,, \quad \rho_0 = 0\,, \rho_{N-4} = \infty\,. \tag{3.19}$$

There is another class of simplicial coordinates which will be very useful in what follows. Let us start from the Fourier-Mellin coordinates, and single out $z_i$. Then there is always a set of simplicial coordinates $(t_1^{(i)}, \ldots, t_{N-5}^{(i)})$ such that $t_i^{(i)} = z_i$. Indeed, from (3.13) we see that we can define these coordinates by

$$(\mathbf{x}_1, \ldots, \mathbf{x}_{N-2}) \to (\infty, t_1^{(i)}, \ldots, t_i^{(i)}, 0, 1, \ldots, t_{N-5}^{(i)})\,. \tag{3.20}$$

We will refer to these coordinates as *simplicial coordinates based at $z_i$*.

### 3.1.3. Single-Valued Multiple Polylogarithms

We have determined that scattering amplitudes in the multi-Regge limit of planar $\mathcal{N} = 4$ SYM must be built up out of iterated integrals on $\mathfrak{M}_{0,N-2}$, which can be written in terms of multiple polylogarithms [143]. Note, however, that physics imposes additional constraints on our amplitudes, since massless scattering amplitudes can have branch points at most when a Mandelstam invariant vanishes or becomes infinite. This requirement puts strong constraints on the first entry of the symbol of the polylogarithms which may appear [144]. Dual conformal invariance implies that the first entry must be built up of cross ratios $U_{ijkl}$. In a Mandelstam region where some of the external particles have negative energies, the branch points are determined by products of cross ratios that become equal to 0, 1 or $\infty$. In other words, in a Mandelstam region the first entry is either a cross ratio $U_{ijkl}$ or $(1 - \prod_{ijkl} U_{ijkl}^{a_{ijkl}})$ with some $a_{ijkl} \in \mathbb{Z}$. From the kinematical considerations in Section 3.1.1 one can show that in the multi-Regge limit such entries become absolute values squared of cross ratios in $\mathbb{CP}^1$ [1]. To start, we note that there are $N(N-5)/2$ multiplicatively independent cross ratios, which we may choose as

$$\begin{aligned} u_{1i}\,, u_{2i}\,, u_{3i}\,, &\quad 1 \le i \le N-5\,, \\ U_{ij}\,, &\quad 2 \le i \le j-4 \le N-5\,, \end{aligned} \tag{3.21}$$

where the $u_{ij}$ have been defined in (3.5) and

$$U_{ij} \equiv U_{i,j+1,j,i+1}\,. \tag{3.22}$$

The multi-Regge limit of $(u_{1i}, u_{2i}, u_{3i})$ was analysed in Section 3.1.1. The $U_{ij}$ tend to 1 in the multi-Regge limit. We introduce new reduced cross ratios which have a finite multi-Regge limit,

$$\tilde{U}_{ij} \equiv \frac{1 - U_{ij}}{\prod_{k=i-1}^{j-4}(1 - u_{1k})} \to \left| \frac{\mathbf{x}_i - \mathbf{x}_{j-1}}{\mathbf{x}_i - \mathbf{x}_{i+2}} \prod_{k=i+1}^{j-3} \frac{\mathbf{x}_k - \mathbf{x}_{k+1}}{\mathbf{x}_k - \mathbf{x}_{k+2}} \right|^2 . \tag{3.23}$$

Recall the equivalence between the multi-Regge and multi-soft limits. The multi-soft limit is approached sequentially according to $\epsilon_2 \ll \epsilon_3 \ll \ldots \ll \epsilon_{N-4}$, where $\epsilon_i$ are the small parameters introduced in (3.15). We see from (3.15) that the cross ratios $u_{1i} = 1 + \mathcal{O}(\epsilon_{i+1})$, whereas $U_{ij} = 1 + \mathcal{O}(\epsilon_i \ldots \epsilon_{j-4})$, and so all the $U_{ij}$ approach 1 at a different speed. Let us now first look at the case where the first entry is $U_{ijkl}$. It is sufficient to analyse the multiplicatively independent cross ratios in (3.21). They all tend to 1, except for $u_{2i}$ and $u_{3i}$, which we may exchange for the corresponding reduced cross ratios $\tilde{u}_{2i}$ and $\tilde{u}_{3i}$. The latter reduce to absolute values squared of cross ratios in $\mathbb{CP}^1$, see (3.11).

Next, let us analyse the case of an entry of the type $(1-\prod_{ijkl} U_{ijkl}^{a_{ijkl}})$. It is sufficient to assume that the factors in the product are taken from (3.21). If one of the factors goes to zero in MRK, then the claim is true, because we have for example,

$$\log(1-u_{2i}^a\,U) \to \begin{cases} a\,\log u_{2i} + \log U\,, & \text{if } a<0\,,\\ 0\,, & \text{if } a>0\,, \end{cases} \tag{3.24}$$

where $U$ is any product of cross ratios that tend to 1 in MRK. If all the factors in the product $\prod_{ijkl} U_{ijkl}^{a_{ijkl}}$ tend to 1, then we know that one of the factors tends to one much slower than the others. Hence, up to terms that are power-suppressed in MRK, we only need to keep this factor. The claim then follows from (3.23). As the first entries describe the branch points of the function (2.33) and we find these entries to be positive for all values of $\mathbf{x}_i$, we conclude that the resulting function should be single-valued.

Since above we show that our results are required to be single-valued and we wish to write them in a way which is manifestly single-valued, we will consider linear combinations of multiple polylogarithms such that their branch cuts cancel. More specifically, we can associate to every multiple polylogarithm[3] $G_{a_1,\ldots,a_n}(z)$ a so-called *single-valued multiple polylogarithm (SVMPL)* $\mathcal{G}_{a_1,\ldots,a_n}(z)$. The SVMPL is a linear combination of multiple polylogarithms in the variable $z$ and its complex conjugate $\bar{z}$ such that it is single-valued and it obeys the same holomorphic differential equation as the original multiple polylogarithm

$$\partial_z \mathcal{G}_{a_1,\ldots,a_n}(z) = \frac{1}{z-a_1}\mathcal{G}_{a_2,\ldots,a_n}(z)\,. \tag{3.25}$$

One can show [145, 146] that there is a one-to-one map which sends each MPL to its single-valued analogue

$$\mathbf{s}\left(G_{a_1,\ldots,a_n}(z)\right) \equiv \mathcal{G}_{a_1,\ldots,a_n}(z)\,. \tag{3.26}$$

This map is given by

$$\mathbf{s} = \mu(\tilde{S}\otimes \mathbb{I})\Delta\,, \tag{3.27}$$

where $\tilde{S}$ is defined as a modified antipode

$$\tilde{S}(G_{\vec{a}}(z)) \equiv (-1)^{|\vec{a}|}\,\overline{S}(G_{\vec{a}}(z))\,, \tag{3.28}$$

with $\overline{S}$ the complex conjugate of the antipode. This purely combinatoric map can be used to construct SVMPLs in a straightforward way. Its action on

[3] In what follows, we use the shorthand $G_{a_1,\ldots,a_n}(z) \equiv G(a_1,\ldots,a_n;z)$.

weight-one and -two multiple polylogarithms is given by

$$\mathbf{s}\left(G_a(z)\right) = \mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z})\,, \tag{3.29}$$

$$\begin{aligned}\mathbf{s}\left(G_{a,b}(z)\right) = \mathcal{G}_{a,b}(z) &= G_{a,b}(z) + G_{\bar{b},\bar{a}}(\bar{z}) + G_b(a)G_{\bar{a}}(\bar{z}) + G_{\bar{b}}(\bar{a})G_{\bar{a}}(\bar{z}) \\ &\quad - G_a(b)G_{\bar{b}}(\bar{z}) + G_a(z)G_{\bar{b}}(\bar{z}) - G_{\bar{a}}(\bar{b})G_{\bar{b}}(\bar{z})\,. \end{aligned}\tag{3.30}$$

As an example, let us have a closer look at the weight-one case

$$\mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z}) = \log\left(1-\frac{z}{a}\right) + \log\left(1-\frac{\bar{z}}{\bar{a}}\right) = \log\left|1-\frac{z}{a}\right|^2. \tag{3.31}$$

Since the argument of the logarithm on the right-hand side is positive-definite, we see explicitly that the function is single-valued. The proof that $\mathcal{G}_{\vec{a}}(z)$ is single-valued now proceeds by induction in the weight. Let us now assume that all functions $\mathcal{G}$ are single-valued up to a certain weight $n$, and let us show that a function $\mathcal{G}_{\vec{a}}(z)$ of weight $n+1$ is still single-valued. First let us mention - using once again the notation defined in (2.28) - that the coproduct and the single-valuedness map compose in the following way

$$\Delta\mathbf{s}(I(a_0;\vec{a};z)) = \sum_{\emptyset\subseteq\vec{c}\subseteq\vec{b}\subseteq\vec{a}} \mathbf{s}(I_{\vec{c}}(a_0;\vec{b};z)) \otimes \left[\tilde{S}(I(a_0;\vec{c};z))\, I_{\vec{b}}(a_0;\vec{a};z)\right]\,, \tag{3.32}$$

which can be shown from (3.27) explicitly as was done in [1]. Crucially, we see that the first factors of the coproduct are given by SVMPLs. Now we apply the coproduct to the discontinuity of a generic $\mathcal{G}_{\vec{a}}(z)$ of weight $n+1$. Since the discontinuity operator only acts in the first factor of the coproduct (2.33) and we claim all lower-weight SVMPLs to be single-valued, only the term with the weight-$(n+1)$ function in the first factor may contribute and we get

$$\Delta\mathrm{Disc}(\mathcal{G}_{\vec{a}}(z)) = (\mathrm{Disc}\otimes\mathbb{I})\Delta(\mathcal{G}_{\vec{a}}(z)) = \mathrm{Disc}(\mathcal{G}_{\vec{a}}(z))\otimes 1\,. \tag{3.33}$$

From (2.35) we obtain

$$0 = \mu(\mathbb{I}\otimes S)\Delta\mathrm{Disc}(\mathcal{G}_{\vec{a}}(z)) = \mathrm{Disc}(\mathcal{G}_{\vec{a}}(z))\,\cdot S(1) = \mathrm{Disc}(\mathcal{G}_{\vec{a}}(z))\,, \tag{3.34}$$

and so $\mathcal{G}_{\vec{a}}(z)$ is single-valued.

Single-valued multiple polylogarithms inherit many of the properties of ordinary MPLs. In particular, SVMPLs form a shuffle algebra and satisfy the same functional relations as their multi-valued analogues. Note that we have managed to define the function space of our scattering amplitudes purely by kinematical considerations.

Now that we have explored the consequences of the high-energy limit on kinematics the underlying geometry, we will turn to dynamics. In what follows, we will consider the representation of amplitudes in MRK in $\mathcal{N}=4$ SYM.

# 3.2. Amplitudes in the Multi-Regge Limit

## 3.2.1. Large Logarithms and Resummation

As was stated in the introduction to this chapter, in the multi-Regge limit the appearance of large logarithms forces us to resum the perturbative expansion. Thus, in the multi-Regge limit we consider *logarithmic accuracies.* The *leading logarithmic accuracy (LLA)* is defined as the sum of all terms with the highest powers of the large logarithm at their respective perturbative orders. The sum of all terms with the second highest powers is referred to as the *next-to-leading logarithmic accuracy (NLLA)*, and so on.

What does such a resummation look like? Consider the example of $2 \to 2$ scattering in pure Yang-Mills theory. In the high-energy limit, the tree level amplitude will be dominated by a $t$-channel exchange shown in Figure 3.3.

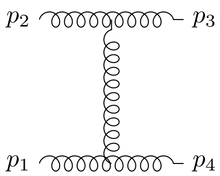



**Figure 3.3:** The dominant tree-level amplitude in multi-Regge kimenatics.

At one-loop, the dominant diagrams are given by Figure 3.4

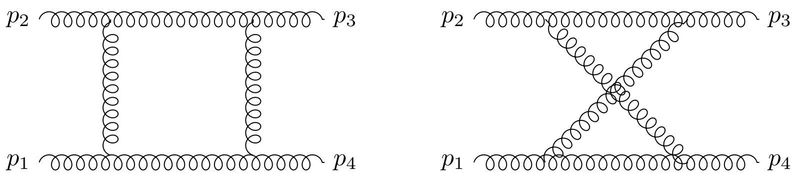



**Figure 3.4:** The dominant one-loop amplitudes in multi-Regge kimenatics.

which through *Cutkosky rules* [147] can be associated to the tree-level amplitude by putting propagators on-shell

$$\begin{aligned}\Im \mathcal{A}_4^{(1)}(s,t) = \int \frac{d^4k}{(2\pi)^2} \delta((p_1-k)^2)\delta((p_2+k)^2) \\ \times \mathcal{A}_4^{(0)}(s,k^2)\overline{\mathcal{A}}_4^{(0)}(s,(k-q)^2)\,,\end{aligned} \tag{3.35}$$

as shown in Figure 3.5. For the one-loop result to leading logarithmic accuracy, one obtains

$$\mathcal{A}_4^{(1)} \simeq \mathcal{A}_4^{(0)} \log\left(\frac{s}{|t|}\right)\alpha(t)\,, \tag{3.36}$$

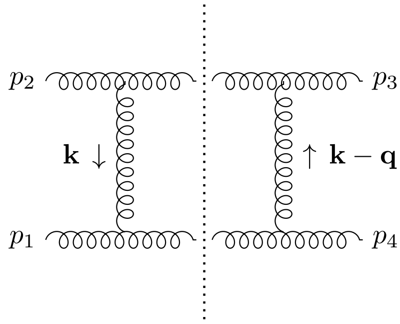



**Figure 3.5:** The cut one-loop gives the diagram shown in Figure 3.3.

where

$$\alpha(t) = -\frac{\alpha_S N_c}{4\pi} \int -\mathbf{q}^2 \frac{\mathrm{d}^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2} , \tag{3.37}$$

is called the leading order *gluon Regge trajectory.* Continuing this pattern to higher orders, one obtains similar diagrams which can be cut to be described in terms of their lower-loop constituents. Studying the first few orders, a pattern emerges and one sees that the leading logarithmic contribution is given by

$$\mathcal{A}_4^{\mathrm{LLA}} \sim \mathcal{A}_4^{(0)} \left(\frac{s}{|t|}\right)^{\alpha(t)} . \tag{3.38}$$

This can be interpreted as the exchange of an effective, resummed propagator in the t-channel as represented in Figure 3.6.

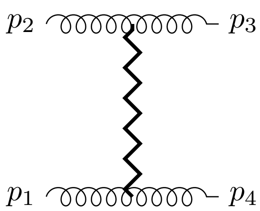



**Figure 3.6:** The effective resummed diagram at leading logarithmic order in the multi-Regge limit.

The resummed propagator defined above captures virtual corrections of the cut diagrams. However, to consider real corrections, one must consider emissions of gluons from this resummed propagator. To this end, via a similar procedure a resummed vertex can be introduced which is called the *Lipatov vertex* as shown in Figure 3.7. The resummed $t$-channel propagator, along with the Lipatov vertex, can be combined into a ladder-shape to form an object governing the leading corrections to $2 \to 2$ scattering called the LLA *BFKL ladder*, which can be shown to obey an integral equation called the *BFKL equation* [116–118] and which will appear in Chapter 4 in the context of the parton-parton cross section.

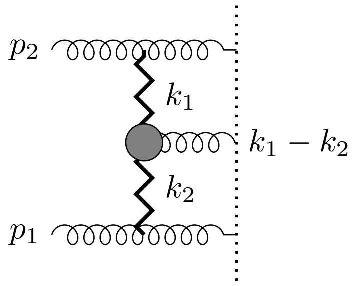



**Figure 3.7:** An extra real emission from the resummed propagator.

In the planar limit of $\mathcal{N} = 4$ SYM, the BDS Ansatz defined in (2.54) determines four and five-particle scattering to all orders. Taking the multi-Regge limit of said Ansatz, one can identify a similar factorisation to what was described above, and thus analogous building blocks - a resummed propagator associated to an effective particle called the *Reggeon* and vertices governing its interaction with gluons - can be extracted. For scattering of six or more particles, these building blocks are also sufficient to describe the amplitude to all orders in the Euclidean region where the energies of all produced particles are positive [122–126]. In other kinematic regions, additional contributions appear [122, 123]. For the remainder of this chapter, these contributions will be our main focus.

### 3.2.2. The MRK Ratio Function

We now turn to our representation of amplitudes in MRK in $\mathcal{N} = 4$ SYM. Helicity must be conserved among the gluons going very forward, so that we only distinguish between different helicity configurations $(h_1, \dots h_{N-4})$ of the gluons emitted along the $t$-channel propagator, also referred to as the *ladder*. We define the *MRK ratio*

$$e^{i\Phi_{h_1,\dots,h_{N-4}}}\, \mathcal{R}_{h_1,\dots,h_{N-4}} \equiv \left[\frac{A_N(-,+,h_1,\dots,h_{N-4},+,-)}{A_N^{\text{BDS}}(-,+,\dots,+,-)}\right]_{|\text{MRK}}, \qquad (3.39)$$

where $A_N(-,+,h_1,\dots,h_{N-4},+,-)$ is the (colour-ordered) amplitude for the production of $N-4$ gluons emitted along the ladder, and $A_N^{\text{BDS}}(-,+,\dots,+,-)$ is the corresponding BDS amplitude. The finite term $\mathcal{R}_{h_1,\dots,h_{N-4}}$ is related to the well-known remainder and ratio functions as defined in the previous chapter (2.54)-(2.55). For example, note that the MHV and $\overline{\text{MHV}}$ ratios are equal to each other, and are also alternatively described by their logarithm, the remainder function,

$$\mathcal{R}_{\underbrace{+\ldots+}_{N-4}} = \mathcal{R}_{\underbrace{-\ldots-}_{N-4}} \equiv e^{R_N} \,. \tag{3.40}$$

The factor $e^{i\Phi_{h_1,\ldots,h_{N-4}}}$ is a phase factor such that in the Euclidian region where all external particles have positive energies we have $\mathcal{R}_{h_1,\ldots,h_{N-4}} = 1$. When performing an analytic continuation to a Mandelstam region where some particle energies can be negative, $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ picks up contributions called Regge cuts [122, 123, 131–133, 141, 148–150]. In the Mandelstam region where the energy components of all produced particles are negative, it can be written as a dispersion integral of a product of building blocks. In what follows, we will discuss the form of this dispersion integral and its perturbative expansion.

### 3.2.3. The Dispersion Integral

In [6], it was conjectured that in the Mandelstam region where all produced particles have negative energies, $\mathcal{R}_N \equiv \mathcal{R}_{h_1,\ldots,h_{N-4}}$ is given by the relation

$$\mathcal{R}_N e^{i\delta_N} = 1 + a i\pi \mathcal{F}_{N-5}\left[\left(\prod_{k=1}^{N-5} e^{-L_k\omega_k}\right)\chi_1^{h_1}\left(\prod_{k=2}^{N-5} C_{k-1,k}^{h_k}\right)\chi_{N-5}^{-h_{N-4}}\right], \tag{3.41}$$

where $\mathcal{F}_m$ denotes the $m$-fold inverse Fourier-Mellin transform,

$$\mathcal{F}_m\Big[f(\{\nu_i,n_i\})\Big] \equiv \prod_{k=1}^{m}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu_k}{2\pi}|z_k|^{2i\nu_k} f(\{\nu_i,n_i\})\,. \tag{3.42}$$

The dependence on the variables $\tau_i$ and $z_i$, $1 \le i \le N-5$, is kept implicit on the left hand side of (3.41). Furthermore, we denote the large logarithms $L_k = \log\tau_k + i\pi$, $\tau_k \equiv \sqrt{u_{2k}u_{3k}} \xrightarrow[\text{MRK}]{} 0$ where the $u_{ij}$ were defined in (3.8), $a$ is the 't Hooft coupling, and

$$\omega(\nu_i,n_i) \equiv \omega_i = -a(E_i + aE_i^{(1)} + \mathcal{O}(a^2)), \tag{3.43}$$
$$\chi^{\pm}(\nu_i,n_i) \equiv \chi_i^{\pm} = \chi_{0,i}^{\pm}(1 + a\kappa_1^{\pm} + \mathcal{O}(a^2)), \tag{3.44}$$
$$C^{\pm}(\nu_i,n_i,\nu_j,n_j) \equiv C_{i,j}^{\pm} = C_{0,i,j}^{\pm}(1 + ac_{1,i,j}^{\pm} + \mathcal{O}(a^2)), \tag{3.45}$$

are the all-order *BFKL eigenvalue*, the all-order *impact factor* and the all-order *central emission block* respectively. We shall collectively refer to these objects as the *BFKL building blocks*. The BFKL eigenvalue and impact factor are known to all orders from [127]. The all-order central emission block is presented in [6]. Their explicit form up to NLO can be found in Appendix A.

The BDS phase $\delta_N$ is given by

$$\begin{aligned}\delta_N &= \pi\Gamma \log\left|\frac{\mathbf{x}_{32}\mathbf{x}_{(N-2)1}\mathbf{x}_{(N-3)(N-2)}\mathbf{x}_{21}}{\mathbf{x}_{31}\mathbf{x}_{(N-2)2}\mathbf{x}_{(N-3)1}\mathbf{x}_{2(N-2)}}\right|^2\,,\\ &= \pi\Gamma\log\left|\frac{\rho_1}{(\rho_1-1)(\rho_{N-5}-1)}\right|^2\,,\end{aligned} \tag{3.46}$$

the factor $\Gamma = \gamma_K/8$ is proportional to the all-order *cusp anomalous dimension* determined in [78] and given to second order by

$$\gamma_K = 4a - 4\zeta_2 a^2 + \mathcal{O}(a^3)\,. \tag{3.47}$$

Note that this conjectural dispersion integral reproduces the known Fourier-Mellin representation of the six-point MHV and NMHV amplitudes in MRK to LLA [122, 123, 148], and also of the seven-point MHV amplitude to LLA [148].

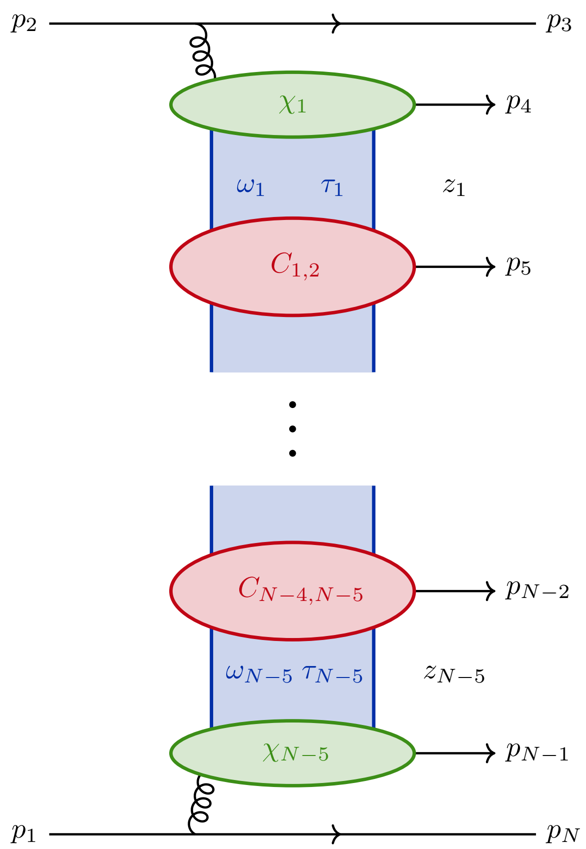



**Figure 3.8:** The structure of the $N$-point amplitude in MRK.

At this point we should comment on some discrete symmetries of the MRK ratio. The first such symmetry is *parity* $P$ which corresponds to spatial reflection. This transformation reverses particle helicities, and acts on the Fourier-Mellin variables as follows

$$P : z \leftrightarrow \bar{z} \,. \tag{3.48}$$

This implies that, for the MRK ratio

$$\begin{aligned} P\mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1, z_1, \ldots ,\tau_{N-5}, z_{N-5}) = \\ \mathcal{R}_{-h_1,\ldots,-h_{N-4}}(\tau_1, z_1, \ldots , \tau_{N-5}, z_{N-5}) \,. \end{aligned} \tag{3.49}$$

From (3.40) we can see that the MHV ratio is invariant under parity transformation. The second discrete symmetry we should consider is *target-projectile symmetry* $F$. This amounts to exchanging the two initial-state gluons and reversing their orientation. More generally it is given by $p_i \to -p_{N+3-i}$ where indices are cyclically identified $p_i \simeq p_{i+N}$. At the level of the variables on which the MRK ratio depends, this translates to

$$F : \begin{cases} \tau_i \leftrightarrow \tau_{N-4-i} \,, \\ z_i \leftrightarrow 1/z_{N-4-i} \end{cases} . \tag{3.50}$$

The reversal of orientation also flips helicities. From this we conclude that

$$\begin{aligned} F\mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1, z_1, \ldots ,\tau_{N-5}, z_{N-5}) = \\ \mathcal{R}_{-h_{N-4},\ldots,-h_1}(\tau_1, z_1, \ldots , \tau_{N-5}, z_{N-5}) \,. \end{aligned} \tag{3.51}$$

To uniquely define (3.41), we must comment on the contours of integration implicitly defined in this equation. In [1, 2, 6], it was shown that this contour is fixed by demanding the MRK ratio $\mathcal{R}_{h_1,\ldots,h_{N-4}}$ to have the correct soft behaviour. Under soft limits, amplitudes reduce to lower-point results multiplying so-called universal soft factors, schematically

$$\lim_{p_i \to 0} \mathcal{M}_N(p_1, \ldots , p_N) \equiv \mathcal{M}_{N-1}(p_1, \ldots , p_{i-1}, p_{i+1}, \ldots , p_N) \times (\text{Soft})_i \,. \tag{3.52}$$

Since the soft behaviour is captured by the BDS Ansatz, the soft limit of the MRK ratio is given by a lower-point MRK ratio.

Due to the strong ordering imposed by the multi-Regge limit, there are no collinear singularities or soft singularities associated with the two final-state particles at the ends of the ladder. Hence, an amplitude in MRK has singularities only in the limits where one of the momenta $k_i$, $1 \le i \le N - 4$, vanishes. The vanishing of $k_i$ implies that in particular its transverse component $\mathbf{k}_i$ goes to zero. The limit $\mathbf{k}_i \to 0$ corresponds to the limit where some of the cross ratios $z_k$ take a special value, as can be seen explicitly from relation (3.13). There are three distinct cases to consider:

1. If $\mathbf{k}_1 \to 0$, $z_1 \to 0$, and all other cross ratios remain finite.
2. If $\mathbf{k}_{N-4} \to 0$, $z_{N-5} \to \infty$, and all other cross ratios remain finite.
3. If $\mathbf{k}_i \to 0$, for $2 \leq i \leq N-5$, $z_i \to 0$ and $z_{i-1} \to \infty$, but the product $z_{i-1}z_i$ remains finite. All other cross ratios remain finite.

Let us study the implications of these soft limits on the integration contour case by case in number of particles, starting with the simplest example of six-particle scattering.

We restrict ourselves to the MHV case. For six external particles, (3.41) reduces to

$$\mathcal{R}_{++}e^{i\delta_6} = 1 + a\, i\pi\, \mathcal{F}\left[\chi^+ e^{-L\omega}\chi^-\right] , \tag{3.53}$$

with soft limits given by

$$\begin{aligned} \lim_{z\to 0} \mathcal{R}_6 e^{i\delta_6} &= |z|^{2\pi i\Gamma} , \\ \lim_{z\to \infty} \mathcal{R}_6 e^{i\delta_6} &= |z|^{-2\pi i\Gamma} , \end{aligned} \tag{3.54}$$

stemming purely from the soft limits of the BDS phase (3.46), since the MRK ratio is trivial for five external particles. From these limits, we can conclude that the integrand of (3.53) should have simple poles at $\nu = \pm\pi\Gamma$ when $n = 0$ (terms where $n \neq 0$ are suppressed in the soft limit). Furthermore, the residues are given by

$$\mathrm{Res}_{\nu=\pm\pi\Gamma}\left(\chi^+ e^{-L\omega}\chi^-\right) = \pm\frac{1}{a\pi} , \tag{3.55}$$

which matches what is known from the *exact bootstrap conditions* [151]

$$\omega(\pm\pi\Gamma, 0) = 0 , \qquad \mathrm{Res}_{\nu=\pm\pi\Gamma}\left(\tilde{\Phi}(\nu, 0)\right) = \pm\frac{1}{a\pi} , \tag{3.56}$$

where $\tilde{\Phi}(\nu, n) = \chi^+(\nu, n)\chi^-(\nu, n)$. Now let us turn to the definition of the integration contour. The integral (3.53) diverges if the poles are located on the integration contour and so we deform said contour to avoid the poles on the real axis. Given the form of the Fourier-Mellin transform (3.42) we need to close the contour on the lower (upper) half-plane in $\nu$ for $z$ small (large), so it is also clear that the integration contour should run above the pole at $\pi\Gamma$ and below the pole at $-\pi\Gamma$. This contour is depicted in Figure 3.9.

