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Single Top production at Next-to-Next-to-Leading order: an analytic approach

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It seems to be the natural end of a journey, but maybe it is just the beginning...

Contents

Chapter 1

Introduction

Modern physics understands the presence of four fundamental interactions governing phenomena we observe in the universe: gravity, electro-magnetic force, weak and strong nuclear forces. The Standard Model (SM) is the theory developed through the last century to describe all of these forces except for gravity. Gravity however is not expected to play a role at current collider energies, where the dominant interactions are governed by the strong and electro-weak forces.
The SM has been successfully confirmed in many of its aspects through an impressive amount of experimental tests conducted at particle colliders. Nevertheless, there are many open questions and issues in particle physics, to which the SM may or may not give an answer. The hope of the particle physics community to find such answers lies nowadays in the Large Hadron Collider (LHC) at CERN, which allows us to investigate the fundamental nature of matter at smaller distances than ever before. With its astonishing luminosity records and unprecedented energies, the LHC has already led to the discovery of a scalar particle which shows all the properties of a SM-like Higgs boson. However, along with the importance of such big achievement, the lesson we have learned so far is that new physics, namely beyond the SM (BSM), does not show up in a spectacular way, rather it hides beyond a huge amount of 'ordinary' SM background processes. A reliable understanding of the SM processes which constitute a background to new physics is thus mandatory in order to perform any serious search at the LHC.

Since the LHC is a hadron collider, the main background is provided by processes whose dynamics is governed by the strong force, which is described at quantum level by the theory of Quantum Chromodynamics (QCD). In order for new physics to be discovered, it is therefore necessary to have full control over QCD driven effects. At high energies, QCD becomes asymptotically free, thus allowing the use of perturbation theory. However, higher order effects in perturbative QCD are generally large and must be included to a sufficient degree for comparisons between

theory and experiment to be meaningful.
On the theory side, the bottleneck in including these QCD higher-order effects is represented by the complexity and difficulty of computations which need to be carried out in order to assess such effects. Prompted by the LHC program, the theory community has made impressive breakthroughs in the last twenty years, and now computations which were considered almost impossible can be done with relative ease. As a result, the panorama of physical observables relevant for the LHC which have been computed at higher order in perturbative QCD has grown larger and larger. The Next-to-Leading-Order (NLO) sector is now fully automated and many $2 \to 2$ processes are now available also at Next-to-Next-to-Leading-Order (NNLO).

One important type of process at the LHC is the production of single top (or anti-top) quarks. Top quarks were discovered for the first time at Tevatron in 1995 by both the CDF and D0 experiments by studying top pair production. In this reaction, pairs of top and anti-top quarks are produced via the strong interaction. This is the dominant mode to produce top quarks at hadron colliders, but it is not the only one. Another possibility is represented by the so-called 'Single Top' reaction, which takes place via the weak interaction and yields in the final state just one top or anti-top quark. Compared with the $t\bar{t}$-production, Single Top production has a smaller cross-section both at Tevatron and at LHC. This, together with a difficult background, made the search for Single Top at Tevatron extremely difficult. This is not the case at the LHC, where, thanks to higher energies of the proton beams and luminosity, Single Top production has a sizeable cross-section. Single Top production is the main subject of this dissertation. The investigation of this production mechanism is crucial for precision study of the SM and could very well be a gateway to BSM physics. Here we report some of the main motivations for studying Single Top process.

The coupling between the Higgs boson and the top quark is described by a Yukawa coupling, y_t. The corresponding mass term in the Lagrangian density for the top quark can be expressed in terms of y_t and the electroweak symmetry breaking scale v, namely

$$m_t = \frac{y_t v}{\sqrt{2}}. \tag{1.1}$$

A very recent (2014) combined measure from ATLAS, CMS, D0, and CDF yields $m_t = 173\text{GeV}$ [1]. Given this value of the top mass and given $v = 246\text{GeV}$, we see that the top-Yukawa coupling is of order unity. This means that the top quark couples strongly to the Higgs field. This makes the top quark very important in studies of the Higgs sector.
According to the SM, the coupling of the top quark to the W-field is of the form

$$\frac{g_w}{\sqrt{2}} V_{tq} (\bar{t}\gamma^\mu q_L) W_\mu^+, \tag{1.2}$$

where V_{tq} is the CKM matrix element describing the mixing between a top quark and another d-type quark, and g_w is the electroweak coupling constant. We observe that the predicted coupling is flavour changing and purely left-handed. This means that the top quark in a single-top process should be polarized when produced. A further advantage is represented by the top quark mass, which is so large that, in contrasts to other quarks in the SM, it decays before it has the chance to hadronize. This means that by studying the decay products of the top quark, the suggested polarization can be measured directly. Single Top process measurements would then allow for verification of top polarization as predicted by the SM.
The top quark couples to the Z-field through a flavour preserving coupling of the form

$$\frac{g_w}{4\cos\theta_w}\bar{t}\left[\left(1-\frac{8}{3}\sin^2\theta_w\right)\gamma^\mu-\gamma^\mu\gamma^5\right]tZ_\mu. \tag{1.3}$$

In a potential scenario of flavour changing neutral currents, the top quark could couple to a, hitherto unknown, flavour changing field W' with a coupling of the form

$$G_{FC}\bar{t}\left(k_1\gamma^\mu+k_2\gamma^\mu\gamma^5\right)uW'. \tag{1.4}$$

Since the W' boson would be much heavier than the W and Z, this coupling may only occur at sufficiently high energies. Again, a direct measurements of the Single Top cross-section would allow for verification of the existence of such a channel.
Last but not least, Single Top production allows for a direct extraction of the CKM matrix element V_{tb}.

At tree-level, three clearly distinct channels contribute to Single Top production. Among these channels, we focus our attention on Single Top in t-channel, where an off-shell W-boson is exchanged in t-channel between two hadronic currents $q \to W^* + q'$ and $b + W^* \to t$, thus producing the reaction $b + q \to t + q'$. This channel has the largest cross-section among the three production modes and is best measured by the experiments.
The work developed in this thesis is part of a more general project whose aim is that of improving the precision of the theoretical prediction for Single Top production in t-channel by taking into consideration QCD quantum effects at higher orders. At present, analytical calculations of quantum corrections to t-channel Single Top inclusive cross-section are available both at NLO-QCD ($\mathcal{O}(\alpha_s)$) and NLO-EW ($\mathcal{O}(\alpha_{ew})$). A fully numerical computation of NNLO-QCD ($\mathcal{O}(\alpha_s^2)$) contribution has recently become available too. Analytical computation of these corrections is still lacking. Analytic results for NNLO-QCD corrections are of interests since they would provide a robust cross-check to the numerical computation mentioned above and would constitute the core for a fast numerical evaluation of the cross-section. In this dissertation we present results for the set of Master Integrals describing

the $\mathcal{O}(\alpha_s^2)$ corrections to the partonic process $b + W^* \to t + X$ and more in general to Form Factors for Charged-Current-DIS with a massive particle in the final state. In the following we explain briefly how this set of Master Integrals happens to constitute a partial result towards the achievement of analytical NNLO-QCD corrections to inclusive Single Top in t-channel.

We consider QCD corrections to Single Top in t-channel within a Structure Function approach. This means that we take into account only those corrections which do not connect the two hadronic currents. We call this type of quantum corrections *factorizable*, since they affect only one single hadronic current at a time. We neglect all those corrections which involve cross-talk between the two currents. In this approximation, all the information about QCD higher-order contributions to the single hadronic current is encoded in the Structure Functions (or Form Factors) describing the current. Analytic expressions for Form Factors describing the massless current $q \to W^* + q'$ are already available up to NNLO-QCD, whereas the ones for the massive ($m_t \neq 0$!) current $b + W^* \to t$ are available only up to NLO-QCD. Our effort is thus concentrate on the analytic calculation of the inclusive cross-section for the partonic subprocess $b + W^* \to t + X$ at $\mathcal{O}(\alpha_s^2)$, where X accounts for possible extra radiation in the final state. This cross-section then, convoluted with Parton Distribution Functions (PDFs), will yield the massive Form Factors describing the hadronic reaction $p + W^* \to t$, which sees a proton and on off-shell W producing a top in the final state. The technique we use to compute the cross-section analytically for $[b + W^* \to t + X]_{\mathcal{O}(\alpha_s^2)}$ is that of Master Integrals, which has been developed in the last twenty years and proved to be successful in many difficult computations.
The original work contained in this thesis thus consists of the determination and computation of the entire set of Master Integrals needed to describe at NNLO in QCD the Single Top massive current $b + W^* \to t + X$.

The structure of the thesis is thus the following. Chapter 2 is dedicated to the physics of top quark, seen from both a theoretical and experimental point of view. Particular attention is given to top quark production mechanisms and especially to Single Top. Theoretical and experimental benchmarks are provided for production cross-sections and for some other relevant observables in top physics.
Chapter 3 illustrates the basics of perturbative QCD, specially the structure of an observable computed at fixed-order in QCD. In this chapter we also introduce the DIS-like picture applied to Single Top in t-channel and we explain in detail which partonic channels contribute at NNLO and thus which partonic subprocesses we need to compute analytically in order to extract the necessary Form Factors.
In chapter 4, we focus on the technique of Master Integrals. First we generically review Feynman integrals and their properties. Then we introduce the defining properties and concepts of Master Integrals, explain the basic principles on which this technique relies. Finally, chapter 5 contains our original work. We report the final set of Master Integrals obtained for the Charged Current(CC)-DIS Massive

Form Factors and go in detail through the explicit calculation of such set of integrals.
In order not to make the presentation of our work too heavy, the complete list of intermediate and final results for the Masters is reported in the Appendix.
Finally, we close our dissertation with an outlook on the to-do-list which constitutes work in progress and which we will accomplish hopefully in the near future in order to achieve our final result, namely t-channel Single Top cross-section at NNLO-QCD.

Chapter 2

Top Physics

As we will show in detail throughout this chapter, although Single Top is not the dominant top quark production mechanism, it yields a sizeable fraction of top quark events, especially at the LHC. Moreover, Single Top is an interesting framework for SM parameters precision test and for the discovery of possible new BSM physics. While the theoretical status of top pair production is advanced, competitive theoretical predictions for Single top are still in a very early stage. These considerations, together with an experimental precision expected to increase in the coming years, makes Single top an interesting field from the point of view of both QCD fixed-order perturbative computations and resumed calculations.
Particularly appealing is the t-channel mode. It has the largest cross-section among single top channels, it is the best measured channel from experiments, and its quantum corrections happen to be pretty small, so that, by combining the different available corrections (QCD and EW), it could easily become one of the most precisely predicted processes in the SM.
In this first chapter we aim to provide the pieces of information, both from a theoretical and experimental point of view, which support these statements. The reader is introduced to the fundamentals of Top Physics which constitute the necessary background in order to understand the physics underlying the project developed in this thesis and the reasons why we focus our attention on Single Top in t-channel.

2.1 Top Quark Physics Overview - Theory

The framework in which nowadays Physics describes nature in its most fundamental aspects is the Standard Model (SM). In this theory, three out of the four fundamental forces present in nature are unified, namely electric, weak and strong forces. The formal framework in which the SM is formulated is that of Quantum

Field Theory. In particular, the SM belongs to the category of gauge theories, with the gauge group $SU(3) \times SU(2) \times U(1)$, where $SU(3)$ is the gauge group of strong interactions and $SU(2) \times U(1)$ that of the electro-weak interactions. In the following a basic knowledge of the SM will be taken for granted, and attention will be focused directly on Top Quark Physics, which is the general area of interest of this thesis[1]. First we review general top quark properties and interactions in the SM. Then, we focus our attention on top quark production and provide a list of the most recent theoretical predictions and benchmarks for top production cross-sections. Finally, we spend the last two subsections to briefly explain the privileged role that top quark has in probing the SM Higgs sector and many BSM physics scenarios.

2.1.1 Top quark properties and interactions in the Standard Model

The top quark is the up-type quark of the third family in the SM. Each family of quarks consists of an up- and down-type quark, which have electric charge $Q_{up} = 2/3$ and $Q_{down} = -1/3$ respectively. On top of that, each family represents a weak-isospin doublet, so that its up- down-type members have weak quantum numbers $T_{up} = +1/2$ and $T_{down} = -1/2$ respectively. Quarks are charged under the strong interactions, being triplets in the $SU(3)$ group.
The most striking feature of the top quark is that, although this particle appears to be point-like, its mass is huge, roughly of the order of a gold nucleus. Top quark phenomenology is mainly driven by its large mass. Being heavier than a W boson, it is the only quark that decays semi-weakly into a W boson and a b quark before hadronization can occur. On top of that, it is the only quark whose Yukawa coupling to the Higgs boson is of order one, meaning that the top quark plays a dominant role in the running of the Higgs mass. Because of this tight link to the electroweak symmetry breaking sector, a deep understanding of all top quark properties, from its quantum numbers to its interactions with the strong, weak and Higgs sectors, is a cornerstone for our understanding of nature at the smallest distance scales. In general we can recall the following points, which are nowadays driving searches in Top Physics.

- Accurate knowledge of top-quark mass is a fundamental input to precision electroweak analyses.

- The Yukawa coupling of the top is proportional to the ratio m_t/v, where v is the vacuum expectation value of the Higgs boson. Since this ratio approaches unity from below, the coupling happens in turn to be very close to unity. Furthermore, the proportionality of the Yukawa to the top mass is in itself an

[1]For the reader who might need or want to dig more in detail in the basis of the SM, we suggest references [99], [102].

interesting feature, since the top mass is the quark mass known with the best accuracy. These two properties together makes the study of such coupling a priority in top physics, since its precise measurement would provide a stringent test of the electro-weak symmetry breaking (EWSB) sector of the SM.

- Unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) constrains the CKM-matrix element $|V_{tb}|$ to be close to one, so that an accurate measurement of this CKM-matrix element is also very important. The extraction of $|V_{tb}|$ can be achieved by studying top production and decays in the SM.
- Top physics is a possible window on Beyond-the-Standard-Model (BSM) physics. Indeed, events containing top quarks are backgrounds to certain new physics processes, so that a precise assessment of such background becomes fundamental in indirect searches for BSM particles.

In this section an overview of interactions and processes in which top quark is involved in the SM is given, with particular attention to the use that can be made of such processes in order to gain better knowledge of certain Standard Model parameters and top properties. At the end of the section theoretical benchmarks for production cross-section are reported.

Top Strong Interactions

The main production mechanisms for (anti)top quarks at hadron colliders are through quark-antiquark annihilation and gluon-gluon fusion (see diagrams in Fig.2.1), which both take place via the strong interaction. At Tevatron, the quark-antiquark annihilation subprocess dominates, whereas at the LHC gluon fusion dominates. The reason for this resides mainly in the shape of Parton Distribution Functions (PDF). Given a collision between two protons carrying momentum P_1 and P_2, the square of the total energy of the partonic subprocess (in the partonic center-of-mass frame) s is related to the energy of the hadronic collision S through

$$s = (x_1 P_1 + x_2 P_2)^2 \simeq 2 x_1 x_2 P_1 \cdot P_2 = x_1 x_2 S. \tag{2.1}$$

with x_1 and x_2 being the fractions of momentum carried by the two initial partons. The threshold for $t\bar{t}$ production is of course $s \geq 4m_t^2$. It follows from Eq.(2.1) that

$$x_1 x_2 = \frac{s}{S} \geq \frac{4m_t^2}{S}. \tag{2.2}$$

If we now take $x_1 = x_2 = x$ in Eq.(2.2), we get the condition $x \geq 2m_t/\sqrt{S}$. This translates into the numerical values $x \geq 0.05(0.025)$ for a $\sqrt{S} = 7(14)$TeV LHC, and $x \geq 0.2$ at Tevatron. Fig.2.2 shows PDFs, which contain information about the probability of finding a given parton species with momentum fraction between

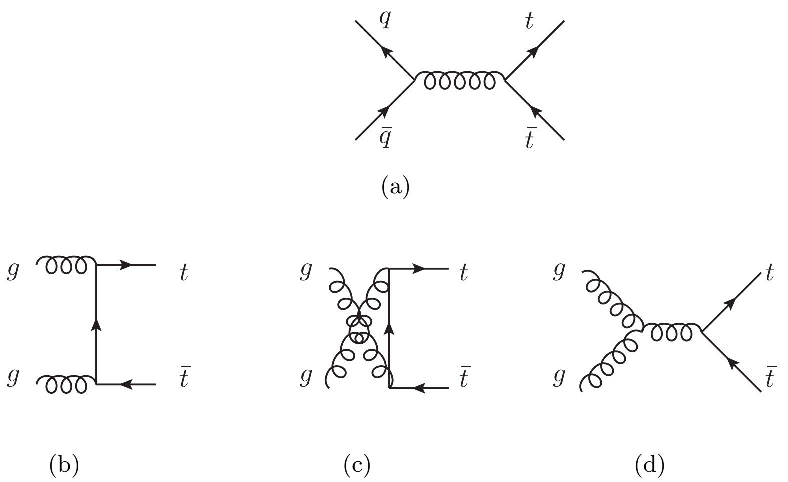


Figure 2.1: Tree-level diagrams for strong production of $t\bar{t}$ pair.

x and $x + dx$. By looking at this plot, it is clear that at the LHC the gg contribution will be the dominant one, whereas at Tevatron the situation is reversed and the $q\bar{q}$ will give largest contribution to the cross-section.

Top quark pair production can be experimentally classified according to the decays of the W bosons coming from the decay of the two top quarks (see below for more information about top decay). In the dilepton channel, namely when both W's decay leptonically, the experimental signature consists of two high-p_T leptons, large missing transverse energy $\not{E}_T$ and at least two b-jets. The branching fraction is comparatively small, but the backgrounds, mostly Z+jets, are also fairly small. This makes the dilepton channel an ideal place to obtain a very clean sample of $t\bar{t}$ events. On the other hand, the hadronic channel, where both W's decay hadronically and thus the experimental signature is at least six jets, two of them b-jets, suffers from a huge background of QCD multi-jet events. This makes measurements of $t\bar{t}$ production in this channel difficult, despite the large branching fraction. Finally the lepton+jets channel, where one W decays hadronically and the other leptonically, has both large branching ratio and moderate background (mostly W+jets). For this reason it is often referred to as the *golden channel.* Its signature is one high p_T lepton, $\not{E}_T$ and at least four jets. In both the lepton+jets and dilepton channel, one typically considers only decays into electrons or muons (including those from leptonic tau decays), whereas final states with hadronically decaying taus are experimentally much more challenging and are often studied

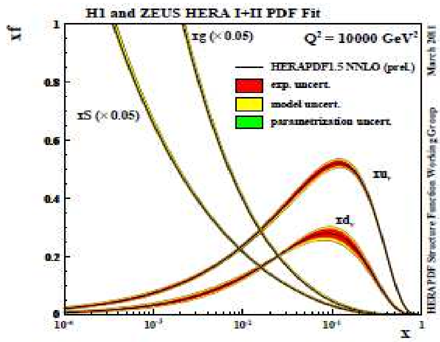
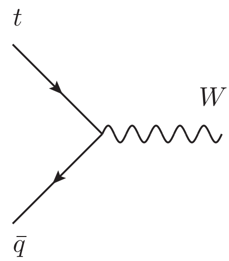


Figure 2.2: HERAPDF1.5NNLO PDF evaluated at $\mu^2 = 10000$ GeV. The behaviour for $\mu^2 = m_t^2 \sim 30000$ GeV is similar.

separately.

Top Weak Interactions

Top quarks also interact weakly. The charged-current weak interaction connects a top with a down, strange or bottom quark, with an amplitude proportional to the corresponding CKM matrix element, respectively V_{td}, V_{ts} or V_{tb} (Fig.2.3). Through charged-current weak interaction both production and decay of top quarks can take place, as described in the following.



$$= -i\frac{g}{2\sqrt{2}}V_{tq}\gamma^{\mu}(1-\gamma_5)$$

Figure 2.3: Top weak current.

The weak production mechanism goes under the name of *Single Top production*, since it allows for production of one single top or antitop in association with a light quark, a b quark or a W, as it is illustrated in the tree-level diagrams in Fig.2.4. Starting from the left, the first diagram shows the W boson exchanged in t-channel, the second one in s-channel whereas in the third one a W is produced in association with the top.

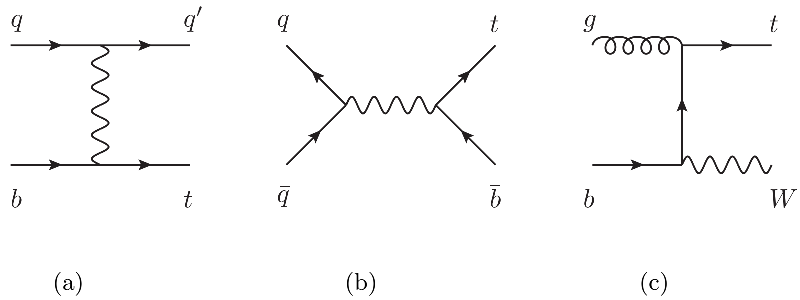


Figure 2.4: Tree-level Feynman diagrams for Single Top production.