For seven external particles, (3.41) reduces to

$$\mathcal{R}_{+++}e^{i\delta_7} = 1 + a\, i\pi\, \mathcal{F}_2\left[e^{-L_1\omega_1}e^{-L_2\omega_2}\chi_1^+ C_{1,2}^+ \chi_2^-\right] , \tag{3.57}$$

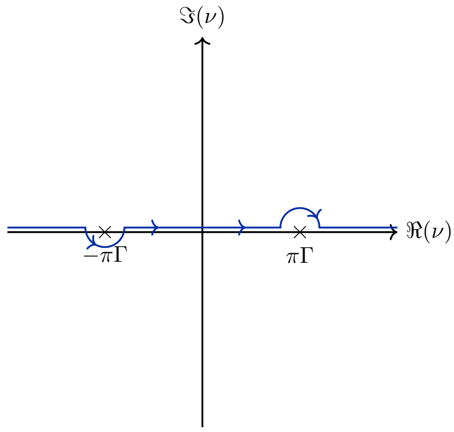



**Figure 3.9:** The integration contour for six-particle scattering in MRK. In this and all following images only poles on the real axis are plotted.

with soft limits given by

$$\lim_{z_1 \to 0} \mathcal{R}_7 e^{i\delta_7} = |z_1|^{2\pi i\Gamma} \left(\mathcal{R}_6 e^{i\delta_6}\right)(z_2)\,, \tag{3.58}$$

$$\lim_{z_2 \to 0, z_1 z_2 \text{ fixed}} \mathcal{R}_7 e^{i\delta_7} = \left(\mathcal{R}_6 e^{i\delta_6}\right)(-z_1 z_2)\,, \tag{3.59}$$

$$\lim_{z_2 \to \infty} \mathcal{R}_7 e^{i\delta_7} = |z_2|^{-2\pi i\Gamma} \left(\mathcal{R}_6 e^{i\delta_6}\right)(z_1)\,, \tag{3.60}$$

as can be inferred from (3.52). From the behavior in the soft limits and taking into account the six-point case considered above, we can determine that the r.h.s. of (3.57) has a pole $\nu_1 = \pi\Gamma - i0$ for the $n_1 = 0$ term in the sum, a pole at $\nu_2 = -\pi\Gamma + i0$ for $n_2 = 0$, as well as at $\nu_1 = \nu_2 + i0$ (or equivalently at $\nu_2 = \nu_1 - i0$) for $n_1 = n_2$. This fixes the integration contours to be as given in Figure 3.10. Furthermore, the residues are given by

$$\operatorname*{Res}_{\nu_1 = \pi\Gamma} \left(\chi^+(\nu_1, 0) C^+(\nu_1, 0, \nu_2, n_2) \chi^-(\nu_2, n_2)\right) = i\chi_2^+ \chi_2^-\,, \tag{3.61}$$

$$\operatorname*{Res}_{\nu_2 = -\pi\Gamma} \left(\chi^+(\nu_1, n_1) C^+(\nu_1, n_1, \nu_2, 0) \chi^-(\nu_2, 0)\right) = -i\chi_1^+ \chi_1^-\,, \tag{3.62}$$

$$\operatorname*{Res}_{\nu_1 = \nu_2} \left(\chi^+(\nu_1, n_2) C^+(\nu_1, n_2, \nu_2, n_2) \chi^-(\nu_2, n_2)\right) = -i(-1)^{n_2} e^{i\pi\omega_2} \chi_2^+ \chi_2^-\,, \tag{3.63}$$

where (3.56) was taken into account. We now define

$$\tilde{C}^h(\nu_1, n_1, \nu_2, n_2) = \frac{C^h(\nu_1, n_1, \nu_2, n_2)}{\chi^-(\nu_1, n_1)\chi^+(\nu_2, n_2)}, \tag{3.64}$$

and

$$I^h(\nu, n) \equiv \frac{\chi^h(\nu, n)}{\chi^+(\nu, n)} = \begin{cases} 1, & h = + \\ H(\nu, n), & h = - \end{cases}, \tag{3.65}$$

where

$$H(\nu, n) \equiv \frac{\chi^-(\nu, n)}{\chi^+(\nu, n)}\,, \tag{3.66}$$

is the *helicity flip kernel* in Fourier-Mellin space [127]. Now note that $\tilde{C}^h$ must be regular at $n_1 = 0, \nu_1 = \pi\Gamma$ and $n_2 = 0, \nu_2 = -\pi\Gamma$ for the soft limits to hold (e.g. a pole would lead to additional $\log z_i$ dependence that is incompatible with these limits). From this, we can conclude that

$$\tilde{C}^+(\pi\Gamma, 0, \nu_2, n_2) = i\pi a\, I^+(\nu_2, n_2)\,, \tag{3.67}$$

$$\tilde{C}^+(\nu_1, n_1, -\pi\Gamma, 0) = -i\pi a\, \bar{I}^+(\nu_1, n_1)\,, \tag{3.68}$$

$$\operatorname*{Res}_{\nu_1=\nu_2} \tilde{C}^+(\nu_1, n_2, \nu_2, n_2) = \frac{-i(-1)^{n_2} e^{i\pi\omega(\nu_2, n_2)}}{\tilde{\Phi}(\nu_2, n_2)}\,, \tag{3.69}$$

with analogous relations for opposite helicities following from (3.40) which implies

$$\tilde{C}^-(\nu_1, n_1, \nu_2, n_2) = \overline{H}(\nu_1, n_1) C^+(\nu_1, n_1, \nu_2, n_2) H(\nu_2, n_2)\,. \tag{3.70}$$

Note furthermore, that the regularity of the entire integrand at $n_1 = 0, \nu_1 = -\pi\Gamma$ and $n_2 = 0, \nu_1 = \pi\Gamma$ implies

$$C^h(-\pi\Gamma, 0, \nu_2, n_2) = C^h(\nu_1, n_1, \pi\Gamma, 0) = 0\,. \tag{3.71}$$

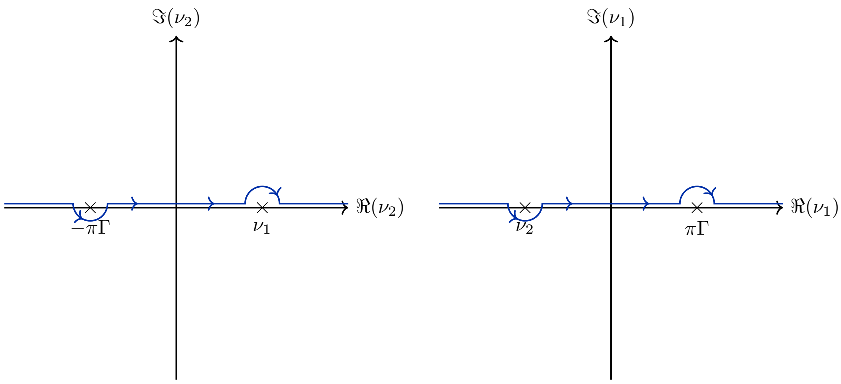



**Figure 3.10:** The integration contours for seven-particle scattering in MRK.

Having now studied the six- and seven-particle cases, we turn to a generic number of particles. In this case the soft limits are given by

$$\lim_{z_1 \to 0} \mathcal{R}_N e^{i\delta_N} = |z_1|^{2\pi i\Gamma} \left(\mathcal{R}_{N-1} e^{i\delta_{N-1}}\right)(z_2, \dots)\,, \tag{3.72}$$

$$\lim_{z_i \to 0, z_{i-1} z_i \text{ fixed}} \mathcal{R}_N e^{i\delta_N} = \left(\mathcal{R}_{N-1} e^{i\delta_{N-1}}\right)(\dots, z_{i-2}, -z_{i-1} z_i, z_{i+1}, \dots)\,, \tag{3.73}$$

$$\lim_{z_{N-5} \to \infty} \mathcal{R}_N e^{i\delta_N} = |z_{N-5}|^{-2\pi i\Gamma} \left(\mathcal{R}_{N-1} e^{i\delta_{N-1}}\right)(\dots, z_{N-6})\,. \tag{3.74}$$

We can infer that the integrand of (3.41) has poles at $\nu_1 = \pi\Gamma - i0$ for $n_1 = 0$, at $\nu_i = \nu_{i-1} + i0$ for $n_{i-1} = n_i$, and at $\nu_{N-5} = -\pi\Gamma + i0$ for $n_{N-5} = 0$ and for $2 \leq i \leq N-5$. The resulting integration contours are given in Figure 3.11.

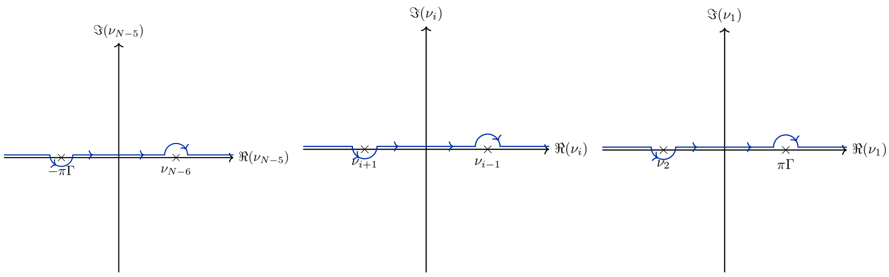


**Figure 3.11:** The integration contours for $N$-particle scattering in MRK.

### 3.2.4. Perturbative Expansion

In this section we will discuss the perturbative expansion of (3.41). Before performing this expansion however, we must first comment on the results of the previous section. Recall that for six-particle scattering as described in Figure 3.9, for $n = 0$ the integration contour in the $\nu$-plane passes in between two poles at $\nu = \pm\pi\Gamma$. Since $\Gamma = \mathcal{O}(a)$, in the weak coupling limit the contour will become pinched between the two poles on the real line and the integral will diverge. Note however, that this pinching need not be a problem for all terms in the integrand. As an example, let's once again consider the six-point MRK ratio given in (3.53). The $\ell$-loop MHV leading logarithmic contributions will be of the form

$$\sum_{n=0}^{\infty} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} |z|^{2i\nu} \chi_0^+ E^{\ell-1} \chi_0^- \,. \tag{3.75}$$

Since pinching occurs at $n = 0$ we are concerned with

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} |z|^{2i\nu} \chi_0^+(\nu, 0) E^{\ell-1}(\nu, 0) \chi_0^-(\nu, 0) \,. \tag{3.76}$$

To make the divergence stemming from the pinching manifest, we will define the integral (3.76) with a contour running along the real axis in the $\nu$-plane and introduce an $\epsilon$-perscription to shift the poles from the real axis as follows

$$\chi_0^+(\nu, 0) = \frac{1}{i\nu - \epsilon} \,, \qquad \chi_0^-(\nu, 0) = \frac{1}{i\nu + \epsilon} \,. \tag{3.77}$$

In this perscription, it is clear that for $\ell = 1$,

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} |z|^{2i\nu} \chi_0^+(\nu, 0) \chi_0^-(\nu, 0) \sim \frac{i}{2\epsilon} \,, \tag{3.78}$$

and thus the divergence in the limit $\epsilon \to 0$ is made manifest. Note, however, that when $\ell > 1$ the divergence is regulated by the presence of the LO BFKL eigenvalue, since we see from the explicit form in Appendix A that

$$E(i\epsilon, 0) = \mathcal{O}(\epsilon^2)\,. \tag{3.79}$$

So at leading logarithmic accuracy, whenever the integrand contains an insertion of a LO BFKL eigenvalue, there is no pinch singularity and no contour deformation is needed. This condition is satisfied by all leading logarithmic terms at two loops or higher. Beyond leading logarithmic accuracy, additional treatment is needed to have a consistent weak coupling limit.

To have a consistent weak-coupling Fourier-Mellin description of the ratio for terms in which pinching may arise, we will now illustrate how to subtract residues from (3.41) so the contour is deformed in such a way as to avoid pinching. To facilitate this subtraction, one may rewrite the combination of impact factors and central emission block as

$$\chi_1^{h_1}\left[\prod_{k=2}^{N-5} C_{k-1,k}^{h_k}\right]\chi_{N-5}^{-h_{N-4}} = I_1^{h_1}\tilde{\Phi}_1\left[\prod_{k=2}^{N-5}\tilde{C}_{k-1,k}^{h_k}\tilde{\Phi}_k\right]\bar{I}_{N-5}^{h_{N-4}}\,, \tag{3.80}$$

where $\bar{I}_i^h$ is the conjugate of $I_i^h$ defined in (3.65). The poles at $\nu_i = \pm\pi\Gamma$ for $2 \le i \le N-6$ and the poles at $\nu_1 = -\pi\Gamma$ and at $\nu_{N-5} = \pi\Gamma$ that are seemingly introduced by the $\tilde{\Phi}$ in between the pairs of $C$'s are spurious, as the central emission block vanishes for these values via (3.71). The poles on the real line are given by the ordered list

$$\nu_1 = \pi\Gamma\,, \qquad \nu_1 = \nu_2\,, \qquad \ldots \qquad \nu_{N-6} = \nu_{N-5}\,, \qquad \nu_{N-5} = -\pi\Gamma\,, \tag{3.81}$$

The behaviour of the building blocks in our dispersion integral at these poles has been discussed in the previous section in (3.56), (3.67)-(3.71). One can now perform the necessary contour deformations by adding/subtracting residues of alternating poles in (3.81) starting with the pole at $\nu_1 = \pi\Gamma$, thus pushing the contour above/below said poles. We will refer to this choice of contour as the *unpinched contour*. From the form of the residues, one can see that the terms which arise from this deformation will typically correspond to MRK ratios for lower numbers of particles. For example, for seven external particles we get [2]

$$\begin{aligned}\mathcal{R}_{h_1h_2h_3}e^{i\delta_7} = &|z_1|^{2\pi i\Gamma}\mathcal{R}_{h_2h_3}(z_2)e^{i\delta_6(z_2)} + |z_2|^{-2\pi i\Gamma}\mathcal{R}_{h_1h_2}(z_1)e^{i\delta_6(z_1)}\\ &-\frac{|z_1|^{2\pi i\Gamma}}{|z_2|^{2\pi i\Gamma}} + 2\pi i f_{h_1h_2h_3}\,,\end{aligned} \tag{3.82}$$

where

$$f_{h_1 h_2 h_3} = \frac{a}{2}\left[\prod_{k=1}^{2}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{\frac{n_k}{2}}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}\right] \\ \times\left[\prod_{k=1}^{2}e^{-L_k\omega_k}\right] I_1^{h_1}\tilde{\Phi}_1\tilde{C}_{1,2}^{h_2}\tilde{\Phi}_2\,\bar{I}_2^{h_3}, \tag{3.83}$$

is integrated over the unpinched contours shown in Figure 3.12. The case of eight external particles can be found in [3], a generic algorithm for any number of particles is presented in [6].

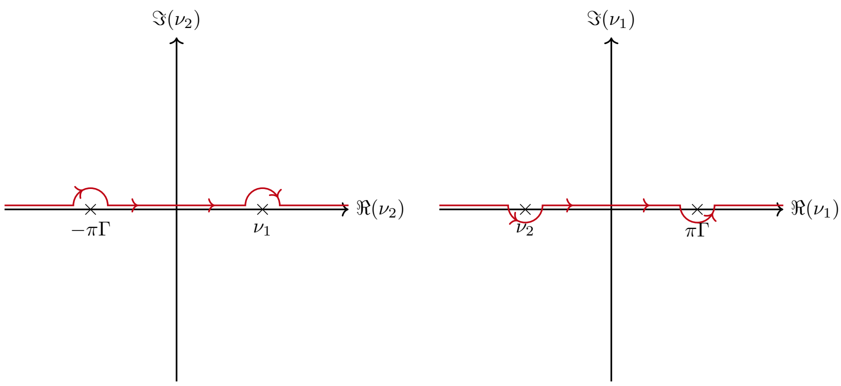



**Figure 3.12:** The unpinched integration contours for seven-particle scattering.

Now that we have found a consistent way to expand the MRK ratio given in (3.41) perturbatively, we will turn our attention to the *perturbative coefficients* $\tilde{g}_{h_1,\ldots,h_{N-4}}^{(\ell;i_1,\ldots,i_{N-5})}$ and $\tilde{h}_{h_1,\ldots,h_{N-4}}^{(\ell;i_1,\ldots,i_{N-5})}$, defined by expanding the MRK ratio in the coupling and large logarithms

$$\mathcal{R}_{h_1,\ldots,h_{N-4}}(\{z_i,\tau_i\}) = 1 + 2\pi i\sum_{\ell=1}^{\infty}\sum_{i_1,\ldots,i_{N-5}=0}^{\ell-1}a^\ell\left(\prod_{k=1}^{N-5}\frac{1}{i_k!}\log^{i_k}\tau_k\right) \\ \times\left(\tilde{g}_{h_1,\ldots,h_{N-4}}^{(\ell;i_1,\ldots,i_{N-5})}(\{z_i\}) + 2\pi i\,\tilde{h}_{h_1,\ldots,h_{N-4}}^{(\ell;i_1,\ldots,i_{N-5})}(\{z_i\})\right). \tag{3.84}$$

## 3.3. Amplitudes through Fourier-Mellin Convolutions

### 3.3.1. Perturbative Coefficients at Leading Logarithmic Accuracy

In this section we will study the perturbative coefficients defined in (3.84) at leading logarithmic accuracy $\sum_{k=1}^{N-5} i_k = \ell - 1$. More specifically, we will discuss a method developed in [1] through which these coefficients can be computed efficiently. We will see that the study of the geometry undertaken in Section 3.1 plays a crucial role in this method, and that the understanding of this mechanism in its turn leads to a further understanding of the structure of amplitudes in MRK at LLA.

First, note that the LLA remainder function is purely imaginary and hence

$$\sum_{k=1}^{N-5} i_k = \ell - 1 \quad \to \quad \tilde{h}^{(\ell;i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}} = 0\,. \tag{3.85}$$

We need only concern ourselves with the imaginary contributions. In what follows, we introduce a notation for the product of leading order impact factors and central emission blocks, which we call the *vacuum ladder*

$$\varpi_N \equiv \varpi_N^{h_1\ldots h_{N-4}} = \chi^{h_1}_{0,1}\left(\prod_{j=1}^{N-6} C^{h_{j+1}}_{0,j,j+1}\right)\chi^{-h_{N-4}}_{0,N-5}\,, \tag{3.86}$$

and where we will often drop explicit dependence on the helicities. Comparing the expansion (3.84) to the Fourier-Mellin integral which describes the MRK ratio (3.41), we see that they are given by

$$\tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}} = -\frac{1}{2}\mathcal{F}_{N-5}\left[\left(\prod_{k=1}^{N-5} E_k^{i_k}\right)\varpi_N\right]. \tag{3.87}$$

In the following sections we will discuss how to obtain analytic expressions for the coefficients given in (3.87).

#### 3.3.1.1. MHV Amplitudes in MRK

To study the computation of perturbative coefficients at LLA in MRK, we will once again turn to the test case of six-particle scattering for the MHV configuration. In this case, the $\ell$-loop perturbative coefficients are given by

$$\tilde{g}^{(\ell;\ell-1)}_{++} = -\frac{1}{2}\mathcal{F}\left[E^{\ell-1}\varpi_6^{++}\right]. \tag{3.88}$$

Immediately we see that the coefficients at different loop orders only differ by additional insertions of a LO BFKL eigenvalue into the integrand. Now note that the Fourier-Mellin transformation maps products into convolutions, so that for $\mathcal{F}[F] = f$ and $\mathcal{F}[G] = g$ we have

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g\,, \tag{3.89}$$

where the convolution is given by

$$(f * g)(z) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 w}{|w|^2}\, f(w)\; g\left(\frac{z}{w}\right) . \tag{3.90}$$

This implies that

$$\tilde{g}_{++}^{(\ell;\ell-1)} = \tilde{g}_{++}^{(\ell-1;\ell-2)} * \mathcal{F}[E] . \tag{3.91}$$

We see that increasing the number of loops is equivalent to convoluting the lower loop result with the Fourier-Mellin transform of the BFKL eigenvalue. Using this method, we could compute the perturbative coefficients recursively.

Let's investigate (3.91) in more detail for a low-loop example. In order to do this, we need to know $\tilde{g}_{++}^{(\ell;\ell-1)}$ analytically for some value of $\ell$. This can easily be achieved by performing explicitly the Fourier-Mellin transform for $\ell = 1$ or $\ell = 2$, cf., e.g., [134],

$$\begin{aligned} \mathcal{F}\left[\varpi_6^{++}\right] &= \mathcal{G}_1(z) - \frac{1}{2}\,\mathcal{G}_0(z)\,, \\ \mathcal{F}\left[E_{\nu n}\,\varpi_6^{++}\right] &= \frac{1}{2}\mathcal{G}_{0,1}(z) + \frac{1}{2}\mathcal{G}_{1,0}(z) - \mathcal{G}_{1,1}(z)\,, \end{aligned} \tag{3.92}$$

We also need the Fourier-Mellin transform of the LO BFKL eigenvalue, which can easily be obtained by noting that the functions $\chi_0^{\pm}(\nu, n)$ defined in Appendix A are related to derivatives in $z$-space,

$$z\,\partial_z \mathcal{F}\left[\chi_0^{+}(\nu,n)\,F(\nu,n)\right] = \mathcal{F}\left[F(\nu,n)\right] . \tag{3.93}$$

A similar relation holds when replacing $z$ by $\bar{z}$ and $\chi_0^+$ by $\chi_0^-$. From (3.92), we see the Fourier-Mellin transform of the LO BFKL eigenvalue is then given by

$$\mathcal{E}(z) \equiv \mathcal{F}\left[E_{\nu n}\right] = z\,\bar{z}\,\partial_z\,\bar{\partial}_z \mathcal{F}\left[E_{\nu n}\,\varpi_6^{++}\right] = -\frac{z+\bar{z}}{2\,|1-z|^2} . \tag{3.94}$$

The only remaining piece of the puzzle is how to evaluate the convolution integral given in (3.91). Note that our starting point is written in terms of single-valued multiple polylogarithms. In [146] it was shown that convolution integrals of this type can be computed using residues. To see how this works, consider a function $f(z)$ that consists of single-valued hyperlogarithms and rational functions with singularities at $z = a_i$ and $z = \infty$. Close to any of these singularities, $f$ can be expanded into a series of the form

$$\begin{aligned} f(z) &= \sum_{k,m,n} c_{k,m,n}^{a_i} \log^k \left|1 - \frac{z}{a_i}\right|^2 (z-a_i)^m\,(\bar{z}-\bar{a}_i)^n\,, \quad z \to a_i\,, \\ f(z) &= \sum_{k,m,n} c_{k,m,n}^{\infty} \log^k \frac{1}{|z|^2}\,\frac{1}{z^m}\,\frac{1}{\bar{z}^n}\,, \quad z \to \infty\,. \end{aligned} \tag{3.95}$$

The *holomorphic residue* of $f$ at the point $z = a$ is then defined as the coefficient of the simple holomorphic pole without logarithmic singularities,

$$\text{Res}_{z=a} f(z) \equiv c^a_{0,-1,0} \,. \tag{3.96}$$

The integral of $f$ over the whole complex plane, if it exists, can be computed in terms of its holomorphic residues [146]. More precisely, if $F$ is a single-valued anti-holomorphic primitive of $f$, $\bar{\partial}_z F = f$, then

$$\int \frac{d^2 z}{\pi}\, f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z) \,. \tag{3.97}$$

This result is essentially an application of Stokes' theorem to the punctured complex plane. Note that the anti-holomorphic primitive is only defined up to an arbitrary holomorphic function. It was shown in [145] that every single-valued polylogarithm has a single-valued primitive, and the sum of residues is independent on the choice of the primitive [146]. It is clear that we can repeat the previous argument by reversing the roles of holomorphic and anti-holomorphic functions.

Now we apply this method to the results given in (3.92). Our starting point is

$$\begin{aligned} \mathcal{F}[E_{\nu n}\, \varpi_6^{++}] &= \mathcal{F}\left[\varpi_6^{++}\right] * \mathcal{E}(z) \\ &= \int \frac{d^2 w}{\pi} \underbrace{\left[\frac{1}{2}\, \mathcal{G}_0(w) - \mathcal{G}_1(w)\right] \frac{\bar{w} z + w \bar{z}}{2\, |w|^2\, |w-z|^2}}_{=f(w)} \,. \end{aligned} \tag{3.98}$$

First, we need to compute the anti-holomorphic primitive

$$F(w) = \int d\bar{w}\, f(w) \,. \tag{3.99}$$

Since[4]

$$\mathcal{G}_0(w) = \mathcal{G}_0(\bar{w}) \quad \text{and} \quad \mathcal{G}_1(w) = \mathcal{G}_1(\bar{w}) \,, \tag{3.100}$$

and single-valued hyperlogarithms satisfy the same (holomorphic) differential equations as their non-single-valued analogues, we obtain

$$\begin{aligned} F(w) = {} & \frac{1}{2w\,(w-z)} \int d\bar{w} \left[\frac{1}{2}\, \mathcal{G}_0(\bar{w}) - \mathcal{G}_1(\bar{w})\right] \frac{\bar{w} z + w \bar{z}}{\bar{w}\,(\bar{w} - \bar{z})} \\ = {} & \frac{1}{4(w-z)} \left[2\mathcal{G}_{0,z}(w) - 4\mathcal{G}_{1,z}(w) - \mathcal{G}_{0,0}(w) + 2\mathcal{G}_{1,0}(w) \right. \\ & \left. -4\mathcal{G}_1(w)\mathcal{G}_0(z) + 4\mathcal{G}_1(w)\mathcal{G}_1(z) + 2\mathcal{G}_0(z)\mathcal{G}_z(w) - 4\mathcal{G}_1(z)\mathcal{G}_z(w)\right] \\ & + \frac{1}{4w} \left[-\mathcal{G}_{0,z}(w) + 2\mathcal{G}_{1,z}(w) + 2\mathcal{G}_1(w)\mathcal{G}_0(z) - 2\mathcal{G}_1(w)\mathcal{G}_1(z) \right. \\ & \left. -\mathcal{G}_0(z)\mathcal{G}_z(w) + 2\mathcal{G}_1(z)\mathcal{G}_z(w)\right] \,. \end{aligned} \tag{3.101}$$

[4]Note that for higher weights the relation between $\mathcal{G}_{\vec{a}}(w)$ and $\mathcal{G}_{\vec{a}}(\bar{w})$ will not be so straighforward.

We see that $F(w)$ has potential poles at $w = 0$, $w = z$ and $w = \infty$. The residue at $w = 0$ vanishes. The residue at $w = z$ is given by

$$\begin{aligned} \text{Res}_{w=z} F(w) &= -\frac{1}{4}\mathcal{G}_{0,0}(z) - \mathcal{G}_{0,1}(z) - \frac{1}{2}\mathcal{G}_{1,0}(z) + 2\mathcal{G}_{1,1}(z) - \mathcal{G}_{1,z}(z) \\ &= -\frac{1}{4}\mathcal{G}_{0,0}(z) - \frac{1}{2}\mathcal{G}_{1,0}(z) + \mathcal{G}_{1,1}(z)\,, \end{aligned} \tag{3.102}$$

where the last step follows from the identity

$$\mathcal{G}_{1,z}(z) = \mathcal{G}_{1,1}(z) - \mathcal{G}_{0,1}(z)\,. \tag{3.103}$$

Finally, the residue at infinity is obtained by letting $w = 1/u$ (and including the corresponding Jacobian) and expanding the result around $u = 0$. We find

$$\text{Res}_{w=\infty} F(w) = \frac{1}{2}\mathcal{G}_{0,1}(z) - \frac{1}{4}\mathcal{G}_{0,0}(z)\,. \tag{3.104}$$

Hence,

$$\begin{aligned} \mathcal{F}\left[E_{\nu n}\,\varpi_6^{++}\right] &= \text{Res}_{w=\infty} F(w) - \text{Res}_{w=z} F(w) \\ &= \frac{1}{2}\mathcal{G}_{0,1}(z) + \frac{1}{2}\mathcal{G}_{1,0}(z) - \mathcal{G}_{1,1}(z)\,, \end{aligned} \tag{3.105}$$

which is indeed the correct result, as we see from (3.92). Using this method recursively, we can build from the result given above to compute the three-loop coefficient and so on. Thus, we have a method to compute MHV coefficients at LLA at arbitrarily high loop orders, bypassing the evaluation of the Fourier-Mellin integrals entirely.

This six-point example can straightforwardly be extended to an arbitrary number of particles through

$$\tilde{g}_{+\dots+}^{(\ell;i_1,\dots,i_k+1,\dots,i_{N-5})} = \mathcal{E}(z_k) * \tilde{g}_{+\dots+}^{(\ell;i_1,\dots,i_{N-5})}\,. \tag{3.106}$$

The starting point of this recursion is the two-loop MHV remainder function in MRK, which is known at LLA for an arbitrary number $N$ of external legs [148, 152]

$$g_{+\dots+}^{(2;0,\dots,0,1,0,\dots,0)}(\rho_1,\dots,\rho_{N-5}) = \frac{1}{4}\,\log|1-\rho_k|^2\,\log\left|1-\frac{1}{\rho_k}\right|^2\,. \tag{3.107}$$

While for many external particles a direct evaluation of the Fourier-Mellin transform in terms of multiple sums of residues becomes prohibitive because the number of sums increases with the number of external legs, the recursion (3.106) requires the evaluation of a single convolution integral at every loop order, independently of the number of external legs. This is one of the key properties why the convolution integral combined with Stokes' theorem gives rise to an efficient algorithm to compute scattering amplitudes in MRK.

#### 3.3.1.2. Beyond MHV

So far we have only considered MHV amplitudes. In this section we show how the convolution method can be applied to also obtain results beyond MHV. Let us start by analysing what happens if we start from an MHV amplitude and we flip the helicity on an impact factor. In Fourier-Mellin space, this amounts to replacing $\chi_0^+(\nu,n)$ by $\chi_0^-(\nu,n)$,

$$\begin{aligned}\mathcal{F}[\chi_0^+(\nu,n)\,F(\nu,n)] &\longrightarrow \mathcal{F}\left[\chi_0^-(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\frac{\chi_0^-(\nu,n)}{\chi_0^+(\nu,n)}\right] * \mathcal{F}\left[\chi_0^+(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right] * \mathcal{F}\left[\chi_0^+(\nu,n)\,F(\nu,n)\right]\,.\end{aligned} \tag{3.108}$$

We see that flipping the helicity on a LO impact factor amounts to convoluting with the evaluated leading order helicity-flip kernel (3.66)

$$\mathcal{H}_0(z) = \mathcal{F}\left[H_0\right] = \mathcal{F}\left[\frac{\chi_0^-(\nu,n)}{\chi_0^+(\nu,n)}\right] = \mathcal{F}\left[\frac{i\nu+\frac{n}{2}}{i\nu-\frac{n}{2}}\right]\,. \tag{3.109}$$

The functional form of $\mathcal{H}_0(z)$ can easily be obtained by performing explicitly the Fourier-Mellin transform. The integrand has only a simple pole at $i\nu = n/2$, and so we find

$$\mathcal{H}_0(z) = \mathcal{H}_0(1/z) = -\frac{z}{(1-z)^2}\,. \tag{3.110}$$

Note that flipping helicities represents an involution - flipping a negative helicity to a postive one and then flipping that helicity back should give the same result - and so

$$\mathcal{H}_0(z) * \mathcal{H}_0(\bar{z}) = \mathcal{F}[1] = \pi\,\delta^{(2)}(1-z)\,. \tag{3.111}$$

If we flip the helicity on one of the leading order central emission blocks, using (3.70)

$$\frac{C_0^-(\nu_1,n_1,\nu_2,n_2)}{C_0^+(\nu_1,n_1,\nu_2,n_2)} = \frac{\chi_0^+(\nu_1,n_1)\,\chi_0^-(\nu_2,m_2)}{\chi_0^-(\nu_1,n_1)\,\chi_0^+(\nu_2,n_2)}\,, \tag{3.112}$$

we obtain

$$\begin{aligned}&\mathcal{F}\Big[C_0^+(\nu_1,n_1,\nu_2,n_2)F(\nu_1,n_1,\nu_2,n_2)\Big]\\ &\quad\longrightarrow \mathcal{F}\left[C_0^-(\nu_1,n_1,\nu_2,n_2)F(\nu_1,n_1,\nu_2,n_2)\right]\\ &\quad= \mathcal{F}\left[\frac{C_0^-(\nu_1,n_1,\nu_2,n_2)}{C_0^+(\nu_1,n_1,\nu_2,n_2)}\right] * \mathcal{F}\left[C_0^+(\nu_1,n_1,\nu_2,n_2)F(\nu_1,n_1,\nu_2,n_2)\right]\\ &\quad= \mathcal{H}_0(\bar{z}_1) * \mathcal{H}_0(z_2) * \mathcal{F}\left[C_0^+(\nu_1,n_1,\nu_2,n_2)F(\nu_1,n_1,\nu_2,n_2)\right]\,.\end{aligned} \tag{3.113}$$

We see that the flipping of the helicity on a leading order central emission block is controlled by the same kernels as for the impact factor.

Thus, by convoluting with the helicity flip kernel we can obtain any helicity configuration from the $\ell$-loop results obtained recursively in the previous section. The combination of convolutions with $\mathcal{E}$ and $\mathcal{H}_0$ give us an algorithm to obtain any perturbative coefficient at LLA for an arbitrary number of particles from the known two-loop result. The computation of the MRK ratio at a certain loop order at LLA, however, can be made even more efficient. In the next section we will discuss a result which allows us to relate perturbative coefficients for different numbers of external particles.

### 3.3.2. The Factorisation of Perturbative Coefficients at LLA

In [1], beyond the convolution formalism we have discussed above, a factorisation of perturbative coefficients was presented which greatly simplifies the algorithm to compute all perturbative coefficients at LLA for a certain number of loops and legs. We will present the factorisation and its consequences here and review its proof in Appendix B.

In order to state the factorisation theorem, it is useful to introduce the graphical representation for the perturbative coefficients in Figure 3.13. We work with

$$g^{(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} =$$

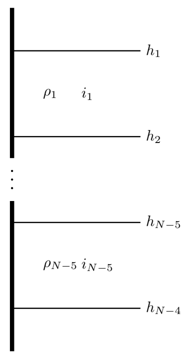



**Figure 3.13:** The graphical representation of LLA perturbative coefficients.

the simplicial MRK coordinates $\rho_k$ defined in Section 3.1.2. Every outgoing line is labeled by its helicity $h_k$. In addition, to every face we do not only associate its coordinate $\rho_k$ but also the index $i_k$.

Using this graphical representation of the perturbative coefficients the factorisation theorem takes the simple form given in Figure 3.14. In other words, whenever the graph representing a perturbative coefficient contains a face with index

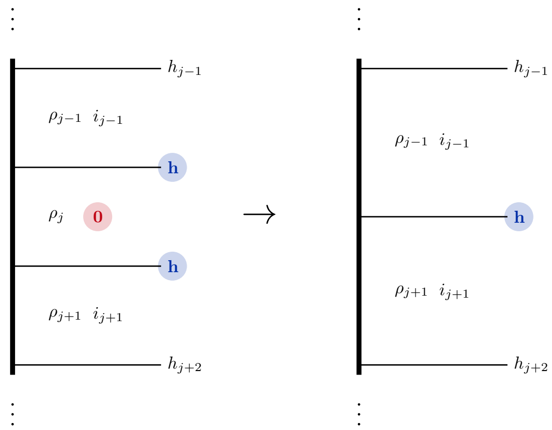



**Figure 3.14:** The factorisation of LLA perturbative coefficients. Faces with vanishing indices and matching neighbouring helicities can be deleted.

$i_b = 0$ and the lines adjacent to this face have the same helicity, then this perturbative coefficient is equal to the coefficient where this face has been deleted. We stress that the factorisation theorem holds for arbitrary helicity configurations and is not restricted to MHV amplitudes. In the MHV case, the factorisation theorem implies that we can drop all the faces labeled by a zero,

$$g_{+\ldots+}^{(\ell;0,\ldots,0,i_{a_1},0,\ldots,0,i_{a_2},0,\ldots,0,i_{a_k},0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{+\ldots+}^{(\ell;i_{a_1},i_{a_2},\ldots,i_{a_k})}(\rho_{i_{a_1}},\rho_{i_{a_2}},\ldots,\rho_{i_{a_k}})\,. \tag{3.114}$$

Since the sum of all indices is related to the loop number, we see that for a fixed number of loops there is a maximal number of non-zero indices, and so there is only a finite number of different perturbative coefficients at every loop order. This generalises the factorisation observed for the two-loop MHV amplitude in MRK to LLA [148, 152, 153]. Indeed, if all indices are zero except for one, say $i_a$, then (3.114) reduces to

$$g_{+\ldots+}^{(\ell;0,\ldots,0,i_a,0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{++}^{(\ell;i_a)}(\rho_a)\,, \tag{3.115}$$

and so at two loops the amplitude completely factorises, in agreement with [148, 152, 153],

$$\mathcal{R}^{(2)}_{+\ldots+}\bigg|_{\text{LLA}} = \sum_{1\le i\le N-5} \log\tau_i\, g_{++}^{(2;1)}(\rho_i)\,. \tag{3.116}$$

As anticipated in [148], the amplitude no longer factorises completely beyond two loops. However, we find that at every loop order only a finite number of different functions appear. For example, at three-loop order at most two indices are non-zero, and so we have

$$\mathcal{R}^{(3)}_{+\ldots+}\Big|_{\rm LLA} = \frac{1}{2}\sum_{1\leq i\leq N-5} \log^2\tau_i\, g^{(3;2)}_{++}(\rho_i) + \sum_{1\leq i<j\leq N-5} \log\tau_i\,\log\tau_j\, g^{(3;1,1)}_{+++}(\rho_i,\rho_j)\,. \tag{3.117}$$

The only new function appearing at three loops that is not determined by the six-point amplitude is $g^{(1,1)}_{++}$, which is determined by the three-loop seven-point MHV amplitude. At four loops we have

$$\begin{aligned}\mathcal{R}^{(4)}_{+\ldots+}\Big|_{\rm LLA} &= \frac{1}{6}\sum_{1\leq i\leq N-5} \log^3\tau_i\, g^{(4;3)}_{++}(\rho_i) \\ &+ \frac{1}{2}\sum_{1\leq i<j\leq N-5}\Big[\log^2\tau_i\,\log\tau_j\, g^{(4;2,1)}_{+++}(\rho_i,\rho_j) \\ &\qquad + \log\tau_i\,\log^2\tau_j\, g^{(4;1,2)}_{+++}(\rho_i,\rho_j)\Big] \\ &+ \sum_{1\leq i<j<k\leq N-5} \log\tau_i\,\log\tau_j\,\log\tau_k\, g^{(4;1,1,1)}_{++++}(\rho_i,\rho_j,\rho_k)\,.\end{aligned} \tag{3.118}$$

The four-loop answer is determined for any number of external legs by the six, seven and eight-point amplitudes through four loops. Similar equations can be obtained for higher-loop amplitudes. In general, at $L$ loops the LLA MHV MRK ratio is determined for any number of legs by the MHV amplitudes involving up to $(L+4)$ external legs.

Beyond MHV, the factorisation is less straightforward. Indeed, only vanishing indices where the neighbouring helicities are equal can be deleted. Consider the seven-point two-loop NMHV amplitude $\mathcal{R}^{(2)}_{-++}$. We can write

$$\begin{aligned}\mathcal{R}^{(2)}_{-++}\Big|_{\rm LLA} &= \mathcal{H}_0(z_1) * \mathcal{R}^{(2)}_{+++}\Big|_{\rm LLA} \\ &= \log\tau_1\,\mathcal{H}(z_1) * g^{(2;1)}_{++}(\rho_1) + \log\tau_2\,\mathcal{H}(z_1) * g^{(2;1)}_{++}(\rho_2)\,.\end{aligned} \tag{3.119}$$

It is easy to check that the first term behaves as expected,

$$\mathcal{H}_0(z_1) * \mathfrak{g}^{(2;1)}_{++}(\rho_1) = \mathfrak{g}^{(2;1)}_{-+}(\rho_1)\,. \tag{3.120}$$

The second term, however, also depends on $\rho_2$, and we would expect that this term should be a function of both $\rho_1$ and $\rho_2$. By explicit computation, one establishes that this is indeed the case, and so we obtain a new non-MHV building block with a vanishing index,

$$\mathcal{R}^{(2)}_{-++}\Big|_{\rm LLA} = \log\tau_1\,\mathfrak{g}^{(2;1)}_{-+}(\rho_1) + \log\tau_2\,\mathfrak{g}^{(2;0,1)}_{-++}(\rho_1,\rho_2)\,. \tag{3.121}$$

As a consequence of these terms with vanishing indices, unlike for MHV amplitudes, the number of building blocks is no longer finite at each loop order in the non-MHV case.

As (3.114) is no longer valid for non-MHV amplitudes, the number of different coefficients is no longer bounded. In particular, there should be an infinite tower of different non-MHV coefficients already at two loops, because the factorisation theorem does not allow us to reduce the coefficients corresponding to alternating helicities to simpler functions.

In [1], the methods described above allowed for the computation of the LLA MHV amplitude for any number of particles to five loops, the LLA amplitude for eight or less particles for any helicity configuration up to four loops. The methods described here are not limited to LLA however, and in the following section we will turn to the application of convolution products to NLLA.

### 3.3.3. Beyond Leading Logarithmic Accuracy

So far we have focussed on perturbative coefficients at leading logarithmic accuracy. Let us now consider the next logarithmic order. Recall from the discussion in Section 3.2.4 that when there is no insertion of an LO BFKL eigenvalue into the integrand, a pinching singularity may appear which needs to be regulated. At NLLA, two-loop terms contain no LO BFKL eigenvalue and thus require additional treatment. To this end, we introduce a *regularised Fourier-Mellin transform*. At two loops NLLA we define

$$\begin{aligned}\mathcal{F}_{N-5}\left[\mathcal{I}^{(2)}_{\text{NLLA}}\right] &= \mathcal{F}^{\text{Reg}}_{N-5}\left[\mathcal{I}^{(2)}_{\text{NLLA}}\right] \\ &= \prod_{k=1}^{3}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{\frac{n_k}{2}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu_k}{2\pi}|z_k|^{2i\nu_k}\mathcal{I}^{(2)}_{\text{NLLA}} \\ &\quad + P^{(2;0,\dots,0)}_{h_1\dots h_{N-4}} + 2\pi i\, Q^{(2;0,\dots,0)}_{h_1\dots h_{N-4}}\,,\end{aligned} \tag{3.122}$$

where the contours of integration on the r.h.s. correspond to the unpinched contour, $\mathcal{I}^{(2)}_{\text{NLLA}}$ is the two-loop NLLA contribution to the integrand of (3.41) and where $P^{(2;0,\dots,0)}_{h_1\dots h_{N-4}}$ and $Q^{(2;0,\dots,0)}_{h_1\dots h_{N-4}}$ are the two-loop contributions of the lower-point amplitudes arising from the contour deformation discussed in Section 3.2.4, see e.g. (3.82). Their exact forms for seven and eight particles can be found in [2] and [3] respectively. Now that we have a consistent Fourier-Mellin description at NLLA, let us turn to the form of the perturbative coefficients. For ease of use, we will introduce some additional notation.

At NLLA, (i.e. for $\sum_{k=1}^{N-5} i_k = \ell - 2$), we write

$$\begin{aligned}
\tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} &= \sum_{j=1}^{N=5} i_j \tilde{g}^{j;(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} + \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{\varpi;h_1\ldots h_{N-4}}\,,\\
\tilde{h}^{(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} &= \sum_{j=1}^{N-5} \tilde{h}^{j;(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} + \tilde{h}^{(\ell;i_1,\ldots,i_{N-5})}_{\varpi;h_1\ldots h_{N-4}}\,,
\end{aligned} \tag{3.123}$$

where the perturbative coefficients with extra indices are called *corrected perturbative coefficients*. Perturbative coefficients with an additional upper index correspond to insertions of the NLO corrections to the BFKL eigenvalue and perturbative coefficients with an additional lower index correspond to the insertion of the corrections to the vacuum ladder. This implies

$$\tilde{g}^{j;(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} = \frac{1}{2}\,\mathcal{F}_{N-5}\left[\varpi_N\, E^{(1)}_j \prod_{k=1}^{N-5} E_k^{i_k-\delta_{kj}}\right], \tag{3.124}$$

$$\tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{\varpi;h_1\ldots h_{N-4}} = \frac{1}{2}\,\mathcal{F}_{N-5}\left[\varpi_N\left(\kappa^{h_1}_{1,1} + \sum_{i=1}^{N-6}\Re\left(c^{h_{i+1}}_{1,i,i+1}\right) + \kappa^{-h_{N-4}}_{1,N-5}\right)\prod_{k=1}^{N-5} E_k^{i_k}\right],$$

for the imaginary contributions and

$$\begin{aligned}
\tilde{h}^{j;(\ell;i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}} &= -\frac{1}{4}\,\mathcal{F}_{N-5}\left[\varpi_N\, E_j \prod_{k=1}^{N-5} E_k^{i_k}\right],\\
\tilde{h}^{(\ell;i_1,\ldots,i_{N-5})}_{\varpi;h_1\ldots h_{N-4}} &= \frac{1}{4\pi}\,\mathcal{F}_{N-5}\left[\varpi_N\left(\sum_{i=1}^{N-6}\Im\left(c^{h_{i+1}}_{1,i,i+1}\right)\right)\prod_{k=1}^{N-5} E_k^{i_k}\right],
\end{aligned} \tag{3.125}$$

for the real contributions, where $\Re$ and $\Im$ denote the real and imaginary parts, respectively. As an example of how the convolution method can be applied here, let's examine the case of seven external particles.

Our starting points are the corrected perturbative coefficients at two loops, $g^{(2;0,0)}_{\varpi;+++}$ and $h^{(2;0,0)}_{\varpi;+++}$ which can be obtained from the two-loop MHV ratio at NLLA $\mathcal{R}^{(2)}_N|_{\text{NLLA}}$ known for any number of particles [2, 153]:

$$\tilde{g}^{(2;0,0)}_{+++} = \frac{1}{2\pi i}\left(\mathcal{R}^{(2)}_7|_{\text{NLLA}} - i\delta^{(2)}_7\right) = \tilde{g}^{(2;0,0)}_{\varpi;+++}\,, \tag{3.126}$$

$$\tilde{h}^{(2;0,0)}_{+++} = \frac{(i\delta^{(1)}_7)^2}{2(2\pi i)^2} = \sum_{j=1}^{3}\tilde{h}^{j;(2;0,0)}_{+++} + \tilde{h}^{(2;0,0)}_{\varpi;+++}\,. \tag{3.127}$$

The aim is to lift these objects to three loops by convoluting with the integration kernels $\mathcal{E}_k$. Note however that due to the regularisation procedure in (3.122), we must be careful performing the convolution. The regularised Fourier-Mellin integral at two loops consists of a combination of an integral transform with a modified contour and lower point terms that originate from the contour deformation. As only the former naturally obeys the convolution structure, we need to subtract these lower point contributions from the perturbative coefficients before inserting additional building blocks via convolutions. Since performing this convolution preserves the contour of the original integral transform, the resulting expression with the inserted building block will have the same, deformed contour as the two-loop integral transform we started out from. This contour, however is not the same as the one in the Fourier-Mellin transform prescribed by $\mathcal{F}_2$ at three loops. In order to obtain the desired building block we must add the corresponding three-loop lower point contributions to restore the integration contour. Concretely, the relations between the two- and three loop corrected perturbative coefficients are given by

$$\begin{aligned}\tilde{g}^{(3;\delta_{k,1},\delta_{k,2})}_{\varpi;+++} &= \mathcal{E}(z_k) * \left(\tilde{g}^{(2;0,0)}_{\varpi;+++} - P^{(2;0,0)}_{+++}\right) + P^{(3;\delta_{k,1},\delta_{k,2})}_{+++}, \\ \tilde{h}^{(3;\delta_{k,1},\delta_{k,2})}_{\varpi;+++} &= \mathcal{E}(z_k) * \left(\tilde{h}^{(2;0,0)}_{\varpi;+++} - Q^{(2;0,0)}_{+++}\right) + Q^{(3;\delta_{k,1},\delta_{k,2})}_{+++},\end{aligned} \tag{3.128}$$

where the $P^{(3;\delta_{k,1},\delta_{k,2})}_{+++}$ and $Q^{(3;\delta_{k,1},\delta_{k,2})}_{+++}$ indicate the three-loop lower-point terms needed to restore the original contour. Studying these terms in more detail, we find that

$$P^{(3;\delta_{k,1},\delta_{k,2})}_{+++} = \mathcal{E}(z_k) * P^{(2;0,0)}_{+++}\,, \tag{3.129}$$

$$Q^{(3;\delta_{k,1},\delta_{k,2})}_{+++} = \mathcal{E}(z_k) * \left(Q^{(2;0,0)}_{+++} - \frac{1}{32}\mathcal{G}_0(z_k)^2\right)\,, \tag{3.130}$$

so that, in practice, we need only subtract lower point contributions for the real part. Thus we see that, taking special care in going from two to three loops, the convolution method also works at NLLA. The remaining terms at three loops

$$\tilde{g}^{k;(3;\delta_{k,1},\delta_{k,2},\delta_{k,3})}_{++++}(\rho_1,\rho_2,\rho_3) = \tilde{g}^{1;(3;1)}_{++}(\rho_k), \tag{3.131}$$

correspond to six-point corrected perturbative coefficients through an extension of the factorisation beyond LLA [6] and can thus be obtained from known six-particle NLLA results [134].

Now that we have observed how to obtain the three-loop MHV NLLA seven-point MRK ratio from the two-loop result by convolutions, let us move beyond MHV and see how the helicity flip formalism works beyond the LLA. In this context, we will expand the all-order helicity flip kernel (3.66)

$$\mathcal{H}(z) = \mathcal{F}\left[\frac{\chi_i^-}{\chi_i^+}\right] = \mathcal{H}^{(0)}(z_i) + a\mathcal{H}^{(1)}(z_i) + \mathcal{O}(a^2)\,, \tag{3.132}$$

with

$$\mathcal{H}^{(1)}(z_i) = \frac{1}{4}\left(\mathcal{G}_1(z_i) + \frac{z_i}{(1-z_i)}\mathcal{G}_0(z_i) + \frac{z_i}{(1-z_i)^2}\mathcal{G}_{0,0}(z_i)\right) . \tag{3.133}$$

When flipping the helicity of an all-order impact factor, we then find

$$\begin{aligned}\mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right] &\longrightarrow \mathcal{F}\left[\chi^-(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{F}\left[\frac{\chi^-(\nu,n)}{\chi^+(\nu,n)}\right] * \mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right]\\ &= \mathcal{H}(z) * \mathcal{F}\left[\chi^+(\nu,n)\,F(\nu,n)\right] .\end{aligned} \tag{3.134}$$

The same kernel can also be used to flip the helicity of one of the central emission blocks, as can be seen from (3.70), which leads to

$$\begin{aligned}&\mathcal{F}\big[C^+(\nu_1,n_1,\nu_2,n_2)\,F(\nu_1,n_1,\nu_2,n_2)\big]\\ &\quad\longrightarrow \mathcal{F}\left[C^-(\nu_1,n_1,\nu_2,n_2)\,F(\nu_1,n_1,\nu_2,n_2)\right]\\ &\quad= \mathcal{H}(\bar{z}_1) * \mathcal{H}(z_2) * \mathcal{F}\left[C^+(\nu_1,n_1,\nu_2,n_2)\,F(\nu_1,n_1,\nu_2,n_2)\right] .\end{aligned} \tag{3.135}$$

At NLLA the helicity flip kernel contributes up to NLO and thus the computation consists of two types of contributions: a leading-order helicity flip kernel $\mathcal{H}^{(0)}$ of the NLLA perturbative coefficient and a next-to-leading order helicity flip kernel $\mathcal{H}^{(1)}$ of a LLA perturbative coefficient. Flipping the helicity of the first radiated gluon for example we get

$$\begin{aligned}\tilde{g}_{-+++}^{(3;\delta_{k,1},\delta_{2,k})}(\rho_1,\rho_2) = \tilde{g}_{++++}^{(3;\delta_{k,1},\delta_{2,k})}(\rho_1,\rho_2) * \mathcal{H}^{(0)}(z_1)\\ + \tilde{g}_{++++}^{(2;\delta_{k,1},\delta_{2,k})}(\rho_1,\rho_2) * \mathcal{H}^{(1)}(z_1) ,\end{aligned} \tag{3.136}$$

for $k \in \{1,2\}$. Thus we see that the helicity flip formalism can also be applied beyond LLA. These methods have been applied to compute the seven-point amplitude at NLLA through four loops for the MHV case, and through three loops for the NMHV helicity configurations [2]. Subsequently, the eight-point NLLA amplitude for any helicity configuration up to three loops was computed [3].

## 3.4. The Function Space and Transcendentality

### 3.4.1. Maximal Transcendentality at LLA

In this chapter we have explored the multi-Regge limit of planar $\mathcal{N} = 4$ SYM. We have found that a thorough understanding of the geometry and function space which arises from the scattering amplitudes in this setup has allowed us to develop very powerful computational techniques. Since we have an algorithmic way of computing higher-order results, we can turn to the central topic of this thesis. It is natural to wonder if we can show the maximal transcendentality of planar $\mathcal{N}=4$ SYM in this limit using the methods developed here. In this section we show that the recursive structure of LLA scattering amplitudes in the multi-Regge limit implies that they can always be expressed in terms of single-valued multiple polylogarithms of maximal and uniform weight, independently of the loop number and helicity configuration.

We first consider the MHV case. The factorisation at LLA allows us to state that any MHV perturbative coefficient can be computed from a low-loop low-leg starting point using two operations: adding particles and convolution with $\mathcal{E}(z_1)$. Clearly, factorisation implies that adding particles can not change the weight or functional form of the perturbative coefficient, so it suffices to show that convolution with $\mathcal{E}(z_1)$ has the same property. The proof proceeds by induction. Assume that $g_{+\cdots+}^{(\ell;i_1,\ldots,i_k)}$ is a pure function of uniform weight $\omega=1+i_1+\ldots+i_k$, and let us show that $g_{+\cdots+}^{(\ell;i_1+1,\ldots,i_k)}=\mathcal{E}(z_1)*g_{+\cdots+}^{(\ell;i_1,\ldots,i_k)}$ is a pure function of uniform weight $\omega+1$. We have, in terms of simplicial coordinates based at $z_1$,

$$\begin{aligned}\mathcal{E}(z_1)\,&*g_{+\cdots+}^{(\ell;i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5})\\ &=-\int\frac{\mathrm{d}^2w}{2\pi}g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(w,t_2,\ldots,t_{N-5})\frac{\bar{w}t_1+w\bar{t}_1}{|w|^2|w-t_1|^2}\\ &=-\int\frac{\mathrm{d}^2w}{2\pi}g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}(w,t_2,\ldots,t_{N-5})\,\frac{1}{w(w-t_1)}\left(\frac{w+t_1}{\bar{w}-\bar{t}_1}-\frac{w}{\bar{w}}\right)\,.\end{aligned}\tag{3.137}$$

We evaluate the integral in terms of residues. As $g_{+\cdots+}^{(\ell;i_1,\ldots,i_{N-5})}$ is assumed pure by the induction hypothesis and all the denominators are linear in $\bar{w}$, the anti-holomorphic primitive is a pure function (seen as a function of $\bar{w}$) of uniform weight $\omega+1$. The convolution in (3.137) can then be written in the form

$$\begin{aligned}\mathcal{E}(z_1)\,&*g_{+\cdots+}^{(\ell;i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5})\\ &=-\int\frac{\mathrm{d}w}{2\pi}\left[\frac{1}{w}\,F_1(w,t_2,\ldots,t_{N-5})+\frac{1}{w-t_1}\,F_2(w,t_2,\ldots,t_{N-5})\right]\,,\end{aligned}\tag{3.138}$$

where $F_1$ and $F_2$ are pure single-valued polylogarithmic functions of weight $\omega+1$. As all the poles are simple, the holomorphic residues can be computed by simply evaluating the pure functions of weight $\omega+1$ at $w=0$, $w=t_1$ and $w=\infty$ (and dropping all logarithmically divergent terms). Hence, $\mathcal{E}(z_1)*g_{+\cdots+}^{(i_1,\ldots,i_{N-5})}$ is a pure polylogarithmic function of weight $\omega+1$.

To extend this argument beyond MHV, we must first note that non-MHV amplitudes are in general not pure but contain rational prefactors. The form of these rational prefactors, however can be derived and classified from the convolution algorithm [1]. The aim is to show that beyond MHV, the pure functions multiplying the rational prefactors are always pure polylogarithmic functions of uniform weight $\omega = 1 + i_1 + \ldots + i_k$. To show this, analogously to the MHV case it suffices to show that the claim remains true when doing three operations: adding particles, convoluting with $\mathcal{E}(z_1)$ and flipping the first helicity. Once again, factorisation guarantees that we may always add particles with equal neighbouring helicities. Let us then show that also in the non-MHV case a convolution with $\mathcal{E}(z_1)$ will increase the weight by one unit. From the classification of rational prefactors in [1] we know that all poles in $z_1$ are simple and either holomorphic or anti-holomorphic. In the following we discuss the anti-holomorphic case, the extension to the holomorphic case is trivial. The integrand of the convolution integral in the non-MHV case may have additional poles in $\bar{w}$ at points where rational prefactors become singular. However, none of these additional poles are located at $\bar{w} = 0$ or $\bar{w} = z_1$, causing no interplay between the poles of the integration kernel and the poles of the rational prefactors and so all the anti-holomorphic poles entering the convolution integral are simple. We can thus repeat the same argument as in the MHV case, and the anti-holomorphic primitive will be a pure polylogarithmic function of weight $\omega + 1$. Moreover, there are no additional holomorphic poles in $w$ introduced by the rational prefactors, and so we can compute all the holomorphic residues by evaluating the pure functions of weight $\omega + 1$ at $w \in \{0, t_1, \infty\}$. Hence, a convolution with $\mathcal{E}(z_1)$ produces pure polylogarithmic functions of weight $\omega + 1$ also in the non-MHV case. To complete the argument, we need to show that flipping the leftmost helicity does not change the weight of the functions. For this, we can make use of an alternative method to compute the leftmost helicity flip.