Even though t- and s-channel modes are enhanced by the Phase Space, thanks to the presence of just one heavy particle in the final state, in general single top topologies involve fewer jets compared to $t\bar{t}$ production, and the signal to background ratio is generally significantly smaller, so that $t\bar{t}$ production remains the dominant mechanism to produce top quarks both at Tevatron and LHC.
Turning to analyse single channels, both at Tevatron and LHC, t-channel is the dominant contribution, at nearly one third of the $t\bar{t}$ cross-section. The next largest cross-section at Tevatron is from the s-channel subprocess whereas at the LHC it comes from associated tW production. This might seem surprising because, having two heavy particles in the final state, this subprocess is clearly suppressed with respect to the t- and s-channel. But, at the LHC, the gluon PDF gives a strong enhancement to this subprocess, such as to compensate the Phase Space suppression.
We observe that in the t-channel subprocess there is a b in the initial state. The b-PDF is mostly driven by gluon radiation, meaning that the b-quarks are most likely to originate from a splitting $g \to b\bar{b}$. For this reason, the b distribution function cannot be extracted directly from global fits of experimental data as it happens for light quarks, but is calculated from the initial condition $b(x) = 0$ at $\mu = m_b$ and evolved to higher values of the factorization scale μ_R via the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations (for an introduction to DGLAP equations, see Chapter 3). A more detailed discussion of

Single Top production involving a bottom in the initial/final state will be given in Chapter 3, in the context of a comparison between the 4-flavour and 5-flavour scheme.

Cross-section measurements for t-channel production have been provided both by LHC and Tevatron, by searching for a signature given by one or more jets originating from the light quark recoiling against the top, and a b-jet plus $\not{E}_T$ from the W decay. The main background is given by W+jets.
The tW channel has not been observed at Tevatron, due to its very small cross-section at Tevatron energies. This mode is indeed challenging also at the LHC, since it interferes at Next-to-Leading-Order (NLO) in QCD with top quark pair production. Some methods have been implemented in current MC generators to allow an unambiguous signal definition. According to the decays of the two W-bosons, this channel can be studied via different final state signatures: dilepton channel, where both W decay into a charged lepton and a neutrino, or in the lepton-jets channel, where one of W decays into lepton and neutrino, while the other one decays hadronically into jets.
Finally, the s-channel has never been observed individually at Tevatron, since only a combined t- plus s-channel cross-section has been measured. At the LHC, despite the small cross-section, this channel is interesting for indirect searches of various new physics models and an initial search has already been carried out by the ATLAS collaboration. The final state signature is given by a b-jet plus either a lepton and $\not{E}_T$ if the W decays weakly or by additional jets if the W decays hadronically. Backgrounds include mainly W+jets, QCD multi-jet production and $t\bar{t}$ production.

Top decay is illustrated in Fig.2.5.

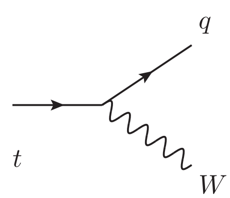


Figure 2.5: Top quark decay into quark and W boson.

The top quark decays almost exclusively as $t \to bW$. Since $|V_{tb}| \gg |V_{td}|, |V_{ts}|$, the decays $t \to W(d, s)$ are strongly suppressed. At NLO QCD, the total width of the top quark Γ_t depends on four parameters, namely $G_F, m_t, m_W, |V_{tb}|$. A top mass

of 172.5 GeV gives $\Gamma_t = 1.33$GeV, whose inverse gives a lifetime $\tau_t \sim 5 \cdot 10^{-25}$s. As it will be discussed further in this Section, such short lifetime implies that top quark decays before it can hadronize, so that *toponium* $t\bar{t}$ bound state cannot form.
The branching ratios for the decay of the top quark into light quarks s, d are strongly suppressed in the SM (BR($t \to Ws$) $\sim 0.2\%$, BR($t \to Wd$) $\sim 0.005\%$). Given the unitarity of the CKM matrix, the denominator of the ratio R defined as

$$R = \frac{|V_{tb}|^2}{|V_{tb}|^2 + |V_{ts}|^2 + |V_{td}|^2} \tag{2.3}$$

is equal to 1. Thus, if unitarity of the CKM matrix holds, a measurement of R provides constraints on $|V_{tb}|$. Measurements of R at the LHC will be briefly discussed in the next subsection, dedicated to measurements of properties of top quark.

2.1.2 Role of QCD in theoretical predictions and benchmarks

The increasing precision of experimental measurements in top physics introduces now a new challenge also on the theory side. Let us take the clearest example, namely inclusive cross-section for $t\bar{t}$ production. The most precise measurements, listed in Fig.2.13a, have reached an unprecedented precision of $\sim 4.5\%$ which, as underlined in [33], challenges the current theory benchmark precision (to be discussed in a while). For top quark weak production, the scenario is slightly different. Experimental precision is still quite larger then theoretical one even in t-channel, which is the best measured among the three weak production modes (see Fig.2.15a). The situation is expected to change in the future though. With the LHC Run III, thanks to the higher energies and increased luminosities, experimental error bars will decrease, even for difficult topologies like single top ones.
Given this picture, if a meaningful comparison between theory and experiments is to be done, theoretical predictions for both signal and background processes need in turn to be updated with increasing precision. Within the framework of a perturbative approach to the SM, this means taking into account quantum corrections at higher orders in the expansion around the coupling constants. Being the LHC a proton-proton collider, and being, at the LHC energies, the strong coupling constant α_s larger than the electromagnetic and weak ones, the most significant enhancement in precision is given by QCD corrections.
While the details of how a cross-section for an hadron-initiated process is computed will be given in the next chapter, we provide here a list of theoretical benchmarks for top production mechanisms.

With top pair production being the dominant production mode at the LHC, efforts of the theory community have concentrated mainly on this process, so that it is now one of the most precisely predicted LHC standard processes. The NLO-QCD ($\mathcal{O}(\alpha_s^3)$) corrections are known since more than 20 years [97], [98], [17], the mixed QCD-weak corrections of $\mathcal{O}(\alpha_s^2\alpha)$ were computed in [16], [23], [25], [24], [85], [86], and the mixed QCD-QED in [79]. There are also calculations of $t\bar{t}$ production at NLO-QCD which include the top quark decays and the correlations between production and decay, such as the information on the top quark spin. These results have been obtained in the narrow-width approximation for top quarks produced on shell ([21], [22], [91], [26], [39]). The NLO-QCD differential cross sections for the production of $t\bar{t}$ in association with 1 and 2 extra jets are available [57], [58], [92], [90], [27], [28]. Probably the most notable progress is, in 2013, the first complete calculation of the inclusive and fully differential top-pair cross-section at NNLO-QCD, reported in [54], where it was also directly supplemented with the previously computed Next-to-Next-to-Leading-Log (NNLL) resumed result (see references in the original paper [54]). The results for the LHC cross-section at NNLO-NNLL as a function of the center of mass energy are plotted in Fig.2.6, together with experimental measurements at 7 and 8 TeV. More recently, results have also become

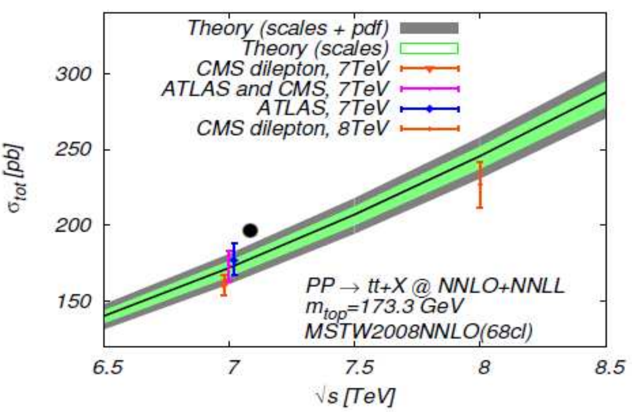


Figure 2.6

available for the approximate N3LO (NNNLO) top pair cross-section. For instance the NNNLO soft-gluon corrections have been released in [84] during 2014, and in 2015 a N3LO approximated cross-section has been obtained in [96] by matching two contributions computed respectively in the high- and low-energy limit.

Computations for Single Top cross-section are instead in a less advanced stage, even though theoretical results for single top quark production are available at an ever increasing level of sophistication. These include NLO-QCD and EW predictions in four- or five-flavour scheme for both stable ([31], [117], [74], [15]) and decaying ([40], [], [43], [41], [101], [103], [61], [60]) top quarks, resummations ([83], [82], [126]) and fixed order computations matched to parton shower ([64], [65], [62]). Focusing on NLO-QCD corrections, it is interesting to note that they are small, of the order of a few percent, for the t-channel production. On the other hand, they are large for the s-channel, of the order of fifty percent, both at Tevatron and LHC. Finally, corrections to tW associated production are known to be moderate at both colliders [43]. In this panorama, the most striking and up-to-date result is the numerical computation, achieved in 2014, of the bulk of NNLO-QCD corrections to t-channel single top production ([36]). The numerical approach has made it possible to address both the NNLO inclusive and fully differential cross-section. Setting the renormalization and factorization scales $\mu = m_t$, it is found for inclusive single top t-channel production that

- $\sigma_t^{LO} = 53.8pb$,
- $\sigma_t^{NLO} = 55.1pb$,
- $\sigma_t^{NNLO} = 54.2pb$.

It is interesting to observe that NLO and NNLO corrections are of the same order of magnitude, namely a few percent. This confirms that in the case of Single Top production too it is worth the effort to go beyond NLO. For a more detailed discussion and interpretation of these numbers, and for the complete list of numerical results, including t-channel anti-top production and transverse momentum distributions, we refer the reader to the original paper (above-mentioned).

2.1.3 Top quark and Higgs boson

As already mentioned briefly, the top quark is closely related to the detection and study of the Higgs boson, mainly because of its large mass. The paragraph that follows is dedicated to describe the topic more in detail.

As a loop-particle, the top quark plays an important role in electroweak precision analyses, as we briefly illustrate in the following. There are five independent SM parameters (including gauge, matter and Higgs sectors) : the three gauge couplings g_s,g,g', respectively related to strong, and electro-weak interactions, the Higgs vacuum-expectation value v and the Higgs self-interaction coupling λ. At tree-level the independent quantities reduce to just three, g, g', v, which are related to three very well measured quantities by

$$\alpha_{em} = \frac{1}{4\pi}\frac{g^2 g'^2}{g^2 + g'^2}$$

$$G_F = \frac{1}{\sqrt{2}v^2}$$
$$M_Z = \frac{1}{2}\sqrt{g^2 + g'^2}. \tag{2.4}$$

From these three quantities all other electroweak quantities can be predicted at tree-level, including the well known relation between the W mass and the Z mass, α and G_F, Eq.(2.5).

$$M_W^2 = \frac{1}{4}g^2v^2 = \frac{1}{2}M_Z^2\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}G_F M_Z^2}}\right) \tag{2.5}$$

Eq.(2.5) can then be reformulated as

$$M_W^2 = \frac{\pi\alpha/(\sqrt{2}G_F)}{\sin^2\theta_w} \text{ with } \sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}, \tag{2.6}$$

Eq.(2.6) holds at tree-level, but when one wants to go at one loop , the presence of an extra term Δr, which takes into account one loop corrections, modifies Eq.(2.6) into

$$M_W^2 = \frac{\pi\alpha/(\sqrt{2}G_F)}{\sin^2\theta_w(1-\Delta r)}. \tag{2.7}$$

The contribution of top quark to Δr is given through the $t\bar{b}$-loop in the W self-energy and the $t\bar{t}$-loop in the Z self energy (see diagrams in Fig.(2.7)).

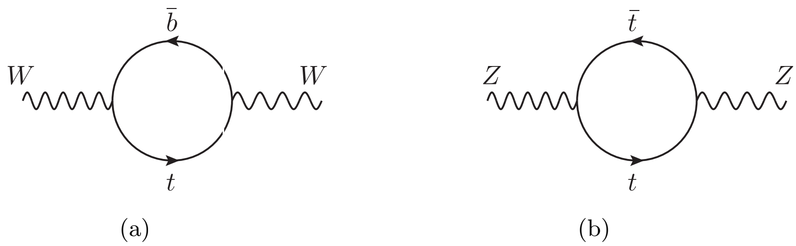


Figure 2.7

By computing 1-loop diagrams in Fig.(2.7), one obtains the expression Eq.(2.8).

$$(\Delta r)_{top} \simeq -\frac{3G_F m_t^2}{8\sqrt{2}\pi^2}\frac{1}{\tan^2\theta_W}. \tag{2.8}$$

In the same way, also the Higgs boson gives its contribution Eq.(2.9) to the 1-loop correction Δr through the diagrams in Fig.(2.8).

(a) (b)

Figure 2.8

The resulting contribution $(\Delta r)_{higgs}$ is given by

$$(\Delta r)_{higgs} \simeq \frac{11 G_F M_Z^2 \cos^2\theta_W}{24\sqrt{2}\pi^2} \ln\frac{m_h^2}{M_Z^2}. \qquad (2.9)$$

By inspection of Eq.(2.8), (2.9), we see that the main contribution is given by the top loop, whose expression depends quadratically on m_t, whereas the Higgs loops only contribute a logarithmic dependence on m_{higgs}. Nonetheless, both these heavy particles contributions to Δr have to be taken into account in order to predict M_Z at 1-loop, implying that, at this perturbative order, there are five independent SM parameters , namely $\alpha_{em}, G_F, M_Z, m_t, m_h$. These relations can also be used the other way around, namely to predict m_h starting from precise measurements of $\alpha_{em}, G_F, M_Z, m_t, M_W$. This explains how electroweak analyses can produce an estimate for the mass of the Higgs boson. Needless to say that, the more accurate the input parameters are, the narrower will be the range the Higgs boson mass will be constrained to. This is one of the main reason why an accurate measurement of the top quark mass is important in the context of precision tests of the Standard Model.

Top and Higgs are also strongly related from the point of view of Higgs boson production through processes mediated by the strong interactions. Such processes are *Higgs production in association with a $t\bar{t}$ pair* and *gluon fusion*. The former is obtained by remembering the Yukawa couplings of the Higgs boson to fermions Fig.(2.9), which allows for radiation of an extra Higgs from a top or antitop in diagrams contributing to $t\bar{t}$ production (Fig.2.1). The latter is instead a process initiated by two gluons which, through a quark loop, produce a Higgs in the final state, as illustrated at tree-level in Fig.2.10.

While associated production with a $t\bar{t}$ pair is suppressed by the presence of three heavy particles in the final state, gluon fusion happens to be the dominant Higgs

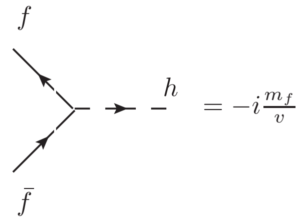


Figure 2.9: Yukawa coupling of the Higgs boson to fermions.

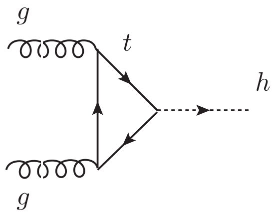


Figure 2.10: Higgs production via gluon fusion.

production mechanism at the LHC, as can be seen in the theoretical predictions shown in Fig.2.11. In the gluon fusion process, the role of top quark is dominant. In principle, any type of quark can circulate in the fermion loop, but since the Higgs coupling to fermions is proportional to the squared mass m_q^2 of the fermion, contributions from lighter quarks propagating in the loops are suppressed proportionally to m_q^2. The main role is then acted by the top quark circulating in the loop. The leading top quark contribution can be evaluated, with a good approximation, in the limit $m_t \to \infty$, by matching the Standard Model to an effective field theory where the amplitude is evaluated from an effective Lagrangian which describes the point-like effective coupling of the Higgs to the gluons. The role of m_t remains of great importance in this scenario, since, in order to assess the validity of such effective approach, the effective result is to be compared with approximated calculations of the m_t dependence based on asymptotic expansions. Gluon fusion is the theoretically best known channel among those contributing to Higgs production. Fixed order analytical QCD corrections are available up to NNLO and since 2015 up to NNNLO (computed by summing the first 37 orders of the threshold expansion of the cross-section, see [6]). Electroweak radiative corrections are computed at NLO, whereas virtual corrections are available up to 2-loops. Also mixed QCD-electroweak corrections at order $\mathcal{O}(\alpha\alpha_s)$ have been calculated. QCD corrections at NLO and NNLO have been improved through the matching to the results which takes into account the resummation of soft-gluon contributions at full next-to-next-to-leading logarithmic (NNLL) accuracy. References can be found in

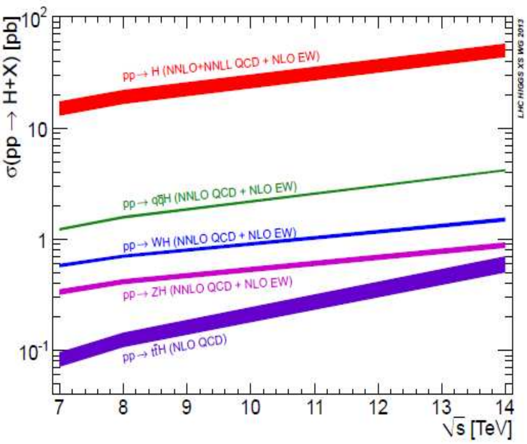


Figure 2.11: The SM Higgs boson production cross sections as a function of the hadronic center of mass energy, $\sqrt{S}$, for pp collisions. Bands indicate theoretical uncertainties.

[99].

Given its sensibly smaller cross-section at the LHC, the state of the art for Higgs associated production with a $t\bar{t}$ pair is much less advanced. Currently, NLO QCD corrections and interfaces between NLO QCD and parton-shower Monte Carlo are available for this process (references are given in [99]). These programs provide, up to now, the most flexible tools for the computation of differential distribution, of $pp \to t\bar{t}H$, including experimental cuts.
Both gluon fusion and production in association with a $t\bar{t}$ pair, can provide important information on the top-Higgs Yukawa coupling. As already mentioned, one striking feature of the SM Higgs boson is its strong coupling to the top quark relative to the other SM fermions. Based on its large mass, the top-quark Yukawa coupling is expected to be of order one. Since the top quark is heavier than the Higgs boson, its coupling cannot be assessed by measuring Higgs boson decays to top quarks. However, this coupling can be experimentally constrained through measurements involving the gluon fusion production mechanisms, assuming there is no physics beyond the SM contributing to the loop. On top of that, the top quark Yukawa coupling can be probed directly through a process that involves both a Higgs and top quarks explicitly reconstructed via their final-state decay products. $Ht\bar{t}$ associated production satisfies these requirements, so that a measurement of the rate of $Ht\bar{t}$ production provides a direct test of the coupling between the top quark and the Higgs boson.

2.1.4 Top physics as a window on Beyond-the-SM physics

We recap very briefly the BSM scenarios that have been explored up to now with processes involving (anti)top quarks. Since it goes beyond the purpose of this thesis and given the proliferation of BSM models and searches, we do not report directly bounds found by experiments at Tevatron and mainly at LHC in single searches, but refer the reader to [111], where all bounds are listed and explained in detail.

- *$t\bar{t}$ invariant mass distribution* - Many extensions of the SM predict new interactions, leading to new particles that would decay predominantly into $t\bar{t}$ pairs and may then show up, in the simplest scenario, as resonances in the top quark pair invariant mass distribution. New particles coupling predominantly to top quarks could be realized in many different ways. They could be spin-0 scalars or pseudo-scalars in supersymmetric (SUSY) or Two-Higgs-Doublet (2HDM) models, as well as spin-1 vector or axial-vector particles, for instance a leptophobic topcolor Z' boson. The shape of the $t\bar{t}$ invariant mass distribution is studied separately in the low-energy (below 1TeV) and high-energy (above 1TeV) regime, since for large masses, the top quarks decay products tend to be collimated and a dedicated reconstruction algorithm is necessary. In Fig.(2.12), the shape at Tevatron (left) and at LHC 7 TeV (right) are shown.

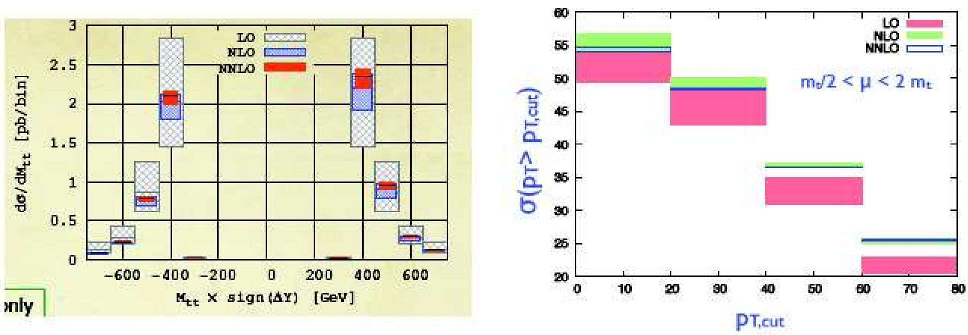


Figure 2.12: Left plot: Tevatron results for the invariant mass distribution M_{tt} for top quark pair production at NNLO accuracy. Right plot: NNLO cross-section for the production of single top quarks at the LHC 7 TeV, as a function of the cut of the p_T of the top quark.

- *Flavour changing neutral currents (FCNC)* - In the SM at tree-level FCNC are forbidden, and also at loop level are suppressed with respect to the dominant decay mode. Interestingly, many SM extensions allows for tree-level FCNC, which is translated into increased predicted branching fractions for FCNC decays of top quarks. Such extensions can be for instance 2HDM, MSSM (minimal supersymmetric Standard Model), topcolor assisted technicolor, super-symmetry with R-parity violation. The privileged process for searches of upper bounds on the branching fractions $\mathrm{BR}(t \to q\gamma), \mathrm{BR}(t \to qZ), \mathrm{BR}(t \to qg)$ is Single Top production. Since, in particular, the decay mode $t \to qg$ is very difficult to disentangle from the QCD multi-jet background, a first dedicated search was also presented by ATLAS for anomalous Single Top production $qg \to t \to bW$, where better sensitivity should be achieved for the anomalous coupling $qg \to t$.

- *Anomalous $\not{E}_T$ in $t\bar{t}$ production* - In some models like SUSY models with R-parity conservation and little Higgs models, partners of the top quark with masses below around 1TeV appear. In such scenarios, pair-produced exotic top partners $T\bar{T}$ can decay each into an ordinary top and a neutral weakly interacting particle A_0, thus giving rise to the final state $T\bar{T} \to t\bar{t}A_oA_o$. From the point of view of experimental detection, this final state has the same signature as a normal $t\bar{t}$ production, but with increased missing $\not{E}_T$. A first search for such a final state has already been performed by ATLAS.

- *Same sign top quark pair production* - Some models predict FCNC in the top quark sector mediated by the t-channel exchange of a new boson Z'. This type of interaction would also give rise to same-sign top quark pair production.

- *Charged Higgs production* - In 2HDM or SUSY models the existence of light charged Higgs boson $H^{\pm}$ is predicted. Such particles can be produced for instance in the decay of a quark top, through the $t \to H^+b$ or $\bar{t} \to H^-\bar{b}$, with subsequent decay of the charged Higgs bosons $H^{\pm} \to \tau\nu_\tau$. Searches have been performed using both top pair production and Single Top events.

2.2 LHC as a top factory, experimental ‘state of the art‘

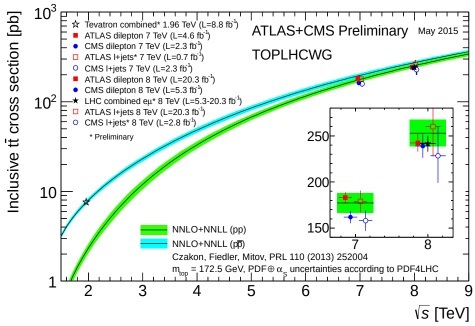
This section is dedicated to give an overview on precision top physics at the LHC from an experimental perspective. In the entire section, particular emphasis is put on top production mechanisms, especially on the weak production of single (anti-)top, which represents the subject of this thesis.
The section is divided into two parts. In the first subsection we review the experimental successes in measuring top quark production cross-sections achieved thanks to the high energies available at the LHC. In the second part of the section, we report up-to-date measurements of some SM observables related to Top physics which can be indirectly extracted in experimental searches. In particular we address measurements of top quark quantum numbers and of the CKM matrix element $|V_{tb}|$.

2.2.1 LHC experimental benchmarks: top quark production

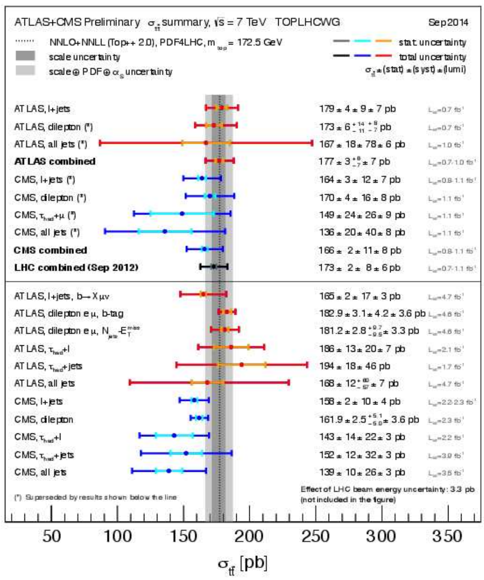
Measurements for $\mathbf{t\bar{t}}$ cross-section.
As it was already mentioned in the previous section, the strong interaction mediates the leading production mechanism for top quarks, which reads $pp(p\bar{p}) \to t\bar{t}$ at LHC(Tevatron). The final states for pair-production at leading-order in QCD, can be the following (where the quarks in the final states evolve then into jets of hadrons):

1. *all-jets channel* : $t\bar{t} \to W^+b\, W^-\bar{b} \to q\bar{q}'b\, q''\bar{q}'''\bar{b}$,
2. *lepton+jets channel* : $t\bar{t} \to W^+b\, W^-\bar{b} \to q\bar{q}'b\, l^-\bar{\nu}_l\bar{b} + l^+\nu_l b\, q''\bar{q}'''\bar{b}$,
3. *dilepton channel* : $t\bar{t} \to W^+b\, W^-\bar{b} \to \bar{l}\,\nu_l\, b\, l'\bar{\nu}_{l'}\,\bar{b}$.

Although the symbol $l(\bar{l})$ refers in general to a lepton(anti-lepton), so that it could be e, μ or τ, most of the analyses distinguish the e and μ from the τ channel, which is more difficult to reconstruct. Otherwise, in the following, the symbol l and the generic word ‘lepton’ will refer to the leptonic flavours e, μ, unless specified. The reason why τ leptons are more challenging is that they have a very short lifetime (approximately $2.9 \cdot 10^{-13}s$), so that most of them decay before leaving the beam pipe. They decay either leptonically into a lighter lepton (e or μ) and the corresponding flavour neutrino (BR $\sim$ 35%) or into hadrons (BR $\sim$ 65%). Since the cross-section for the production of hadronic jets is much larger than the one for the production of jets coming from τ leptons, the challenge lies in rejecting the jets faking τ candidates. When the τ leptons decay leptonically, this is usually counted as signal in the dilepton or lepton+jets final states. Dedicated analyses are instead carried out for hadronically decaying taus. In this case, two kinds of analyses are usually distinguished. On one side, there is the dilepton channel with one τ, where both W bosons decay leptonically, one into a light lepton e or μ and the other one into a τ, the last one giving then rise to hadrons (hadronic τ). On the other side, also the τ+jets case is considered, where one W boson decays directly hadronically, and the other one leptonically into a hadronic τ. Analyses involving hadronic τ leptons, as well as the all-jets channel are so difficult obviously because of the huge QCD multi-jets background. Finally, it must be stressed that the number of jets in the final state could be actually greater than the number of quarks listed in the final states above, due to QCD extra radiation that can possibly lead to extra-jets. The most precise results are thus provided by the dilepton and in particular the lepton+jets channels, because of the best ratio between signal and background. The production of top pair has been observed since the discovery of top quark, both at Tevatron and LHC. The first measurements were made during Run I at Tevatron at $\sqrt{S} = 1.8$TeV and then made more precise during Run II at $\sqrt{S} = 1.96$TeV. Finally, since beginning of 2010, measurements have been taken at LHC at $\sqrt{S} = 7$TeV and $\sqrt{S} = 8$TeV. It is interesting to measure the total production cross-sections in all possible final states, since the impact of new physics could affect different channels in different ways. Indeed, all possible final states involving leptons, jets and missing transverse energy have been measured by the two colliders, except for final states involving two hadronically decaying taus [33]. The results are in agreement with benchmark theoretical predictions, which will be in turn discussed in the following. We report the most up-to-date and precise measurements, which come from the LHC experiments ATLAS and CMS, stressing that the measurement of top pair production, because of its final state involving essentially all physics objects and thus being so complex, represented a major milestone achieved in the LHC Run I and Run II program. Experimental results are shown in Fig.(2.13a), (2.13b), (2.14a), (2.14b). The first important feature, visible in all plots, is the striking accord between measurements and theory predictions, regardless of the final state channel in which the analyses are performed. The worst experimental performance is, as already anticipated, in all-jets and τ+jets channels, but also in this case measurements and theory are compatible (see Fig.(2.13b)). Such accord between theory and experiments provide a stringent test of the SM and in particular of our theory of strong interactions.

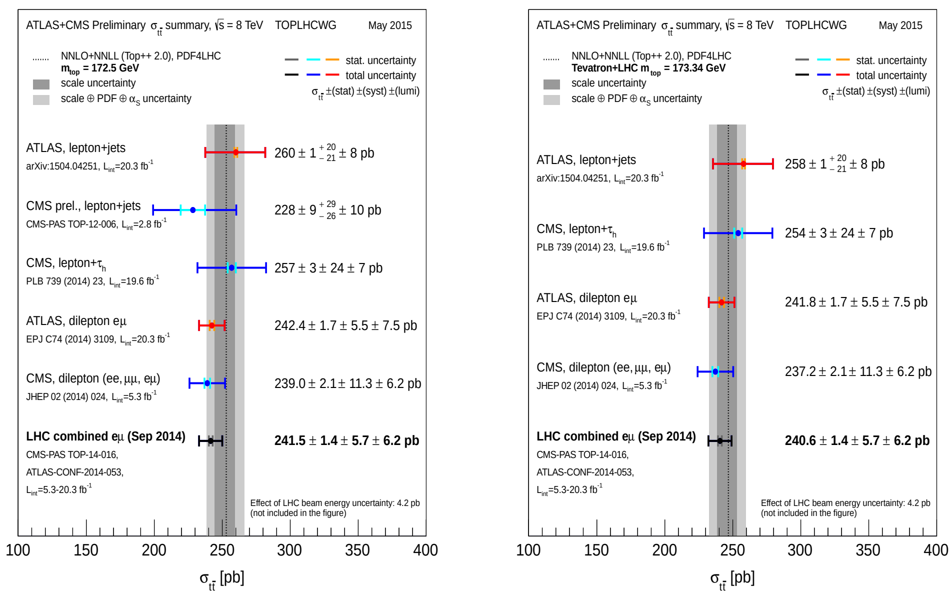


(a) Summary of LHC and Tevatron measurements of the top-pair production cross-section as a function of the centre-of-mass energy compared to the NNLO QCD calculation complemented with NNLL resummation, quoted at $m_t = 172.5$GeV. The theory band represents uncertainties due to renormalization and factorisation scale, parton density functions and the strong coupling.



(b) Top-pair cross-section measurements at 7 TeV by the ATLAS and CMS collaborations. The band shows the NNLO-QCD+NNLL resummation. The theory band represents uncertainties due to renormalisation and factorisation scale, parton density functions and the strong coupling.The upper part of the figure shows early LHC measurements and their combination. The lower part summarizes measurements performed after the LHC cross-section combination ($m_t = 172.5$GeV).

Figure 2.13



(a) (b)

Figure 2.14: Summary of measurements of the top-pair production cross-section at 8 TeV compared to the exact NNLO QCD calculation complemented with NNLL resummation. The theory band represents uncertainties due to renormalisation and factorisation scale, parton density functions and the strong coupling.
In Fig.(a) results quoted at $m_t = 172.5\text{GeV}$, whereas in Fig.(b) at the current world average $m_t = 173.5\text{GeV}$.

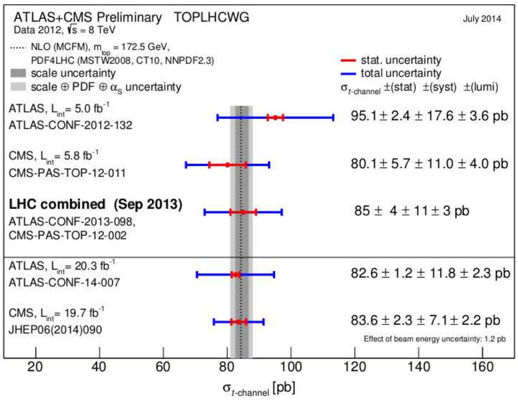
Measurements for Single Top cross-section.

At the LHC, single top in t-channel and Wt production have been observed, whereas only an upper bound has been put on s-channel cross-section. Though for single top topologies the signal to background ratio is generally less favourable than for top pair production, at the LHC such ratio is enhanced with respect to Tevatron.
The cleanest signature belongs to t-channel, where a light quark jet recoils against the top quark, which decays in turn into a W and a b, thus originating a b-jet. The W can decay either leptonically, thus originating an isolated lepton and missing energy or hadronically, thus giving rise to additional jets. The signature of t-channel requires then at least two jets, among which one has two be b-tagged, and missing energy plus an isolated lepton in case of a leptonically decaying W. The tW associated production is particularly challenging, since at NLO-QCD it happens to share with $t\bar{t}$ the same final state, so that the two processes can interfere and also on a theoretical level it is not clear how to define the signal. To overcome this problem, two schemes have been proposed to define the tW signal in [120], [65]. The final states classification is indeed the same as for top pair production, namely dilepton, lepton+jets, all-jets. Finally, concerning the s-channel, its signature is characterized by one charged lepton, missing energy and two b-tagged jets. At present, it has never been measured directly at Tevatron, and the LHC experiments only managed to put an upper bound, as can be seen from Fig.2.15b. Due to the huge background to single top topologies composed mainly by QCD multijets, W+jets, $t\bar{t}$, Z+jets, Drell-Yan, ATLAS and CMS used, on top of usual *cut and count* analyses, also Neural-Network (NN) and Boosted Decision Tree (BDT) analyses, in order to increase the efficiency in extracting the signal from background (see [109]).

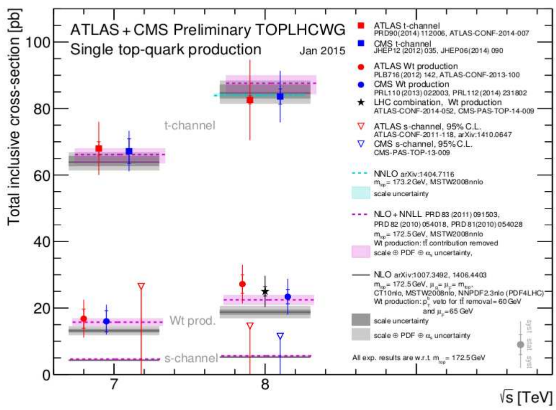
To conclude the discussion about experimental benchmarks, we would like to mention, without entering into details, two recent major achievements.

- The recently claimed observation at the LHC ([112]) of $t\bar{t}W$ and $t\bar{t}Z$ associated production, which are of fundamental importance to test top quark electroweak couplings.

- The possibility to measure, thanks to the great abundance of top quarks produced at the LHC, not only inclusive cross-sections, but also differential distributions $d\sigma_{t\bar{t}}/dX$ and $d\sigma_t/dX$, where X is some relevant quantity describing the kinematics of the top(anti-top) or the $t\bar{t}$ system. Differential distributions provide even more stringent tests of QCD, can be used to validate the Monte Carlo models and finally provide an interesting framework for the detection of new physics, which could manifest itself in deviations from the expected SM shape of the distributions. Further information can be retrieved in [55], [111].



(a)



(b)

Figure 2.15: (a): Summary of the ATLAS and CMS Collaboration measurements of the single top production cross-sections in the t-channel at 8 TeV. The measurements are compared to a theoretical calculation based on NLO QCD complemented with NNLL resummation computed assuming a top mass of 172.5 GeV. In the lower part, best measurements are reported.
(b) : Summary of ATLAS and CMS measurements of the single top production cross-sections in various channels as a function of the center of mass energy. For the *s*-channel only an upper limit is shown. The measurements are compared to theoretical calculations based on: NLO QCD, NLO QCD complemented with NNLL resummation and NNLO QCD (t-channel only).

2.2.2 Measures of top properties and SM observable from Top physics

Top quark mass, spin and charge.
The top mass is the best measured property of top quark, since, as it will become clear in the rest of this section, it has a fundamental role in precision tests of the Standard Model. Since the dominant and most precise measured cross-section both at LHC and Tevatron, is $t\bar{t}$ pair production, usually measurements of m_t are extracted from this process. Nonetheless, it is worth mentioning that thanks to LHC high energies and thus increased number of events, a first attempt to extract m_t also from a combined signal of $t\bar{t}$ and t-channel Single Top data has recently been done in [59]. Restricting attention to the $t\bar{t}$ production, the lepton+jets channel yields the most precise measurements because of its good signal to background ratio and the presence of a single neutrino in the final state.
The combination of both LHC and Tevatron analyses ([1]) gave the most up-to-date value of $m_t = 173.34 \pm 0.76$, where m_t is the parameter which identifies the top mass in the Monte Carlo simulator used to extract the measurement. One open theoretical issue regards the interpretation of such result , since the MC mass, namely the parameter used in the Monte Carlo to identify the top mass, is not a renormalized field theory mass and it is not even clear how it is related to the top mass as defined for instance in the $\overline{\text{MS}}$ or in the on-shell scheme. Discussion of such topic goes beyond the scope of this text, but the interested reader might find out more in [78].
To conclude the discussion about top mass, it should be mentioned that another possibility, which allows to by pass this interpretation issue, is that of extracting m_t from the measured cross-section using the theoretical relation between the mass and the production cross-section. Such possibility has been widely explored and some references can be found in [99].

Thanks to its huge mass, and consequently its very short life time ($\Gamma_t^{-1} \simeq (1.5\text{GeV})^{-1}$), the top quark is the only quark which decays before its spin can be flipped by the strong interactions. Indeed, the evolution of a heavy quark produced with definite spin in a high-energy hadronic collision, can be roughly explained as follows. After a time-scale of $\mathcal{O}(\Lambda_{QCD}^{-1})$ (with $\Lambda_{QCD} \simeq 200\text{MeV}$) after it has been produced, the heavy quark is likely to pick up a light quark of opposite spin from the vacuum and hadronize into a meson. The interaction between the two opposite spin brings the meson into a spin-zero state after a typical time of $\mathcal{O}((\Lambda_{QCD}^2/m_Q)^{-1})$, being m_Q the mass of the heavy quark. This implies that after this typical time, the heavy quark is depolarized. Given the value of m_t, one can see that top quark actually decays before the depolarization mechanism comes into play, so that its spin is observable in the angular distribution of its decay products.
In $t\bar{t}$ production, and in general in unpolarized QCD reactions, top quarks are

produced unpolarized. The argument is very simple and makes use of parity being a symmetry of QCD to show that, if the initial partons are unpolarized (namely the $p\bar{p}$ collision is unpolarized), then the probability to produce positive-helicity or negative-helicity top quark must be the same, namely top quarks cannot be produced polarized. Despite all this, the spin of the $t\bar{t}$ pair are correlated, which can be translated by saying that the rate for opposite-helicity $t\bar{t}$ production is greater than that of same-helicity $t\bar{t}$ production. Spin correlations have now been conclusively measured at LHC by both ATLAS and CMS. In gluon fusion production mode, the angular distribution between the two leptons in $t\bar{t}$ decays to dileptons is sensitive to the degree of spin correlations (see references given in [99]).
On the other hand, when the top quark is produced via the weak interaction, it is 100% polarized and its spin orientation stays encoded in the angular distribution of its decay products. Therefore, observables which are sensitive to such information can be designed and directly measured. Focusing attention on t-channel Single top production, one possibility is given by the forward-backward asymmetry A in the top quark rest frame, which is defined by

$$A = \frac{N\left(\cos\theta_{l,q}^{(top)} > 0\right) - N\left(\cos\theta_{l,q}^{(top)} < 0\right)}{N\left(\cos\theta_{l,q}^{(top)} > 0\right) + N\left(\cos\theta_{l,q}^{(top)} < 0\right)} = \frac{1}{2}P_t\alpha_l. \tag{2.10}$$

The angle $\theta_{l,q}^{(top)}$ is the angle between the lepton coming from top decay and the light quark produced together with the top quark. The polarization P_t denotes the alignment of the top quark spin with the light quark momentum, whereas the spin analysing power α_l quantifies the alignment of the lepton with the top-quark spin. Theoretical expected values for these quantities are $P_t = 0.98$ and $\alpha_l = 1$. The most up-to-date measurement from CMS is based on data recorded during pp collisions at $\sqrt{S} = 8$TeV at the LHC and yields $P_t = 0.82 \pm 0.34$ and $\alpha_l = 1$.

The top quark charge is $+2/3e$ in the SM or $-4/3e$ in some exotic models, such as Mirror Quark Doublet Models ([52], [50], [51]). Top quark is the the only quark whose electric charge has not been measured through production at threshold in e^+e^- production. Luckily, its charge can be inferred from its decay products. However, this is not straightforward, due to the fact that the original top charge gets ‘diluted’ in the case where the quarks coming from its decay hadronize into jets and also the charge observed in the decay products has to be matched to either at top or an antitop. Measurements at the Tevatron have excluded the hypothesis that the top has an exotic charge of $-4/3e$ at the 95% CL. In the same way, ATLAS and CMS presented measurements of the top charge in the lepton+jets channel ($t\bar{t}$ production mode) and the exotic $-4/3e$ charge could be excluded with high significance by both of them. Another possibility to access the top quark charge is the measurement of the cross-section of the production of top pairs in association

with a photon, which is clearly sensitive to the electromagnetic coupling of the top. The cross-section for $t\bar{t}\gamma$ production, thanks to the enhancement due to the presence of a γ in the final state, is already accessible during Run I at the LHC, namely for energies of $\sqrt{S} = 7$TeV (ATLAS already presented a first measurement of such cross-section at the LHC). However, more detailed tests of the couplings at the $t\bar{t}\gamma$ vertex are only possible with larger integrated luminosities.
For further information and references about the top quark properties discussed above and other properties that can be measured in the context of a Standard Model top quark, we redirect the interested reader to [111].