Flipping the first helicity on an MHV LLA coefficient

$$\begin{aligned} \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{-+\ldots+} &= \mathcal{H}(z_1) * \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{+\ldots+} \\ &= -\int \frac{\mathrm{d}^2 w}{\pi} \, \frac{z_1}{\bar{w}(w-z_1)^2} \, \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{+\ldots+}(w, z_2 \ldots, z_{N-5}) \,. \end{aligned} \tag{3.139}$$

We can evaluate (3.139) in terms of residues. Let us denote by $F$ the anti-holomorphic primitive,

$$F(w, z_2 \ldots, z_{N-5}) \equiv \int \frac{\mathrm{d}\bar{w}}{\bar{w}} \, \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{+\ldots+}(w, z_2 \ldots, z_{N-5}) \,. \tag{3.140}$$

Then the result is obtained by summing over all the holomorphic residues of $F$. As MHV amplitudes are pure functions, they have no poles, and so $F$ has no poles either. Furthermore, it is easy to check that there is no pole at infinity, and so the only residue we need to take into account comes from the double pole at $w = z_1$ in (3.139),

$$\begin{aligned} \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{-+\ldots+} &= \text{Res}_{w=z_1} \frac{z_1\, F(w, z_2 \ldots, z_{N-5})}{(w-z_1)^2} \\ &= z_1\, \partial_{z_1} F(z_1, z_2 \ldots, z_{N-5}) \\ &= z_1 \partial_{z_1} \int \frac{\mathrm{d}\bar{w}}{\bar{w}} \tilde{g}^{(\ell;i_1,\ldots,i_{N-5})}_{+\ldots+}(w, z_2 \ldots, z_{N-5})\,. \end{aligned} \tag{3.141}$$

Now, the statement that for the rational prefactors all poles in $z_1$ are simple and either holomorphic or anti-holomorphic implies that we can in fact use this method on any helicity configuration, taking the (anti)holomorphic primitive if poles are (anti)holomorphic. Since all poles are simple, the integration will increase the weight by one unit. This effect is compensated by the differentiation, so that the total weight of the functions remains unchanged. We find that our algorithmic method to compute scattering amplitudes in the multi-Regge limit has allowed us to show that to LLA, they can always be expressed in terms of single-valued multiple polylogarithms of maximal and uniform weight, independently of the loop number and helicity configuration.

### 3.4.2. Beyond Leading Logarithmic Accuracy

#### 3.4.2.1. The Transcendentality of MHV Amplitudes

Now let us see what can be said about transcendentality beyond leading logarithmic accuracy. From the all-order conjecture for the MRK ratio given in (3.41) and the form of the all-order building blocks [6, 127], one can deduce that MHV amplitudes are made up of terms of the form

$$\mathcal{F}\left[A\, \varpi_N\right] \tag{3.142}$$

where $A$ is an arbitrary product of *elementary building blocks*[5] $\{E, V, N, M\}$ defined in Appendix A and (repeated) derivatives $D_\nu^n$ thereof. Let us examine these terms more carefully.

Terms of the form (3.142) can be computed by starting from the Fourier-Mellin transform of the MHV vacuum ladder (3.86) given by

$$\mathcal{F}\left[\varpi_N\right] = \frac{1}{4}\left(\mathcal{G}_0(\rho_1) - \mathcal{G}_1(\rho_1) - \mathcal{G}_1(\rho_{N-5})\right)\,, \tag{3.143}$$

[5]These elementary building blocks are the objects out of which the BFKL building blocks of (3.41) are built to all orders.

and acting upon this object by repeated convolutions with the Fourier-Mellin transforms of the elementary building blocks and their (higher) derivatives. Now, noting that (3.94) and

$$\mathcal{V}(z,\bar{z}) \equiv \mathcal{F}\left[V(\nu,n)\right] = \frac{2-z-\bar{z}}{2|1-z|^2}\,, \tag{3.144}$$

$$\mathcal{N}(z,\bar{z}) \equiv \mathcal{F}\left[N(\nu,n)\right] = \frac{z-\bar{z}}{|1-z|^2}\,, \tag{3.145}$$

have the form described in Lemma C.2 in Appendix C, we can see that convolution of a pure function with such objects will result in a pure function of one weight higher. For derivatives of such kernels, we may use the property that for a generic Fourier-Mellin transform,

$$\mathcal{F}\left[D_\nu X\right] = -\mathcal{G}_0(z)\mathcal{F}\left[X\right]\,. \tag{3.146}$$

This implies that for $\phi(z,\bar{z})$, a generic pure function of SVMPLs with uniform transcendental weight $n$,

$$\begin{aligned} \mathcal{F}\left[D_\nu X\right](z,\bar{z}) * \phi(z,\bar{z}) &= -\int \frac{d^2w}{|w|^2}\mathcal{G}_0(w/z)\mathcal{F}\left[X\right]\left(\frac{w}{z},\frac{\bar{w}}{\bar{z}}\right)\phi(w,\bar{w})\,,\\ &= \int \frac{d^2w}{|w|^2}\mathcal{F}\left[X\right]\left(\frac{w}{z},\frac{\bar{w}}{\bar{z}}\right)\Big(\mathcal{G}_0(z)-\mathcal{G}_0(w)\Big)\phi(w,\bar{w})\,, \end{aligned} \tag{3.147}$$

which is the convolution of $\mathcal{F}\left[X\right]$ with a pure function of SVMPLs with uniform transcendental weight $n+1$. From this we may conclude that for convolutions with (higher) derivatives of building blocks which obey the form of Lemma C.2 the result can always be written as a pure combination of SVMPLs with uniform weight, and that the weight is raised by one for every derivative. Thus, we can remove all building blocks from the Fourier-Mellin transform except for $M$ and its derivatives and we know that reinserting them via convolutions will preserve the function space and will raise the weight uniformly. This leaves us to determine what effect the remaining building block $M$ and its derivatives have.

In order to study the effect of insertions of $M$ into the Fourier-Mellin integrand it is useful to rewrite it as

$$M(\nu_k,n_k,\nu_l,n_l) = \frac{D_{\nu_k}C_0^+(\nu_k,n_k,\nu_l,n_l)}{C_0^+(\nu_k,n_k,\nu_l,n_l)} + F(\nu_k,n_k) \tag{3.148}$$

$$= -\frac{D_{\nu_l}C_0^+(\nu_k,n_k,\nu_l,n_l)}{C_0^+(\nu_k,n_k,\nu_l,n_l)} + F(\nu_l,-n_l) - N(\nu_l,n_l)\,. \tag{3.149}$$

Where $F(\nu,n)$ is a new building block defined as

$$F(\nu,n) = -2\psi(1) + \psi\left(1+i\nu-\frac{n}{2}\right) + \psi\left(1-i\nu-\frac{n}{2}\right)\,. \tag{3.150}$$

Inserting the decomposition of $M$ into the Fourier-Mellin transform we have

$$\begin{aligned}\mathcal{F}\left[\varpi_N M(\nu_k,n_k,\nu_{k+1},n_{k+1})\right] = \mathcal{F}\left[\varpi_N \frac{D_k C_0^+(\nu_k,n_k,\nu_{k+1},n_{k+1})}{C_0^+(\nu_k,n_k,\nu_{k+1},n_{k+1})}\right] \\ + \mathcal{F}\left[\varpi_N F(\nu_k,n_k)\right].\end{aligned} \tag{3.151}$$

Now note that the first term can be rewritten using partial integration, which for $k>1$ gives us

$$\begin{aligned}\mathcal{F}\left[\varpi_N \frac{D_k C_0^+(\nu_k,n_k,\nu_{k+1},n_{k+1})}{C_0^+(\nu_k,n_k,\nu_{k+1},n_{k+1})}\right] = &- \mathcal{G}_0(z_k)\mathcal{F}\left[\varpi_N\right] \\ &+ \mathcal{F}\left[\varpi_N M(\nu_{k-1},n_{k-1},\nu_k,n_k)\right] \\ &+ \mathcal{F}\left[\varpi_N N(\nu_k,n_k)\right] \\ &- \mathcal{F}\left[\varpi_N F(\nu_k,-n_k)\right] .\end{aligned} \tag{3.152}$$

From (3.151) and (3.152), using $\mathcal{F}[F(\nu,n)] = \mathcal{F}[F(\nu,-n)]$ we can conclude that

$$\begin{aligned}\mathcal{F}\left[\varpi_N M(\nu_k,n_k,\nu_{k+1},n_{k+1})\right] = \mathcal{F}\left[\varpi_N M(\nu_{k-1},n_{k-1},\nu_k,n_k)\right] \\ + \mathcal{F}\left[\varpi_N N(\nu_k,n_k)\right] - \mathcal{G}_0(z_k)\mathcal{F}\left[\varpi_N\right] ,\end{aligned} \tag{3.153}$$

and so we can recursively shift the building block $M$ to the first set of Fourier-Mellin variables. The end point of the recursion is

$$\begin{aligned}\mathcal{F}\left[\varpi_N M(\nu_1,n_1,\nu_2,n_2)\right] = \mathcal{F}\left[\varpi_N \frac{D_1 C_0^+(\nu_1,n_1,\nu_2,n_2)}{C_0^+(\nu_1,n_1,\nu_2,n_2)}\right] \\ + \mathcal{F}\left[\varpi_N F(\nu_1,n_1)\right] .\end{aligned} \tag{3.154}$$

Using partial integration for the first term on the r.h.s., we obtain

$$\begin{aligned}\mathcal{F}\left[\varpi_N \frac{D_1 C_0^+(\nu_1,n_1,\nu_2,n_2)}{C_0^+(\nu_1,n_1,\nu_2,n_2)}\right] &= \mathcal{F}\left[D_1\varpi_N\right] - \mathcal{F}\left[\varpi_N \frac{D_1 \chi_0^+(\nu_1,n_1)}{\chi_0^+(\nu_1,n_1)}\right] , \\ &= \mathcal{F}\left[D_1\varpi_N\right] + i\mathcal{F}\left[\varpi_N \chi_0^+(\nu_1,n_1)\right] .\end{aligned} \tag{3.155}$$

Since $i\chi_0^+(\nu_i,n_i) = V(\nu_i,n_i) - \frac{1}{2}N(\nu_i,n_i)$, this means

$$\begin{aligned}\mathcal{F}\left[\varpi_N \frac{D_1 C_0^+(\nu_1,n_1,\nu_2,n_2)}{C_0^+(\nu_1,n_1,\nu_2,n_2)}\right] = \mathcal{G}_0(z_1)\mathcal{F}\left[\varpi_N\right] + \mathcal{F}\left[\varpi_N V(\nu_1,n_1)\right] \\ - \frac{1}{2}\mathcal{F}\left[\varpi_N N(\nu_1,n_1)\right] .\end{aligned} \tag{3.156}$$

Since

$$\mathcal{F}\left[F(\nu,n)\right] = \frac{|z|^2}{|z+1|^2} , \tag{3.157}$$

obeys the form dictated by Lemma C.2, the extra terms we introduce are pure combinations of SVMPLs of uniform weight. Our discussion has shown that the Fourier-Mellin transform of the building block $M$ multiplied by the vacuum ladder can be rewritten into a linear combination of Fourier-Mellin transforms known to resolve to pure functions one step higher in weight by shifting the variable dependence of $M$ to the left and resolving it in the first Fourier-Mellin variables. For multiple insertions of $M$ in the Fourier-Mellin transform, we can build on this result. Note that we managed to replace the insertion of $M$ in the Fourier-Mellin integrand by two types of terms:

$$\mathcal{F}[\varpi_N X_i] \qquad \text{and} \qquad \mathcal{G}_0(z_i)\mathcal{F}[\varpi_N]\,, \tag{3.158}$$

with $X_i \in \{F_i, N_i, V_i\}$. Let us consider the effect of adding another $M$ to the Fourier-Mellin integrand of these terms, to know how products of $M$ can be resolved. We introduce the shorthand $M_{ij} = M(\nu_i, n_i, \nu_j, n_j)$. The first term can be treated using the commutativity and associativity of the convolution product

$$\begin{aligned}\mathcal{F}[\varpi_N X_i] * \mathcal{F}[M_{kl}] &= \mathcal{F}[\varpi_N] * \mathcal{F}[X_i] * \mathcal{F}[M_{kl}]\,,\\ &= (\mathcal{F}[\varpi_N] * \mathcal{F}[M_{kl}]) * \mathcal{F}[X_i]\,,\end{aligned} \tag{3.159}$$

the term in brackets in (3.159) can then be treated as was described above. In the second case defined by

$$(\mathcal{G}_0(z_i)\mathcal{F}[\varpi_N]) * \mathcal{F}[M_{kl}] \tag{3.160}$$

the situation is a bit trickier. For $i \neq k$ and $i \neq l$ the logarithm is immaterial from the point of view of the convolution integral and we can treat $M$ as before. For $i = k$ or $i = l$, we can use

$$(\mathcal{G}_0(z)\mathcal{F}[X]) * \mathcal{F}[Y] = \mathcal{G}_0(z)(\mathcal{F}[X] * \mathcal{F}[Y]) + \mathcal{F}[X] * \mathcal{F}[D_\nu Y] \tag{3.161}$$

where we have used (3.146). We therefore have

$$\begin{aligned}(\mathcal{G}_0(z_k)\mathcal{F}[\varpi_N]) * \mathcal{F}[M_{kl}] =& \,\mathcal{G}_0(z_k)(\mathcal{F}[\varpi_N] * \mathcal{F}[M_{kl}])\\ &+ \mathcal{F}[\varpi_N] * \mathcal{F}[D_k M_{kl}],\\ (\mathcal{G}_0(z_l)\mathcal{F}[\varpi_N]) * \mathcal{F}[M_{kl}] =& \,\mathcal{G}_0(z_l)(\mathcal{F}[\varpi_N] * \mathcal{F}[M_{kl}])\\ &+ \mathcal{F}[\varpi_N] * \mathcal{F}[D_l M_{kl}],\end{aligned} \tag{3.162}$$

and are left with a term which can be treated as before and an insertion of a derivative of $M$ into the vacuum ladder.

This leads us to the final ingredient we need to describe MHV amplitudes, namely derivatives of the building block $M$. To deal with such terms, we may once again use integration by parts. Take the term

$$D_k^n M(\nu_k, n_k, \nu_l, n_l) \tag{3.163}$$

as the result for $D_l^n$ is completely analogous. We may then recursively reduce the power of $D_k$ by using

$$\begin{aligned}\mathcal{F}[\varpi_N D_k^n M_{kl}] &= \mathcal{F}[D_k\left(\varpi_N D_k^{n-1} M_{kl}\right)] - \mathcal{F}[(D_k\varpi_N)D_k^{n-1}M_{kl}] \\ &= \mathcal{G}_0(z_k)\mathcal{F}[\varpi_N D_k^{n-1}M_{kl}] \\ &\quad - \mathcal{F}\left[\varpi_N \frac{D_k C^+_{0,jk}}{C^+_{0,jk}}\right] * \mathcal{F}[D_k^{n-1}M_{kl}] \\ &\quad + \mathcal{F}\left[\varpi_N \frac{D_k C^+_{0,kl}}{C^+_{0,kl}}\right] * \mathcal{F}[D_k^{n-1}M_{kl}]\,,\end{aligned} \tag{3.164}$$

where we have identified $j = k-1$ and $l = k+1$ for compactness. The final two terms on the r.h.s. of (3.164) can be rewritten using (3.152). Using (3.164) we may recursively eliminate all appearances of derivatives of $M$. The extra terms generated in each step of this recursion all preserve the function space and raise the weight by one, as does the multiplication by $\mathcal{G}_0(z_k)$, so we can conclude that insertions of derivatives $D_k^n M_{kl}$ of $M$ raise the weight by $n+1$.

To summarize: MHV amplitudes are made up of terms of the form $\mathcal{F}\left[A\,\varpi_N\right]$ where $A$ is an arbitrary product of elementary building blocks and their derivatives. To determine such terms, we can make use of the convolution product to introduce each factor in the product $A$ to the integrand one by one. First, however, we will make use of (3.164) to eliminate all derivatives acting on $M$, resulting in a sum of terms of the form $\mathcal{F}\left[A'\,\varpi_N\right]$ with prefactors that may contain SVMPLs $\mathcal{G}_0(z_i)$, where the $A'_i$ are again arbitrary products of elementary building blocks and their (repeated) derivatives $D_\nu^n$, but the derivative never acts on $M$. Note that from (3.164) we see that for every power of $\mathcal{G}_0(z_i)$ in the prefactor, the total number of factors in $A'$ is reduced by one. Now, we use the convolution structure to introduce each factor in $A'$. We start by introducing factors of $M$, for which the above discussion shows that we obtain a pure function of uniform weight, where the weight is raised by one for each factor of $M$ which is introduced. Introducing all other building blocks $X \in \{E, V, N\}$ and their derivatives $D_i^n X$ is guaranteed to result in pure functions of uniform weight by the result of Lemma C.2, and will raise the weight by one and $n+1$, respectively.

In conclusion, $\mathcal{F}\left[A\,\varpi_N\right]$ evaluates to a pure function of uniform transcendental weight, the weight of which is determined by the total number of factors (also counting derivates) in $A$. From (3.41) and an analysis of the all-order BFKL building blocks, one may then conclude that MHV amplitudes in the multi-Regge limit of planar $\mathcal{N} = 4$ SYM must be pure functions of uniform transcendental weight to all orders in the large logarithm and all orders in the coupling constant [6].

### 3.4.2.2. The All-Order Helicity Flip

Let us now turn our attention beyond MHV. As we have discussed before, the only new ingredient is the all-order helicity flip kernel. After expanding the helicity flip kernel (3.66)

$$H(\nu,n) = H^{(0)}(\nu,n) + aH^{(1)}(\nu,n) + a^2H^{(2)}(\nu,n) + \dots \tag{3.165}$$

$$= H^{(0)}(\nu,n)\left(1 + ah^{(1)}(\nu,n) + a^2h^{(2)}(\nu,n) + \dots\right), \tag{3.166}$$

from the form of the all-order impact factor, we can see that all corrections $h^{(k)}$, $k>1$ consist of combinations of the elementary building blocks $E, N, V$ and their derivatives. Therefore, from Lemma C.2 we again see that the convolution of any pure function of uniform weight with $\mathcal{F}[h^{(k)}(\nu,n)]$ will again be a pure function of uniform weight. The perturbative coefficients in any helicity configuration can be obtained by repeated convolutions with all-order helicity flip kernels. Due to commutativity of the convolution we have

$$\mathcal{F}[X] * \mathcal{H}_{i_1}^{(\ell_{i_1})} * \mathcal{H}_{i_2}^{(\ell_{i_2})} * \dots = \mathcal{F}[X h_{i_1}^{(\ell_{i_1})} h_{i_2}^{(\ell_{i_2})} \dots] * \mathcal{H}_{i_1}^{(0)} * \mathcal{H}_{i_2}^{(0)} * \dots, \tag{3.167}$$

such that any non-MHV amplitude can be expressed as a series of leading order helicity flips of a pure function of uniform weight. Furthermore, note that $h^{(k)}(\nu,n)$ raises the weight by $2k$, such that all-order helicity flips lead to results of uniform transcendental weight at fixed orders. For example, if we consider the expansion at NLLA for seven particles in (3.136), the NLO component of the helicity flip kernel will increase the weight by two. We see that this pattern holds to all orders.

We have shown that amplitudes beyond MHV in multi-Regge kinematics can be computed by applying the leading-order helicity flip kernel to a pure function of uniform weight. From our discussion at LLA, we know that the LO helicity flip does not change the transcendental weight upon convolution, and so we may conclude by stating the following theorem [6].

**Theorem 3.1.** *In the multi-Regge limit of planar $\mathcal{N}=4$ super Yang-Mills theory at N$^k$LLA in the Mandelstam region where the energy components of all produced particles are negative, the $\ell$-loop N-point MRK ratio is given by linear combinations of cross ratios multiplying pure combinations of SVMPLs and powers of $2\pi i$ of uniform weight $k+\ell+1$. The form of the cross ratios multiplying the pure functions is fixed by the helicity configuration. In particular, the $\ell$-loop MHV MRK ratio at N$^k$LLA is given by pure combinations of SVMPLs of weight $k+\ell+1$.*

In this chapter, we have seen that knowledge of the geometry of the configuration space, along with an all-order form for the amplitude in terms of an integral transform allowed us to gain tremendous insight on the functions which may appear and their analytic properties, in particular with regard to the transcendental weight. We have seen that our recursive way of computing higher order corrections offers a unique handle on transcendental weight and allows us to show results to all orders. Note, however, that we have been working in a highly symmetric theory with very special properties. In the following chapter, we will move beyond this theory and consider the possibility of less-than-maximal supersymmetry. We will aim to apply the formalism developed here to an object defined in the high-energy limit for gauge theories with generic matter content, and see what connection can be made between the matter content on the one hand and the analytic properties of the result on the other.

# 4 | The High-Energy Limit for Generic Gauge Theories

Having explored the high-energy limit of planar $\mathcal{N} = 4$ SYM, we now aim to apply the developed techniques to a more general setting to study transcendentality in the multi-Regge regime in a generic sense. In particular, we wish to understand if the maximal transcendentality that was observed in the previous chapter is an artefact of the special theory which was considered, or if we may find it in the high energy limit for a larger set of theories. As a starting point, we will consider the high-energy limit of parton scattering in QCD. It is known from the BFKL formalism that in this case the total cross section factorises into impact factors which couple to the external partons and an effective propagator called *the BFKL ladder*. This object will be the central topic of this chapter, reviewing the work done in [7]. Using the techniques developed in the previous chapter, we will compute this BFKL ladder to NLLA, first in QCD but then turning our attention to theories with generic matter content, in order to study how the matter content can be tuned to enforce maximal transcendentality of our result.

In this chapter, we start with quick overview of the BFKL formalism. After reviewing the BFKL equation in QCD and its solutions in terms of the BFKL ladder, we recap its well-known LLA part. Turning to NLLA, we first discuss the procedure to extract the BFKL ladder at this order and then show its perturbative expansion. After discussing the methods by which this perturbative expansion can be computed, we generalise our discussion to generic gauge theories and study the relation between the matter content and transcendentality of the BFKL ladder at NLLA.

# 4.1. The BFKL Formalism

## 4.1.1. The BFKL Equation

As can be seen from the optical theorem

$$\sigma_{\rm tot} \sim {\rm Disc}(i\overline{\mathcal{M}}(s,t=0))\,, \tag{4.1}$$

where $\overline{\mathcal{M}}$ is the forward amplitude summed over final colour and helicities and averaged over initial ones, and the factorisation of amplitudes in the high-energy limit, the total cross section for parton scattering in the high-energy limit factorises,

$$\sigma(s) \simeq \int \frac{{\rm d}^2q_1\,{\rm d}^2q_2}{(2\pi)^2\,q_1^2\,q_2^2}\,\Phi_A(q_1)\,\Phi_B(q_2)\,f(q_1,q_2,\log(s/s_0))\,, \tag{4.2}$$

where $s$ is the center-of-mass energy and

$$s_0 \equiv \sqrt{q_1^2\,q_2^2} \tag{4.3}$$

is the geometric mean of the two transverse momenta, $\Phi_{A/B}$ denote the *impact factors* and $f$ is the *BFKL ladder*. The BFKL ladder given here is related to the $t$-channel exchange of a colour singlet state. A schematic representation of (4.2) can be found in Figure 4.1.

The BFKL ladder can be written as,

$$f(q_1,q_2,y) = \int_C \frac{{\rm d}\omega}{2\pi i}\,e^{y\,\omega}\,f_\omega(q_1,q_2)\,, \tag{4.4}$$

where $y$ denotes the large logarithm, the integration contour $C$ is a straight vertical line such that all poles in $\omega$ are to the right of the contour and $f_\omega$ is a solution to *the BFKL equation*,

$$\omega\,f_\omega(q_1,q_2) = \frac{1}{2}\,\delta^{(2)}(q_1-q_2) + (\mathcal{K}\star f_\omega)(q_1,q_2)\,, \tag{4.5}$$

with the convolution defined by

$$(\mathcal{K}\star f_\omega)(q_1,q_2) \equiv \int {\rm d}^2k\,K(q_1,k)\,f_\omega(k,q_2)\,, \tag{4.6}$$

and $K(q_1,q_2)$ *the BFKL kernel.* This kernel is real and symmetric, $K(q_1,q_2) = K(q_2,q_1)$, and so the integral operator $\mathcal{K}$ is hermitian and its eigenvalues are real.

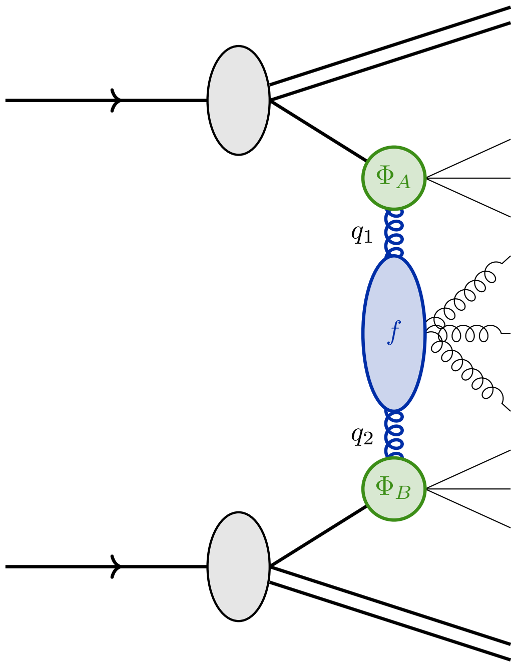



**Figure 4.1:** A graphical representation of the factorised cross section in the high-energy limit in terms of impact factors and the BFKL ladder.

To solve (4.5), we find a complete and orthonormal set of eigenfunctions of the BFKL integral operator

$$(\mathcal{K} \star \Phi_{\nu n})(q) = \omega_{\nu n}\, \Phi_{\nu n}(q)\,. \tag{4.7}$$

The eigenvalue $\omega_{\nu n}$ is referred to as the *singlet BFKL eigenvalue* and is distinct from the eigenvalue introduced in the previous chapter in (3.41). The eigenfunctions are labeled by $(\nu, n)$, where $\nu$ is a real number and $n$ an integer. As they are orthonormal and complete, they satisfy

$$\begin{aligned} 2\int \mathrm{d}^2 q\, \Phi_{\nu n}(q)\, \Phi^*_{\nu' n'}(q) &= \int_0^\infty \mathrm{d}q^2 \int_0^{2\pi} \mathrm{d}\theta\, \Phi_{\nu n}(q)\, \Phi^*_{\nu' n'}(q) \\ &= \delta(\nu - \nu')\, \delta_{nn'}\,, \end{aligned} \tag{4.8}$$

and

$$\sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \Phi_{\nu n}(q)\, \Phi^*_{\nu n}(q') = \frac{1}{2}\, \delta^{(2)}(q - q')\,. \tag{4.9}$$

We can then solve the BFKL equation by stating

$$f_\omega(q_1,q_2) \;=\; \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \frac{1}{\omega-\omega_{\nu n}}\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)\,. \tag{4.10}$$

Indeed, we have

$$\begin{aligned}(\mathcal{K}\star f_\omega)(q_1,q_2) \;&=\; \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \frac{1}{\omega-\omega_{\nu n}}\, (\mathcal{K}\star\Phi_{\nu n})(q_1)\, \Phi^*_{\nu n}(q_2)\\ &=\; \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \frac{\omega_{\nu n}}{\omega-\omega_{\nu n}}\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)\\ &=-\frac{1}{2}\delta^{(2)}(q_1-q_2)+\omega\, f_\omega(q_1,q_2)\,,\end{aligned} \tag{4.11}$$

where in the last step we used the completeness of the eigenfunctions. Finally, inserting (4.10) into (4.4), we find

$$f(q_1,q_2,y) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)\, e^{y\,\omega_{\nu n}}\,. \tag{4.12}$$

In the following we are interested in the perturbative expansion of the BFKL ladder. The kernel of the integral equation admits the expansion,

$$K(q_1,q_2) = \overline{\alpha}_\mu \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, K^{(l)}(q_1,q_2)\,, \tag{4.13}$$

where $\overline{\alpha}_\mu = N_c\,\alpha_S(\mu^2)/\pi$ is the renormalised strong coupling constant evaluated at an arbitrary scale $\mu^2$. The leading order (LO) BFKL kernel [116–118, 154],

$$K^{(0)}(q_1,q_2) = \frac{1}{\pi}\frac{1}{(q_1-q_2)^2}\,, \tag{4.14}$$

leads to the resummation of the terms of $\mathcal{O}\left((\overline{\alpha}_\mu y)^n\right)$, i.e., terms at LLA, and the NLO kernel $K^{(1)}$ [119, 121] resums the terms at NLLA, i.e. of $\mathcal{O}\left(\overline{\alpha}_\mu(\overline{\alpha}_\mu y)^n\right)$, and so forth. The BFKL singlet eigenvalue and eigenfunctions also admit an expansion in the strong coupling,

$$\omega_{\nu n} = \overline{\alpha}_\mu \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, \omega_{\nu n}^{(l)} \quad \text{and} \quad \Phi_{\nu n}(q) = \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, \Phi_{\nu n}^{(l)}(q)\,. \tag{4.15}$$

The truncated eigenvalue and eigenfunctions,

$$\omega_{\nu n}^{N^kLO} = \overline{\alpha}_\mu \sum_{l=0}^{k} \overline{\alpha}_\mu^l\, \omega_{\nu n}^{(l)} \quad \text{and} \quad \Phi_{\nu n}^{N^kLO}(q) = \sum_{l=0}^{k} \overline{\alpha}_\mu^l\, \Phi_{\nu n}^{(l)}(q)\,, \tag{4.16}$$

are eigenvalues and eigenfunctions of the truncated BFKL integral operator,

$$\begin{gathered}\left(\mathcal{K}^{N^kLO} \star \Phi_{\nu n}^{N^kLO}\right)(q) = \omega_{\nu n}^{N^kLO}\,\Phi_{\nu n}^{N^kLO}(q) + \mathcal{O}(\overline{\alpha}_\mu^{k+1})\,, \\ \text{with } \mathcal{K}^{N^kLO} = \overline{\alpha}_\mu \sum_{l=0}^{k} \overline{\alpha}_\mu^l\,\mathcal{K}^{(l)}\,. \end{gathered} \tag{4.17}$$

In the remainder of this chapter we consider the first two terms in the expansion of the BFKL ladder,

$$f(q_1, q_2, y) = f^{LL}(q_1, q_2, \eta_\mu) + \overline{\alpha}_\mu\, f^{NLL}(q_1, q_2, \eta_\mu) + \dots\,. \tag{4.18}$$

Note that $f^{N^kLL}$ is defined as an expansion in $\eta_\mu = \overline{\alpha}_\mu\, y$ , implying they are expressions of fixed logarithmic order. The LO term $f^{LL}(q_1, q_2, \eta_\mu)$ is the BFKL ladder at LLA, and the NLO term $f^{NLL}(q_1, q_2, \eta_\mu)$ is the ladder at NLLA. We will focus on the LLA ladder for the remainder of this section, before turning our attention to NLLA subsequently.