CKM matrix element $|\mathbf{V_{tb}}|$.
In the SM, the CKM matrix is predicted to be unitary. Once the unitarity of the CKM matrix is assumed, a measurement of the ratio R defined in Eq.(2.3) provides a direct measure of the CKM matrix element $|V_{tb}|$. The quantity R can be measured by measuring the probability of a top quark to decay into a W and a first, second, or third generation quark. In other words,

$$R = \frac{\mathrm{BR}(t \to Wb)}{\mathrm{BR}(t \to Wq)}, \text{ with } \mathrm{BR}(t \to Wq) = \mathrm{BR}(t \to Wd) + \mathrm{BR}(t \to Ws) + \mathrm{BR}(t \to Wb), \tag{2.11}$$

so that a measurement of R boils down to measuring the branching ratios BR for the possible decay channels. This can be achieved by studying the decays of top quarks produced in pair via the strong interactions, or singly produced via the weak interaction modes. We report in the following values obtained by the LHC, with data collected at $\sqrt{S} = 8$TeV.
In $t\bar{t}$ production, CMS measures $R = 1.023^{+0.036}_{-0.034}$ and $R > 0.945$ at 95% C.L., by comparing the number of events with 0,1,2 tagged b-jets in the lepton+jets channel and also in the dilepton channel.
In single top production, at the LHC only the t-channel and the Wt associated production are accessible, and a measure of $|V_{tb}|$ is extracted for each of these channels separately. In the t-channel, whose cross-section at the LHC is more than three times as large as s-channel and Wt combined, ATLAS find $|V_{tb}| = 1.04^{+0.1}_{-0.11}$ with $|V_{tb}| > 0.80$ at 95% C.L. We stress that a significant discrepancy of R (or equally $|V_{tb}|$) from unity would imply space for BSM physics, but all measurements performed at LHC up to now are in very good agreement with the SM predictions.

Chapter 3

pQCD and CC-DIS

3.1 Basics of perturbative QCD

In this section the basic notions of the theory of strong interactions (QCD) are presented. First the QCD Lagrangian and corresponding Feynman rules are introduced. Then, within the framework of perturbative QCD, we introduce the formula that allows to compute the cross-sections for hadron-initiated processes. Finally, we review the current panorama of QCD fixed-order predictions.

3.1.1 QCD Lagrangian, quantum numbers and Feynman rules

Strong interactions are described by a $SU(3)$ Yang-Mills theory with n_f quark fields transforming in the fundamental representation of the gauge group. The degree of freedom associated to the $SU(3)$ group is called *color*, so that, in a more general non-abelian theory with gauge group $SU(N_c)$, the quarks will carry color index a with $a = 1, ..., N_c$. The QCD Lagrangian can be written as the sum of two pieces

$$\mathcal{L} = \mathcal{L}_q + \mathcal{L}_g. \tag{3.1}$$

$\mathcal{L}_q$ and $\mathcal{L}_g$ include respectively the quark and gluon kinetic term. In addition to that, $\mathcal{L}_q$ describes the interaction of quarks with gauge fields whereas $\mathcal{L}_g$ the gauge fields self-interactions. These two sectors take the following form

$$\mathcal{L}_q = \bar{\psi}_a(i(\not{D})_{ab} - m)\psi_b \tag{3.2}$$

$$\mathcal{L}_g = -\frac{1}{2}\mathrm{Tr}(F_A^{\mu\nu} F_{\mu\nu}^A), \tag{3.3}$$

where the trace is in color space and ψ is the fermion field in the fundamental representation of $SU(N_c)$. The covariant derivative $\not{D}$ and the gauge field strength F are defined as

$$(\not{D})_{ab} = \gamma^\mu \partial_\mu \delta_{ab} - i g_s \gamma^\mu G_\mu^A t_{ab}^A \tag{3.4}$$

$$F_{\mu\nu}^A = \frac{i}{g}\left[D_\mu, D_\nu\right] = \left(\partial_\mu G_\nu^A - \partial_\nu G_\nu^A + g_s f^{ABC} G_\mu^B G_\nu^C\right) t^A. \tag{3.5}$$

The f^{ABC} are the structure constants of $SU(3)$. They are defined through the commutators of the generators t^A of the group (first definition in Eq.(3.6))

$$\left[t^A, t^B\right] = i f^{ABC} t_C, \quad \mathrm{Tr}\left[t^a t^b\right] = T_R \delta^{ab}. \tag{3.6}$$

The second definition in Eq.(3.6) gives instead the normalization of the trace of a product of generators, which is commonly chosen to be $T_R = 1/2$. The main difference with the theory of electromagnetic interaction (QED) and more in general with abelian gauge theories is given by the presence of the term $g_s f^{ABC} G_\mu^B G_\nu^C$, which has to be inserted in order to preserve the gauge invariance of the theory under local $SU(3)$ transformations

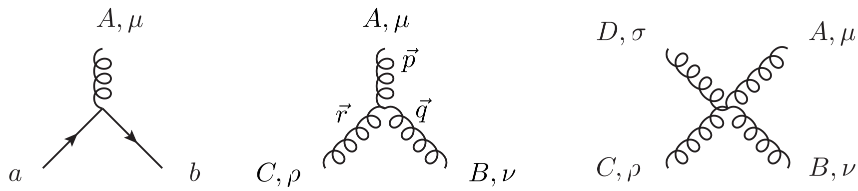
$$\begin{aligned} &\psi_a \to e^{i\theta_C(x) t_{ab}^C} \psi_b \\ &G^C t^C \to e^{i\theta^D(x) t^D} \left(G^C t^C - \frac{1}{g_s} \partial_\mu \theta^C(x) t^C\right) e^{-i\theta^E(x) t^E} \end{aligned} \tag{3.7}$$

where $\theta^C(x)$ are eight arbitrary real functions of the space-time position x. Such non-abelian term in the Lagrangian is responsible for the self-interactions of gluon fields.

Perturbation theory applied to QCD relies on the idea of an order-by-order expansion in a small coupling $\alpha_s = \frac{g_s^2}{4\pi} \ll 1$. In this framework, some given observable f, can then be predicted as

$$f = f_0 + f_1 \alpha_s + f_2 \alpha_s^2 + f_3 \alpha_s^3 + ... \tag{3.8}$$

where one might calculate just the first one or two terms of the series, with the understanding that remaining ones should be small.
The principal technique to calculate the coefficients f_i of the above series is through the use of Feynman diagrammatic (or other related) techniques. The interaction vertices arising from the QCD Lagrangian, are reported in Fig.3.1. It is well known that in order to perform perturbation theory with the Yang-Mills Lagrangian Eq.(3.1), (3.3) one needs to choose a gauge. As a consequence, a gauge fixing



$$-ig_s t^A_{ba}\gamma_\mu \qquad -g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}] \qquad -ig_s^2 f^{XAC}F^{XBD}[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\gamma}] + (C,\gamma)\leftrightarrow(D,\rho) + (B,\nu)\leftrightarrow(C,\gamma)$$

Figure 3.1: Feynman rules of QCD.

term and a ghost Lagrangian enter Eq.(3.1), (3.3). The final QCD Lagrangian thus reads

$$\mathcal{L}_{QCD} = \mathcal{L} + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost}. \tag{3.9}$$

There are different possible choices for the gauge fixing and ghost Lagrangian, but we limit ourselves to quote the two most popular classes of gauge fixing. The most commonly used gauges are the so-called *covariant gauges*, defined by

$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2\lambda}(\partial^\mu G^a_\mu)^2, \quad \mathcal{L}_{ghost} = \partial_\mu(\eta^a)^\dagger\left(D^\mu_{ab}\eta^b\right), \tag{3.10}$$

where λ is an arbitrary gauge parameter and η is a complex scalar field in the adjoint representation which obeys Fermi statistics. In a covariant gauge the gluon propagator is given by

$$\Delta^{ab}_{\mu\nu}(p) = \delta^{ab}\frac{i}{p^2}\left(-g_{\mu\nu} + (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right), \tag{3.11}$$

which becomes particularly simple for the *Feynman gauge* choice $\lambda = 1$. Another convenient choice are the so-called *axial gauges*, defined by

$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2\lambda}\left(n^\mu G^a_\mu\right)^2, \tag{3.12}$$

where again λ is an arbitrary parameter and n an arbitrary vector. The nice property of axial gauges is that ghost fields are not required. On the other hand, the gluon propagator has a more complicated form

$$\Delta^{ab}_{\mu\nu}(p) = \delta^{ab}\frac{i}{p^2}\left(-g_{\mu\nu} + \frac{n_\mu p_\nu}{n\cdot p} - \frac{(n^2+\lambda p^2)p_\mu p_\nu}{(n\cdot p)^2}\right). \tag{3.13}$$

Again a simplification occurs if one chooses a *light-cone gauge*, defined by the conditions $n^2 = 0, \lambda = 0$. In this gauge, the gluon propagator happens to contain only a sum over polarization of physical states, or, in other words, only transverse gluon polarizations (namely physical ones) propagates.
Axial gauges are usually quite useful when studying general properties and behaviour of QCD amplitudes, whereas covariant gauges are very convenient for real computations. Indeed, the gauge choice we do for our NNLO computation is Feynman gauge, so that we can deal with a very simple gluon propagator, but we have to include also ghost diagrams.

3.1.2 pQCD@Hadron Colliders

Colliders like Tevatron and LHC are designed to investigate phenomena involving high-momentum transfers (more precisely large transverse momenta), say in the range 50 GeV to 5 TeV. It is well known that in this energy regime the QCD coupling is small, and we would then hope to apply perturbation theory. Yet, the initial state involves protons, at whose mass scale, $m_p \simeq 1$GeV, the coupling is not weak. And the final states of collider events involve the presence of lots of hadrons, which are not perturbative either. We are then faced with the problem that exact perturbative methods can't deal with low momentum scales that inevitably enter the description of a collision, nor with the high multiplicities that events have. Despite all this, it turns out that we are reasonably successful in making predictions for collider events. In the following paragraphs, we briefly illustrate the formalism that allows us to reach this goal, by explaining the structure that QCD cross-sections[1] assume in such formalism and in particular how we deal with the presence of hadrons in the initial state[2].

The simplest observables in QCD are those that do not involve initial-state hadrons and that are fully inclusive with respect to details of the final state. One example is the total cross-section for $e^+e^- \to$ hadrons, for which one can avoid caring about the difficulties coming from the presence of hadrons in the initial state. This cross-section, which we will address in the following simply as σ is formulated as a perturbative series in α_s. If one aims at computing the terms of this series beyond LO, the first conceptual issue that must be taken into account is the *running of* α_s. Indeed, most higher-order computations are carried out within Dimensional Regularization (see [119] for a complete treatment of DR), in order to handle the ultraviolet divergences appearing in loop diagrams and possibly also the infra-red divergences arising from phase-space integrations. In the process of going from 4 to $d = 4 - 2\epsilon$ dimensions, one introduces an arbitrary scale μ, having dimensions

[1] In the context of this thesis, we take into consideration only fully inclusive cross-sections, but material on the structure of differential distribution can be found in [99].

[2] The ideas and formalism regarding the treatment of hadrons in the final state go beyond the purpose of this thesis but more material can be found in [110].

$[E]^1$ in energy units, in order to keep consistent dimensions for all quantities. If one wants to compare his theoretical prediction for σ with experimental data, the dependence on such unphysical scale μ has to be eliminated. This is achieved by introducing an energy *renormalization scale* μ_R, large enough such that the coupling is small, and by fixing the value of α_s at this new chosen scale. This can be done by meaning of the well-known *renormalization* procedure, which we do not discuss in this context since it goes beyond the purpose of this thesis[3]. Let us consider the renormalized $e^+e^- \to$ *hadrons* cross-section σ at all orders(we maintain the same notation for simplicity). This total cross-section cannot depend on the conventions chosen to fix the renormalization point. This consideration allows to conclude that σ must obey a Callan-Symanzik equation Eq.(3.14).

$$\left[\mu_R \frac{\partial}{\partial \mu_R} + \beta(g)\frac{\partial}{\partial g}\right] \sigma\left(s, \mu_R, \alpha_s\right) = 0. \tag{3.14}$$

By dimensional analysis, one can write

$$\sigma = \frac{c}{s} f\left(\frac{s}{\mu_R^2}, \alpha_s\right) \tag{3.15}$$

with c a dimensionless constant. Then, Eq.(3.14) implies that f depends on its argument only through a running coupling constant $\alpha_s(Q^2) = \bar{g}^2/4\pi$, evaluated at $Q^2 = s$. The coupling constant $\bar{g}$ is defined to satisfy the renormalization group equation

$$\frac{d}{d\log(Q/\mu_R)}\bar{g} = \beta(\bar{g}), \tag{3.16}$$

supplied by the initial condition $\alpha_s(\mu_R) = \alpha_s$. The β function is in turn a perturbative quantity and admits thus an α_s-expansion

$$\beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + ...). \tag{3.17}$$

At this point we take the chance to make a little digression to analyse the quantities the govern the running of α_s and how they are related to the well-known phenomena of *confinement* and *asymptotic freedom*. Now, for abelian theories, b_0 happens to be negative, so that the coupling constant increases its strength with the energy. In the case instead of non-abelian theories, the sign of b_0 depends on the chosen $SU(N_c)$ group, i.e. on the field content of the theory. For the particular case of QCD, the first two coefficients of the β-function read

$$b_0 = \frac{11C_A - 2n_f}{12\pi}, \quad b_1 = \frac{17C_A^2 - 5C_An_f - 3C_Fn_F}{24\pi^2}, \tag{3.18}$$

[3]Renormalization is discussed in any standard QFT text-book, such as for instance [102]

where $C_A = N_c$ and n_f is the number of 'light' flavours, namely those whose mass is lower than μ_R. Such terms arise from respectively the gluon and massless fermions contributions to the gluon self-energy at 1-loop. Given $N_c = 3$, we get $b_0 = \frac{33-2n_f}{12\pi}$, which implies $b_0 > 0$ (and thus $\beta(\alpha_s)_{1-loop} < 0$) if $n_f < 33/2$. Given that in the SM, we have three generations of quarks, namely no more than six flavours, the QCD β-function can't happen to be positive. On the other side, for abelian gauge theories the situation is reversed. Since the term C_A is zero (there are no self-interactions of gauge boson fields, so gauge bosons do not contribute loops to their own self-energy!), the first term of the β-function reads $b_0 = -n_f/3$ and it is thus always negative.
Now, the QCD β-function being negative, implies that the strong coupling α_s becomes weaker at higher energies, i.e. the theory almost becomes a free theory, in which quarks and gluons do not interact. This behaviour is commonly known as *asymptotic freedom*. Conversely, at low momentum scales the coupling grows strong, giving rise to the so-called *confinement*, namely quarks and gluons being tightly bound into hadrons. Such behaviour of the running of α_s also tells very clearly that a perturbative approach to QCD has a limited range of validity, namely at energies higher than a certain scale. This scale is determined by solving the Renormalization Group Equation (RGE) EQ.(3.16) and is found to be $\Lambda_{QCD} \sim 200\text{MeV}$. This is the energy regime at which the strong coupling diverges. In conclusion, we can say that the Callan-Symanzik or RGE equation instructs us to replace the fixed renormalized coupling α_s with the running coupling constant $\alpha_s(Q)$, with Q of the order of the hard scale governing the process in order for the all-order cross-section to be independent on the choice of renormalization scheme.

Now, what happens in everyday life, is that we never deal with a cross-section containing an infinite number of terms in the α_s-expansion. A realistic theoretical prediction for QCD cross-section will contain indeed only the very first terms in the expansion. Given this, it is interesting to see what happens when we replace the fixed renormalized coupling with the running coupling in a theoretical prediction for the cross-section σ truncated at a certain order in α_s. The renormalized cross-section at NLO can be written

$$\sigma_{NLO} = \sigma_{LO}(1 + c_1\alpha_s(\mu_R)), \tag{3.19}$$

where c_1 contains both real and virtual 1-loop corrections. Given an expansion of the running coupling

$$\alpha_s(\mu_R) = \alpha_s(Q) - 2b_0\alpha_s^2(Q)\ln\left(\frac{\mu_R}{Q}\right) + \mathcal{O}(\alpha_s^3), \tag{3.20}$$

we can rewrite Eq.(3.19) as

$$\sigma_{NLO}(\mu_R) = \sigma_{LO}\left(1 + c_1\alpha_s(Q) - 2c_1b_0\alpha_s^2(Q)\ln\left(\frac{\mu_R}{Q}\right) + \mathcal{O}(\alpha_s^3)\right). \qquad (3.21)$$

This tells us that as we vary the renormalization scale for a prediction up to $\mathcal{O}(\alpha_s)$ (NLO), we effectively introduce $\mathcal{O}(\alpha_s^2)$ (NNLO) pieces into the calculation: by generating some fake set of NNLO terms, we are probing the uncertainty of the cross-section associated with the missing full NNLO correction.
If we calculate the actual NNLO cross-section for general μ_R it will have a form

$$\sigma_{NNLO}(\mu_R) = \sigma_{LO}\left(1 + c_1\alpha_s(\mu_R) + c_2(\mu_R)\alpha_s^2(\mu_R)\right). \qquad (3.22)$$

We observe then that the c_2 coefficient now depends on μ_R. This is necessary because the second-order coefficient must cancel the $\mathcal{O}(\alpha_s^2)$ ambiguity due to the scale choice in Eq.(3.21). This constraints how c_2 depends on μ_R

$$c_2(\mu_R) = c_2(Q) + 2c_1b_0\alpha_s^2(Q)\ln\left(\frac{\mu_R}{Q}\right). \qquad (3.23)$$

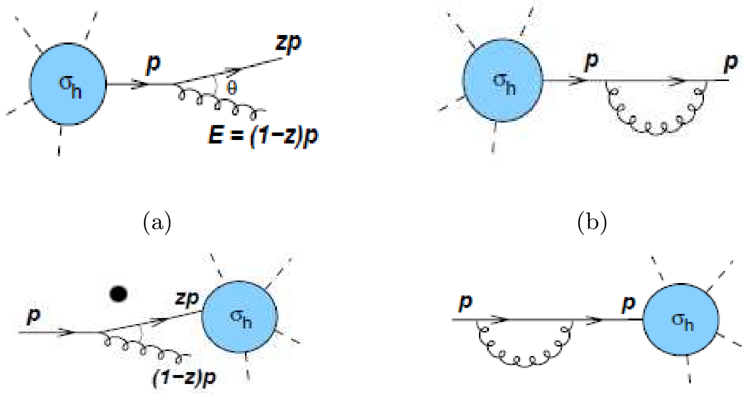
If we now expressed σ_{NNLO} in terms of $\alpha_s(Q)$, we would find that the residual dependence on μ_R appears entirely at $\mathcal{O}(\alpha_s^3(Q))$, namely one order further than in Eq. (3.21). In other words, when we truncate the expansion at a given order α_s^n and substitute the fixed renormalized coupling with the running coupling, we are always left with a residual dependence on μ_R of order α_s^{n+1}. Given this fact, a first consideration to one can do is that the μ_R dependence that affects a cross-section at a given perturbative order α_s^n is a probe of the impact of the missing α_s^{n+1} term. On top of that, one should consider that the choice of the value for μ_R is totally arbitrary. In principle, in order to obtain a realistic value for the cross-section, it seems a sensible choice to set $\mu_R = Q$, where Q is the typical hard scale of the process. But again, this is just an arbitrary choice and other choices might be equally good. If we stick to our example and consider NLO real corrections to $e^+e^- \to q\bar{q}$, namely the process $e^+e^- \to q\bar{q}g$, the most energetic gluon that could be produced would have energy $E = Q/2$, so maybe we should choose $\mu_R = Q/2$. On the other hand, if we consider NLO virtual corrections, in loop diagrams we would integrate over gluon energies that go beyond Q, so maybe $\mu_R = 2Q$ would be as reasonable. It is clear then that the μ_R residual dependence translates into an uncertainty, which inevitably affects fixed-order theoretical predictions. If we had an arbitrarily large number of terms in the α_s expansion, the scale dependence would disappear exactly. In practice this never happens and we always deal with a finite number of terms. As a consequence, a residual μ_R dependence, and thus an uncertainty related to the choice of μ_R, will always affect our prediction. But, if the perturbative series is converging, we can expect then such uncertainty to shrink as we compute more and more terms in the expansion.