### 4.1.2. The BFKL Ladder at LLA

The LO eigenfunctions are given by

$$\Phi_{\nu n}^{LO}(q) \equiv \varphi_{\nu n}(q) = \frac{1}{2\pi}\,(q^2)^{-1/2+i\nu}\,e^{in\theta}\,. \tag{4.19}$$

The LO singlet eigenvalue is given by [117, 118]

$$\omega_{\nu n}^{(0)} \equiv \chi_{\nu n} = -2\gamma_E - \psi\left(\frac{|n|+1}{2} + i\nu\right) - \psi\left(\frac{|n|+1}{2} - i\nu\right)\,, \tag{4.20}$$

where $\gamma_E = -\Gamma'(1)$ is the Euler-Mascheroni constant and $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function. We thus have

$$(\mathcal{K}^{LO} \star \varphi_{\nu n})(q) = \chi_{\nu n}\,\varphi_{\nu n}(q)\,. \tag{4.21}$$

The LO eigenvalue is symmetric under $\nu \to -\nu$. Using these expressions of the eigenvalue and eigenfunctions, we can determine the BFKL ladder at LLA

$$f^{LL}(q_1, q_2, \eta_\mu) \quad = \quad \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\nu\, e^{\eta_\mu\,\chi_{\nu n}}\,\varphi_{\nu n}(q_1)\,\varphi_{\nu n}^*(q_2)\,. \tag{4.22}$$

The dependence of the strong coupling on the renormalisation scale $\mu^2$ in (4.22) is immaterial, since the effect of changing the scale contributes at NLLA. We expand $f^{LL}$ in powers of $\eta_\mu$,

$$f^{LL}(q_1, q_2, \eta_\mu) = \frac{1}{2}\delta^{(2)}(q_1 - q_2) + \frac{1}{2\pi\,\sqrt{q_1^2\,q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_\mu^k}{k!}\,f_k^{LL}(z)\,. \tag{4.23}$$

The coefficients of the expansion depend on a single complex variable $z$ defined by

$$z \equiv \frac{\tilde{q}_1}{\tilde{q}_2}\,, \qquad \text{with } \tilde{q}_k \equiv q_k^x + i q_k^y\,. \tag{4.24}$$

These coefficients can then be cast in the form of a Fourier-Mellin transform,

$$f_k^{LL}(z) = \mathcal{F}\left[\chi_{\nu n}^k\right] \equiv \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi}\, |z|^{2i\nu}\, \chi_{\nu n}^k\,. \tag{4.25}$$

In [134] it was shown that the natural space of functions to which Fourier-Mellin transforms of this type evaluate are the single-valued multiple polylogarithms introduced in Section 3.1.2. In [155] a (conjectural) generating functional of the BFKL ladder was given that allows one to express each coefficient $f_k^{LL}(z)$ as a linear combination of SVMPLs with singularities at $z = 0$ and $z = 1$. Writing

$$f_k^{LL}(z) = \frac{|z|}{2\pi\,|1-z|^2}\, F_k(z)\,, \tag{4.26}$$

we can express the first few coefficients as [155],

$$\begin{aligned} F_1(z) &= 1\,,\\ F_2(z) &= 2\,\mathcal{G}_1(z) - \mathcal{G}_0(z)\,,\\ F_3(z) &= 6\,\mathcal{G}_{1,1}(z) - 3\,\mathcal{G}_{0,1}(z) - 3\,\mathcal{G}_{1,0}(z) + \mathcal{G}_{0,0,0}(z)\,,\\ F_4(z) &= 24\,\mathcal{G}_{1,1,1}(z) + 4\,\mathcal{G}_{0,0,1}(z) + 6\,\mathcal{G}_{0,1,0}(z) - 12\,\mathcal{G}_{0,1,1}(z)\\ &\quad + 4\,\mathcal{G}_{1,0,0}(z) - 12\,\mathcal{G}_{1,0,1}(z) - 12\,\mathcal{G}_{1,1,0}(z) - \mathcal{G}_{0,0,0}(z) + 8\,\zeta_3\,. \end{aligned} \tag{4.27}$$

The conjecture of [155] implies that the functions $F_k$ have a particularly simple form: at any loop order, they are pure [156]. More precisely, the $F_k$ are pure combinations of SVHPLs[1] of uniform weight $(k-1)$ with singularities at most at $z = 0$ or $z = 1$. This claim follows from Lemma C.1, the proof of which can be found in Appendix C. Note that this proof does not make use of the conjectural generating function, but is based on the observation that the coefficients of the leading logarithmic BFKL ladder at different orders are related by convolutions

$$f_k^{LL}(z) = \mathcal{F}\left[\chi_{\nu n}^k\right] = \mathcal{F}\left[\chi_{\nu n}\right]^{*k}\,, \tag{4.28}$$

and thus can be constructed recursively. An inductive argument then shows that $F_k$ has uniform weight.

Thus, we see that also in this setup, the convolution formalism allows us to study transcendentality in a profound way using a recursive argument. At LLA however, only the LO BFKL kernel which is conformally-invariant contributes

[1] And the multiple zeta values associated to them [157].

and the result does not depend on the matter content of the theory. In what follows, we will move to NLLA, where the matter content does play a role. We will use the convolution formalism to compute high-loop expressions for the BFKL ladder and then study how their transcendentality relates to the chosen matter content. As a first case, however, we will consider the NLLA BFKL ladder in QCD.

## 4.2. The BFKL Ladder to NLLA

### 4.2.1. The Chirilli-Kovchekov Procedure

At NLLA, we must take into account the NLO corrections to the BFKL kernel. These were obtained in QCD in [119]. The corresponding NLO corrections to the BFKL singlet eigenvalue were computed in [119] for $n = 0$ and in [74, 158] for arbitrary $n$, albeit in the approximation that the NLO eigenfunctions are identical to the LO eigenfunctions given in (4.19). In other words, the NLO corrections $\delta_{\nu n}$ to the BFKL eigenvalue of [74, 158] are defined by the equation,

$$(\mathcal{K}^{NLO} \star \varphi_{\nu n})(q) \equiv \overline{\alpha}_S(q^2) \left( \chi_{\nu n} + \overline{\alpha}_S(q^2) \frac{\delta_{\nu n}}{4} \right) \varphi_{\nu n}(q) + \mathcal{O}(\overline{\alpha}_S^3(q^2)) \,. \quad (4.29)$$

The NLO corrections to the eigenvalue $\delta_{\nu n}$ in QCD are given in this approximation by [74, 119, 158],

$$\begin{aligned}
\delta_{\nu n} &= 6\zeta_3 - \frac{1}{2}\beta_0\, \chi_{\nu n}^2 + 4\gamma_K^{(2)}\, \chi_{\nu n} + \frac{i}{2}\beta_0\, \partial_\nu \chi_{\nu n} + \partial_\nu^2 \chi_{\nu n} \\
&- 2\Phi(n,\gamma) - 2\Phi(n, 1-\gamma) - \frac{\Gamma(\gamma)\Gamma(1-\gamma)}{2i\nu} \left[\psi\left(\gamma\right) - \psi\left(1-\gamma\right)\right] \\
&\times \left[ \delta_{n0} \left( 3 + \left(1 + \frac{N_f}{N_c^3}\right) \frac{2 + 3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)} \right) \right. \\
&\left. - \, \delta_{|n|2} \left( \left(1 + \frac{N_f}{N_c^3}\right) \frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)} \right) \right] ,
\end{aligned} \quad (4.30)$$

with $\gamma = 1/2 + i\nu$, $\beta_0$ the one-loop beta function and $\gamma_K^{(2)}$ *the two-loop cusp anomalous dimension* for QCD in the dimensional reduction scheme,

$$\beta_0 = \frac{11}{3} - \frac{2N_f}{3N_c} \,, \qquad \gamma_K^{(2)} = \frac{1}{4}\left(\frac{64}{9} - \frac{10N_f}{9N_c}\right) - \frac{\zeta_2}{2} \,. \quad (4.31)$$

The function $\Phi(n,\gamma)$ is defined as,

$$\begin{aligned}\Phi(n,\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+\gamma+|n|/2} \Big\{ & \psi'(k+|n|+1) - \psi'(k+1) \\ & + (-1)^{k+1}[\beta'(k+|n|+1) + \beta'(k+1)] \\ & - \frac{1}{k+\gamma+|n|/2}[\psi(k+|n|+1) - \psi(k+1)] \Big\} ,\end{aligned} \tag{4.32}$$

with

$$\beta'(z) = \frac{1}{4}\left[\psi'\left(\frac{1+z}{2}\right) - \psi'\left(\frac{z}{2}\right)\right] . \tag{4.33}$$

While the NLO eigenvalue in (4.30) was derived under the assumption that the eigenfunctions are the same at LO and NLO, we do not expect this to hold for a generic theory. In fact, since it is the eigenvalue of a hermitian operator, the true NLO eigenvalue must be real and independent of $q^2$. The $\delta_{\nu n}$ fails to meet either criterion: the right-hand side of (4.29) depends on $q^2$ through the strong coupling constant and (4.30) contains the term $i\beta_0\, \partial_\nu \chi_{\nu n}$, which is imaginary. In [119], it was hinted that one could get rid of the undesired properties of $\delta_{\nu n}$ by using a different set of eigenfunctions. This was made explicit by Chirilli and Kovchegov [159, 160]. In the remainder of this section we shall review the Chirilli-Kovchegov procedure, and construct the NLO eigenfunctions and the corresponding NLO eigenvalue for any value of $n$. Our goal is to construct functions $\omega^{(1)}_{\nu n}$ and $\Phi^{(1)}_{\nu n}(q)$ such that

$$\begin{aligned}\left[\mathcal{K}^{NLO} \star \left(\varphi_{\nu n} + \overline{\alpha}_\mu \Phi^{(1)}_{\nu n}\right)\right] = & \left(\overline{\alpha}_\mu \chi_{\nu n} + \overline{\alpha}^2_\mu \omega^{(1)}_{\nu n}\right) \left[\varphi_{\nu n}(q) + \overline{\alpha}_\mu \Phi^{(1)}_{\nu n}(q)\right] \\ & + \mathcal{O}(\overline{\alpha}^3_\mu) .\end{aligned} \tag{4.34}$$

We parametrise the NLO eigenvalue $\omega^{(1)}_{\nu n}$ in terms of $\delta_{\nu n}$ and an unknown function $c_{\nu n}$ as

$$\omega^{(1)}_{\nu n} = \frac{\delta_{\nu n}}{4} + c_{\nu n} = i\frac{\beta_0}{8}\, \partial_\nu \chi_{\nu n} + \Delta_{\nu n} + c_{\nu n} , \tag{4.35}$$

where $\Delta_{\nu n}$ collects all the terms in (4.30) that are symmetric under $\nu \to -\nu$, which is a symmetry we expect of the function $\omega^{(1)}_{\nu n}$. Inserting the parametrisation in (4.35) into (4.34) and using (4.29) and the running of the strong coupling,

$$\overline{\alpha}_S(q^2) = \overline{\alpha}_\mu \left[1 - \overline{\alpha}_\mu \frac{\beta_0}{4} \ln \frac{q^2}{\mu^2} + \mathcal{O}(\overline{\alpha}^2_\mu)\right] , \tag{4.36}$$

we obtain

$$\left(\mathcal{K}^{LO} \star \Phi^{(1)}_{\nu n}\right)(q) = \left(c_{\nu n} + \frac{\beta_0}{4}\, \chi_{\nu n} \log \frac{q^2}{\mu^2}\right) \varphi_{\nu n}(q) + \chi_{\nu n}\, \Phi^{(1)}_{\nu n}(q) . \tag{4.37}$$

Following Chirilli and Kovchegov [159, 160], since (4.37) must be satisfied for arbitrary values of $q$, we must take $\Phi^{(1)}_{\nu n}$ proportional to $\varphi_{\nu n}$. We therefore make the following Ansatz,

$$\Phi^{(1)}_{\nu n}(q) = \left( a_0 + a_1 \ln \frac{q^2}{\mu^2} + a_2 \ln^2 \frac{q^2}{\mu^2} \right) \varphi_{\nu n}(q)\,, \tag{4.38}$$

where $a_j$ for $j = 0, 1, 2$ are arbitrary complex coefficients which depend on $\nu$ and $n$. Inserting (4.38) into (4.37), one finds

$$\begin{aligned} a_2 &= i\,\frac{\beta_0}{8}\,\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}}\,, \\ c_{\nu n} &= -i\,a_1\,\partial_\nu \chi_{\nu n} + i\,\frac{\beta_0}{8}\,\frac{\chi_{\nu n}\,\partial^2_\nu \chi_{\nu n}}{\partial_\nu \chi_{\nu n}}\,. \end{aligned} \tag{4.39}$$

The expressions in (4.39) only hold for $\nu \neq 0$, because the denominator has a simple pole for $\nu = 0$, $\partial_\nu \chi_{\nu n|\nu=0} = 0$. To regulate this singularity, we can make use of the principal value prescription

$$\int_{-\infty}^{+\infty} \mathrm{d}\nu \left( P\frac{1}{\nu} \right) f(\nu) \equiv \lim_{\epsilon \to 0} \left( \int_{-\infty}^{-\epsilon} \frac{\mathrm{d}\nu}{\nu} f(\nu) + \int_{\epsilon}^{+\infty} \frac{\mathrm{d}\nu}{\nu} f(\nu) \right)\,. \tag{4.40}$$

Upon imposing this prescription, the solutions in (4.39) for arbitrary $\nu$ are given by

$$\begin{aligned} a_2 &= i\,\frac{\beta_0}{8}\,P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}}\,, \\ c_{\nu n} &= -i\,a_1\,\partial_\nu \chi_{\nu n} + i\,\frac{\beta_0}{8}\,P\frac{\chi_{\nu n}\,\partial^2_\nu \chi_{\nu n}}{\partial_\nu \chi_{\nu n}}\,. \end{aligned} \tag{4.41}$$

With these solutions at hand the eigenfunctions can be written as

$$\Phi_{\nu n}(q) = \varphi_{\nu n}(q) \left[ 1 + \overline{\alpha}_\mu \left( a_0 + a_1 \ln \frac{q^2}{\mu^2} + i\,\frac{\beta_0}{8}\,P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}} \ln^2 \frac{q^2}{\mu^2} \right) \right]\,, \tag{4.42}$$

with the coefficients $a_0$ and $a_1$ still to be determined. Note that we require the eigenfunctions to form a complete and orthonormal set. This places additional constraints on (4.42). Imposing the completeness relation (4.9), the NLO eigenfunction can be written as,

$$\begin{aligned} \Phi_{\nu n}(q) = \varphi_{\nu n}(q) \Bigg[ 1 + \overline{\alpha}_\mu \Bigg( & \frac{1}{2}\,\partial_\nu \Im[a_1] + i\,\Im[a_0] + i\,\Im[a_1]\,\ln \frac{q^2}{\mu^2} \\ & + \frac{\beta_0}{8} \ln \frac{q^2}{\mu^2}\,\partial_\nu P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}} + i\,\frac{\beta_0}{8}\,\ln^2 \frac{q^2}{\mu^2}\,P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}} \Bigg) \Bigg]\,. \end{aligned} \tag{4.43}$$

The orthogonality condition in (4.8) is now automatically fulfilled through NLO and does not add any new constraints. The NLO eigenfunctions are now determined up to two unknown real parameters, $\Im[a_0]$ and $\Im[a_1]$. These parameters can be absorbed into the phase of the eigenfunctions and by using the translation invariance of the $\nu$ integral, respectively. This implies the following result for the NLO eigenfunctions,

$$\Phi_{\nu n}(q) = \varphi_{\nu n}(q)\left[1+\overline{\alpha}_\mu\frac{\beta_0}{8}\,\ln\frac{q^2}{\mu^2}\,\left(\partial_\nu P\frac{\chi_{\nu n}}{\partial_\nu\chi_{\nu n}}+i\,\ln\frac{q^2}{\mu^2}\,P\frac{\chi_{\nu n}}{\partial_\nu\chi_{\nu n}}\right)\right], \tag{4.44}$$

in agreement with [160]. With this choice of eigenfunctions, the NLO eigenvalue becomes

$$\omega^{(1)}_{\nu n} = \Delta_{\nu n} = \frac{\delta_{\nu n}}{4} - i\frac{\beta_0}{8}\,\partial_\nu\chi_{\nu n}\,, \tag{4.45}$$

where $\delta_{\nu n}$ is given in (4.30). The new eigenvalue is indeed real and independent of $q^2$, as we would expect. Now that we have defined appropriate NLO eigenvalues and -functions, let us turn to the BFKL ladder.

### 4.2.2. The BFKL Ladder at NLLA

To determine the BFKL ladder through NLLA according to our newly found eigenvalue and -functions, we start by expanding the product of two eigenfunctions through NLO. We have

$$\Phi_{\nu n}(q_1)\,\Phi^*_{\nu n}(q_2) = \varphi_{\nu n}(q_1)\,\varphi^*_{\nu n}(q_2)\left[1+\overline{\alpha}_\mu\frac{\beta_0}{4}\,\ln\frac{s_0}{\mu^2}\,X_{\nu n}\left(\frac{q_1^2}{q_2^2}\right)\right], \tag{4.46}$$

where we defined

$$X_{\nu n}(x) = \partial_\nu P\frac{\chi_{\nu n}}{\partial_\nu\chi_{\nu n}} + i\,\ln x\,P\frac{\chi_{\nu n}}{\partial_\nu\chi_{\nu n}}\,. \tag{4.47}$$

From (4.12), we can conclude

$$\begin{aligned} f(q_1,q_2,y) = &\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\nu\,e^{y\,\omega_{\nu n}}\,\varphi_{\nu n}(q_1)\,\varphi^*_{\nu n}(q_2)\\ &\times\left(1+\overline{\alpha}_\mu\frac{\beta_0}{4}\,\ln\frac{s_0}{\mu^2}\,X_{\nu n}(q_1^2/q_2^2)+\mathcal{O}(\overline{\alpha}_\mu^2)\right). \end{aligned} \tag{4.48}$$

The term involving $X_{\nu n}$ can be simplified using integration by parts, giving

$$\begin{aligned} f(q_1,q_2,y) = &\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\nu\,e^{y\,\omega_{\nu n}}\,\varphi_{\nu n}(q_1)\,\varphi^*_{\nu n}(q_2)\\ &\times\left(1-\overline{\alpha}_\mu^2\frac{\beta_0}{4}\,\ln\frac{s_0}{\mu^2}\,y\,\chi_{\nu n}+\mathcal{O}(\overline{\alpha}_\mu^3)\right). \end{aligned} \tag{4.49}$$

Now note that the term involving the beta function can be absorbed by writing (4.49) in terms of the renormalised coupling with renormalisation scale $s_0$, see

$$\begin{aligned}
&\exp y\left[\overline{\alpha}_S(s_0)\,\chi_{\nu n}+\overline{\alpha}_S(s_0)^2\,\Delta_{\nu n}+\mathcal{O}(\overline{\alpha}_S^3)\right]\\
&\quad=\exp y\left[\overline{\alpha}_\mu\left(1-\overline{\alpha}_\mu\frac{\beta_0}{4}\log\frac{s_0}{\mu^2}\right)\chi_{\nu n}+\overline{\alpha}_\mu^2\,\Delta_{\nu n}+\mathcal{O}(\overline{\alpha}_\mu^3)\right]\\
&\quad=e^{y\,\omega_{\nu n}}\left(1-\overline{\alpha}_\mu^2\frac{\beta_0}{4}\log\frac{s_0}{\mu^2}\,y\,\chi_{\nu n}+\mathcal{O}(\overline{\alpha}_\mu^3)\right).
\end{aligned}\tag{4.50}$$

Through NLLA, this means we can cast (4.49) in the equivalent form,

$$\begin{aligned}
f(q_1,q_2,y)\big|_{\text{NLLA}}=&\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\nu\,\varphi_{\nu n}(q_1)\,\varphi_{\nu n}^*(q_2)\\
&\times e^{y\,\overline{\alpha}_S(s_0)[\chi_{\nu n}+\overline{\alpha}_S(s_0)\Delta_{\nu n}]}\,.
\end{aligned}\tag{4.51}$$

Thus, if we choose the scale of the strong coupling to be the geometric mean of the transverse momenta, $\mu^2=s_0=\sqrt{q_1^2q_2^2}$, we can use the LO eigenfunctions instead of the NLO ones. This is a very valuable trick, as it will allow us to recycle parts of the analysis done at LLA.

### 4.2.3. The Fourier-Mellin Representation

Having determined the NLLA BFKL ladder, we now turn to its perturbative expansion. We set the renormalisation scale to the geometric mean of the two transverse momenta. Define

$$f^{NLL}(q_1,q_2,\eta_{s_0})=\frac{1}{2\pi\sqrt{q_1^2q_2^2}}\sum_{k=1}^{\infty}\frac{\eta_{s_0}^k}{k!}\,f_{k+1}^{NLL}(z)\,,\tag{4.52}$$

with $\eta_{s_0}=y\,\overline{\alpha}_S(s_0)$. The perturbative coefficients are given by the Fourier-Mellin transform,

$$f_k^{NLL}(z)=\mathcal{F}\left[\Delta_{\nu n}\,\chi_{\nu n}^{k-2}\right]=\sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar z}\right)^{n/2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{2\pi}\,|z|^{2i\nu}\,\Delta_{\nu n}\,\chi_{\nu n}^{k-2}\,.\tag{4.53}$$

Our aim is to compute these perturbative coefficients to high orders. In order to facilitate our discussion, it will be useful to split the NLO eigenvalue $\Delta_{\nu n}$ into a sum of terms,

$$\Delta_{\nu n}=\frac{1}{4}\,\delta_{\nu n}^{(1)}+\frac{1}{4}\,\delta_{\nu n}^{(2)}+\frac{1}{4}\,\delta_{\nu n}^{(3)}+\frac{3}{2}\,\zeta_3+\gamma_K^{(2)}\chi_{\nu n}-\frac{1}{8}\beta_0\chi_{\nu n}^2\,,\tag{4.54}$$

where we singled out terms proportional to powers of the LO eigenvalue, because their Fourier-Mellin transform at any order will evaluate to the known coefficients appearing in the expansion of the BFKL ladder at LLA, cf. (4.25). In QCD, the remaining terms are given by

$$\delta_{\nu n}^{(1)} = \partial_\nu^2 \chi_{\nu n}\,, \tag{4.55}$$

$$\delta_{\nu n}^{(2)} = -\,2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)\,, \tag{4.56}$$

$$\begin{aligned}\delta_{\nu n}^{(3)} =& -\frac{\Gamma(\gamma)\Gamma(1-\gamma)}{2i\nu}\left[\psi\left(\gamma\right)-\psi\left(1-\gamma\right)\right] \\ &\times\left[\delta_{n0}\left(3+A\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)-\delta_{|n|2}\left(A\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\right)\right]\,,\end{aligned} \tag{4.57}$$

with

$$A = \left(1+\frac{N_f}{N_c^3}\right)\,. \tag{4.58}$$

The coefficients in (4.53) can then be written as

$$\begin{aligned}f_k^{NLL}(z) =& \frac{1}{4}\,C_k^{(1)}(z)+\frac{1}{4}\,C_k^{(2)}(z)+\frac{1}{4}\,C_k^{(3)}(z) \\ &+\frac{3}{2}\,\zeta_3\,f_{k-2}^{LL}(z)+\gamma_K^{(2)}\,f_{k-1}^{LL}(z)-\frac{1}{8}\,\beta_0\,f_k^{LL}(z)\,,\end{aligned} \tag{4.59}$$

where we set $f_0^{LL}(z) = \mathcal{F}[1] = \pi\,\delta^{(2)}(1-z)$. The only unknowns in (4.59) are the functions $C_k^{(i)}$, which are defined by

$$C_k^{(i)}(z) = \mathcal{F}\left[\delta_{\nu n}^{(i)}\,\chi_{\nu n}^{k-2}\right]\,, \tag{4.60}$$

with $k \geq 2$. In the remainder of this section we discuss the computation of each of these quantities in turn.

### 4.2.4. The Contribution from $\delta^{(1)}$

We first turn our attention to the contributions $C_k^{(1)}(z)$. Note that inspired by the results of Section 3.3.1.1, we aim to compute it to high orders using a loop recursion

$$C_{k+1}^{(1)}(z) = \left(\mathcal{X} * C_k^{(1)}\right)(z)\,, \tag{4.61}$$

with $k \geq 2$, and where we defined

$$\mathcal{X}(z) \equiv \mathcal{F}\left[\chi_{\nu n}\right] = f_1^{LL}(z) = \frac{|z|}{2\pi\,|1-z|^2}\,. \tag{4.62}$$

The starting point of the recursion is the two-loop coefficient. In order to compute it, we make use of the following trick

$$\mathcal{F}\left[\partial_\nu A_{\nu n}\right] = -i\,\mathcal{F}\left[A_{\nu n}\right]\,\log|z|^2 = -i\,\mathcal{F}\left[A_{\nu n}\right]\,\mathcal{G}_0(z)\,. \tag{4.63}$$

Hence, we find

$$C_2^{(1)}(z) = \mathcal{F}\left[\partial_\nu^2 \chi_{\nu n}\right] = -\mathcal{X}(x)\,\log^2|z|^2 = -\frac{|z|}{\pi\,|1-z|^2}\mathcal{G}_{0,0}(z)\,. \tag{4.64}$$

Since $\mathcal{X}$ and $C_2^{(1)}$ only have isolated singularities at $z = 0$ and $z = 1$, the formalism developed in Section 3.3.1.1 straightforwardly applies. In [7], this was applied to compute the functions $C_k^{(1)}(z)$ up to $k \leq 5$. Furthermore, up to a rational prefactor, the terms $C_k^{(1)}$ are pure functions of weight $k$. This can be shown to all orders from the convolution integral and is a straightforward application of Lemma C.1 in Appendix C.

### 4.2.5. The Contribution from $\delta^{(2)}$

We now turn our attention to the contributions $C_k^{(2)}(z)$. Once again, we aim to use the recursion

$$C_{k+1}^{(2)}(z) = \left(\mathcal{X} * C_k^{(2)}\right)(z)\,. \tag{4.65}$$

Unlike in the previous case, the two-loop coefficient can not be obtained straightforwardly from known LLA results. This means that to obtain the two-loop contribution, one must compute the Fourier-Mellin integral. This is done by closing the integration contour in the $\nu$-plane and summing over residues. The resulting multiple sums can then be associated to polylogarithms. In this way, one obtains

$$C_2^{(2)}(z) = \mathcal{F}\left[\delta_{\nu n}^{(2)}\right] = C_2^{(2,1)}(z) + C_2^{(2,2)}(z)\,, \tag{4.66}$$

with

$$\begin{aligned} C_2^{(2,1)}(z) =& \frac{|z|\,(z-\bar{z})}{2\pi\,|1+z|^2|1-z|^2}\left[\mathcal{G}_{1,0}(z) - \mathcal{G}_{0,1}(z)\right]\,, \\ C_2^{(2,2)}(z) =& \frac{|z|\,(1-|z|^2)}{2\pi\,|1+z|^2|1-z|^2}\left[\mathcal{G}_{1,0}(z) + \mathcal{G}_{0,1}(z) - G_{-1,0}\left(|z|^2\right) - \zeta_2\right]\,. \end{aligned} \tag{4.67}$$

We see that unlike $C_2^{(1)}(z)$, $C_2^{(2)}(z)$ is not a pure function up to a rational prefactor, but it can be separated into two expressions $C_2^{(2,1)}(z), C_2^{(2,2)}(z)$ for which this statement does hold. Furthermore, the function content for $C_2^{(2)}(z)$ is more involved, while $C_2^{(2,1)}$ is a linear combination of SVMPLs with singularities at most at $z = 0$ and $z = 1$, $C_2^{(2,2)}$ is expressed in terms of both SVMPLs and ordinary HPLs evaluated at $|z|^2$. Though $C_2^{(2,2)}$ can not be written in terms of SVMPLs, it is still single-valued, as the argument of $G_{-1,0}\left(|z|^2\right)$ is positive-definite and the function has no branch cut on the positive real axis.

To use the convolution formalism, we must first have a detailed understanding of such functions and their properties. In particular, to apply Stokes' theorem (3.97) we must be able to compute the single-valued primitive of these objects. To this end, we consider a larger class of functions which envelops the studied SVMPLs. In [161] Schnetz defined a more general class of single-valued multiple polylogarithms in one complex variable with singularities at

$$z = \frac{\alpha\,\bar{z} + \beta}{\gamma\,\bar{z} + \delta}\,, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}\,. \tag{4.68}$$

These functions reduce to the SVMPLs of Section 3.1.2 in the case where the singularities are at constant locations. Since the two-loop result (4.66) has a singularity at $z = -1/\bar{z}$, we expect that the coefficients $\mathcal{C}_k^{(2,1)}$ and $\mathcal{C}_k^{(2,2)}$ can be expressed in terms of Schnetz' generalised SVMPLs (gSVMPLs). Let us review the definition of these objects. Consider a set of functions $\mathscr{G}(a_1, \ldots, a_n; z)$ defined by the following conditions,

1. The functions $\mathscr{G}(a_1, \ldots, a_n; z)$ are single-valued.
2. They form a shuffle algebra.
3. They satisfy the holomorphic differential equation,

$$\partial_z \mathscr{G}(a_1, \ldots, a_n; z) = \frac{1}{z - a_1}\,\mathscr{G}(a_2, \ldots, a_n; z)\,. \tag{4.69}$$

4. They vanish for $z = 0$, except if all $a_i$ are 0, in which case we have

$$\mathscr{G}(\underbrace{0, \ldots, 0}_{n \text{ times}}; z) = \frac{1}{n!}\log^n |z|^2\,. \tag{4.70}$$

5. The singularities in (4.69) are antiholomorphic functions of $z$ of the form,

$$a_i = \frac{\alpha\,\bar{z} + \beta}{\gamma\,\bar{z} + \delta}\,, \text{ for some } \alpha, \beta, \gamma, \delta \in \mathbb{C}\,. \tag{4.71}$$

Every linear combination $f$ of such functions with singularities at most for $z = a_i$, with $a_i$ defined in (4.71), has both a single-valued holomorphic and antiholomorphic primitive [161], which allows us to perform the convolution integral. For the purpose of computing $C_k^{(2)}(z)$ we need only consider gSVMPLs with $a_i \in \{-1, 0, 1, -1/\bar{z}\}$. These functions can be constructed recursively in weight [161] as was done in [7] up to weight five. Note that the algebra generated by these gSVMPLs contains two natural subalgebras,

1. If $a_i \neq -1/\bar{z}$, the gSVMPL reduces to an ordinary SVMPL,
$$\mathscr{G}(a_1, \ldots, a_n; z) = \mathcal{G}(a_1, \ldots, a_n; z)\,, \qquad \text{if } a_i \in \{-1, 0, 1\}\,. \tag{4.72}$$
2. If $a_i \neq \pm 1$, the gSVMPL reduces to an ordinary HPL evaluated at $|z|^2$,
$$\mathscr{G}(a_1, \ldots, a_n; z) = G(\bar{z}\, a_1, \ldots, \bar{z}\, a_n; |z|^2)\,, \qquad \text{if } a_i \in \{0, -1/\bar{z}\}\,. \tag{4.73}$$

These subalgebras cover the class of functions encountered in (4.67). An example of a function that cannot be reduced to these two subalgebras is given by

$$\begin{aligned}\mathscr{G}_{-1/\bar{z},1}(z) =& G_1(\bar{z})\, G_{-1/\bar{z}}(z) + G_{-1/\bar{z},1}(z) \\ &- G_{-1,1}(\bar{z}) + \log(2)\, G_{-1}(\bar{z}) - \log(2)\, G_1(\bar{z})\,.\end{aligned} \tag{4.74}$$

Since Schnetz' formalism allows us to compute the single-valued antiholomorphic primitive, we are able to compute the convolution integral in (4.65) using (3.97). In [7], this was applied to compute the functions $C_k^{(2)}(z)$ up to $k \leq 5$. The results can be written in the form,

$$C_k^{(2)}(z) = \frac{|z|\,(z-\bar{z})}{2\pi\,|1+z|^2|1-z|^2}\mathcal{C}_k^{(2,1)} + \frac{|z|\,(1-|z|^2)}{2\pi\,|1+z|^2|1-z|^2}\mathcal{C}_k^{(2,2)}\,, \tag{4.75}$$

where the functions $\mathcal{C}_k^{(2,i)}$ are pure and have uniform weight $k$. Once again it follows from Lemma C.1 in Appendix C that this structure holds at any loop order. Finally, it is interesting to note that at two and three loops, the results can be expressed in terms of SVHPLs and ordinary HPLs evaluated at $|z|^2$. Starting from four loops, one obtains genuine gSVMPLs that can no longer be expressed in terms of HPLs.

### 4.2.6. The Contribution from $\delta^{(3)}$

The final contributions left to consider are those coming from $\delta^{(3)}$. We separate these contribution as follows

$$C_k^{(3)} = C_k^{(3,0)}(z) + C_k^{(3,2)}(z)\,, \tag{4.76}$$

where $C_k^{(3,i)}(z)$ is due to the terms proportional to $\delta_{|n|i}$. Since the dependence of $\delta_{\nu n}^{(3)}$ on $n$ is only through Kronecker deltas, the Fourier-Mellin transform reduces to an ordinary Mellin integral,

$$\begin{aligned} C_k^{(3)}(z) &= -\int_{-\infty}^{+\infty} \frac{\mathrm{d}\nu}{2\pi} \, |z|^{2i\nu} \, \frac{\Gamma(\gamma)\Gamma(1-\gamma)}{2i\nu} \left[\psi(\gamma) - \psi(1-\gamma)\right] \\ &\quad \times \left[\chi_{\nu 0}^{k-2} \, A_0(\nu) + \left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right) \chi_{\nu 2}^{k-2} \, A_2(\nu)\right], \end{aligned} \tag{4.77}$$

with

$$A_0(\nu) = 3 + A \, \frac{2 + 3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}, \tag{4.78}$$

$$A_2(\nu) = -A \, \frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}. \tag{4.79}$$

This integral can be evaluated by closing the contour in the upper half-plane and summing up the residues at $\nu = i\left(\frac{1}{2} + m\right)$, $m \in \mathbb{N}$. The sum of residues can be performed using the techniques of [162–164], expressing the result in terms of MPLs of the type $G(a_1, \ldots, a_n; |z|)$, with $a_k \in \{-i, 0, i\}$. This was done in [7] up to $k \leq 5$. Clearly these functions are single-valued functions of the complex variable $z$, because they have no branch cut on the positive real axis.

Complex conjugation acts in a simple and natural way on this class of functions: it leaves $|z|$ invariant and exchanges the purely imaginary arguments. It is then natural to decompose the functions into real and imaginary parts. For example, we can write

$$G(0, i; |z|) \pm G(0, -i; |z|) = \begin{cases} \frac{1}{2} G(0, -1, |z|^2), \\ 2i \, \mathrm{Ti}_2(|z|), \end{cases} \tag{4.80}$$

where $\mathrm{Ti}_n(z)$ are the *inverse tangent integrals*,

$$\mathrm{Ti}_n(z) = \Im(\mathrm{Li}_n(i\,z)) = \Im(\,G(\vec{0}_{n-1}, i;\, z)). \tag{4.81}$$

The results for generic $C_k^{(3)}$ can be written in terms of the gSVMPLs of the previous section, and *generalised inverse tangent integrals*,

$$\mathrm{Ti}_{m_1,\ldots,m_k}(|z|) = \Im(\mathrm{Li}_{m_1,\ldots,m_k}(\sigma_1, \ldots, \sigma_{k-1}, i\,\sigma_k\,|z|)), \tag{4.82}$$

where $\sigma_j = \mathrm{sign}(m_j)$ and $\mathrm{Li}_{m_1,\ldots,m_k}$ denotes the sum representation of MPLs,

$$\begin{aligned} \mathrm{Li}_{m_1,\ldots,m_k}(z_1, \ldots, z_k) &= \sum_{0<n_1<n_2<\cdots<n_k} \frac{z_1^{n_1} \ldots z_k^{n_k}}{n_1^{m_1} \ldots n_k^{m_k}} \\ &= (-1)^k G\Big(\underbrace{0, \ldots, 0}_{m_k-1}, \frac{1}{z_k}, \ldots, \underbrace{0, \ldots, 0}_{m_1-1}, \frac{1}{z_1 \ldots z_k}; 1\Big). \end{aligned} \tag{4.83}$$

Neither of the functions $C_k^{(3,i)}$ is uniform in transcendental weight, but both $C_k^{(3,0)}$ and $C_k^{(3,2)}$ involve functions of weight $0 \leq w \leq k$.

## 4.3. From Particle Content to Transcendentality

### 4.3.1. The NLLA BFKL Ladder in QCD

In the previous section we have discussed computational methods which may be invoked to compute the BFKL ladder in QCD at NLLA. In [7], these methods were used to compute the various contributions through five loops. Furthermore, we have discussed the analytic properties of said contributions. In particular, we have observed that the full result is polylogarithmic and involves terms of various transcendental weight. As expected, the QCD result is not a maximal weight function. Let us analyse the weights of the various contributions in (4.59) for QCD.

We classify the contributions which give rise to functions of weight $k$ in momentum space. Using the result of Appendix C, we can show that the functions $f_k^{LL}$ and $C_k^{(i)}$ for $i \in \{1, 2\}$ have uniform weight $k-1$ and $k$ respectively. Hence, we see that terms of weight strictly less than $k$ in (4.59) arise from the following places.

1. The term $\beta_0\, f_k^{LL}(z)$ involving the beta function has weight $k-1$, and so it is always of lower weight.
2. Because the cusp anomalous dimension in QCD is not of maximal weight, $\gamma_K^{(2)}\, f_{k-1}^{LL}(z)$, involves a mixture of weights $k-2 \leq w \leq k$.
3. Using the explicit results through five loops of [7], we see that in QCD the functions $C_k^{(3)}$ involve terms of weight $0 \leq w \leq k$.

It is instructive to compare this to the corresponding result in $\mathcal{N} = 4$ SYM, which we do expect to have uniform weight. First of all, for $\mathcal{N} = 4$ SYM the beta function vanishes and the cusp anomalous dimension has uniform weight. From [158] it can be seen that $\mathcal{N} = 4$ SYM has no contribution $C_k^{(3)}$ to the NLLA BFKL ladder. Thus we can conclude that indeed the maximal transcendentality-violating terms do not appear for $\mathcal{N} = 4$ SYM. Note however that $C_k^{(3)}$ also has terms of maximal weight, and so the maximal weight part of QCD does not correspond to the $\mathcal{N} = 4$ SYM result. In what follows, we will rewrite (4.59) for a generic gauge theory, and see what constraints maximal transcendentality places on the matter content.

### 4.3.2. Generic Gauge Theories

We study the transcendental weight properties of the BFKL ladder at NLLA in a generic $SU(N_c)$ gauge theory with scalar or fermionic matter in arbitrary representations. Our starting point is the singlet BFKL eigenvalue at NLO in a generic theory [158]. This object depends solely on the matter content of the theory. To match our discussion of QCD (4.59), we write the BFKL eigenvalue in a generic theory as

$$\begin{aligned}\Delta_{\nu n} =& \frac{1}{4}\delta^{(1)}_{\nu n} + \frac{1}{4}\delta^{(2)}_{\nu n} + \frac{1}{4}\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) \\ &+ \frac{3}{2}\zeta_3 + \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s)\,\chi_{\nu n} - \frac{1}{8}\beta_0(\tilde{n}_f, \tilde{n}_s)\,\chi^2_{\nu n}\,.\end{aligned} \tag{4.84}$$

The quantities $\delta^{(1)}_{\nu n}$ and $\delta^{(2)}_{\nu n}$ are independent of the theory under consideration, and so they are the same as in QCD [158], given by (4.55) and (4.56). The one-loop beta function and the two-loop cusp anomalous dimension in the dimensional reduction scheme are given by

$$\begin{aligned}\beta_0(\tilde{n}_f, \tilde{n}_s) &= \frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c}\,, \\ \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s) &= \frac{1}{4}\left(\frac{64}{9} - \frac{10\,\tilde{n}_f}{9N_c} - \frac{4\,\tilde{n}_s}{9N_c}\right) - \frac{\zeta_2}{2}\,,\end{aligned} \tag{4.85}$$

where we defined

$$\tilde{n}_f = \sum_R N_f^R T_R \quad \text{and} \quad \tilde{n}_s = \sum_R N_s^R T_R\,, \tag{4.86}$$

where the sum runs over all irreducible representations $R$ of $SU(N_c)$, and $N_f^R$ and $N_s^R$ denote the number of Weyl fermions and real scalars transforming in the representation $R$. The index $T_R$ of the representation is defined through $\mathrm{Tr}(T_R^a T_R^b) = T_R\,\delta^{ab}$, with $T_R^a$ the infinitesimal generators of the representation $R$. We fix the normalisation of the structure constants of $SU(N_c)$ such that for the fundamental representation $T_F = 1/2$. The contribution from $\delta^{(3)}_{\nu n}$ is determined entirely by the matter content of the theory [158]. We find

$$\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) = \delta^{(3,1)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) + \delta^{(3,2)}_{\nu n}(\tilde{N}_f, \tilde{N}_s)\,, \tag{4.87}$$

with

$$\tilde{N}_x = \frac{1}{2}\sum_R N_x^R\, T_R\,(2C_R - N_c)\,, \qquad x = f, s\,, \tag{4.88}$$

and $C_R = T^a_R T_{R\,a}$ is the *quadratic Casimir* of the representation $R$. The functions $\delta^{(3,i)}_{\nu n}$ are given by

$$\begin{aligned}\delta^{(3,1)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) =& \frac{f(\gamma)}{8}\left[\delta_{n0}\left(2\tilde{N}_s + 12\tilde{N}_f - 30N_c^2\right)\right.\\ &\left. + \delta_{|n|2}\left(N_c^2 - 2\tilde{N}_f + \tilde{N}_s\right)\right],\\ \delta^{(3,2)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) =& \frac{f(\gamma)}{8}\left[\frac{(3\delta_{|n|2} - 2\delta_{n0})(2\gamma-1)}{2(2\gamma-3)(2\gamma+1)}\left(N_c^2 - 2\tilde{N}_f + \tilde{N}_s\right)\right],\end{aligned} \tag{4.89}$$

with

$$f(\gamma) = \frac{1}{4\pi^2(1-2\gamma)}\Gamma(1-\gamma)\Gamma(\gamma)\Big[\psi(1-\gamma) - \psi(\gamma)\Big]. \tag{4.90}$$

In [7] it was checked explicitly through five loops and conjectured to all orders that Fourier-Mellin transforms of the type $\mathcal{F}\left[\delta^{(3,1)}_{\nu n}\,\chi^k_{\nu n}\right]$ give rise to functions of uniform weight $k+2$, while the remaining contributions from $\delta^{(3,2)}_{\nu n}$ only produce lower weight terms.

Using the above analysis, we can define a set of necessary and sufficient conditions for a theory to have a BFKL ladder at NLLA of uniform transcendental weight. Since the NLO singlet BFKL eigenvalue depends only on the matter content of the theory, we obtain constraints only in terms of these parameters. They are given below.

1. The one-loop beta function vanishes,
$$\frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c} = 0\,. \tag{4.91}$$

2. The two-loop cusp anomalous dimension is proportional to $\zeta_2$,
$$\frac{16}{9} - \frac{5\,\tilde{n}_f}{18N_c} - \frac{\tilde{n}_s}{9N_c} = 0\,. \tag{4.92}$$

3. The contribution from $\delta^{(3,2)}_{\nu n}$ vanishes, which implies
$$2\tilde{N}_f = N_c^2 + \tilde{N}_s\,. \tag{4.93}$$

In the next section we will restrict ourselves to a limited set of matter representations in order to classify the solutions to these constraints. In particular, we will consider only matter in the adjoint and fundamental representation of the gauge group. Note, however, that just by studying these constraints we can make an important observation. If (4.93) is satisfied, the term proportional to $\delta_{|n|2}$ is absent from $\delta^{(3,1)}_{\nu n}$,

$$\delta^{(3,1)}_{\nu n}(\tilde{N}_f, 2\tilde{N}_f - N_c^2) = 2f(\gamma)\,\delta_{n0}\,\left(\tilde{N}_f - 2N_c^2\right)\,. \tag{4.94}$$

The Fourier-Mellin integral of this missing term has maximal weight, and so by requiring all non-maximal weight terms to vanish, we will force some of the maximal terms to vanish as well. This implies that there is no theory such that the BFKL ladder at NLLA has uniform and maximal weight and agrees with the maximal weight terms in QCD.

### 4.3.3. Adjoint and Fundamental Matter Content

In this section we restrict ourselves to the case of theories with matter in the fundamental and adjoint representations. The indices and Casimir operators of the adjoint and fundamental representations are

$$T_A = C_A = N_c \quad \text{and} \quad C_F = T_F\,\frac{N_c^2-1}{N_c}\,. \tag{4.95}$$

We can then write

$$\begin{aligned} \tilde{n}_x &= T_A N_x^A + T_F\, N_x^F\,, \\ \widetilde{N}_x &= \frac{1}{2} T_A\,(2C_A - N_c) N_x^A + \frac{1}{2}\,T_F\,(2C_F - N_c) N_x^F\,, \end{aligned} \tag{4.96}$$

where $x = s, f$. We aim to keep the number of colours arbitrary and thus will solve (4.91)-(4.93) for generic $N_c$. Inserting (4.96) into (4.93), we find

$$T_F\,(2C_F - N_c)\,N_f^F + N_c^2\,N_f^A = N_c^2 + \frac{1}{2} N_c^2\,N_s^A + \frac{1}{2}\,T_F\,(2C_F - N_c)\,N_s^F\,. \tag{4.97}$$

Since we are looking for solutions that are valid for an arbitrary number of colours, this implies

$$2\,N_f^F = N_s^F \quad \text{and} \quad 2\,N_f^A = 2 + N_s^A\,. \tag{4.98}$$

Now recall that we are counting in terms of Weyl fermions and real scalars. This means that (4.98) implies that the number of bosonic degrees of freedom equals the number of fermionic ones. We learned in Section 1.1.3 that this is one of the key features of supersymmetric theories. Thus, in what follows, we can represent the solutions of (4.98) in terms of supersymmetric field theories. Note that we do not claim that any theory solving (4.98) must have a certain degree of supersymmetry, but only that their matter content corresponds to that of a supersymmetric one.

We parameterise the matter content in terms of a gauge multiplet with $\mathcal{N}$ supersymmetries and $N_F \equiv N_f^F$ chiral multiplets[2] in the fundamental representation

[2] We consider chiral multiplets in $\mathcal{N} = 1$ supersymmetry, consisting of a Weyl fermion and two real scalar degrees of freedom, see Figure 1.2.

and $N_A \equiv N_f^A - \mathcal{N}$ chiral multiplets in the adjoint representation. In terms of the parameters $\mathcal{N}$, $N_F$ and $N_A$, the matter content is then cast in the form,

$$\begin{aligned} N_f^A &= \mathcal{N} + N_A\,, \\ N_f^F &= N_F\,, \\ N_s^A &= 2(\mathcal{N}-1) + 2N_A\,, \\ N_s^F &= 2N_F\,, \end{aligned} \tag{4.99}$$

which is consistent with (4.98). Inserting these reparameterisations of the matter content into (4.91) and (4.92), they both reduce to the equation,

$$N_A + \frac{N_F}{2N_c} + \mathcal{N} = 4\,. \tag{4.100}$$

This equation has only four positive integer solutions which are shown in Table 4.1. For these theories, the BFKL eigenvalue at NLO takes the form,

$$\Delta_{\nu n} = \frac{1}{4}\delta^{(1)}_{\nu n} + \frac{1}{4}\delta^{(2)}_{\nu n} + \frac{3}{2}\zeta_3 + \frac{\zeta_2}{2}\chi_{\nu n} + f(\gamma)\,(N_c^2+1)\,(N_f^A - 4)\,\delta_{n0}\,. \tag{4.101}$$

| | | | | |
|---|---|---|---|---|
| $\mathcal{N}$ | 4 | 2 | 1 | 1 |
| $N_A$ | 0 | 0 | 0 | 2 |
| $N_F$ | 0 | $4N_c$ | $6N_c$ | $2N_c$ |

**Table 4.1:** The four solutions to (4.100)

Let us have a look at these solutions in detail. The first solution is given by $(\mathcal{N}, N_A, N_F) = (4, 0, 0)$ and thus has the same field content as $\mathcal{N} = 4$ SYM. It is unsurprising that we find this solution, as $\mathcal{N} = 4$ SYM is conjectured to be maximally transcendental and we had already discussed the vanishing of lower weight terms for its BFKL ladder in the previous section.

Of more interest are the other solutions. The solution $(\mathcal{N}, N_A, N_F) = (2, 0, 4N_c)$ has the field content of $\mathcal{N} = 2$ SQCD at the conformal point[3], introduced in Section 1.3.2 and also refered to as $\mathcal{N} = 2$ SCQCD. The solution $(\mathcal{N}, N_A, N_F) = (1, 0, 6N_c)$ corresponds to $\mathcal{N} = 1$ SQCD for $N_f = 3$, implying a vanishing one-loop beta function. Put otherwise, it corresponds to $\mathcal{N} = 1$ SQCD at the lower end of the conformal window. The final solution $(\mathcal{N}, N_A, N_F) = (1, 2, 2N_c)$ can be interpreted as the naive[4] matter content of an "$\mathcal{N} = 3$ gauge multiplet"

[3]Note that for our discussion here, we need not differentiate between fundamental and anti-fundamental matter, leading to the apparent "mismatch" in field content w.r.t. Sections 1.3.1 and 1.3.2.

[4]The $\mathcal{N} = 3$ multiplet in 4 dimensions must naturally be enlarged to an $\mathcal{N} = 4$ multiplet to be consistent with CPT invariance.

with added fundamental matter to enforce the vanishing of the one-loop beta function.

In conclusion, we find that demanding maximal transcendentality for the BFKL ladder at NLLA leads us to consider supersymmetric theories with vanishing one-loop beta functions. As further logarithmic orders will involve higher orders of the beta function, this hints towards a connection between maximal transcendentality and superconformal symmetry. A natural question to ask at this point, is whether this connection only holds in this specific kinematic regime. In the next chapter, we will study the transcendentality of an amplitude in $\mathcal{N} = 2$ S(C)QCD in full kinematics to see whether this connection holds in a testing case beyond the Regge limit.

# 5 | Half-maximal SUSY in General Kinematics

In this chapter we study a scattering amplitude in full kinematics in $\mathcal{N} = 2$ SQCD. This theory has several ingredients which make it a valuable testing ground. As a theory with extended supersymmetry, one hopes part of the simplicity observed for $\mathcal{N} = 4$ SYM would survive. On the other hand, it contains fundamental matter and this tuneable matter content implies that the theory is only superconformal for a certain choice of $N_f$, see (1.38). In this sense, $\mathcal{N} = 2$ SQCD interpolates QCD and $\mathcal{N} = 4$ SYM and it is interesting to see how many properties of $\mathcal{N} = 4$ SYM survive for this Goldilocks theory. In particular, with regard to the discussion of the previous chapter, we would like to see whether the maximal transcendentality manifested by the BFKL ladder of the conformal limit of this theory at NLLA breaks down when going beyond the high-energy regime. In previous work, the leading-colour contributions to one-loop $N$-gluon scattering in $\mathcal{N} = 2$ SCQCD were computed [165] and shown to have maximal transcendental weight. At two-loop order however, the four-gluon amplitude was found to break maximal transcendentality through the presence of a single MZV of lower weight [166, 167]. We wish to investigate this peculiar structure beyond leading-colour and beyond the conformal point.

To this end, we study the full-colour two-loop four-gluon amplitude in $\mathcal{N} = 2$ SQCD, which was computed in [8] starting from the integrand derived in [168]. We review the structure of said integrand, before describing the integration procedure used to obtain the final result. Then we investigate the ultraviolet and infrared structure of the integrated amplitude, before turning to a study of the transcendentality of the result, at first for generic $N_c$ and $N_f$ and consequently in the conformal point.

## 5.1. Four-Gluon Scattering in $\mathcal{N} = 2$ SQCD

### 5.1.1. The Colour-Dual Integrand

Consider the $N$-gluon amplitude in $\mathcal{N} = 2$ SQCD. Prior to renormalisation, we perturbatively expand $N$-point amplitudes in terms of the *bare coupling* $\alpha_S^0$,

$$\mathcal{A}_N = (4\pi\alpha_S^0)^{\frac{N-2}{2}} \sum_{L=0}^{\infty} \left(\frac{\alpha_S^0 S_\epsilon}{4\pi}\right)^L \mathcal{A}_N^{(L)}\,, \tag{5.1}$$

where $S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}$ anticipates dimensional regularisation in $D = 4 - 2\epsilon$ dimensions. In what follows we restrict ourselves to multiplicity four, and will always use kinematics defined by $s > 0$; $t, u < 0$, where we recall $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, $u = (p_1 + p_3)^2$, with $s + t + u = 0$, and all momenta outgoing. Recall, furthermore, that our discussion of the supersymmetric Ward identities in Section 2.1.1 implies we need only consider the MHV contribution.

We are interested in computing the four-gluon two-loop amplitude $\mathcal{A}_4^{(2)}$. Our starting point is the two-loop integrand computed in [168]. Let us review some of the important properties of the integrand before turning to the integration itself. The integrand of [168] consists of nineteen independent numerator contributions. Of these only ten give rise to non-vanishing integrals. Each of these numerators can be represented by a diagram, and they are shown in Figure 5.1.

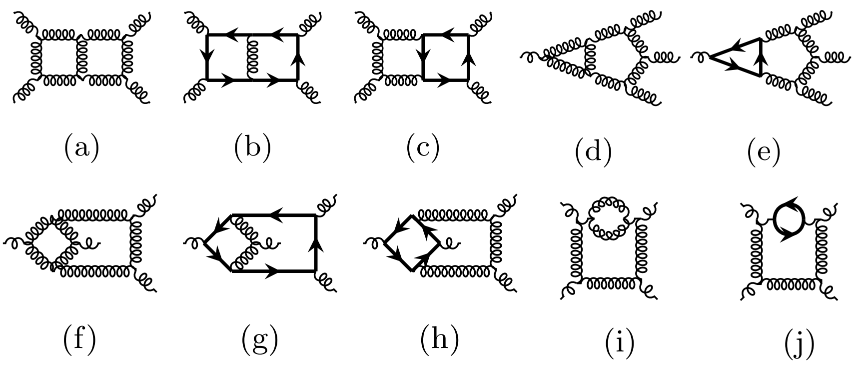



**Figure 5.1:** The ten cubic diagrams that describe the four-gluon two-loop amplitude in $\mathcal{N} = 2$ SQCD.

Using these ten diagrams (a–j) the $\mathcal{N} = 2$ SQCD amplitude is assembled, with an $S_4$ permutation sum over external particle labels, as

$$i\mathcal{A}_4^{(2)} = e^{2\epsilon\gamma} \sum_{S_4} \sum_{i\in\{\mathrm{a},\ldots,\mathrm{j}\}} \int \frac{\mathrm{d}^{2D} l}{(i\pi^{D/2})^2} \frac{(N_f)^{|i|}}{S_i} \frac{n_i c_i}{D_i}\,, \tag{5.2}$$

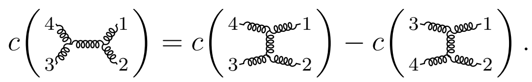
where $\mathrm{d}^{2D}l \equiv \mathrm{d}^D l_1 \mathrm{d}^D l_2$ is the two-loop integration measure and $|i|$ is the number of matter loops in the given diagram.

The integrand implicitly defined in (5.2) was constructed to be *colour-dual*, which means that it satisfies *colour-kinematics duality* [169–171]. Note that the colour factors $c_i$ are naturally related by linear relations due to the algebraic structure of the gauge group $\mathrm{SU}(N_c)$. Exploiting commutation relations, we expect to find identities of the form

$$c_i = c_j - c_k \, , \tag{5.3}$$

as can be readily observed from the colour factor for tree level scattering of four gluons

$$f^{a_1 a_2 b} f^{b a_3 a_4} = f^{a_4 a_1 b} f^{b a_2 a_3} - f^{a_2 a_4 b} f^{b a_3 a_1} \, , \tag{5.4}$$

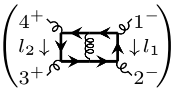



If the integrand is colour-dual, the kinematic numerators $n_i$ are expected to obey the same relations as the colour factors

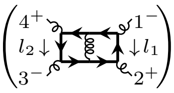
$$c_i = c_j - c_k \iff n_i = n_j - n_k \, . \tag{5.5}$$

Through these relations it was shown in [168] that the numerators of diagrams (e–j) are completely determined by the four planar diagrams (a–d). The independent numerators were computed by fixing an Ansatz through graph symmetries and $(D \leq 6)$-dimensional unitarity cuts, along with various symmetries related to properties of the $\mathcal{N} = 2$ theory such as the similarity of the fundamental and anti-fundamental hypermultiplet and the relation of $\mathcal{N} = 2$ SQCD to $\mathcal{N} = 4$ SYM. Using higher-dimensional cuts is required to be consistent with dimensional regularisation $D = 4 - 2\epsilon$ upon integration. The external momenta are restricted to a four-dimensional subspace, whereas the extra-dimensional components of the loop momenta are interpreted as scalar parameters of the solution.