We are now ready to move on to a more complicated case, namely cross-sections for hadron-initiated processes.
Let us start with a very naive picture of a hadronic collision. At very high energy, most of the collisions between hadrons will involve only soft interactions of the constituents quarks and gluons. Such interactions cannot be treated using perturbative QCD, because α_s is large when the momentum transfer is small. In some collisions, however, two quarks or gluons will exchange a large momentum p_T perpendicular to the collision axis. Then, the elementary interaction takes place very rapidly compared to the internal time scale of the hadron wave-functions, so that we can think of describing this 'hard' collision between two of the constituents of the colliding protons in perturbation theory.
The general underlying idea is thus that whenever we have a partonic process governed by a typical scale Q^2 which satisfies the condition $m_h^2/Q^2 \ll 1$, with m_h being the mass of the initial-state hadrons, we can think of factorizing the description of our process into two parts. On one side we have the partonic *hard scattering*, which takes place at a scale Q^2 where α_s is small and can then be described in perturbative QCD. On the other hand, we have instead the internal structure of the initial hadron, say proton at the LHC, which is governed by a typical scale of the order of the mass of the hadron. The coupling constant at $\mathcal{O}(m_p^2)$, with m_p being the mass of a proton, blows up, and we enter in confinement regime, where we cannot use anymore the tools of perturbative QCD to describe the internal dynamics of the proton.
Given this, the first ingredient we need is indeed the perturbative computation of partonic cross-sections, which can be carried out at the desired order in the α_s-expansion. On top of that, in order to get predictions for hadron-initiated cross-sections, we need some other ingredient which connect the hadron- to the parton-level description and describe the non-perturbative internal structure of the hadron. This second ingredient is represented by Parton Distribution Functions (PDFs), which cannot be computed in perturbation theory and contain the information about the (non-perturbative) structure of the proton. The inclusive cross-section for the production of a final state $\mathcal{V}$ in the collision of two hadrons h_1, h_2 will then look like

$$\sigma(h_1 h_2 \to \mathcal{S} + X) = \sum_{n=0}^{\infty} \alpha_s^n(\mu_R^2) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1}\left(x_1, \mu_F^2\right) f_{j/h_2}\left(x_2, \mu_F^2\right) \times \\ \hat{\sigma}_{ij\to\mathcal{S}+X}\left(x_1 x_2 S, \mu_R^2, \mu_F^2, \mathcal{S}_{\mathcal{V}}\right) \times \left(1 + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)\right). \tag{3.24}$$

In this expression S is the center-of-mass energy of the hadronic collision, $f_{i/h}$ is the probability distribution for parton i in hadron h, x_1, x_2 are the parton momentum fractions, and $\mathcal{S}_{\mathcal{V}}$ is the set of kinematic variables describing the final state $\mathcal{V}$. The parton-level cross-section $\hat{\sigma}_{ij\to\mathcal{S}+X}$ is technically called *coefficient function* and it contains all the information about the hard scattering. Last but

not least, μ_R is the already mentioned renormalization scale, whereas μ_F is the so-called *factorization scale*, whose presence we quickly motivate in the following lines.
The majority of the emissions that modify a parton momentum are collinear to that parton and do not depend on the fact that the parton will interact with another parton via a hard scattering. It is thus natural to view these emissions as modifying the proton structure rather than being part of the coefficient function for the parton hard interaction. Technically, one uses a procedure called *collinear factorization* to give a well-defined meaning to this distinction. This factorization between PDFs and coefficient function happens through the introduction of a new unphysical scale μ_F whose meaning can be understood roughly as follows: emissions with transverse momenta above μ_F are 'hard emissions' and, as such, they are included in the coefficient function, whereas 'soft emissions' (with transverse momenta below μ_F) are considered part of the proton structure description and thus they are accounted for within the PDFs.



(a) (b)

(c) (d)

Figure 3.2: Single real and virtual QCD radiation from the initial and final state of a generic process h.

Let us discuss a bit more in detail how this happens.
We consider a generic hard process h with cross-section σ_h and examine in particular the cross-section for h with an extra gluon in the final state, σ_{h+g}. If we parametrize momenta as in Fig.(3.2a), we can write

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_T^2}{k_T^2}, \tag{3.25}$$

with $k_T = E\sin\theta \simeq E\theta$. If we avoid distinguishing a collinear $q+g$ pair from a plain quark (namely we measure an IR-safe observable), it is well known that the IR divergent part of the gluon emission contribution always cancels with a related virtual correction sketched in Fig.(3.2b) and given by

$$\sigma_{h+V} \simeq -\sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_T^2}{k_T^2}. \tag{3.26}$$

Now let us examine what happens for the initial-state splitting, where the hard process occurs after the splitting. In this case the momentum entering the hard process is modified as $p \to zp$ (Fig.(3.2c) and we can write

$$\sigma_{g+h}(p) \simeq \sigma_h(zp) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_T^2}{k_T^2}, \tag{3.27}$$

where it is assumed that σ_h is governed by a hard scale $Q \gg k_T$, so that we can ignore the extra transverse momentum entering σ_h and retain only the dependence of σ_h on the longitudinal component zp. For virtual terms, the momentum entering the process is unchanged (Fig.(3.2d)), so that the virtual cross-section reads

$$\sigma_{g+h}(p) \simeq -\sigma_h(p) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_T^2}{k_T^2}. \tag{3.28}$$

The total cross-section thus gets contributions two kinds of contributions, proportional to either $\sigma_h(p)$ or $\sigma_h(zp)$.

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \int_0^1 \frac{dz}{1-z} \left[\sigma_h(zp) - \sigma_h(p)\right]. \tag{3.29}$$

It is important to stress that the integration over k_T^2 has an upper bound which is fixed by the hard scale of the hard scattering process: the approximations on which all this argument relies are valid as long as the transverse momentum emitted in the initial state is much smaller than the momentum transfers Q present in the hard process. The integration over z is finite because in the region of the soft divergence $z \to 1$, the difference of the hard cross sections $\sigma_h(zp) - \sigma_h(p)$ tends to zero (in presence of radiation going soft, a Born-like kinematic is recovered).
In contrast to that, the k_T integral diverges in the collinear limit: the cross-section with an incoming parton (and virtual corrections) appears not to be collinear safe. This is a general feature of processes with incoming partons. In order to bypass this issue, it makes sense to introduce a new scale, μ_F, which acts as a cut-off in separating the perturbative region, where the hard process takes place from the non-perturbative region to which the description of the proton internal structure belongs. In other words, as mentioned at the beginning of the discussion on divergences, μ_F separates the 'soft' emissions which occur at $k_T \leq \mu_F$ and are

thus factorized into the proton structure from the 'hard' emissions, which occur instead at $k_T \geq \mu_F$ and belong then to the hard scattering process.
The presence of a non-integrable divergence that somehow needs to be regulated and absorbed with a scale choice into some 'constant' of the theory (here the PDFs), reminds of the renormalization for the coupling constant. The difference are that here we are faced with a infra-red (collinear) divergence rather than with an ultraviolet one, and that, unlike the coupling, the PDFs are not fundamental parameters of the theory. Nevertheless, as for the coupling, the freedom in choosing the scale entering the regularization, here μ_F, implies that the dependence on μ_F of both PDFs and coefficient functions is fixed by a group of differential equations which go under the name of Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations. They read at LO

$$\mu_F^2 \frac{\partial f_{i/p}(x,\mu_F^2)}{\partial \mu_F^2} = \sum_j \frac{\alpha_s(\mu_F^2)}{2\pi} \int_x^1 \frac{dz}{z} P^{(1)}_{i\to j}(z) f_{j/p}\left(\frac{x}{z},\mu_F^2\right). \tag{3.30}$$

The function $P^{(1)}_{i\to j}(z)$ to be convoluted with the PDFs is the first-order (LO) term of the perturbative expansion in α_s of the so-called *splitting function* $P_{i\to j}(z)$ and can be interpreted as the probability for a propagating parton i to emit another parton and continue propagating after the emission as parton j. After LO, the coefficient functions are also μ_F dependent.
The choice of factorization scale is arbitrary, but if one has infinite number of terms in the perturbative series, the μ_F-dependences of the coefficient functions and PDFs will fully compensate each other. Given only N terms of the series, a residual $\mathcal{O}(\alpha_s^{N+1})$ uncertainty is associated with the ambiguity in the choice of μ_F.
The picture that emerges from these arguments is that the generic hadronic cross-section computed at order α_s^N will always be affected by an uncertainty of order α_s^{N+1} connected to the residual dependence on μ_F and μ_R .
This is why *scale variation* has become a standard procedure to assess this type of uncertainties: by convention one fixes $\mu_R = \mu_F = Q$, and vary then separately μ_F and μ_R in the range $Q/2 < \mu_i < 2Q$. The envelope containing the curves obtained through these variations provides the final uncertainty band due to scale-dependence which accompanies the fixed-order prediction.

It is then clear that LO predictions for QCD observables are in general not very accurate, since they are plagued by large uncertainties, coming from missing higher-order corrections. This often results in a strong dependence of the predictions on the renormalization and factorisation scales, and moreover in many cases , Higgs production in gluon fusion ([73], [8], [104]) being the most famous , perturbative higher-order corrections can be large and may completely invalidate the LO approximation. That's why we definitely need to go beyond LO in perturbation theory in order for our theoretical fixed-order predictions to be reliable.

The structure and features of QCD observables discussed in the previous paragraphs holds at all-order in perturbation theory and provide the general framework and consistency checks for any fixed-order computation in QCD.
In the next subsection, we will dedicate some space to review the status of the art for QCD fixed-order predictions, quoting in particular some of the most recent and striking results.

3.1.3 Panorama of fixed-order computations results

NLO computations have been carried out over a period of about 30 years. Over the last few years, a lot of progress has been made towards automation of such kind of computation, and by now NLO sector has been almost completely automated. The main difficulty, as it is well known, is represented by the fact that one has to consider both virtual and real corrections, which are affected by different kinds of singularities. Ultra-violet (UV) singularities affect only virtual corrections and are removed through renormalization of the coupling, masses and wave-functions. Infra-red (IR) singularities are instead present in both virtual and real corrections. For inclusive cross-sections soft IR singularities cancel between real and virtual diagrams, and the same happens to final-state collinear singularities. Initial-state collinear singularities instead have to be factorized into the PDFs. The final physical observable must be a finite quantity, and the general requirement (for both inclusive and exclusive observables) is that it has to be infra-red and collinear safe. General methods exist to handle and cancel IR singularities at NLO [63],[48]. For many years the bottleneck has been the computation of the relevant one-loop amplitudes, but in the last years this issue has undergone an enormous progress. The traditional approach based on Feynman diagrams is now complemented with new powerful methods based on recursion relations and unitarity. The general one-loop amplitude is expressed as a sum of known 1-loop scalar 4-,3-,2-point functions (namely *boxed, triangles, bubbles*), plus a finite remainder term. The coefficients of these integrals can be computed by taking suitable multiple cuts (see [32]). These progress led to the 'NLO revolution', namely the complete automation of NLO computations, through the release of a number of general packages and codes meant to compute automatically NLO amplitudes and/or cross-sections. Among these tools, we cite `GoSam` ([122]), `OpenLoops` ([47]), `Helac-NLO` ([81]), `NJet` ([10]), `BlackHat` ([20]), `MadGraph5_aMC@NLO` ([3]). Among the most recent and striking NLO-QCD results, we quote the computation of inclusive cross-sections and some differential distributions for Higgs production in association with up to three jets ($pp \to H + 3j$, [53]), five jets production ($pp \to 5j$, [11]), W production in association with up to five jets ($pp \to W + 5j$, [19]), unified $t\bar{t}$ and associated Wt production in 4F-scheme ([46]).
Even in this panorama, where NLO computations are in such an advanced stage, NNLO calculations are still needed and become particularly useful in some specific

cases:

- processes whose NLO corrections are comparable to LO contributions, for instance Higgs production at hadron colliders,
- benchmark processes measured with high experimental accuracy (e.g. α_s measurements from e^+e^- event shape variables, W and Z production, heavy quark hadroproduction),
- processes relevant to determine PDFs or that can hide new physics signal (e.g. high $\not{E}_T$ jet hadroproduction),
- important background processes (e.g. vector boson pair production).

The difficulties affecting a NNLO computation are of the same nature of those already described for a NLO computation, namely the renormalization of UV singularities and cancellation of the IR ones, but the patterns of renormalization and specially of IR poles cancellation is much more involved than at NLO. Over the last twenty years, analytical computations, with explicit cancellation of IR singularities, have become available for inclusive cross-sections of some standard processes: DIS structure functions ([131], [133], [132]), single hadron production ([106], [108], [107], [93]), DY lepton pair production ([72]), Higgs boson production ([73], [8], [104]). On top of this, an analytical computation for the dilepton rapidity distribution in Drell-Yan process has been carried out in [5], by modelling the phase space constraint with an 'effective' propagator. Recently, more results have been achieved also in the computation at NNLO of cross-sections for processes which are described by more than one dimensionless scale. We cite here some of them:

- *top pair production*: as already mentioned in chapter 2 ([54], 2013), both the inclusive and fully differential cross-section at NNLO were computed numerically and matched to the NNLL resumed result;
- *diphoton production*: the fully differential cross-section for $pp \to \gamma + \gamma + X$ has been computed numerically in [49] (2011);
- *photon and vector boson associated production*: the next logical step after diphoton production was the computation of $pp \to V + \gamma + X$ cross-section where V is either a W or a Z; parton-level amplitudes for double-virtual and real-virtual contributions were computed over more than twenty years in ([67], [4], [56], [42]) and the computation was finally completed with double real parton amplitudes and numeric integration over the phase space or double-real and real-virtual contributions in 2012 ([71]), thus yielding the final inclusive and differential cross-sections;
- *vector bosons production*: the inclusive cross-section for $pp \to VV' + X$ has been obtained very recently through a completely analytical computation in

[68], [69], [77], [70] and there is at the moment work in progress towards the fully differential cross-section.

We would like to stress that the computation of differential distributions at NNLO is in general a formidable task and that a lot of effort has been done in understanding how the singularities of double-real, real-virtual and double virtual contributions are structured and how the calculation can be organized into finite pieces that can be integrated numerically.
We would like to conclude this brief review of the status of the art for fixed-order computations by quoting one of the most recent and probably most impressive results, namely the computation, already mentioned in chapter 2, of inclusive Higgs production in gluon fusion at NNNLO-QCD. This is the first calculation which goes beyond the NNLO barrier. It is based on a method to perform a series expansion of the partonic cross-section around the threshold limit to an arbitrary order. The expansion is performed to sufficiently high order as to obtain the value of the hadronic cross-section at NNNLO in the large top mass limit. For a more detailed description of technical details and results we send the reader to the original reference [6].

In such rich and dynamical panorama of higher-order QCD predictions, is integrated our computation for Single Top inclusive cross-section in t-channel, which we finally address in the next section.

3.2 t-channel Single Top in a DIS-like approach

We introduce in this section the project which constitutes the main subject of this thesis, namely the computation of NNLO-QCD corrections to Single Top production in t-channel. First, we explain how this process can be described in a Charged-Current Deep-Inelastic-Scattering (CC-DIS) framework, and we introduce CC-DIS Form Factors, which are the objects of our computation. We discuss which kind of diagrams is neglected at NNLO in this picture and the numerical importance of such contributions. Then, we give the detailed structure of massless and massive Form Factors which build up the t-channel Single Top cross-section up to NNLO-QCD and we discuss how the computation of the NNLO corrections to these Form Factors is naturally organized.
Finally, we dedicate some space to specific issues that may arise in general in the computation of higher-orders corrections, and that affect also our particular case.

3.2.1 CC-DIS picture of t-channel Single Top

[4] Let us consider the tree-level partonic cross-section for Single Top production (Fig.(3.3)). This can be easily interpreted as the interaction of two weak currents,

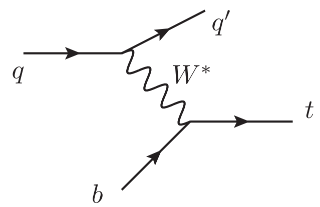


Figure 3.3: Tree-level diagram for Single Top production in t-channel (in a 5-flavour scheme).

taking place via the exchange of a W boson in t-channel. At LO-QCD, these two weak currents describe the subprocesses

$$q(p_1) \to q'(p_2) + W^*(q), \quad b(p_b) + W^*(q) \to t(p_t). \tag{3.31}$$

The first subprocess in Eq.(3.31) only contains massless quarks, whereas the second one contains a massive top. Thus we will refer to these subprocesses respectively as *light* and *massive current*.

[4]For the reader who is already familiar with DIS-like approach to higher-order computation, we suggest to go directly at the end of this subsection, where the expression for the hadronic cross-section in terms of structure functions is given.

The momentum conservation relation for the initial $2 \to 2$ process $p_1 + p_b \to p_2 + p_t$ gets 'split' into two momentum conservation relations for the $2 \to 1$ subprocesses

$$p_1 = p_2 + q, \quad p_b + q = p_t. \tag{3.32}$$

Given these relations, it is clear that each of these subprocesses is described by only two independent momenta. Since we are considering an inclusive cross-section, namely we are integrating over momenta of particles in the final state, the 'fixed' momenta on which the two subprocesses will finally depend will be respectively $\{p_1, q\}$ and $\{p_b, q\}$. This holds also at higher orders, because again, momenta k_i of extra emitted quarks or gluons will be integrated out. The set $\mathcal{V}_l = \{p_1, q\}$ is sufficient to describe the light current, because no other dimensional scale is involved in the subprocess $q \to q' + W^*$. In the case of the massive current, on top of the independent momenta p_b, q, there is also m_t^2 playing the role of independent dimensional scale. Thus, the massive current depends upon the quantities $\mathcal{V}_m = \{p_b, q, m_t^2\}$. Since, as it will become clear in a few lines, form factors, namely the objects we use to encode information about higher orders corrections, are scalar quantities (they have no free Lorentz indexes), it convenient to switch to set of scalar variables. The equivalent 'scalar' sets we use are respectively $\mathcal{V}_l = \{2p_1 \cdot q, q\}$ and $\mathcal{V}_m = \{(p_b + q)^2, Q^2, m_t^2\}$ (for simplicity we kept the same names for these sets). Since $(p_b + q)^2$ is the squared energy in the center of mass of the collision between the virtual W and the b quark, we will set $s = (p_b + q)^2$. The mathematical description of this tree-level partonic process fully reflects this idea of factorization into two weak subprocesses. The cross-section Eq.(3.33), differential with respect to the W virtuality $Q^2 = -q^2 > 0$, is given by the contraction of two rank-2 tensors $C(\mathcal{V}_l)$ and $C(\mathcal{V}_m)$ in Eq.(3.34), containing the results of traces respectively over the light and massive fermionic lines.

$$\frac{d\sigma}{dQ^2} = \frac{G_f^2}{64\pi\hat{s}^2} C^{\alpha\beta}(\mathcal{V}_l) \left(-g_{\alpha\mu} + \frac{q_\alpha q_\mu}{m_W^2}\right) \frac{1}{Q^2 + m_W^2} \left(-g_{\beta\nu} + \frac{q_\beta q_\nu}{m_W^2}\right) \frac{1}{Q^2 + m_W^2} C^{\mu\nu}(\mathcal{V}_m), \tag{3.33}$$

with

$$\begin{aligned} C^{\alpha\beta}(\mathcal{V}_l) &= 4(2p_1 \cdot q) g^{\alpha\beta} + 16 p_1^\alpha p_1^\beta - 8i\epsilon^{\alpha\beta\mu\nu} p_{1_\mu} p_{1_\nu} - 8(p_1^\alpha q^\beta + p_1^\beta q^\alpha) \\ C^{\mu\nu}(\mathcal{V}_m) &= -4(2p_b \cdot q) g^{\mu\nu} + 16 p_b^\mu p_b^\nu + 8i\epsilon_{\mu\nu\rho\gamma} p_b^\rho q^\gamma + 8(q^\mu p_b^\nu + p_b^\mu q^\nu) \end{aligned} \tag{3.34}$$

and $\hat{s} = (p_1 + p_b)^2$.

What happens if we consider now higher-order QCD corrections to the tree-level process Fig.(3.3)? In general, given a t-channel process happening via the weak interaction of two fermionic currents, we can think of dividing QCD corrections into two categories

- *factorizable corrections*: they involve QCD real and virtual radiation affecting just one quark line,
- *non-factorizable corrections*: on the opposite, these corrections link the two quark lines through both virtual and real gluon emissions.

For the sake of clarity of the argument we are presenting, we postpone the discussion of the structure and importance of non-factorizable contributions at the end of this subsection. For the moment we limit ourselves to underline that these kind of corrections are pretty small if compared to the factorizable ones. Given this, we can think of making a good approximation by retaining only factorizable corrections at higher orders.
In this perspective, the factorized description of the process that naturally happens at LO (Eq.(3.33), (3.34)), is conserved also at higher orders, the only difference being that the rank-2 tensors $C(\mathcal{V}_l)$ and $C(\mathcal{V}_m)$ describing the two currents need to be generalized in order to take into account information about the now included QCD corrections. We stress that such factorization is possible because, since we are neglecting cross-talks between the two weak currents, both matrix elements and phase space are completely factorized [5].

One of the possible forms in which we can write the most general rank-2 tensor describing one of our fermionic currents is

$$C^{\mu\nu}(\mathcal{V}_i) = -4(2p\cdot q)C_1(\mathcal{V}_i)g^{\mu\nu} + C_2(\mathcal{V}_i)16p^\mu p^\nu + C_3(\mathcal{V}_i)8i\epsilon_{\mu\nu\rho\gamma}p^\rho q^\gamma + 8C_4(\mathcal{V}_i)q^\mu q^\nu + C_5(\mathcal{V}_i)8(q^\mu p^\nu + p^\mu q^\nu), \tag{3.35}$$

with

- $\mathcal{V}_i$ being the set of variables $\mathcal{V}_l$ or $\mathcal{V}_m$ on which the subprocess depends,
- p being the incoming fermion momentum, i.e. p_1 or p_b.