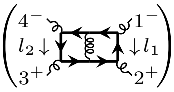
As an example we show numerator contributions to diagram (b)

$n\Big($ 

$\Big) = -\kappa_{12}\mu_{12}\,,$

$n\Big($ 

$\Big) = \dfrac{\kappa_{13}}{u^2}\,\mathrm{tr}_-(1 l_1 2 4 l_2 3)\,,$ (5.6)

$n\Big($ 

$\Big) = \dfrac{\kappa_{14}}{t^2}\,\mathrm{tr}_-(1 l_1 2 3 l_2 4)\,.$

The factor $\kappa_{ij}$ is proportional to the colour-stripped gluon tree level amplitude and captures the helicities of the external particles. For example

$$\kappa_{12} = ist\, A^{(0)}_{(--++)}\,, \tag{5.7}$$

where, employing the spinor-helicity invariants defined in (2.46), the tree level amplitude for the $(--++)$ helicity configuration is given by

$$A^{(0)}_{(--++)} = \frac{-i}{st}\langle 12\rangle^2[34]^2\,. \tag{5.8}$$

The $\mathrm{tr}_\pm$ are chirally projected Dirac traces taken strictly over the four-dimensional parts of the momenta

$$\mathrm{tr}_\pm(i_1 i_2 \dots i_k) = \frac{1}{2}\,\mathrm{tr}((1\pm\gamma_5) i_1 i_2 \dots i_k)\,, \tag{5.9}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the usual antisymmetric element of the Dirac algebra. The scalar products $\mu_{ij} = -l_i^{[-2\epsilon]}\cdot l_j^{[-2\epsilon]}$ absorb the degrees of freedom of the extra-dimensional loop momenta. Numerators of the other diagrams are of comparable simplicity, see [168, 172].

### 5.1.2. The Integration Procedure

Let us now review the steps taken in [8] to promote the integrand (5.2) to a fully integrated two-loop amplitude. First, all contributions were reduced to scalar Feynman integrals using *Schwinger parameterisation.* In this parameterisation, we write the denominators of the original tensor integral as

$$\frac{1}{D_1^{a_1}\dots D_n^{a_n}} = \int \mathcal{D}x\, \exp\left(\sum_{i=1}^{n} x_i D_i\right)\,, \tag{5.10}$$

with

$$\int \mathcal{D}x = \prod_{i=1}^{n} \frac{(-1)^a_i}{\Gamma(a_i)} \int_0^\infty \mathrm{d}x_i x_i^{a_i - 1} \,. \tag{5.11}$$

After reparameterising the loop momenta such that the exponential in (5.10) diagonalises, we have a polynomial in the Schwinger parameters and the loop momenta $l_i$ in the numerator. The diagonalised exponential then leads to Gaussian integrals

$$\int d^D y \, \exp\left(-a y^2\right) = \left(\frac{\pi}{a}\right)^{\frac{D}{2}} . \tag{5.12}$$

This allows one to factor out the tensorial numerator, rewriting the integral as a tensor structure composed of metric tensors multiplying scalar integrals with added factors of $x_i$ or $l_i^2$ in the numerator. Factors of $x_i$ can be interpreted as raising the powers of their associated denominators via (5.11), whereas factors of $l_i^2$ raise the dimension of the scalar integral. The same method can be applied to resolve the extra-dimensional loop integration and resolve factors of $\mu_{ij}$. For example, for an arbitrary function $f$ we get

$$\int \frac{\mathrm{d}^D l_i}{\pi^{D/2}} l_i^\mu l_j^\nu f(l_i^2) = -\frac{1}{2} g^{\mu\nu} \int \frac{\mathrm{d}^{D+2} l_i}{i\pi^{(D+2)/2}} f(l_i^2) \,, \tag{5.13}$$

$$\int \frac{\mathrm{d}^D l_i}{\pi^{D/2}} \mu_{ij} f(l_i^2) = \epsilon \int \frac{\mathrm{d}^{D+2} l_i}{i\pi^{(D+2)/2}} f(l_i^2) \,. \tag{5.14}$$

For a more detailed overview of this technique see e.g. [173, 174].

Thus, all contributions were reduced to scalar-type integrals in shifted dimensions. Next, the higher-dimensional scalar integrals were reduced to $D$ dimensions using dimensional recurrence relations [50, 175–179]. These relations offer a straightforward way to write a scalar integral in $D + 2$ dimensions in terms of scalar integrals of the same family in $D$ dimensions and are implemented in the `Mathematica` package `LiteRed` [180]. The resulting integrals were reduced to a basis of master integrals using integration-by-parts (IBP) relations (2.15), for which we again used `LiteRed`, which handles such relations algorithmically. We have chosen the basis displayed in Figure 5.2.

These master integrals are known analytically [178, 181–183] to the required orders in $\epsilon$ in $D = 4 - 2\epsilon$ dimensions, and their insertion yields the final result. Manipulation of the master integrals expressed in terms of harmonic polylogarithms (2.20) was done using the `Mathematica` package `HPL` [184]. Several checks were performed on this result. First, an alternative representation of the integrand — given also in [168] — was integrated, non-trivially obtaining the same result. Second, the result reproduces the high-energy behaviour expected from the known two-loop Regge trajectory for supersymmetric gauge

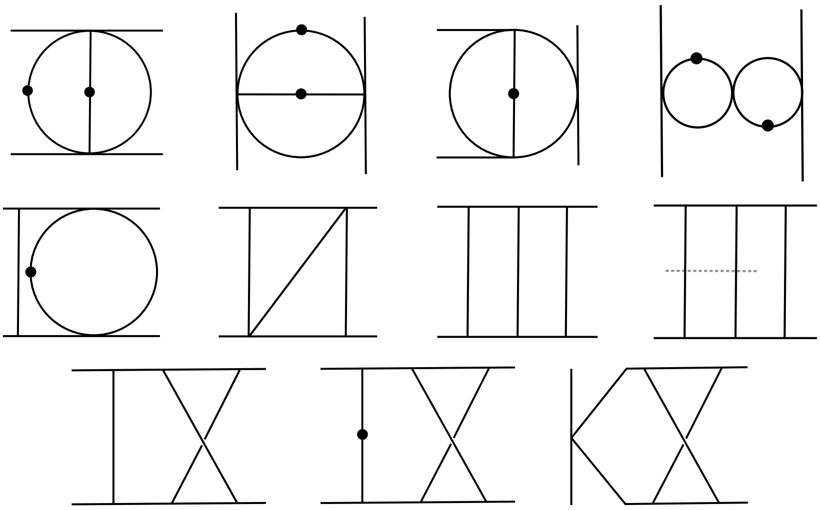


**Figure 5.2:** Our chosen set of scalar master integrals. Momenta are ordered anticlockwise with the top left external leg associated to $p_1$. Dotted propagators are doubled, dashed lines indicate numerators.

theories [158]. Finally, the amplitude is divergent, and it was checked that the amplitude has the correct IR-pole structure after UV renormalisation. We will now discuss this structure in more detail.

## 5.2. Infrared and Ultraviolet Structure

Let us study the singularity structure of the amplitude presented in [8]. Ultraviolet divergences are captured by the beta function. From (1.38) it can be seen that the beta function of $\mathcal{N} = 2$ SQCD is one-loop exact. In this chapter we will use the form

$$\beta(\alpha_S(\mu^2)) = -\alpha_S(\mu^2)\left(2\epsilon + \beta_0 \frac{\alpha_S(\mu^2)}{2\pi}\right), \tag{5.15}$$

where an $\epsilon$-dependence was introduced anticipating dimensional regularisation, $\beta_0 = C_A - T_R N_f$ with $C_A = 2N_c$ and we take $T_R = 1$ to match the conventions used in [168]. Note that this differs from the normalisation used in the previous chapter. Renormalised amplitudes $\widetilde{\mathcal{A}}_N^{(L)}$ are defined as in (5.1), except with the bare coupling $\alpha_S^0 S_\epsilon$ replaced by the *renormalised coupling* $\alpha_S(\mu^2)$ which depends on the *renormalisation scale* $\mu$. In the $\overline{MS}$ *scheme* the renormalised and bare couplings are related by

$$\alpha_S^0 S_\epsilon = \alpha_S(\mu^2)\mu^{2\epsilon} \sum_{L=0}^{\infty} \left(-\frac{\beta_0}{\epsilon}\frac{\alpha_S(\mu^2)}{4\pi}\right)^L . \tag{5.16}$$

At four points, this implies that for the one- and two-loop amplitudes

$$\begin{aligned} \widetilde{\mathcal{A}}_4^{(1)} &= \mu^{2\epsilon}\mathcal{A}_4^{(1)} - \frac{\beta_0}{\epsilon}\mathcal{A}_4^{(0)} , \\ \widetilde{\mathcal{A}}_4^{(2)} &= \mu^{4\epsilon}\mathcal{A}_4^{(2)} - 2\mu^{2\epsilon}\frac{\beta_0}{\epsilon}\mathcal{A}_4^{(1)} + \frac{\beta_0^2}{\epsilon^2}\mathcal{A}_4^{(0)} . \end{aligned} \tag{5.17}$$

With these equations at hand we can turn our attention to the infrared singularities of our amplitude. They are universal and independent of the hard scattering process. We will use a formalism due to Catani [185] to subtract them and define the finite part of our two-loop amplitude. At one-loop order a colour-space operator $\mathbf{I}^{(1)}(\epsilon)$ may be defined that encodes all IR singularities [185],

$$|\widetilde{\mathcal{A}}_N^{(1)}\rangle = |\widetilde{\mathcal{A}}_N^{(1)\mathrm{fin}}\rangle + \mathbf{I}^{(1)}(\epsilon)|\widetilde{\mathcal{A}}_N^{(0)}\rangle , \tag{5.18}$$

where $\widetilde{\mathcal{A}}_N^{(L)\mathrm{fin}}$ is finite as $\epsilon \to 0$. The bra-ket notation implies that the amplitude is interpreted as a vector in colour space. In our case for four external particles we expand the amplitude

$$\widetilde{\mathcal{A}}_4^{(L)} = \langle \mathcal{T}_{\mathrm{col}} | \widetilde{\mathcal{A}}_4^{(L)} \rangle , \tag{5.19}$$

where the *trace basis vector* is given by

$$\langle \mathcal{T}_{\mathrm{col}} | = \begin{pmatrix} \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) \\ \mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ \mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) \\ \mathrm{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) \\ \mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) \\ \mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) \\ \mathrm{tr}(T^{a_1}T^{a_2})\,\mathrm{tr}(T^{a_3}T^{a_4}) \\ \mathrm{tr}(T^{a_1}T^{a_3})\,\mathrm{tr}(T^{a_2}T^{a_4}) \\ \mathrm{tr}(T^{a_1}T^{a_4})\,\mathrm{tr}(T^{a_2}T^{a_3}) \end{pmatrix} , \tag{5.20}$$

and $a_i$ is the colour index related to the particle with momentum $p_i$. In this colour basis for four external gluons, the one-loop operator is given by [186, 187]

$$\mathbf{I}^{(1)}(\epsilon) = \frac{-e^{-\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{\beta_0}{N_c\epsilon}\right) \tag{5.21}$$
$$\times\begin{pmatrix} N(\mathtt{S}+\mathtt{T}) & 0 & 0 & 0 & 0 & 0 & (\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{S}-\mathtt{U}) \\ 0 & N(\mathtt{S}+\mathtt{U}) & 0 & 0 & 0 & 0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{S}-\mathtt{T}) & 0 \\ 0 & 0 & N(\mathtt{T}+\mathtt{U}) & 0 & 0 & 0 & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) \\ 0 & 0 & 0 & N(\mathtt{T}+\mathtt{U}) & 0 & 0 & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) \\ 0 & 0 & 0 & 0 & N(\mathtt{S}+\mathtt{U}) & 0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{S}-\mathtt{T}) & 0 \\ 0 & 0 & 0 & 0 & 0 & N(\mathtt{S}+\mathtt{T}) & (\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{S}-\mathtt{U}) \\ (\mathtt{S}-\mathtt{U}) & (\mathtt{S}-\mathtt{T}) & 0 & 0 & (\mathtt{S}-\mathtt{T}) & (\mathtt{S}-\mathtt{U}) & 2N\mathtt{S} & 0 & 0 \\ 0 & (\mathtt{U}-\mathtt{T}) & (\mathtt{U}-\mathtt{S}) & (\mathtt{U}-\mathtt{S}) & (\mathtt{U}-\mathtt{T}) & 0 & 0 & 2N\mathtt{U} & 0 \\ (\mathtt{T}-\mathtt{U}) & 0 & (\mathtt{T}-\mathtt{S}) & (\mathtt{T}-\mathtt{S}) & 0 & (\mathtt{T}-\mathtt{U}) & 0 & 0 & 2N\mathtt{T} \end{pmatrix},$$

where

$$\mathtt{S} = \left(\frac{\mu^2}{-s}\right)^{\epsilon}, \qquad \mathtt{T} = \left(\frac{\mu^2}{-t}\right)^{\epsilon}, \qquad \mathtt{U} = \left(\frac{\mu^2}{-u}\right)^{\epsilon}. \tag{5.22}$$

This method allows us to obtain the finite part of the one-loop amplitude. Explicit computation of the one-loop amplitude through the integrand given in [188] confirms the structure in (5.18).

The infrared singularities of two-loop amplitudes are encoded into the formula [185, 189–191]

$$\begin{aligned} |\widetilde{\mathcal{A}}_N^{(2)}\rangle = |\widetilde{\mathcal{A}}_N^{(2)\mathrm{fin}}\rangle + \mathbf{I}^{(1)}(\epsilon)|\widetilde{\mathcal{A}}_N^{(1)}\rangle - \Bigg[\frac{1}{2}\mathbf{I}^{(1)}(\epsilon)^2 + \frac{\beta_0}{\epsilon}\mathbf{I}^{(1)}(\epsilon) \\ - e^{-\epsilon\gamma}\,\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\mathbf{I}^{(1)}(2\epsilon) - \mathbf{H}^{(2)}(\epsilon)\Bigg]|\mathcal{A}_N^{(0)}\rangle. \end{aligned} \tag{5.23}$$

The terms $K$ and $\mathbf{H}^{(2)}$ determine the $\mathcal{O}(\epsilon^{-2})$ and $\mathcal{O}(\epsilon^{-1})$ poles respectively and depend on the process under consideration. In our case we have

$$K = -\zeta_2\, C_A + 2(1+2\epsilon)\beta_0\,. \tag{5.24}$$

The tensor $\mathbf{H}^{(2)}(\epsilon)$, which determines the $\mathcal{O}(\epsilon^{-1})$ pole, is split up into a colour-diagonal and a non-diagonal part

$$\mathbf{H}^{(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^{2\epsilon}\left(4\mathrm{H}_g^{(2)}\mathbf{1} + \widehat{\mathbf{H}}^{(2)}\right). \tag{5.25}$$

The colour-diagonal part is fixed by

$$\mathrm{H}_g^{(2)} = C_A^2\frac{\zeta_3}{2} + \beta_0\left(2\beta_0 + C_A\frac{\zeta_2}{4}\right). \tag{5.26}$$

The non-diagonal part $\widehat{\mathbf{H}}^{(2)}$ matches what was found in [187] for QCD and $\mathcal{N}=1$ SYM and is given by

$$\widehat{\mathbf{H}}^{(2)} = 4\log\left(\frac{-s}{-t}\right)\log\left(\frac{-t}{-u}\right)\log\left(\frac{-u}{-s}\right) \tag{5.27}$$
$$\times\begin{pmatrix} 0 & 1 & -1 & -1 & 1 & 0 & N_c & 0 & -N_c \\ -1 & 0 & 1 & 1 & 0 & -1 & -N_c & N_c & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & -N_c & N_c \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & -N_c & N_c \\ -1 & 0 & 1 & 1 & 0 & -1 & -N_c & N_c & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & N_c & 0 & -N_c \\ -N_c & N_c & 0 & 0 & N_c & -N_c & 0 & 0 & 0 \\ 0 & -N_c & N_c & N_c & -N_c & 0 & 0 & 0 & 0 \\ N_c & 0 & -N_c & -N_c & 0 & N_c & 0 & 0 & 0 \end{pmatrix}.$$

The singularities of the computed two-loop amplitude fit the structure given in (5.23), further suggesting the validity of the result.

## 5.3. Transcendentality in $\mathcal{N}=2$ S(C)QCD

Let us now turn to the question which led us to investigate this amplitude in the first place: what are the transcendentality properties of $\mathcal{N}=2$ S(C)QCD in full kinematics? In particular, we wish to investigate if the maximal transcendentality of the BFKL ladder at NLLA in the conformal point $N_f = 2N_c$ indicates maximal transcendentality for $\mathcal{N}=2$ SCQCD, also beyond the high-energy limit.

To study this question, let us first compartmentalise our results as follows

$$\mathcal{A}_N^{(L)} = \mathcal{A}_N^{(L)[\mathcal{N}=4]} + \mathcal{R}_N^{(L)} + (C_A - N_f)\mathcal{S}_N^{(L)}\,, \tag{5.28}$$

dividing the amplitude into the gluon amplitude in $\mathcal{N}=4$ SYM, a *remainder function* $\mathcal{R}_N^{(L)}$ which survives at the conformal point $N_f = 2N_c$, and a term that contributes away from the conformal point $\mathcal{S}_N^{(L)}$. The $\mathcal{N}=4$ SYM result $\mathcal{A}_N^{(L)[\mathcal{N}=4]}$ at one and two loops is known to be maximally transcendental [50, 68, 69], and so to study the transcendentality we restrict ourselves to $\mathcal{R}_N^{(L)}$ and $\mathcal{S}_N^{(L)}$. Note that the colour-dual integrand (5.2) is particularly well suited to this decomposition. In fact, studying the diagram contributions from Figure 5.1, only four diagrams (b,c,g,h) contribute to $\mathcal{R}_4^{(2)}$, and six diagrams (b,c,e,g,h,j) contribute to $\mathcal{S}_4^{(2)}$. Diagrams (a,d,f,i) may be ignored as they only contribute to the two-loop $\mathcal{N}=4$ SYM amplitude.

Studying the full result, we observe that the unrenormalised amplitude in $\mathcal{N}=2$ SQCD for generic $N_f$ and $N_c$ shares the same weight properties as the two-loop QCD amplitude. In particular, the coefficient of $\epsilon^{-k}$ of the function $\mathcal{S}_4^{(2)}$ contains polylogarithms of weight up to $4-k$, including weight zero (note that the smallest value of $k$ is $k=3$, and the coefficient of $1/\epsilon^3$ has weight zero). Let us then turn our attention to the conformal point $N_f = 2N_c$ and consider the transcendentality of $\mathcal{R}_N^{(L)}$.

By definition $\mathcal{R}_N^{(0)}$ vanishes, and the remainder also vanishes at one-loop order in the large-$N_c$ limit for any number of external legs as was shown in [165]. Beyond leading colour, $\mathcal{R}_N^{(1)}$ is non-zero and IR finite. At four points, we define

$$\begin{aligned}\mathcal{R}_4^{(L)} = \sum_{\substack{\text{helicites}\\ \text{colours}}} \sum_{k=0}^{L} & A^{(0)}_{(h_1h_2h_3h_4)} N_c^k \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\, R^{(L)[k]}_{(h_1h_2h_3h_4)} \\ & + A^{(0)}_{(h_1h_2h_3h_4)} N_c^k \operatorname{Tr}(T^{a_1}T^{a_2}) \operatorname{Tr}(T^{a_3}T^{a_4})\, R^{(L)[k]}_{(h_1h_2)(h_3h_4)}\,,\end{aligned} \tag{5.29}$$

summing over all external helicity configurations and the colour traces given in (5.20). The independent helicity configurations of the one-loop remainder for four external gluons at next-to-leading colour are given by [188]

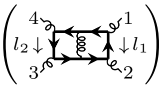
$$R^{(1)[0]}_{(--)(++)} = 2\tau\left[(T-U)^2 + 6\zeta_2\right] + \mathcal{O}(\epsilon)\,, \tag{5.30a}$$

$$R^{(1)[0]}_{(-+)(-+)} = \frac{2\tau}{\upsilon^2} T(T+2i\pi) + \mathcal{O}(\epsilon)\,, \tag{5.30b}$$

where we have introduced the shorthand notation $\tau = -t/s$, $\upsilon = -u/s$, $T = \log(\tau)$, and $U = \log(\upsilon)$. Recall that $s>0$; $t,u<0$, so $T$ and $U$ are real. We see that the one-loop remainders at $\mathcal{O}(\epsilon^0)$ have uniform transcendental weight two, just like the corresponding $\mathcal{N}=4$ SYM amplitudes.

At the next loop order, the two-loop remainder functions are finite and non-zero already at leading colour [166, 167]. Using the integrand in (5.2) we can easily isolate individual diagrams that contribute to $\mathcal{R}_n^{(2)}$ at different orders of $N_c$. For example, at leading colour $\mathcal{O}(N_c^2)$, only graph (b) is needed to compute the SCQCD remainder. The coefficient of $N_c^2 \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ is diagrammatically given in

$$R^{(2)[2]}_{(1234)} = \frac{-i}{\pi^D} e^{2\epsilon\gamma} \sum_{\text{cyclic}} \int \frac{d^{2D} l}{D_{\text{b}}} n \left( \quad \right)\,. \tag{5.31}$$




Recall that the relevant numerators are given in (5.6). As the integral with the $\mu_{12}$ factor begins at $\mathcal{O}(\epsilon)$, we need only integrate the two Dirac traces in the other terms. They are manifestly IR finite[1], giving the leading-colour remainders

$$
\begin{aligned}
R^{(2)[2]}_{(--++)} &= 12\zeta_3 + \frac{\tau}{6}\Bigg[48\mathrm{Li}_4(\tau) - 24T\mathrm{Li}_3(\tau) - 24T\mathrm{Li}_3(\upsilon) + 24TU\mathrm{Li}_2(\upsilon) \\
&+ 24\mathrm{Li}_2(\tau)\,(\zeta_2 + TU) - 24U\mathrm{Li}_3(\tau) - 24S_{2,2}(\tau) + T^4 - 4T^3U + 18T^2U^2 \\
&- 12\zeta_2T^2 + 24\zeta_2TU + 24\zeta_3U - 168\zeta_4 - 4i\pi\big(6\mathrm{Li}_3(\tau) + 6\mathrm{Li}_3(\upsilon) - 6U\mathrm{Li}_2(\tau) \\
&- 6U\mathrm{Li}_2(\upsilon) - T^3 + 3T^2U - 6TU^2 - 6\zeta_2T + 6\zeta_2U\big)\Bigg] + \mathcal{O}(\epsilon)\,, \qquad (5.32a)
\end{aligned}
$$

$$
R^{(2)[2]}_{(-+-+)} = 12\zeta_3 + \frac{1}{6}\frac{\tau}{\upsilon^2}T^2(T+2i\pi)^2 + \mathcal{O}(\epsilon)\,, \qquad (5.32b)
$$

where $\mathrm{Li}_n(z)$ are the classical polylogarithms (2.19) and $S_{n,p}(z)$ are Nielsen generalised polylogarithms defined by

$$
S_{n,p}(z) = (-1)^p\, G(\bar{0}_n, \bar{1}_p; z)\,. \qquad (5.33)
$$

The leading-colour remainder of $\mathcal{N}=2$ SCQCD was presented in [166], and the $R^{(2)[2]}_{(--++)}$ remainder was first published in [167]; we confirm those results independently.

Inspecting the leading-colour results (5.32a)-(5.32b), we see that the remainder does not have uniform weight four. The deviation from uniform weight, however, is very minimal and entirely captured by a constant $12\zeta_3$ up to the tree level amplitude. The striking simplicity of the sub-maximal part at leading colour is suggestive, and one might hope that this points to structure that can be understood to higher orders as well.

Let us now turn to the subleading-colour contributions. At this point the remainder is no longer finite, and we encounter IR divergences which can be subtracted using the Catani formalism

$$
\mathcal{R}^{(2)}_N = \mathcal{R}^{(2)\mathrm{fin}}_N + \mathbf{I}^{(1)}(\epsilon)\mathcal{R}^{(1)}_N\,. \qquad (5.34)
$$

The full sub-leading colour terms violate maximal transcendentality in a worse way than the leading colour, containing terms of weight two through four with all weights appearing in the $\mathcal{O}(\epsilon^0)$ terms. These submaximal weight terms are not given by constants as in the leading colour case, but rather explicitly depend on the kinematic variables.

[1]Note that the integrals defined by (5.31) precisely match the simple IR finite integrals considered in [192].

Computing the finite parts for the subleading-colour two-loop remainder, we obtain

$$
\begin{aligned}
R^{(2)[1]\text{fin}}_{(--)(++)} = \frac{2\tau}{3}\Big[& 96\text{Li}_4(\tau) - 72T\text{Li}_3(\tau) + 24T\text{Li}_3(\upsilon) + 24T\text{Li}_2(\tau)(T-U) \\
& - 24U\text{Li}_2(\upsilon)(T-U) + 96\text{Li}_4(\upsilon) + 24U\text{Li}_3(\tau) - 72U\text{Li}_3(\upsilon) + T^4 + 4T^3U \\
& - 18T^2U^2 + 4TU^3 + U^4 + 24\zeta_2 TU - 12\zeta_2 T^2 - 12\zeta_2 U^2 - 654\zeta_4 \\
& - 4i\pi\big(12\text{Li}_3(\tau) + 12\text{Li}_3(\upsilon) - 12T\text{Li}_2(\tau) - 12U\text{Li}_2(\upsilon) - T^3 - 3T^2U \\
& - 3TU^2 - U^3 - 18\zeta_2 T - 18\zeta_2 U\big)\Big] + \mathcal{O}(\epsilon)\,, \qquad (5.35a)
\end{aligned}
$$

$$
\begin{aligned}
R^{(2)[1]\text{fin}}_{(-+)(-+)} = \frac{2\tau}{3\upsilon^2}\Big[& 48\text{Li}_4(\tau) - 24T\text{Li}_3(\tau) - 24S_{2,2}(\tau) + 24\zeta_2\text{Li}_2(\tau) + T^4 \\
& - 84\zeta_2 T^2 - 102\zeta_4 + 24T\zeta_3 - 8i\pi\big(3T\zeta_2 - T^3\big)\Big] - \frac{8\tau}{3\upsilon^2}\Big[6\tau\text{Li}_3(\tau) - 6\tau\text{Li}_3(\upsilon) \\
& - 6\tau T\text{Li}_2(\tau) + 6\text{Li}_3(\upsilon) - 6\upsilon U\text{Li}_2(\upsilon) + 3\tau TU^2 + 3T\upsilon U^2 - 3TU^2 - 30\tau T\zeta_2 \\
& - 30\upsilon U\zeta_2 - 6\zeta_3 + 3i\pi\big(2(\upsilon-\tau)\text{Li}_2(\tau) + \tau T^2 + 2T\upsilon U + \upsilon U^2 + 2\tau\zeta_2\big)\Big] \\
& + \mathcal{O}(\epsilon)\,. \qquad (5.35b)
\end{aligned}
$$

Note that the situation w.r.t. transcendentality is ameliorated for the finite parts compared to the full sub-leading colour terms. This implies that the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms of $\mathcal{R}_4^{(1)}$ — which are not of uniform weight — interplay with the form of the Catani operator to cancel exactly some of the lower-weight contributions. In particular, we see that the weight-two terms cancel for (5.35b), and for (5.35a) all non-maximal weight terms are subtracted by the Catani formalism (5.34). We observe no such cancellation in the non-conformal case.

From the form of the SU($N_c$) colour algebra, and using the fact that only four diagrams (b,c,g,h) are required to fully specify $\mathcal{R}_4^{(2)}$, one can determine the subsubleading-colour parts of $\mathcal{R}_4^{(2)}$ as follows

$$
R^{(2)[0]}_{(1234)} = R^{(2)[2]}_{(1234)} - R^{(2)[1]}_{(13)(24)} + \frac{1}{2}\left(R^{(2)[1]}_{(12)(34)} + R^{(2)[1]}_{(14)(23)}\right). \qquad (5.36)
$$

From this we see that the subsubleading-colour is entirely determined by the higher orders in $N_c$. These relations were observed to hold using the full two-loop amplitude [8].

The integrand of (5.2) is valid for any gauge group and matter representation, making it possible to consider results for different gauge groups straightforwardly. For abelian U(1) $\mathcal{N}=2$ SQED, only diagram (b) is non-vanishing. This implies that the full amplitude is obtained by summing over all permutations of the external legs in the leading-colour remainders (5.32), multiplied by the respective tree amplitudes. Since the weight-three terms are constants, they cancel in the sum due to a photon-decoupling identity satisfied by the tree-level amplitudes. As a result, the two-loop $\mathcal{N} = 2$ SQED amplitude has uniform weight four, in agreement with what was found in [193]. Furthermore, for gauge group SO(3) and a single fundamental hypermultiplet ($N_f = 1$), the amplitude in $\mathcal{N} = 2$ SQCD is identical to the amplitude in SO(3) $\mathcal{N} = 4$ SYM, and thus has uniform transcendental weight. We expect this equality to hold to any loop order since both the structure constants and fundamental generators of SO(3) are described by rank-three Levi-Civita tensors, so the fundamental hypermultiplet transforms the same way as if it were in the adjoint representation. From this analysis we see that the weight properties are tightly connected not only to the matter content and the symmetries of the theory, but also the choice of gauge group and representation.