The scalar coefficient $C_i(\mathcal{V}_i)$ (called Coefficient Functions (CF)) are extracted by contracting the squared matrix element multiplied by the phase space measure with the projectors P_i given by

$$P_1^{\mu\nu} = \frac{-(2p\cdot q)^2 g^{\mu\nu} - 4q^2 p^\mu p^\nu + 2(2p\cdot q)(p^\mu q^{nu} + q^\mu p^{nu})}{8(1-\epsilon)M^6} \tag{3.36}$$

$$P_2^{\mu\nu} = \frac{1}{8(1-\epsilon)M^8} \times \left[q^2((2p\cdot q)^2 g^{\mu\nu} + 4(3-2\epsilon)q^2 p^\mu p^\nu) - 2(3-2\epsilon)q^2(2p\cdot q)(p^\mu q^\nu + q^\mu p^\nu) + 2(1-\epsilon)(2p\cdot q)^2 q^\mu q^\nu\right] \tag{3.37}$$

$$P_3^{\mu\nu} = \frac{i\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma}{(-4M^4)} \tag{3.38}$$

[5] Factorization of phase space holds at partonic level as long as we are considering the differential cross-section with respect to Q^2.

$$P_4^{\mu\nu} = \frac{p^\mu p^\nu}{2M^4} \tag{3.39}$$

$$P_5^{\mu\nu} = \frac{-(2p\cdot q)^2 g^{\mu\nu} - 4(3-2\epsilon)q^2 p^\mu p^\nu + 2(2-\epsilon)(2p\cdot q)(p^\mu q^\nu + q^\mu p^\nu)}{8(1-\epsilon)M^6} \tag{3.40}$$

(3.41)

with $M = 2p \cdot q$. We stress that, since the computation of higher-order contributions is carried out in Dimensional Regularization, the projectors are given in $d = 4 - 2\epsilon$ dimensions.
Starting from the general form Eq.(3.35), we recover the tree-level tensors in Eq.(3.34) by setting

- $C_1(\mathcal{V}_m) = C_2(\mathcal{V}_m) = C_3(\mathcal{V}_m) = C_5(\mathcal{V}_m) = 1$ and $C_4(\mathcal{V}_m) = 0$ (massive current),
- $C_1(\mathcal{V}_l) = C_3(\mathcal{V}_l) = C_5(\mathcal{V}_l) = -1$, $C_2(\mathcal{V}_l) = 1$ and $C_4(\mathcal{V}_l) = 0$ (massless current).

The $C_i(\mathcal{V}_i)$ are related to the standard Structure Functions (SF) $F_i(\mathcal{V}_i)$ through the convolution with PDFs. Since at higher orders new channels are opened, the organization of the CFs (and consequently of SFs) can be quite involved. We postpone the presentation of this organization to the next subsection, which will be dedicated entirely to this issue.
For the moment we go directly one step further and present the structure of the hadronic cross-section for Single Top in t-channel, after making a brief recap of ideas presented up to now.

All the argument presented up to now can be resumed as follows. The leading order process $q + b \to q' + t$ is analogous to a charged-current deep-inelastic scattering (CC-DIS). In fact, it is a double deep-inelastic scattering: the virtual W boson is probing both the hadron containing the b quark and the hadron containing the light quark q. In general, at higher orders this factorization does not hold exactly anymore, due to cross-talk between currents, but we can still think of neglecting such cross-talks and continuing 'cutting' our process in correspondence of the t-channel W boson (Fig.(3.4)). This allows us to continue exploiting the analogy with CC-DIS, thus computing the QCD corrections in terms of structure functions. Given two colliding protons with momenta P_1, P_2, the differential hadronic cross-section is given by

$$d\sigma = \frac{1}{2S} 4 \left(\frac{g^2}{8}\right)^2 \frac{1}{(Q^2 + m_W^2)^2} W^{\mu\nu}(x_1, Q^2) W_{\mu\nu}(x_2, Q^2, m_t^2)(2\pi)^2 \frac{1}{4S} dQ^2 dW_1^2 dW_2^2, \tag{3.42}$$

where

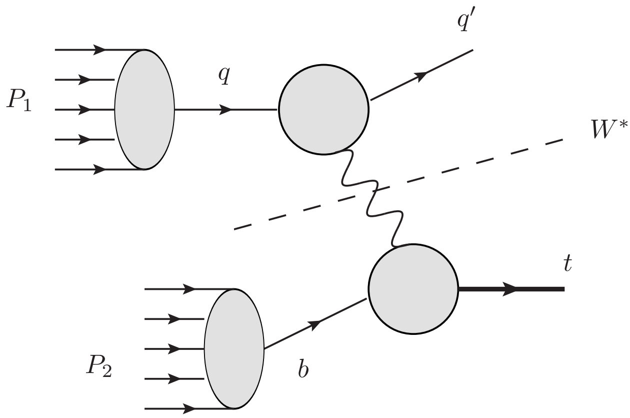


Figure 3.4: Single Top in t-channel in a CC-DIS approach. Grey circular blobs represent corrections to weak currents (factorizable contributions).

- $W_1^2 = (P_1 - q)^2$ and $W_2^2 = (P_2 + q)^2$ are the squared invariants masses of the hadron remnants (including the top quark),
- $S = 2P_1 \cdot P_2$ is the square of the hadronic center-of-momentum energy,
- x_1, x_2 are the 'natural' DIS variables encoding information about the ratio between the hard scale and the energy in the c.o.m of the (sub)process; they are thus defined as

$$x_1 = \frac{Q^2}{2P_1 \cdot (-q)}, \quad x_2 = \frac{Q^2 + m_t^2}{2P_2 \cdot q}; \tag{3.43}$$

by inverting the definition of W_1^2 and W_2^2 as follows

$$2P_1 \cdot (-q) = W_1^2 + Q^2, \quad 2P_2 \cdot q = W_2^2 + Q^2 \tag{3.44}$$

and by substituting these relations into the definitions of the x_i's we get

$$x_1 = \frac{Q^2}{W_1^2 + Q^2}, \quad x_2 = \frac{Q^2 + m_t^2}{W_2^2 + Q^2}. \tag{3.45}$$

In this way all quantities appearing in the hadronic cross-section Eq.(3.42) are expressed in terms of Q^2, W_1^2, W_2^2 and the differential cross-section can

be easily integrated in order to get the inclusive result.
The integration domain is identified as the physical region, defined by the following inequalities

$$
\begin{aligned}
&W_1 \geq 0, \\
&W_2 \geq m_t, \\
&W_1 + W_2 \leq \sqrt{S}, \\
&Q^{2max}_{min} = \frac{1}{2}\left[S - W_1^2 - W_2^2 \pm \lambda^{1/2}(S, W_1^2, W_2^2)\right], \\
&\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc. \qquad (3.46)
\end{aligned}
$$

- $W^{\mu\nu}(x_1, Q^2), W_{\mu\nu}(x_2, Q^2, m_t^2)$ are the so-called *hadronic tensors.*

The hadronic tensors $W_{\mu\nu}$ are strictly related to the tensors $C_{\mu\nu}$ defined at parton-level (Eq.(3.35)). Indeed, the $W_{\mu\nu}$ have exactly the same structure of the $C_{\mu\nu}$ but with the following replacements.

- The coefficient functions C_i's, are replaced by the structure functions F_i's. These are directly obtained as convolutions between the C_i's and the PDFs (see the next subsection for more details).
- Inside the tensor structures upon which the partonic/hadronic tensors are decomposed, the momenta of the incoming partons p_1, p_b are substituted by the momenta of the incoming protons P_1, P_2.

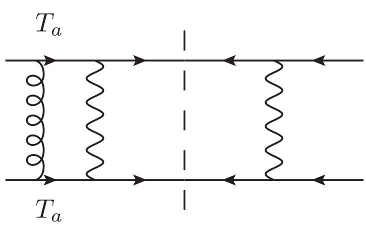
So, the light and massive hadronic tensors are decomposed as

$$
\begin{aligned}
W^{\mu\nu}(x_1, Q^2) =& -4(2P_1\cdot q)F_1(x_1, Q^2)g^{\mu\nu} + F_2(x, Q^2)16P_1^\mu P_1^\nu \\
&+ F_3(x_1, Q^2)8i\epsilon_{\mu\nu\rho\gamma}P_1^\rho q^\gamma + 8F_4(x_1, Q^2)q^\mu q^\nu \\
&+ F_5(x_1, Q^2)8(q^\mu P_1^\nu + P_1^\mu q^\nu) \qquad (3.47) \\
W^{\mu\nu}(x_2, Q^2, m_t^2) =& -4(2P_2\cdot q)F_1(x_2, Q^2, m_t^2)g^{\mu\nu} + F_2(x_2, Q^2, m_t^2)16P_2^\mu P_2^\nu \\
&+ F_3(x_2, Q^2, m_t^2)8i\epsilon_{\mu\nu\rho\gamma}P_2^\rho q^\gamma + 8F_4(x_2, Q^2, m_t^2)q^\mu q^\nu \\
&+ F_5(x_2, Q^2, m_t^2)8(q^\mu P_2^\nu + P_2^\mu q^\nu) \qquad (3.48)
\end{aligned}
$$

One last comment is to be done about the structure functions (or alternatively coefficient functions, to which the same considerations apply) that actually enter our computation. If the quark struck by the W boson and the quark into which it is converted are both massless, then the current with which the W boson interacts is conserved, and one has $q^\mu W_{\mu\nu}(x, Q^2) = q^\nu W_{\mu\nu}(x, Q^2) = 0$. This condition can be realized only if $F_4(x, Q^2) = F_5(x, Q^2) = 0$. If the quark into which the struck quark is converted is massive, such as the top quark, then the current is no longer conserved, and $F_4(x, Q^2, m_t^2), F_5(x, Q^2, m_t^2)$ are non-vanishing, so in principle we have to take them into consideration. But, in our particular case, since we are looking at a massive current interacting with a massless one, F_4 and F_5 never enter the expression for the final cross-section. What happens, is the following. The hadronic cross-section Eq.(3.42) is obtained by contracting the hadronic tensors at each vertex with the square of the W propagator connecting them (as in Eq.(3.33)). Due to current conservation of the light-quark tensor, the $q^\mu q^\nu/m_W^2$ term in the numerator of the W propagator does not contribute, so one simply contracts the two tensors together. Now, F_4 and F_5 are the coefficients of tensors which contain q^μ, q^ν or both and these tensors give vanishing contribution when contracted with the light-quark tensor. Thus we can conclude that, due to current conservation of the light-quark tensor, the structure functions F_4 and F_5, either in their massless or massive version, do not enter our computation and this remains true at all orders.

Non-factorizable contributions.
Before proceeding, we discuss the nature and importance of non-factorizable corrections.

- **NLO**:
  At NLO, diagrams where a gluon is exchanged between the two currents are exactly zero because of the color degree of freedom, since they are all proportional to the trace of a single Gell-Mann matrices.



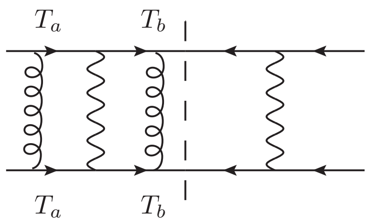
$$\propto \mathrm{Tr}[T_a]\mathrm{Tr}[T_a] = 0$$

Figure 3.5: Example of (virtual) non-factorizable gluon exchange at NLO.

  That's why at NLO the structure function approach gives exactly the correct result, without approximations.

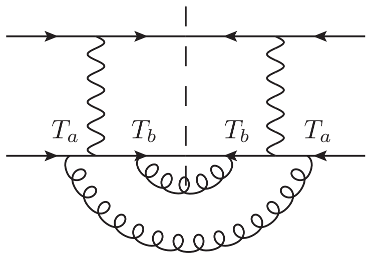
- **NNLO**:
  At NNLO this is not true anymore, because this kind of cross-talks diagrams gives non-zero contribution. But luckily, this class [6] of diagrams is suppressed by a factor $\mathcal{O}(1/N_c^2)$ with respect to the leading factorizable corrections (see examples in Fig.(3.6), (3.7)).

[6] In this context, by 'class' of diagrams, we refer to a gauge invariant, finite subset of diagrams.



$$\propto Tr[T_aT_b]Tr[T_aT_b] = \frac{C_aC_f}{2} = 2$$

Figure 3.6: Example of (double virtual) non-factorizable gluon exchange at NNLO.



$$\propto Tr[T_aT_bT_bT_a]N_c = C_f^2C_aN_c = 16$$

Figure 3.7: Example of (double virtual) factorizable gluon exchange at NNLO.

So, if we adopt a structure function approach, namely we neglect these class of diagrams, we can think of making quite a good and safe approximation of the total NNLO cross-section.
This belief is reinforced by the fact that these diagrams are also suppressed by the kinematic of the process. Indeed, single-top production in t-channel is mediated by the exchange of a virtual W, whose propagator is

$$\frac{1}{t - m_W^2} = \frac{1}{-Q^2 - m_W^2}. \tag{3.49}$$

Q^2 is positive, so that the denominator of this propagator is always negative and the propagator is then maximum when $Q^2 \to 0$. The cross-section is dominated by the small Q^2 region (see [89]), but small Q^2 means that the energy exchanged between the two currents is little and consequently that the interaction between them happens at 'large' distance. In principle it is perfectly possible that a low Q^2 gluon is emitted from a current and such gluon could then interact with the other current. But the hard scale of the process is of order $m_t^2 + Q^2 \gg Q^2$, and this tells us that emissions characterized by $k_T^2 \sim Q^2$ are to be considered 'soft' and as such do not contribute

to the description of the hard process, but rather to the description of the initial-proton structure. This kinematic argument provides a further source of suppression of this category of diagrams.
One last argument, that further supports the reliability of our approximation, is represented by the estimation that was done in [130] of this class of cross-talk diagrams in the case of Higgs production in VBF. Indeed, in that case, they were found to contribute 1% of the total VBF cross-section. Of course, all the arguments provided up to now are purely qualitative and a quantitative estimation of the error introduced by neglecting these diagrams needs to be provided. This will be one of the goal we would like to achieve in the close future.

With these last considerations, we close this introductory subsection where Single Top in t-channel was presented as a double CC-DIS process. In the next subsection we analyse the structure of the objects of our calculation, namely CC-DIS Form Factors (or, equivalently Structure Functions).

3.2.2 Structure of Form Factors up to NNLO-QCD

We briefly review the basic formulae for the CC-DIS structure functions F_i^V with $i = 1, 2, 3$ and $V \in \{W^\pm\}$ [7]. We choose to report these formulae not only for completeness in the treatise of CC-DIS, but also for practical purposes. Indeed, the way we implement Form Factors in our standalone code which computes the total t-channel cross-section is dictated by the way CC-DIS SF are organized on a theoretical level.

As already anticipated, QCD factorization allows to express the structure functions as convolutions of the PDFs in the proton and the coefficient functions C_i, which contain in turn information about the short-distance, hard scattering. The gluon PDF at the factorization scale μ_F is denoted by $g(x, \mu_F)$ and the quark (or anti-quark) PDF by $q_i(x, \mu_F)$ (or $\bar{q}_i(x, \mu_F)$) for a specific quark flavour i. The quark PDFs appear in the following combinations,

$$\begin{aligned} q_s &= \sum_{i=1}^{n_f} (q_i + \bar{q}_i)\,, & q_{ns}^V &= \sum_{i=1}^{n_f} (q_i - \bar{q}_i)\,, \\ q_{ns,i}^+ &= (q_i + \bar{q}_i) - q_s, & q_{ns,i}^- &= (q_i - \bar{q}_i) - q_{ns}^V, \end{aligned} \tag{3.50}$$

as the singlet distribution q_s, the (non-singlet) valence distribution q_{ns}^V as well as flavour asymmetries $q_{ns,i}^\pm$.

[7]NB: We present formulae assuming massless structure functions, but all these results holds for the massive case without introducing any modifications, except obviously for the variables the SF themselves depend upon

For the charged current case with $W^{\pm}$-boson exchange the DIS structure functions $F_i^{W^\pm}$ are given by,

$$
\begin{aligned}
F_i^{W^-}(x,Q^2) = \frac{1}{2} f_i(x) \int_0^1 dz \int_0^1 dy \delta(x-yz) \frac{1}{n_f} \sum_{i=1}^{n_f} \left(v_i^2 + a_i^2\right) \times \\
\times \left\{ \delta q_{ns}^-(y,\mu_F) C_{i,ns}^-(z,Q,\mu_R,\mu_F) + q_s(y,\mu_F) C_{i,q}(z,Q,\mu_R,\mu_F) \right. \\
\left. + g(y,\mu_F) C_{i,g}(z,Q,\mu_R,\mu_F) \right\}, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
F_3^{W^-}(x,Q^2) = \frac{1}{2} \int_0^1 dz \int_0^1 dy \delta(x-yz) \frac{1}{n_f} \sum_{i=1}^{n_f} (2 v_i a_i) \times \\
\times \left\{ +\delta q_{ns}^+(y,\mu_F) C_{3,ns}^+(z,Q,\mu_R,\mu_F) + q_{ns}^V(y,\mu_F) C_{3,ns}^V(z,Q,\mu_R,\mu_F) \right\}, \\
(3.52)
\end{aligned}
$$

where $i = 1, 2$ and the pre-factors in Eq.3.51 are $f_1(x) = 1/2$, $f_2(x) = x$. The asymmetry $\delta q_{ns}^{\pm}$ parametrizes the iso-triplet component of the proton, i.e. $u \neq d$ and so on. It is defined as

$$
\delta q_{ns}^{\pm} = \sum_{i \in u-type} \sum_{j \in d-type} \left\{ (q_i \pm \bar{q}_i) - (q_j \pm \bar{q}_j) \right\}. \qquad (3.53)
$$

The respective results for $F_i^{W^+}$ are obtained from Eq.(3.51), (3.52) with the simple replacement $\delta q_{ns}^{\pm} \to -\delta q_{ns}^{\pm}$. The vector- and axial- vector coupling constants v_i and a_i are given by $v_i = a_i = 1/\sqrt{2}$.
The perturbative expansion of the coefficient functions C_i in the strong coupling α_s up to two loops reads in the non-singlet sector,

$$
\begin{aligned}
C_{i,ns}^{\pm}(x) =& \delta(1-x) + a_s \left\{ c_{i,q}^{(1)} + L_M P_{qq}^{(0)} \right\} \\
&+ a_s^2 \left\{ c_{i,ns}^{(2),\pm} + L_M \left(P_{ns}^{(1),\pm} + c_{i,q}^{(1)} (P_{qq}^{(0)} - \beta_0) \right) + L_M^2 \left(\frac{1}{2} P_{qq}^{(0)} (P_{qq}^{(0)} - \beta_0) \right) \right. \\
&\left. + L_R \beta_0 c_{i,q}^{(1)} + L_R L_M \beta_0 P_{qq}^{(0)} \right\}, \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
C_{3,ns}^{\pm}(x) =& \delta(1-x) + a_s \left\{ c_{3,q}^{(1)} + L_M P_{qq}^{(0)} \right\} \\
&+ a_s^2 \left\{ c_{3,ns}^{(2),\pm} + L_M \left(P_{ns}^{(1),\pm} + c_{3,q}^{(1)} (P_{qq}^{(0)} - \beta_0) \right) + L_M^2 \left(\frac{1}{2} P_{qq}^{(0)} (P_{qq}^{(0)} - \beta_0) \right) \right. \\
&\left. + L_R \beta_0 c_{3,q}^{(1)} + L_R L_M \beta_0 P_{qq}^{(0)} \right\}, \qquad (3.55)
\end{aligned}
$$

where $a_s = \alpha_s(\mu_R)/(4\pi)$ and $i = 1, 2$ in Eq.(3.54). The complete scale dependence, i.e. the towers of logarithms in $L_M = \ln(Q^2/\mu_F^2)$ and $L_R = \ln(\mu_R^2/\mu_F^2)$, has been derived by renormalization group methods (see e.g. ..) in terms of splitting functions $P_{ij}^{(l)}$ and coefficients β_l of the QCD beta function. Given the normalization of the expansion parameter $a_s = \alpha_s/(4\pi)$, the conventions for the running coupling are

$$\frac{d}{d\ln\mu^2}\frac{\alpha_s}{4\pi} = \frac{da_s}{d\ln\mu^2} = -\beta_0 a_s^2 - ..., \qquad \beta_0 = \frac{11}{3}C_a - \frac{2}{3}n_f. \tag{3.56}$$

Note that the valence coefficient function $C_{3,ns}^V$ in Eq. (3.52) is defined as $C_{3,ns}^V = C_{3,ns}^- + C_{3,ns}^s$. However, we have $C_{3,ns}^s \neq 0$ starting at three-loop order only, so that Eq.(3.52) suffices with $C_{3,ns}^V = C_{3,ns}^-$ up to NNLO.
In the singlet sector we have

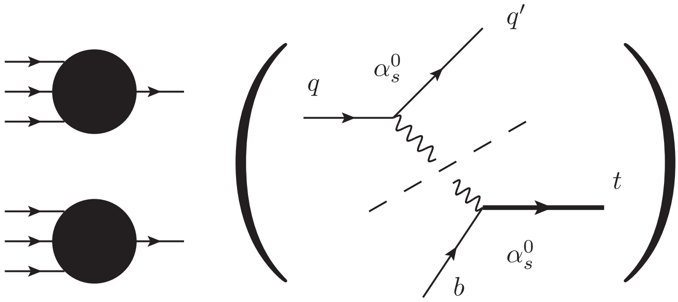
$$\begin{aligned} C_{i,q}(x) =& \delta(1-x) + a_s\left\{c_{i,q}^{(1)} + L_M P_{qq}^{(0)}\right\} \\ &+ a_s^2\left\{c_{i,q}^{(2),\pm} + L_M\left(P_{qq}^{(1),\pm} + c_{i,q}^{(1)}(P_{qq}^{(0)} - \beta_0) + c_{i,g}^{(1)}P_{gq}^{(0)}\right)\right. \\ &+ L_M^2\left(\frac{1}{2}P_{qq}^{(0)}(P_{qq}^{(0)} - \beta_0) + \frac{1}{2}P_{qg}^{(0)}P_{gq}^{(0)}\right) \\ &\left.+ L_R\beta_0 c_{i,q}^{(1)} + L_R L_M \beta_0 P_{qq}^{(0)}\right\}, \end{aligned} \tag{3.57}$$

$$\begin{aligned} C_{i,g}(x) =& a_s\left\{c_{i,g}^{(1)} + L_M P_{qg}^{(0)}\right\} \\ &+ a_s^2\left\{c_{i,g}^{(2),\pm} + L_M\left(P_{qg}^{(1),\pm} + c_{i,q}^{(1)}P_{qg}^{(0)} + c_{i,g}^{(1)}(P_{gg}^{(0)} - \beta_0)\right)\right. \\ &+ L_M^2\left(\frac{1}{2}P_{qq}^{(0)}P_{qg}^{(0)} + \frac{1}{2}P_{qg}^{(0)}(P_{gg}^{(0)} - \beta_0)\right) \\ &\left.+ L_R\beta_0 c_{i,g}^{(1)} + L_R L_M \beta_0 P_{qg}^{(0)}\right\}, \end{aligned} \tag{3.58}$$

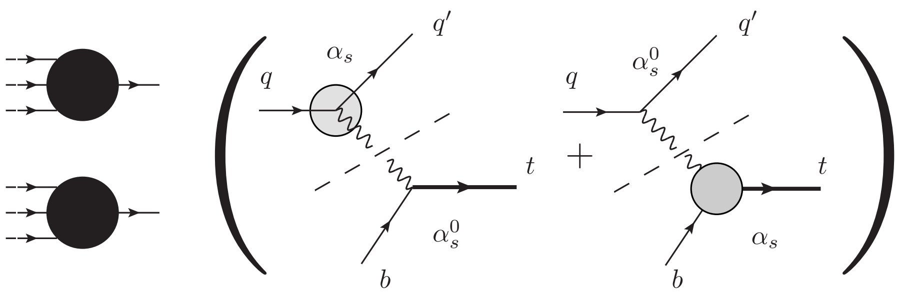
where again $i = 1, 2$ in Eq.3.57. The quark-singlet contribution contains the so-called pure-singlet part, $C_{i,q} = C_{i,ns}^+ + C_{i,ps}$, i.e. $P_{qq}^{(1)} = P_{ns}^{(1),+} + P_{ps}^{(1)}$ and $c_{i,q}^{(2)} = c_{i,ns}^{(2),+} + c_{i,ps}^{(2)}$ in Eq.(3.57). Starting at two-loop order we have $C_{i,ps} \neq 0$. The coefficient functions $c_{i,k}^{(l)}$ in the massless case are known up to NNLO from [123], [133], [132], [94]. NNLO evolution of PDFs has been determined in [95], [124], together with splitting functions $P_{i,j}^{(l)}$. Needless to say that all products in equations from Eq.(3.54) to (3.58) are to be understood as Mellin convolutions.

3.2.3 Organization of a NNLO computation

We are now ready to explain how the computation of t-channel Single top up to NNLO is organized in the framework of a Structure Function approach.



(a) Structure Functions contributions to Single top in t-channel at $\mathcal{O}(\alpha_w^2)$.



(b) Structure Functions contributions to Single top in t-channel at $\mathcal{O}(\alpha_w^2 \alpha_s)$.

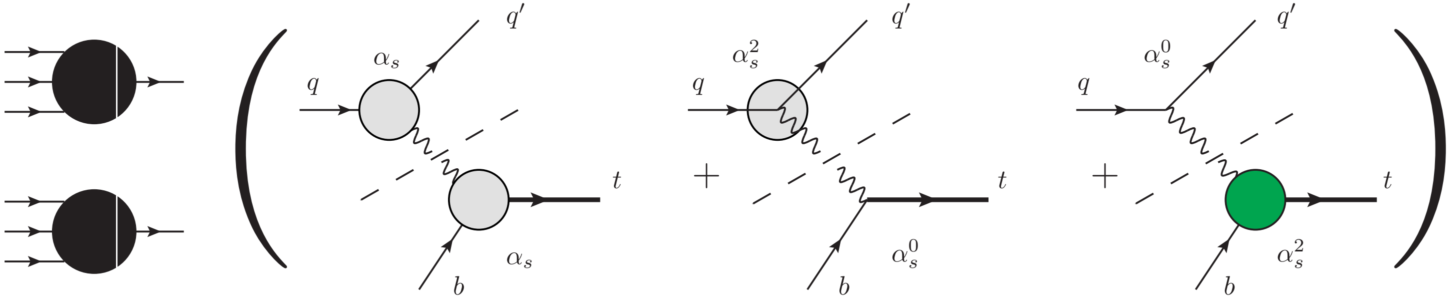


Figure 3.8: Structure Functions contributions to Single top in t-channel at $\mathcal{O}(\alpha_w^2 \alpha_s^2)$.

- **LO**: The LO-QCD contribution (Fig.(3.8a)) to the cross-section is simply given by the product of the hadronic tensors containing SF evaluated at $\mathcal{O}(\alpha_s^0)$. We indicate the hadronic tensor containing SF evaluated at the generic order α_s^i as $\left(W_{\alpha_s^0}\right)^{\mu\nu}$ (the weak coupling is here tacitly implied). So, by considering the initial general formula for the hadronic cross-section Eq.(3.42), we can write

$$d\sigma_{\alpha_s^0} = \frac{1}{2S} 4 \left(\frac{g^2}{8}\right)^2 \frac{1}{(Q^2+m_W^2)^2} \times \left[\left(W_{\alpha_s^0}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^0}\right)_{\mu\nu}(x_2,Q^2,m_t^2)\right] \times \\ \times (2\pi)^2 \frac{1}{4S} dQ^2 dW_1^2 dW_2^2. \tag{3.59}$$

- **NLO**: The NLO-QCD contribution (Fig.(3.8b)) to the cross-section will be the sum of two terms, containing in turn the product of a hadronic tensor for one of the two currents at order α_s times the hadronic tensor for the other current at order α_s^0, as in Eq.(3.60).

$$d\sigma_{\alpha_s^1} = \frac{1}{2S} 4 \left(\frac{g^2}{8}\right)^2 \frac{1}{(Q^2+m_W^2)^2} \times \times \left[\left(W_{\alpha_s^1}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^0}\right)_{\mu\nu}(x_2,Q^2,m_t^2) \right. \\ \left. + \left(W_{\alpha_s^0}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^1}\right)_{\mu\nu}(x_2,Q^2,m_t^2)\right] \times \\ \times (2\pi)^2 \frac{1}{4S} dQ^2 dW_1^2 dW_2^2. \tag{3.60}$$

- **NNLO**: Along the same line, the NNLO-QCD corrections contains the sum of three contributions, namely all the possible ways in which the product of the two hadronic tensors can yield an $\mathcal{O}(\alpha_s^2)$ quantity. So, following Fig.(3.8), the contribution to the cross-section at this order can be written as

$$d\sigma_{\alpha_s^2} = \frac{1}{2S} 4 \left(\frac{g^2}{8}\right)^2 \frac{1}{(Q^2+m_W^2)^2} \times \times \left[\left(W_{\alpha_s^1}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^1}\right)_{\mu\nu}(x_2,Q^2,m_t^2) \right. \\ \left. + \left(W_{\alpha_s^2}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^0}\right)_{\mu\nu}(x_2,Q^2,m_t^2) + \left(W_{\alpha_s^0}\right)^{\mu\nu}(x_1,Q^2)\left(W_{\alpha_s^2}\right)_{\mu\nu}(x_2,Q^2,m_t^2)\right] \times \\ \times (2\pi)^2 \frac{1}{4S} dQ^2 dW_1^2 dW_2^2. \tag{3.61}$$

To compute the analytical cross-section $d\sigma$ up to order α_s^2, we obviously need the analytical results for all the three contributions above listed (Eq.(3.59), (3.60), (3.61)) to be available. As anticipated in the introduction to Chapter 2, in order to reach this goal, we miss just one piece, highlighted in green in Fig.(3.61), namely the massive Form Factors at order $\mathcal{O}(\alpha_s^2)$. The computation of these Form Factors, thanks to the p-QCD master formula Eq.5.14 (introduced in Section 2.1), essentially boils down to the computation of massive Coefficient Functions

for charged-current DIS, which we obtain by computing the specific subprocess $b(p_b) + W^*(q) \to t(p_t) + X$ at the required order α_s^2. Here, X identifies potential extra radiation emitted by the initial and/or final fermion.
Furthermore, we have to take into account that by applying crossing-symmetry to the diagrams for the b-initiated process $b(p_b) + W^*(q) \to t(p_t) + X$, we can obtain diagrams initiated by a gluon $g(k_1) + W^*(q) \to t(p_t) + \bar{b}(p_b) + X$ or by a light quark $q(k_1) + W^*(q) \to t(p_t) + \bar{b}(p_b) + X$. According to the particle which initiates the process together with the virtual W-boson, we will classify diagrams as belonging to *bottom, gluon, or singlet channel.* Inside each channel, diagrams are then organized according to the number of virtual and real emitted gluon or quarks, as

- **Double Real (RR)**: the category of diagrams includes tree-level diagrams containing two extra particles (QCD radiation) in the final state;
- **Real-Virtual (RV)**: these diagrams instead contain one loop (virtual emission) and one extra real particle in the final state;
- **Double Virtual (VV)**: this last category contains two-loop diagrams.

The following table briefly recaps the classification according to channels and type of radiative corrections.

	RR	RV	VV
bottom	$[b + W^* \to t + X]^{0l}_{X=gg,q\bar{q},b\bar{b}}$	$[b + W^* \to t + X]^{1l}_{X=g}$	$[b + W^* \to t]^{2l}$
gluon	$\left[g + W^* \to t + \bar{b} + g\right]^{0l}$	$\left[g + W^* \to t + \bar{b}\right]^{1}$	/
singlet	$\left[q + W^* \to t + \bar{b} + q\right]^{0l}$	/	/

Table 3.1: Organization of the calculation of CC-DIS massive Form Factors at $\mathcal{O}(\alpha_s^2)$.

Each one of the subprocesses in Table.(3.1) is computed within the framework of Dimensional Regularization, which is the tool we choose to deal with IR and UV divergences. Dimensional Regularization is the standard technique used in the majority of analytical higher-order computations, so its effectiveness and consistency have been widely tested over the years and by now safely established. Despite the 'safety' of this technical tool, we have to be particularly careful when using it, due to the presence of γ_5 Dirac matrix in the $bW^* \to t$ vertex.
In general for perturbative calculations at higher-orders the presence of the Dirac matrix γ_5 is a nuisance since it is a purely 4-dimensional object and it can not be continued to d-dimensions in a straightforward way. While computationally-efficient ways to deal with γ_5 in DR exist (see [88]), they are typically complex

and not only transparent. Fortunately, there is a simple way to deal with γ_5 in our case: γ_5 is taken in all diagrams (RR, RV, VV) to be anti-commuting in all d dimensions.
Indeed, since we never get axial anomalies from diagrams involved in our process, any prescription for the $\gamma 5$ is good as long as it is consistently used in all the different pieces of the computation and this is actually what we carefully do!

Before proceeding to illustrate the technique we use perform virtual and real integrations analytically (see next Chapter, n.3) we spend still a few words to discuss the choice of flavour-scheme. Actually, when presenting the organization of our diagrams above, we tacitly assumed the working scheme to be in a 5-flavour scheme, but we never motivated this choice. In the following and last subsection, we address this topic and introduce some ideas which may further improve our 5-flavour NNLO results in the future.

3.2.4 m_b-correction: 4F- versus 5F- scheme

As anticipated in the previous sections, we choose to carry out our computation in a 5F-scheme, where all the quark flavours except for the top are massless, included the b-quark. The reason for choosing this scheme is quite intuitive. The hard scale of Single Top production is m_t^2 [8]. Since we have $m_b^2 \ll m_t^2$, it is a priori a sensible choice to neglect m_b^2, thus considering the b-quark massless. This preliminary and quite obvious statement requires though some further reflection.
Single Top production in t-channel was originally dubbed W-gluon fusion [128], because it was thought of as a virtual W striking a gluon to produce a $t\bar{b}$ pair, as shown in Fig.(3.9). If the $\bar{b}$ in the final state is at high transverse momentum (p_T),

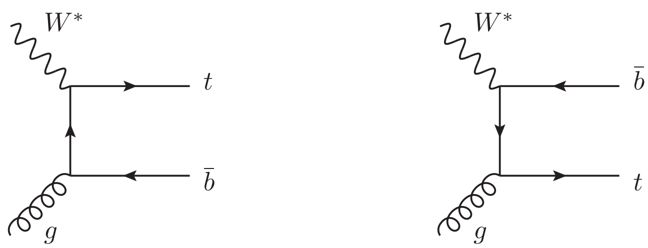


Figure 3.9: Leading-Order diagrams for Single Top production in t-channel in 4F-scheme.

[8]Actually, for the sake of precision, we point out that the correct hard scale is $Q^2 + m_t^2$ (as widely discussed in [118]). But, since the cross-section is peaked around $Q^2 \sim 50$GeV, namely around values of Q^2 much smaller than m_t^2, we can safely take the hard scale to be m_t^2.

the first on the left of these diagrams is indeed the leading-order diagram for this process. If we instead integrate over the p_T of the $\bar{b}$, we obtain an enhancement from the region where the $\bar{b}$ is at low p_T, nearly collinear with the incident gluon. Indeed, if the b-quark is massless, this diagram is not only enhanced, but singular when the final b-quark is collinear with the incoming gluon. This kinematic configuration corresponds to the incoming gluon splitting into a real $b\bar{b}$ pair. The propagator of the internal b-quark in the diagram is therefore on-shell and thus infinite. In reality the b-quark is not massless and its mass regulates the collinear singularity which exists in the massless case (luckily!).
So, if we adopt a 4f-scheme and indeed set $m_b \neq 0$, we will see the collinear singularity manifesting itself in the total cross-section as terms proportional to $\ln[(Q^2 + m_t^2)/m_b^2]$, which we approximate in the following as $\ln(m_t^2/m_b^2)$ (see the argument on the W-boson virtuality Q^2 in the discussion over non-factorizable contribution above). The total cross-section for W-gluon fusion contains these logarithmically enhanced terms $\alpha_s \ln(m_t^2/m_b^2)$, as well as terms of order α_s (both terms also carry a factor of α_w^2, which we omit in the following).
Furthermore, terms of order $\alpha_s^n \ln^n(m_t^2/m_b^2)/n!$ appear at every order in the perturbative expansion in the strong coupling, due to collinear emission of gluons from the internal b-quark propagator. This collinear enhancement is desirable because it yields a larger cross-section, but also it makes the perturbative expansion less convergent.
Fortunately, this can be obviated. The DGLAP evolution equations (introduced in the previous Section 3.1) are indeed the formalism which allows to sum the collinear logarithms to all orders in perturbation theory. Practically speaking, one can sum these collinear logarithms by introducing a b distribution function $b(x, \mu_F^2)$ and calculating its evolution with μ_F (from some initial condition) via the DGLAP equations. Thus the b distribution is at all effects a device to sum collinear logs. Since it is calculated from the splitting of a gluon into a collinear $b\bar{b}$ pair, it is intrinsically of order $\alpha_s \ln(\mu_F^2/m_b^2)$.
Once a b PDF is introduced, it changes the way one orders perturbation theory. The leading order process is now $q + b \to q' + t$, namely the 'usual' leading-order diagram showed in Fig.3.3.
Thus, driven by the need to resum collinear m_b-logs, we have naturally introduced the idea of a 5-flavour scheme, whose spirit is the following. We set $m_b = 0$, reorganize perturbative expansion starting from the LO process $q + b \to q' + t$, and take into account collinear logarithms $\alpha_s^n \ln^n(m_t^2/m_b^2)/n!$ at all orders by convolving this partonic process with a b-PDF.
The LO cross-section for $p+p \to q'+t$ is of order $\alpha_s \ln(m_t^2/m_b^2)$, due to the b-PDF (where the factorization scale is set indicatively equal to the hard scale of the process, namely $\mu_F \sim m_t$). The LO cross-section for W-gluon fusion ($p+p \to q'+t+\bar{b}$) contains terms of both order $\alpha_s \ln(m_t^2/m_b^2)$ and α_s, but the formers, being $m_b = 0$, appear now as effective divergences $\alpha_s 1/\epsilon$ (poles in DR!). Such poles must to be

subtracted manually, since the collinear divergences they represent are already taken into account in the b distribution. After this subtraction has taken place, the W-gluon fusion cross-section contains only $\mathcal{O}(\alpha_s)$ terms. Compared with the leading-order process $p + p \to q' + t$, the W-gluon fusion $p + p \to q'\bar{b} + t$ is thus suppressed by a factor $1/\ln(m_t^2/m_b^2)$, and not by a factor α_s as one might naively think!

The natural conclusion of this discussion is that the choice of a 5F-scheme seems the most sensible one. The condition $m_b = 0$ allows for significant simplification the computation (one dimensional scale less enters the calculation!), and at the same time all the collinear logarithmically enhanced terms are automatically accounted for in the b-PDF. These are the motivations why we decided to use a 5F-scheme.
But, given this, we observed that there is the possibility to do even better. Up to now, we neglected a subtlety: when performing the computation of diagrams Fig.(3.9) keeping $m_b \neq 0$, the final result do not depend only logarithmically on m_b, but also through polynomial dependence (namely power functions $(m_b^2)^n$. The m_b-logs, as already said, are universal, so that they can be predicted at all orders and resumed in PDFs. This is not true for m_b-power corrections. This kind of terms is not universal and one has to compute the process by retaining the full m_b-dependence, in order to be keep them into account. Now, in our case, since the very moment we set $m_b = 0$, these terms are automatically lost.
But, we observed that we still have the possibility to retain m_b-power corrections by performing some manipulations on already existing numerical results. We will talk from now on of *5F-improved scheme*, where with this name we mean a 5F-scheme, enriched by the additional presence of m_b power-corrections. Our aim is that of obtaining a prediction for NNLO Single Top t-channel in this scheme, by calculating analytically the NLO and NNLO-QCD corrections in an ordinary 5F-scheme and, on top of that, estimating the numerical contribution of m_b power-corrections (respectively at NLO and NNLO).
We explain in the following how we can numerically extract such power-like corrections due to the presence of the neglected b-mass.