In conclusion, we see from our investigation of the two-loop four-gluon amplitude in $\mathcal{N} = 2$ SCQCD that maximal transcendentality does not hold in this theory beyond the high energy limit. The violation for the leading colour part is of striking simplicity nonetheless, leaving hope for an improved understanding at higher orders. We note a conspiracy between the lower-weight terms at subleading-colour and the one-loop terms at higher orders in $\epsilon$ through the Catani formalism. Based on these observations one may speculate that conformal symmetry and infrared singularities play a vital role in understanding the detailed transcendental weight of scattering amplitudes. Finally, we note that alternative choices of gauge groups may lead to maximal transcendental weight and we identify two such cases.

# Conclusions and Outlook

In this thesis, we investigated the property of transcendental weight for supersymmetric theories in various contexts.

In Chapter 3, we studied the high-energy limit of planar $\mathcal{N} = 4$ Super Yang-Mills theory. In this highly symmetric setup, in which the geometry of the configuration space of scattering processes can be very well understood, the possible functions which may arise in generic amplitudes can be determined from purely geometric considerations. We find that these functions correspond to single-valued multiple polylogarithms. From an all-order conjectural form of the amplitude in terms of a Fourier-Mellin transform, one can use this knowledge of the function space to compute higher order corrections recursively from known low-loop results via convolution products. Furthermore, this technology allows one to prove a factorisation through which at a certain logarithmic order (in particular, as we show, at LLA), (parts of) scattering amplitudes for an arbitrary number of particles can be related to results for lower multiplicity. These methods were applied to the computation of the LLA MHV amplitude for any number of particles to five loops, the LLA amplitude for eight or less particles for any helicity configuration up to four loops, the seven-point amplitude at NLLA through five loops for the MHV case, and through three and four loops for the two independent NMHV helicity configurations, and the eight-point NLLA amplitude for any helicity configuration at three loops. Beyond mere computational power, the recursive convolution method allows us to prove statements about the analytic properties of scattering amplitudes to all orders inductively. In particular, we show that amplitudes in the multi-Regge limit of planar $\mathcal{N} = 4$ Super Yang-Mills theory are given by pure combinations of SVMPLs of uniform transcendental weight with rational prefactors.

Aiming to apply the convolution method beyond planar $\mathcal{N} = 4$ SYM, in Chapter 4 we turn our attention to the BFKL ladder at NLLA. This object — which is part of the resummed cross-section for parton-parton scattering in the high energy limit — is also described perturbatively by a Fourier-Mellin transform and can be written down for a gauge theory with arbitrary matter content straightforwardly. Using QCD as a testing case, we apply the convolution method to the computation of this object to five loops. We find that the result can be written in terms of generalised single-valued multiple polylogarithms and generalised inverse tangent integrals. Turning to generic gauge theories, we formulate constraints on the matter content which allow us to tune the transcendental weight. From this analysis, we find there is no theory such that the BFKL ladder at NLLA has uniform and maximal weight and agrees with the maximal weight terms in QCD. Restricting ourselves to adjoint and fundamental matter, we solve the constraints for all non-maximal weight terms in the NLLA BFKL ladder to vanish. We find that the matter content which satisfies these constraints naturally corresponds to supersymmetric theories with vanishing one-loop beta functions. This leads us to suspect a connection between maximal transcendentality and superconformal symmetry.

In order to investigate this connection beyond the high-energy limit, in Chapter 5 we turn our attention to amplitudes in general kinematics in $\mathcal{N} = 2$ supersymmetric QCD. For a certain number of fundamental hypermultiplets this theory obeys conformal symmetry, at which point it solves the constraints from the previous chapter and is refered to as $\mathcal{N} = 2$ superconformal QCD. It is known that the leading-colour contributions to $N$-gluon scattering at one-loop order in this theory are uniformally transcendental, whereas the leading-colour four-gluon two-loop contribution was found to break uniform transcendentality by the presence of a single multiple zeta value. Starting from a colour-dual integrand representation, we compute the full-colour four-gluon two-loop amplitude for a generic number of hypermultiplets. We find that the non-conformal contributions violate maximal transcendentality similarly to the QCD result. In the conformal limit, we confirm the known results at leading colour. At next-to-leading colour, uniform transcendentality is broken by terms which – unlike for leading colour – depend on the kinematic variables and have a larger range of weights. Upon subtracting infrared divergences through the Catani formalism however, we note a conspiracy between the lower-weight terms at subleading colour in the conformal point and the one-loop terms at higher orders in $\epsilon$. The subtraction cancels exactly some of the lower-weight contributions at $\mathcal{O}(\epsilon^0)$, even resulting in a maximally transcendental finite part for one of the independent helicity configurations. These findings lead us to speculate that conformal symmetry and infrared singularities play a vital role in understanding the detailed transcendental weight of scattering amplitudes

Looking forward, there are many more interesting questions which can be asked on the transcendental weight properties of scattering amplitudes for supersymmetric theories. Building on the two-loop four-gluon result in $\mathcal{N} = 2$

SQCD discussed in Chapter 5, it would be interesting to study the transcendental weight for external hypermultiplets at the same order, the integrand of which was obtained in [172]. In particular, it would be interesting to investigate if a similar conspiracy between infrared divergences and lower-weight terms can be observed in this case. Recalling the striking simplicity of the lower-weight term for the leading-colour contribution in $\mathcal{N} = 2$ SCQCD, it would be interesting to see if a similar simplicity can be observed for the lower-weight terms at higher orders. Finally, looking at Table 4.1, at this point the transcendental weight of two of the four solutions given by the constraints of the NLLA BFKL ladder have been studied. It would be very interesting to undergo a similar study to what we have done for $\mathcal{N} = 2$ SQCD, in the case of $\mathcal{N} = 1$ SQCD, a theory which also has a conformal regime. In this sense we could investigate the effect of further lowering the degree of supersymmetry on transcendentality.

# A | The BFKL Building Blocks up to NLO

In this appendix we show the BFKL building blocks from (3.41) up to NLO explicitly. The impact factor and BFKL eigenvalue were given to all orders in [127] and up to NLO they are given by

$$\chi^+(\nu,n) = \frac{1}{\nu - \frac{in}{2}}\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 - NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right], \tag{A.1}$$

$$\chi^-(\nu,n) = \frac{1}{\nu + \frac{in}{2}}\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 + NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right], \tag{A.2}$$

$$-\omega(\nu,n) = aE - \frac{a^2}{4}\left(D^2E - 2VDE + 4\zeta_2 E + 12\zeta_3\right) + \mathcal{O}(a^3)\,. \tag{A.3}$$

The central emission block at NLO was first derived in [2] and is given by

$$\begin{aligned} C^+(\nu_1,n_1,\nu_2,n_2) = C^+_{0,1,2}\Big[1 + a\big(\frac{1}{2}\big[&DE_1 - DE_2 + \tfrac{1}{4}(N_1+N_2)^2 \\ &+ E_1E_2 + V_1V_2 + (V_1 - V_2)\big(M - E_1 - E_2\big) \\ &+ i\pi(V_2 - V_1 - E_1 - E_2)\big] - \tfrac{3}{16}(N_1^2 + N_2^2) \\ &- \tfrac{1}{4}(E_1^2 + E_2^2 + N_1V_1 - N_2V_2)\big) + \mathcal{O}(a^2)\Big], \end{aligned} \tag{A.4}$$

where

$$C^+_{0,1,2} = \frac{-\Gamma\big(1 - i\nu_1 - \frac{n_1}{2}\big)\Gamma\big(i\nu_2 + \frac{n_2}{2}\big)\Gamma\big(i\nu_1 - i\nu_2 - \frac{n_1}{2} + \frac{n_2}{2}\big)}{\Gamma\big(1 + i\nu_1 - \frac{n_1}{2}\big)\Gamma\big(-i\nu_2 + \frac{n_2}{2}\big)\Gamma\big(1 - i\nu_1 + i\nu_2 - \frac{n_1}{2} + \frac{n_2}{2}\big)}\,. \tag{A.5}$$

The building blocks are themselves built out of combinations of the objects

$$\begin{aligned}
D_\nu =& -i\partial/\partial\nu\,,\\
V(\nu,n) =& \frac{i\nu}{\nu^2+\frac{n^2}{4}}\,,\\
N(\nu,n) =& \frac{n}{\nu^2+\frac{n^2}{4}}\,,\\
E(\nu,n) =& -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}}+\psi\left(1+i\nu+\frac{|n|}{2}\right)\\
&+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi(1)\,,\\
M(\nu_1,n_1,\nu_2,n_2) =& \psi(1-i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2})\\
&+\psi(i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2})-2\psi(1)\,.
\end{aligned} \tag{A.6}$$

# B | Proof of the LLA Factorisation

In this section we discuss the proof of the factorisation theorem at leading logarithmic accuracy discussed in Section 3.3.2. It will be helpful to formulate and prove the following two Lemmas.

**Lemma B.1.** *At LLA, we can always drop sequences of 0's at either end of the list of indices, i.e.,*

$$\tilde{g}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}_{\ell;+\ldots+}(\rho_1,\ldots,\rho_{N-5}) = \tilde{g}^{(\ell;i_k,\ldots,i_{N-5})}_{+\ldots+}(\rho_k,\ldots,\rho_{N-5})\,, \tag{B.1}$$

*and a similar relation holds if the sequence of indices ends in a 0.*

*Proof.* Target-projectile symmetry implies that it is sufficient to prove Lemma B.1 in the case where the sequence of indices starts with a 0. The proof relies on an analysis of the Fourier-Mellin integral. Let us start from the Fourier-Mellin integral for $g^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}_{+\ldots+}$, and let us concentrate on the terms that depend on $(\nu_1, n_1)$ and $(\nu_2, n_2)$,

$$\begin{aligned}\tilde{g}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}_{+\ldots+} = \ldots\int \frac{d\nu_1}{2\pi} \sum_{n_1=-\infty}^{+\infty} & z_1^{i\nu_1+n_1/2}\,\bar{z}_1^{i\nu_1+n_1/2} \\ & \times z_2^{i\nu_2+n_2/2}\,\bar{z}_2^{i\nu_2+n_2/2}\chi_0^{+}(\nu_1,n_1)\,C_0^{+}(\nu_1,n_1,\nu_2,n_2)\ldots\,,\end{aligned} \tag{B.2}$$

where the dots indicate terms that are independent of $\nu_1$ and $n_1$. Our first goal is to show that the value of the integral is independent of $\rho_1$. From (3.19) we see that only $z_1$ and $z_2$ depend on $\rho_1$, so we do not need to consider the cross ratios $z_i$ with $i > 2$. Due to the symmetry in $z_1 \leftrightarrow \bar{z}_1$, it is sufficient to analyse the holomorphic part, i.e., we let $\bar{z}_1 \to 0$, with $z_1$ held fixed. This corresponds to taking only the residue at $i\nu_1 = n_1/2$ for $n_1 > 0$. After taking this residue, the sum over $n_1$ becomes trivial, and we are left with

$$\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})} \to \ldots (-1)\,\chi_0^+(\nu_2,n_2)\,[\underbrace{(1-z_1)z_2}_{=\rho_2}]^{i\nu_2+n_2/2}\ldots \tag{B.3}$$

We see that the integral does not depend on $\rho_1$. Note the appearance of an impact factor. If there is a second 0, we can iterate the procedure, with $z_1$ replaced by $\rho_2$. The result is

$$\ldots (-1)\,\chi_0^+(\nu_3,n_3)\,[\underbrace{(1-\rho_2)z_3}_{=\rho_3}]^{i\nu_3+n_3/2}\ldots \tag{B.4}$$

Continuing this process, we see that

$$\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5}) = f(\rho_k,\ldots,\rho_{N-5})\,, \tag{B.5}$$

for some function $f$. We still need to show that $f$ is the lower-point perturbative coefficient. This follows from the fact that $f$ must have the correct soft limits. Indeed, we must have

$$\begin{aligned}\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}(\rho_2,\ldots,\rho_{N-5}) &= \lim_{\rho_1\to 0}\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5})\\ &= f(\rho_k,\ldots,\rho_{N-5})\,.\end{aligned} \tag{B.6}$$

Similarly, we must have

$$\begin{aligned}\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}(\rho_3,\ldots,\rho_{N-5}) &= \lim_{\rho_2\to 0}\tilde{g}_{+\ldots+}^{(\ell;0,\ldots,0,i_k,\ldots,i_{N-5})}(\rho_2,\ldots,\rho_{N-5})\\ &= f(\rho_k,\ldots,\rho_{N-5})\,.\end{aligned} \tag{B.7}$$

Continuing this way, we arrive at

$$\begin{aligned}\tilde{g}_{+\ldots+}^{(\ell;i_k,\ldots,i_{N-5})}(\rho_k,\ldots,\rho_{N-5}) &= \lim_{\rho_{k-1}\to 0}\tilde{g}_{+\ldots+}^{(\ell;0,i_k,\ldots,i_{N-5})}(\rho_{k-1},\rho_k,\ldots,\rho_{N-5})\\ &= f(\rho_k,\ldots,\rho_{N-5})\,,\end{aligned} \tag{B.8}$$

which finishes the proof. □

**Lemma B.2.** *If $f(\rho_1,\rho_{i_1},\ldots,\rho_{i_k})$ depends on a subset of simplicial MRK coordinates, then the convolution with some function $g(z_1)$ will depend on the same subset , i.e., we have*

$$g(z_1) * f(\rho_1,\rho_{i_1},\ldots,\rho_{i_k}) = F(\rho_1,\rho_{i_1},\ldots,\rho_{i_k})\,, \tag{B.9}$$

*for some function $F$.*

In the right-hand side of (B.9), the convolution acts on $z_1$, and the simplicial MRK coordinates $\rho_i$ should be interpreted as functions of the MRK coordinates $z_i$. The relation between the two sets of coordinates is given by (3.19).

*Proof.* We proceed by changing variables to simplicial coordinates based at $z_1$, defined in (3.20). The relation between the two sets of coordinates is $z_1 = t_1$ (we write $t_i$ instead of $t_i^{(1)}$) and,

$$\rho_1 = \frac{t_1}{t_{N-5}}\,,\ \rho_2 = \frac{1-t_1}{1-t_{N-5}}\,,\ \rho_i = \frac{t_{i-1}-t_1}{t_{i-1}-t_{N-5}}\,,\ 2 < i < N-5\,. \tag{B.10}$$

We start by proving the claim in the case $i_1 \neq 2$. In that case the convolution integral takes the form

$$\begin{aligned} &g(z_1) * f(\rho_1, \rho_{i_1} \dots, \rho_{i_k}) \\ &\quad = \frac{1}{\pi}\int \frac{d^2\tau}{|\tau|^2}\, g\left(\frac{t_1}{\tau}\right)\, f\left(\frac{\tau}{t_{N-5}}, \frac{t_{i_1-1}-\tau}{t_{i_1-1}-t_{N-5}}, \dots, \frac{t_{i_k-1}-\tau}{t_{i_k-1}-t_{N-5}}\right). \end{aligned} \tag{B.11}$$

Shifting the integration variable $\tau \to t_{N-5}\,\tau$ and writing $x_j = t_j/t_{N-5}$, we see that the integrand only depends on the $x_j$. Hence, there is a function $\widetilde{F}(x_1, x_{i_1}, \dots, x_{i_k})$ such that

$$\begin{aligned} g(z_1) * f(\rho_1, \rho_{i_1} \dots, \rho_{i_k}) &= \frac{1}{\pi}\int \frac{d^2\tau}{|\tau|^2}\, g\left(\frac{x_1}{\tau}\right)\, f\left(\tau, \frac{x_{i_1-1}-\tau}{x_{i_1-1}-1}, \dots, \frac{x_{i_k-1}-\tau}{x_{i_k-1}-1}\right) \\ &\equiv \widetilde{F}(x_1, x_{i_1-1}, \dots, x_{i_k-1})\,. \end{aligned} \tag{B.12}$$

Finally, we can change coordinates back from $t$'s to $\rho$'s. We find

$$x_1 = \frac{t_1}{t_{N-5}} = \rho_1\,, \quad x_j = \frac{t_j}{t_{N-5}} = \frac{\rho_1 - \rho_{j+1}}{1-\rho_{j+1}}\,, \quad j \geq 2\,. \tag{B.13}$$

we see that no new $\rho$-coordinate is introduced, so the claim follows upon identifying $\widetilde{F}(x_1, x_{i_1-1}, \dots, x_{i_k-1})$ with $F(\rho_1, \rho_{i_1}, \dots, \rho_{i_k})$.

To complete the proof, we still need to investigate what happens when $i_1 = 2$. $\rho_2$ is not homogeneous in $t_{N-5}$, and so the function $\widetilde{F}$ will now not only depend on the ratios $x_i$, but also explicitly on $t_{N-5}$,

$$g(z_1) * f(\rho_1, \rho_2, \rho_{i_2} \dots, \rho_{i_k}) = \widetilde{F}(x_1, t_{N-5}, x_{i_2-1}, \dots, x_{i_k-1})\,. \tag{B.14}$$

We know already that the $x_i$'s do not introduce any new $\rho$'s. Adding $t_{N-5}$ will only add $\rho_2$,

$$t_{N-5} = \frac{1-\rho_2}{\rho_1-\rho_2}\,, \tag{B.15}$$

which was already present in the original function $f$. Hence the claim is proven. □

The factorisation theorem for LLA MHV amplitudes (3.114) now follows from Lemma B.1 and B.2. Assume that (3.114) holds for all perturbative MHV coefficients up to a certain number $N-1$ of legs, and let us show that it still holds for coefficients with one more leg. We denote the perturbative coefficient with one more leg by $\tilde{g}_{+\ldots+}^{(\ell;i_1,\ldots,i_{N-5})}(\rho_1,\ldots,\rho_{N-5})$ and we label the non-zero elements in $(i_2,\ldots,i_{N-5})$ by $i_{a_1},\ldots,i_{a_k}$, $2\le a_j\le N-5$. If $i_1=0$, then Lemma B.1 implies that we can drop the first index. The resulting function is an $(N-1)$-point amplitude, where (3.114) applies. So we have

$$\begin{aligned}\tilde{g}_{+\ldots+}^{(\ell;0,i_2,\ldots,i_{N_5})}(\rho_1,\ldots,\rho_{N-5}) &= \tilde{g}_{+\ldots+}^{(\ell;i_2,\ldots,i_{N-5})}(\rho_2,\ldots,\rho_{N-5})\\ &= \tilde{g}_{+\ldots+}^{(\ell;i_{a_1},\ldots,i_{a_k})}(\rho_{a_1},\ldots,\rho_{a_k})\,,\end{aligned} \tag{B.16}$$

in agreement with (3.114). If $i_1\neq 0$, we write the amplitude as a convolution using the recursion (3.106). For the sake of the example, consider the case $i_1=1$. We have,

$$\begin{aligned}\tilde{g}_{+\ldots+}^{(\ell;1,i_2,\ldots,i_{N_5})}(\rho_1,\ldots,\rho_{N-5}) &= \mathcal{E}(z_1)*\tilde{g}_{+\ldots+}^{(\ell;0,i_2,\ldots,i_{N_5})}(\rho_2,\ldots,\rho_{N-5})\\ &= \mathcal{E}(z_1)*\tilde{g}_{+\ldots+}^{(\ell;i_{a_1},\ldots,i_{a_k})}(\rho_{a_1},\ldots,\rho_{a_k})\,,\end{aligned} \tag{B.17}$$

where we have used the fact that we know that (3.114) holds for $i_1=0$. From Lemma B.2 we know that the result of the convolution will only depend on $(\rho_1,\rho_{a_1},\ldots,\rho_{a_k})$. The only thing left to show is that the function $F$ in Lemma B.2 is precisely the perturbative coefficient $\tilde{g}_{+\ldots+}^{(\ell;1,i_{a_1},\ldots,i_{a_k})}$. This follows immediately upon noting that the convolution integral used to compute $\tilde{g}_{+\ldots+}^{(\ell;1,i_{a_1},\ldots,i_{a_k})}$ is exactly the same as the one in (B.17), up to a relabelling of the variables. Repeating exactly the same argument for $i_1>1$, we see that (3.114) holds in general.

Let us now show the LLA factorisation beyond MHV. We proceed by induction in the number of indices. Let us assume that the factorisation in Figure 3.14 holds for up to $k$ indices, and let us show that the theorem still holds for $k+1$ indices. If the first two helicities are not the same, we can factor out a helicity flip operator,

$$\tilde{g}_{-+h_3\ldots h_{k+2}}^{(\ell;i_1\ldots i_{k+1})}(\rho_1,\ldots,\rho_{k+1}) = \mathcal{H}(z_1)*\tilde{g}_{++h_3\ldots h_{k+2}}^{(\ell;i_1\ldots i_{k+1})}(\rho_1,\ldots,\rho_{k+1})\,. \tag{B.18}$$

Hence, it is enough to prove the theorem in the case where the first two helicities are the same, as Lemma B.2 implies that it will remain true in the case where the helicities are different. We therefore assume that the first two helicities are the same. We can use the recursion (3.106) with $z_k=z_1$ to reduce the value of $i_1$ to zero. If $i_1=0$, we can repeat the proof of Lemma B.1 and conclude that we can delete the index $i_1=0$. Indeed, we see from (B.2) that the proof of

Lemma B.1 only relies on the structure of the part of the Fourier-Mellin integral that depends on $(\nu_1, n_1)$ and the first two helicities, and so we can repeat the proof of Lemma B.1 independently of the helicity structure of the rest of the amplitude. After using Lemma B.1 to delete the index $i_1 = 0$, the number of indices has decreased by one, and so the factorisation applies by induction hypothesis. To complete the proof, we need to perform the convolution integrals that were introduced to reduce the value of $i_1$. By Lemma B.2, this does not introduce any new variables except for $\rho_1$, and so the factorisation theorem is proven.

# C | Pure Functions from Convolutions

In this appendix we present and prove two lemmas which will be used in the main text to show that certain convolution products will naturally lead to pure functions of a certain weight.

**Lemma C.1.** *Assume that we have a function $a_{\nu n}$ whose Fourier-Mellin transform evaluates to an expression of the form,*

$$\mathcal{F}[a_{\nu n}] = \frac{|z|A(z)}{(z-b)(\bar{z}-c)}\,, \tag{C.1}$$

*where $A(z)$ is a single-valued pure function of weight $k$, and $b$ and $c$ are complex numbers. We wish to show that $\mathcal{F}[a_{\nu n}\chi_{\nu n}]$ where $\chi_{\nu n}$ is the LO singlet BFKL eigenvalue of* (4.20)*, consists of a single-valued pure function of weight $k+1$ multiplied by the same rational prefactor.*

*Proof.* Using the convolution product, we find

$$\begin{aligned}
&\mathcal{F}[a_{\nu n}\chi_{\nu n}] = \mathcal{F}[a_{\nu n}] * \mathcal{F}[\chi_{\nu n}] \\
&= \int d^2\omega \frac{|z|A(\omega)}{(\omega-b)(\bar{\omega}-c)(\omega-z)(\bar{\omega}-\bar{z})} \\
&= \frac{|z|}{(z-b)(\bar{z}-c)}\int d^2\omega \left(\frac{1}{\omega-z}-\frac{1}{\omega-b}\right)\left(\frac{1}{\bar{\omega}-\bar{z}}-\frac{1}{\bar{\omega}-c}\right)A(\omega)\,.
\end{aligned} \tag{C.2}$$

We can solve this integral in terms of residues as described in Section 3.3.1.1. We start by computing the single-valued primitive,

$$\int_{SV} d\bar{\omega}\left(\frac{1}{\bar{\omega}-\bar{z}}-\frac{1}{\bar{\omega}-c}\right)A(\omega) \equiv \widetilde{A}(\omega)\,. \tag{C.3}$$

Since $A(\omega)$ is assumed to be a pure function of weight $k$, $\widetilde{A}(\omega)$ is a pure function of weight $k+1$. The holomorphic residues are

$$\begin{aligned}\mathcal{F}[a_{\nu n}\chi_{\nu n}] &= \frac{|z|}{(z-b)(\bar{z}-c)}\left(\text{Res}_{\omega=b}\frac{\widetilde{A}(\omega)}{\omega-b} - \text{Res}_{\omega=z}\frac{\widetilde{A}(\omega)}{\omega-z}\right) \\ &= \frac{|z|}{(z-b)(\bar{z}-c)}\left(\text{Reg}_{\omega=b}\widetilde{A}(\omega) - \text{Reg}_{\omega=z}\widetilde{A}(\omega)\right),\end{aligned} \tag{C.4}$$

where $\text{Reg}_{\omega=a}\widetilde{A}(\omega)$ denotes the shuffle-regulated value of $\widetilde{A}(\omega)$ at the point $\omega = a$, defined as follows: since $\widetilde{A}$ has weight $k+1$, close to every point $\omega = a$ it admits an expansion of the type,

$$\widetilde{A}(\omega) = \sum_{i=0}^{k+1} \log^i \left|1-\frac{\omega}{a}\right|^2 \widetilde{A}^{(i)}(\omega)\,, \tag{C.5}$$

where the functions $\widetilde{A}^{(i)}$ are analytic at $\omega = a$, i.e., they admit a Taylor expansion in a neighbourhood of $\omega = a$. The shuffle-regulated value of $\widetilde{A}$ at $\omega = a$ is then defined as

$$\text{Reg}_{\omega=a}\widetilde{A}(\omega) \equiv \widetilde{A}^{(0)}(a)\,. \tag{C.6}$$

Since $\widetilde{A}(\omega)$ is a pure function of weight $k+1$, its shuffle-regulated values remain pure and have the same weight. We have thus shown that

$$\mathcal{F}[a_{\nu n}\chi_{\nu n}](z) = \frac{|z|}{(z-b)(\bar{z}-c)}\left[\widetilde{A}^{(0)}(b) - \widetilde{A}^{(0)}(z)\right], \tag{C.7}$$

where the right-hand side is a pure function of weight $k+1$, which completes the proof. □

**Lemma C.2.** *The Fourier-Mellin convolution of a pure function $\phi$ made up of SVMPLs with uniform transcendental weight $n$ with a kernel $\mathcal{K}$ of the form*

$$\mathcal{K}(z,\bar{z}) = |z|^2 \sum_{i,j} \frac{a_{ij}}{(z-\alpha_i)(\bar{z}-\beta_j)}\,, \tag{C.8}$$

*with $\{a_{ij}, \alpha_i, \beta_j\} \in \mathbb{Q}$ is a pure function of SVMPLs with uniform transcendental weight $n+1$.*

*Proof.* As $\phi$ is a pure function made up of SVMPLs with uniform transcendental weight $n$, it can be written in the form

$$\phi = \sum_{k=0}^{n} \sum_{\bar{a}} b^{\bar{a}}_{n-k}\, \mathcal{G}_{a_1,\ldots,a_k}(z) \tag{C.9}$$

where $b^{\bar{a}}_{n-k}$ is a pure function made up of SVMPLs with uniform transcendental weight $n-k$ which do not depend on $z$. This implies that

$$\phi * \mathcal{K}(z,\bar{z}) = \sum_{k=0}^{n} \sum_{\bar{a}} b^{\bar{a}}_{n-k} \left( \mathcal{G}_{a_1,\ldots,a_k}(z) * \mathcal{K}(z,\bar{z}) \right). \tag{C.10}$$

Due to linearity of the convolution integral, we then only need to show that the convolution

$$f = \mathcal{G}_{a_1,\ldots,a_k}(z) * \left( \frac{|z|^2}{(z-\alpha)(\bar{z}-\beta)} \right), \tag{C.11}$$

is a pure combination of SVMPLs of weight $k+1$. Plugging in the definition of the convolution integral, we get

$$\begin{aligned} f &= \int \frac{d^2w}{|w|^2}\, \mathcal{G}_{a_1,\ldots,a_k}(w) \frac{|w/z|^2}{(w/z-\alpha)(\bar{w}/\bar{z}-\beta)} \\ &= \int d^2w\, \mathcal{G}_{a_1,\ldots,a_k}(w) \frac{1}{(w-z\alpha)(\bar{w}-\bar{z}\beta)}\,. \end{aligned} \tag{C.12}$$

To determine this integral we first compute the single-valued holomorphic primitive $F$ which is given by

$$F = \int_{\mathrm{SV}} dw\, \mathcal{G}_{a_1,\ldots,a_k}(w) \frac{1}{(w-z\alpha)(\bar{w}-\bar{z}\beta)} = \frac{\mathcal{G}_{z\alpha,a_k,\ldots,a_n}(w)}{(\bar{w}-\bar{z}\beta)}\,, \tag{C.13}$$

and then compute its antiholomorphic residues to get

$$f = \mathrm{Res}_{\bar{w}=\bar{z}\beta} \left( \frac{\mathcal{G}_{z\alpha,a_k,\ldots,a_n}(w)}{(\bar{w}-\bar{z}\beta)} \right) = \mathrm{Reg}_{\bar{w}=\bar{z}\beta}\, \mathcal{G}_{z\alpha,a_k,\ldots,a_n}(w) \tag{C.14}$$

where $\mathrm{Reg}_{\omega=a} A(\omega)$ denotes the shuffle-regulated value of $A(\omega)$ at the point $\omega = a$. Since $\mathcal{G}_{z\alpha,a_k,\ldots,a_n}(w)$ has weight $k+1$, its shuffle-regulated values will be pure functions of the same weight. This implies that $\phi * \mathcal{K}(z,\bar{z})$ is a pure function of SVMPLs with uniform transcendental weight $n+1$, which concludes the proof.

□

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