We observe that MCFM ([38]) already contains the exact numerical result for $p+p \to q' + t + \bar{b}$ in a 4F-scheme up to $\mathcal{O}(\alpha_s^2)$ (namely NLO in a 4F- and NNLO in a 5F-scheme) [9]. Starting from this numerical already existent result, we are able to extract the desired m_b power-corrections. We explain in the following how our

[9] We stress again that the perturbative expansion is organized differently in 4F- and 5F-scheme. In a 5F-scheme, the diagrams in Fig.(3.9) belong to NLO, where as in a 4F-scheme they are LO diagrams. When we add the first order of QCD corrections to these diagrams, these will be classified as NNLO and NLO corrections respectively in a 5F- 4F-scheme. To try avoiding confusion, in the following we will refer to QCD corrections by citing the explicit order in α_s, which is obviously scheme-invariant.

idea works by taking structure functions at $\mathcal{O}(\alpha_s)$ as an example. We also carry out a numerical analyses in order to assess the impact of m_b power-corrections at $\mathcal{O}(\alpha_s)$. This is a good way to test if these corrections are indeed small as expected. We start computing analytically the coefficient functions $C^g_{i_{\alpha_s}}(m_t, m_b)$, which are exact in their dependence on both m_t and m_b. This is achieved by computing the cross-section for the partonic sub-process $W * + g \to t + \bar{b}$ keeping both the masses m_t, m_b in the final state. In terms of s, Q^2, m_t^2, m_b^2, these coefficient functions reads:

$$
\begin{aligned}
&C^g_{1_{\alpha_s}}(s, Q^2, m_t^2, m_b^2) = \\
&\frac{1}{(2(Q^2+s)^2)}\Big(\frac{-4m_b^2(m_b^2-m_t^2-Q^2)s}{m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+\frac{(Q^4+s^2)(m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})}{s} \\
&-\frac{4m_t^2(-m_b^2+m_t^2-Q^2)s}{-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+\frac{4m_b^2(m_b^2-m_t^2-Q^2)s}{m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&-\frac{(Q^4+s^2)(m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}]}{s} \\
&+\frac{4m_t^2(-m_b^2+m_t^2-Q^2)s}{-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+(2m_b^4+2m_t^4+Q^4+2m_t^2(Q^2-s)-2m_b^2(2m_t^2+Q^2-s)+s^2) \\
&\times\log[\frac{-((Q^2+s)(m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))}{(-((Q^2+s)(m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})))}] \\
&+(2m_b^4+2m_t^4-2m_t^2Q^2+Q^4+2m_t^2s+s^2-2m_b^2(2m_t^2-Q^2+s)) \\
&\times\log[\frac{((Q^2+s)(-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))}{(((Q^2+s)(-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})))}]\Big),
\end{aligned}
\tag{3.62}
$$

$$
\begin{aligned}
&C^g_{2_{\alpha_s}}(s, Q^2, m_t^2, m_b^2) = \\
&-\frac{1}{(2(Q^2+s)^3)}\Big(\frac{-4m_b^2(m_b^4+(m_t^2+Q^2)^2-2m_b^2(m_t^2+2Q^2))s}{m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&-\frac{Q^2(Q^4-4Q^2s+s^2)(m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})}{s} \\
&-\frac{4m_t^2(m_b^4-2m_b^2m_t^2+m_t^4+2m_b^2Q^2-4m_t^2Q^2+Q^4)s}{-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4m_b^2(m_b^4+(m_t^2+Q^2)^2-2m_b^2(m_t^2+2Q^2))s}{m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+\frac{Q^2(Q^4-4Q^2s+s^2)(m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})}{s} \\
&+\frac{4m_t^2(m_b^4-2m_b^2m_t^2+m_t^4+2m_b^2Q^2-4m_t^2Q^2+Q^4)s}{-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+(2m_b^6-2m_b^4(m_t^2-2Q^2+s)+(m_t^2+Q^2)(2m_t^4+Q^4+2m_t^2(Q^2-s)+s^2) \\
&+m_b^2(-2m_t^4-9Q^4+10Q^2s+s^2+4m_t^2(-4Q^2+s))) \\
&\times\log\frac{-((Q^2+s)(m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})}{-((Q^2+s)(m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))} \\
&-(2m_b^6+2m_t^6+m_t^4(4Q^2-2s)-2m_b^4(m_t^2-2Q^2+s)+Q^2(Q^4+s^2) \\
&+m_t^2(-9Q^4+10Q^2s+s^2)+m_b^2(-2m_t^4+3Q^4-2Q^2s+s^2+4m_t^2(-4Q^2+s))) \\
&\times\log\frac{((Q^2+s)(-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))}{(((Q^2+s)(-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})))}),
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
&C^g_{3_{\alpha_s}}(s,Q^2,m_t^2,m_b^2)= \\
&\frac{1}{2(Q^2+s)^2}(\frac{4m_t^2(-m_b^2+m_t^2-Q^2)s}{-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+\frac{4m_b^2(m_b^2-m_t^2-Q^2)s}{-m_b^2+m_t^2-s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+\frac{4m_b^2(m_b^2-m_t^2-Q^2)s}{m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&-\frac{4m_t^2(-m_b^2+m_t^2-Q^2)s}{-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}} \\
&+(2m_b^4-2m_t^4-2m_t^2Q^2-Q^4+2m_b^2(Q^2-s)+2m_t^2s-s^2) \\
&\times\log\frac{-((Q^2+s)(m_b^2-m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))}{(-((Q^2+s)(m_b^2-m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})))}] \\
&-(2m_b^4-2m_t^4-2m_t^2Q^2+Q^4+2m_b^2(Q^2-s)+2m_t^2s+s^2) \\
&\log\frac{((Q^2+s)(-m_b^2+m_t^2+s+\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)}))}{(((Q^2+s)(-m_b^2+m_t^2+s-\sqrt{m_b^4+(m_t^2-s)^2-2m_b^2(m_t^2+s)})))}).
\end{aligned}
\tag{3.64}
$$

These coefficient functions, combined with a kinematic that keeps into account the non-zero bottom mass, and convoluted with the gluon pdf, give us the gluon structure functions $F^g_{i_{\alpha_s}}(m_t, m_b)$, and thus the $\mathcal{O}(\alpha_s)$ cross-section for $W*+p \rightarrow$

$t + \bar{b}$ (LO in a 4F-scheme).

$$F^g_{i_{\alpha_s}}(m_t, m_b) = \int_{x_b}^1 C^g_{i_{\alpha_s}}(\tau_b, \lambda_b, m_t, m_b) g(\frac{x_b}{\tau_b}), \tag{3.65}$$

where the new variables x_b, τ_b, λ_b have substituted the old x, τ, λ in order to keep into consideration the kinematic effects of $m_b \neq 0$. They are defined as:

$$x_b = \frac{Q^2 + (m_t + m_b)^2}{S + Q^2} \quad , \tau_b = \frac{Q^2 + (m_t + m_b)^2}{s + Q^2} \quad , \lambda_b = \frac{Q^2}{Q^2 + (m_t + m_b)^2}. \tag{3.66}$$

We note that in the limit $m_b \to 0$, these variables become the usual variables x, τ, λ that we will use to parametrize our 5F-coefficient functions (see Chapter 4). By expanding in series $F^g_{i_{\alpha_s}}(m_t, m_b)$ Eq.(3.65) with respect to m_b, we recover at 0-th order the gluon structure functions $F^g_{i_{\alpha_s}}(m_t, 0)$ plus,as expected, the bottom collinear logarithm multiplied for the splitting function $P_{qg}(\tau) \times \log\left(\frac{\mu_F^2}{m_b^2}\right)$ and convoluted with gluon pdf, reminding us that still we are in a 4F-scheme, plus the desired m_b power-corrections.

$$F^g_{i_{\alpha_s}}(m_t, m_b) \overset{m_b \to 0}{\to} \int_x^1 f^g_{i_{\alpha_s}}(\tau, m_t, 0) g(\frac{x}{\tau}) + \log\left(\frac{\mu_F^2}{m_b^2}\right) \int_x^1 P_{qg}(\tau) g(\frac{x}{\tau}) + \mathcal{O}((m_b^2)^n). \tag{3.67}$$

Thus, if we denominate the m_b power-like contributions as Δ_{m_b,α_s}, we can extract them by inverting Eq.(3.67) as follows

$$\Delta_{m_b,\alpha_s} = F^g_{i_{\alpha_s}}(m_t, m_b) - \left(F^g_i(m_t, 0) + \log\left(\frac{\mu_F^2}{m_b^2}\right) \int_x^1 Pqg(\tau) g(\frac{x}{\tau})\right). \tag{3.68}$$

Numerical results for Δ_{m_b,α_s} are listed in Table (3.2). These numerical results are obtained as follows. The process $\left[p + p \to t + \bar{b}\right]_{\alpha_s,4f}$ is run in MCFM and this gives numerical result for the cross-section obtained by using SF with a full m_t, m_b dependence, namely what we called $F^g_{i_{\alpha_s}}(m_t, m_b)$. Then, following Eq.(3.68), we subtract to these numbers other two numerical contributions.

- The collinear m_b-logs, i.e. $\log\left(\frac{\mu_F^2}{m_b^2}\right) \int_x^1 P_{qg}(\tau) g(\frac{x}{\tau})$ in Eq.3.68. This is evaluated thanks to a code kindly provided by the authors of [89].
- The m_b-finite piece $\left[p + p \to t + \bar{b}\right]_{\alpha_s,5f}$ which is nothing but the usual Single Top cross-section obtained with 5-flavour SF, i.e. the $F^g_i(m_t, 0)$ in Eq.(3.68). This is numerically evaluated with a Fortran stand-alone code.

Thus, the final formula we use to extract Δ_{m_b,α_s} is

$$\Delta_{m_b,\alpha_s} = \sigma\left(p+p\to t+\bar{b}\right)_{\alpha_s,4f} - \sigma\left(p+p\to t+\bar{b}\right)_{\alpha_s,5f} - \mathcal{L}_{\alpha_s}\left(m_b,\mu_F\right), \quad (3.69)$$

where we set $\mathcal{L}_{\alpha_s}\left(m_b,\mu_F\right) = \log\left(\frac{\mu_F^2}{m_b^2}\right)\int_x^1 P_{qg}(\tau)g(\frac{x}{\tau})$. As expected, the m_b power-corrections are really small, of the order of 1%.

m_t	$F_{5f}^{1,g}(m_t,0)+\mathcal{L}_{\alpha_s}\left(m_b,\mu_F\right)$	$F_{4f}^{1,g}(m_t,m_b)$	$\Delta_{m_b}^1$
5.	0.580837E+02	0.594740E+02	1.3903
10.	0.450045E+02	0.440248E+02	0.9797
20.	0.307583E+02	0.302614E+02	0.4969
50.	0.136691E+02	0.135635E+02	0.1056
100.	0.504509E+01	0.502624E+01	0.01885
172.	0.170256E+01	0.169942E+01	0.00314
200.	0.119239E+01	0.119040E+01	0.00199
250.	0.671987E+00	0.671188E+00	0.000799
300.	0.402754E+00	0.402380E+00	0.000374
m_t	$F_{5f}^{2,g}(m_t,0)+\mathcal{L}_{\alpha_s}\left(m_b,\mu_F\right)$	$F_{4f}^{2,g}(m_t,m_b)$	$\Delta_{m_b}^2$
5.	0.327305E+03	0.333435E+03	6.13
10.	0.257184E+03	0.262072E+03	4.888
20.	0.190570E+03	0.193535E+03	2.965
50.	0.107625E+03	0.108575E+03	0.95
100.	0.552100E+02	0.554425E+02	0.2325
172.	0.264455E+02	0.265048E+02	0.0593
200.	0.206773E+02	0.207108E+02	0.0335
250.	0.138272E+02	0.138464E+02	0.0192
300.	0.958540E+01	0.960012E+01	0.01472
m_t	$F_{5f}^{3,g}(m_t,0)+\mathcal{L}_{\alpha_s}\left(m_b,\mu_F\right)$	$F_{4f}^{3,g}(m_t,m_b)$	$\Delta_{m_b}^3$
5.	0.696493E+00	-0.735704E+00	-1.432197
10.	0.850068E+01	0.706911E+01	-1.43157
20.	0.120115E+02	0.113304E+02	-0.6811
50.	0.905847E+01	0.889955E+01	-0.15892
100.	0.434350E+01	0.431251E+01	-0.03099
172.	0.172013E+01	0.171415E+01	-0.00598
200.	0.125248E+01	0.124897E+01	-0.00351
250.	0.744482E+00	0.742965E+00	-0.001517
300.	0.464222E+00	0.463485E+00	-0.000737

Table 3.2: m_b power-corrections at $\mathcal{O}(\alpha_s)$ for Single Top in t-channel.

At $\mathcal{O}(\alpha_s^2)$, the argument proceeds exactly the same way, but in this case the m_b logarithmic correction has a more involved structure. So, by generalizing Eq.(3.67), we can write

$$F^g_{i_{\alpha_s^2}}(m_t, m_b) \stackrel{m_b \to 0}{\to} F^g_{i_{\alpha_s^2}}(m_t, 0) + \mathcal{L}_{\alpha_s^2}\left(m_b^2, \mu_F^2, \mu_R^2\right) + \Delta_{m_b,\alpha_s^2}, \tag{3.70}$$

where $F^g_{i_{\alpha_s^2}}(m_t, m_b)$ and $F^g_{i_{\alpha_s^2}}(m_t, 0)$ are respectively the 4F- and 5F- structure functions, $\mathcal{L}_{\alpha_s^2}\left(m_b^2, \mu_F^2, \mu_R^2\right)$ are the corrections depending logarithmically on m_b at order α_s^2 and Δ_{m_b,α_s^2} is the desired m_b power-correction.
The practical formula we will use to extract numerically Δ_{m_b,α_s^2}

$$\begin{aligned}\Delta_{m_b,\alpha_s^2} =& \sigma(p+p \to q' + t + \bar{b} + X)_{\alpha_s^2,4f} - \sigma(p+p \to q' + t + \bar{b})_{\alpha_s,4f} \\ &- \sigma(p \to q' + W^*)_{\alpha_s,5f} \times \sigma(W^* + p \to t + \bar{b})_{\alpha_s,4f} \\ &- \mathcal{L}_{\alpha_s^2}\left(m_b^2, \mu_F^2, \mu_R^2\right) - \sigma\left(p+p \to q' + t + \bar{b} + X\right)_{\alpha_s^2,5f}. \end{aligned} \tag{3.71}$$

Eq.(3.71) is a bit more complex than the equivalent at order α_s (Eq.(3.69)), due to the more involved structure of the cross-section at order α_s^2. Indeed, in order to isolate the m_b power-corrections Δ_{m_b,α_s^2} affecting the massive current, we need to subtract two extra pieces,

- the cross-section in 4-flavour at order α_s, namely $\sigma(p+p \to q' + t + \bar{b})_{\alpha_s,4f}$, which is included by default in MCFM result up to the next perturbative order $\sigma(p+p \to q' + t + \bar{b} + X)_{\alpha_s^2,4f}$;
- the contribution given by the product $\sigma(p \to q' + W^*)_{\alpha_s,5f} \times \sigma(W^* + p \to t + \bar{b})_{\alpha_s,4f}$ of the two $\mathcal{O}(\alpha_s)$ currents, which is indeed affected by m_b power-corrections, but of order α_s, namely Δ_{m_b,α_s}.

Once we will have obtained the exact 5-flavour SF with our analytical computation, we will be able to use them in Eq.(3.71), thus achieving the isolation of Δ_{m_b,α_s^2}.

This concludes the chapter and the presentation of the general framework our computation takes place within. Starting from the next Chapter, we will illustrate in detail the analytical techniques we used to carry it out.

Chapter 4

Master Integrals techniques

4.1 Fundamentals of Feynman integrals and Master integrals

Feynman integrals appear quite naturally in elementary particle physics when quantities such as scattering amplitudes or cross-sections are computed within the framework of perturbation theory beyond the lowest order. The problem of being able to compute Feynman integrals is fundamental to make a bridge between theory and experiments, nowadays even more than in the past. Indeed, the high energies reached by the LHC have opened the era of precision physics, both on the experimental and theoretical side. To achieve this precision goal from the theoretical point of view, computation of physical observables to higher order in perturbation theory is required, thus leading to the crucial problem of evaluating Feynman integrals arising from complicated multi-loop and multi-leg processes.
In this section the definition and fundamental properties of Feynman integrals will be presented ([115]).

4.1.1 Feynman integral definition and properties

Any perturbative physical observable can be written at any given order in perturbation theory as a sum over Feynman graphs, often called also 'diagrams'. To each diagram Γ we can associate, through Feynman rules, a Feynman amplitude

$$G_{\Gamma}\left(q_1, ..., q_{n+1}\right) = (2\pi)^4 i\delta\left(\sum_{i=1}^{n+1} q_i\right) F_{\Gamma}\left(q_i, ..., q_n\right) \tag{4.1}$$

where $q_1, ..., q_n$ are the independent *external momenta*. From a mathematical point of view, external momenta are those flowing into the external legs of the diagram, which correspond in a physical picture to the momenta associated to physical particles involved into the process (both ingoing and outgoing). Through this definition we distinguish external momenta from internal momenta, which are in turns the ones that, mathematically speaking, flow into internal lines of the Feynman graph, thus corresponding to virtual particles produced and annihilated in the 'black box' of the process. We will call from now on *loop momenta* a set of all independent internal momenta.

A Feynman amplitude $F_\Gamma\left(q_1, ..., q_n\right)$, which is a function of the independent external momenta, can be written as a sum of integrals over loop momenta, each one of which is a 'Feynman integral'. In principle integrals at this stage can be tensor integrals, meaning they have free Lorentz indices, but when taking the square modulus of Feynman amplitudes, or more in general when contracting them with appropriate projectors to extract physical predictions, they are turned into scalar integrals. For practical purposes, from now on we will always assume we are dealing with scalar integrals.

In momentum-space language, Feynman integrals are defined as integrals over 4-dimensional space d^4k (or d-dimensional space d^dk, after we will have introduced Dimensional Regularization) over integrands which are rational functions of external and internal momenta given by products of Feynman propagators raised to some powers (either positive or negative). The generic (4-dimensional) Feynman integral thus admits the following representation

$$F(a_1, ..., a_n) = \int ... \int \frac{d^4k_1 ... d^4k_h}{E_1^{a_1} ... E_N^{a_N}} \tag{4.2}$$

where $k_i, i = 1, ..., h$ are loop momenta, a_i are integer indices, and the denominators are given by

$$E_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i \cdot p_j - m_r^2, \tag{4.3}$$

with $r = 1, ..., N$. Momenta p_i are either independent internal (loop) momenta ($p_i = k_i$, $i = 1, ..., h$), or independent external momenta q_i.

The matrix A in Eq.(4.3) normally leads to quadratic propagators, but in some cases also linear propagators can appear. This is for instance the case of propagators left after the expansion of a given Feynman integral in a specific kinematic limit, as will become clear in section 4.3.

We postpone the discussion of how a given amplitude can be written as a sum of such Feynman integrals to the next section, where it will be presented within the frame of the reduction of such amplitude to master integrals.

From now on the definition of Feynman integrals Eq.(4.2) will be our starting point to discuss about Feynman integral properties.

Divergences

Many 4-dimensional Feynman integrals are ill-defined because divergences can arise from one or more regions of the integration domain. Divergences can be of different types. The *ultraviolet* (UV) divergences are those coming from the region of large loop momenta. The degree of UV divergence of an integral can be easily determined via power-counting. Let's take for instance the generic bubble in four dimensions

$$B(q) = \int \frac{d^4k}{[k^2 - m_1^2]^{a_1}[(q-k)^2 - m_2^2]^{a_2}} \tag{4.4}$$

with unit exponents $a_1 = a_2 = 1$. If we send k to infinity, so that k is much larger than any other parameter, namely $k^2 \gg m_1^2$, $k^2 \gg m_2^2$, $k^2 \gg k \cdot q \gg q^2$, we get the asymptotic behaviour

$$B(q) \overset{k\to\infty}{\to} \int \frac{d^4k}{k^4} \tag{4.5}$$

which leads to a logarithmic divergence. Similar power-counting arguments lead to the formula which gives the UV degree of divergence ω of a generic integral

$$\omega = 4h - 2L + \sum_l n_l \tag{4.6}$$

with L being the number of internal momenta, h the number of independent internal (loop) momenta and n_l the degree of the polynomial appearing at numerator. The integral is UV convergent if $\omega < 0$, while it will be logarithmic, linear, quadratic,... UV divergent when $\omega = 0, 1, 2, ...$ respectively.

The other category of divergences that can occur in Feynman integrals are the so-called *infra-red* (IR) divergences. They are generated by regions of the integration domain where loop momenta become small or parallel to certain external momenta. We can distinguish different types of IR divergences: *soft* (*off-shell*, *on-shell* or *threshold*) divergences and *collinear* divergences.
Soft off-shell IR divergences arise when external momenta can assume general value (without being on the mass-shell or at threshold) and the integration momenta become very small. If we take for instance again the generic 1-loop bubble Eq.(4.4) with $a_1 = 2$, $a_2 = 1$, $m_1 = 0$, $m_2 \neq 0$ and $q^2 \neq m_2^2$, we get for small k

$$B(q) \overset{k\to 0}{\to} \frac{1}{q^2 - m_2^2} \int \frac{d^4k}{(k^2)^2} \tag{4.7}$$

which gives again a logarithmic divergent result. There are similarities between the properties of UV and off-shell IR divergences. Also in the latter case, one can define a formula which gives the degree of divergence of the integral. This kind of divergences is absent in theories which actually describes physical phenomena, though they can appear in expansions of Feynman integrals in particular kinematic limits.

Soft on-shell IR divergences appear when external momenta are on mass-shell or at threshold and the Feynman diagram happens to be singular in these particular configurations. We can use again Eq.(4.4) with $a_1 = 1$, $a_2 = 2$, $m_1 = 0$, $m_2 \neq 0$ and we take q on the mass-shell as $q^2 = m_2^2$

$$B(q) = \int \frac{d^4k}{(k^2)(k^2 - 2k \cdot q)^2} \stackrel{k \to 0}{\to} \frac{1}{4} \int \frac{d^4k}{(k^2)(k \cdot q)^2}. \tag{4.8}$$

Once more this integral has clearly a logarithmic divergent behaviour which would be not be present if $q^2 \neq m_2^2$.

The 1-loop bubble can also give rise to a soft threshold divergence if we choose $a_1 = a_2 = 2$, $m_1 = m_2 = m$ and we take the external momentum q at threshold $2m$, namely $q^2 = 4m^2$. We start from

$$B(q) = \int \frac{d^4k}{[k^2 - m^2]^2[(q-k)^2 - m^2]^2} \tag{4.9}$$

and if we make shift of the loop momentum $k \to k + \frac{q}{2}$ (the integration measure is invariant under translations), we get

$$B(q) = \int \frac{d^4k}{[(k + q/2)^2 - m^2]^2[(q/2 - k)^2 - m^2]^2} \tag{4.10}$$

$$= \int \frac{d^4k}{[k^2 + q \cdot k]^2[k^2 - q \cdot k]^2} \stackrel{k \to 0}{\to} \int \frac{d^4k}{(q \cdot k)^4} \tag{4.11}$$

which is logarithmic divergent.

Last but not least, collinear divergences may arise when a loop momentum becomes collinear or anti-collinear to an external one. We can make an example by taking the 1-loop triangle with massless internal propagators and massless independent external momenta p_1 and p_2.

$$T(p_1, p_2) = \int \frac{d^4k}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2}. \tag{4.12}$$

This integral has an on-shell soft divergence for $k \to 0$, with all components scaling to 0 in the same way. But, at the same time, it also has a divergence for $k \neq 0$ but $k^2 \sim 0$ and k is parallel to either p_1 or p_2, provided that p_1 and p_2 be light-like. Indeed, if $k^2 \to 0$, the integrand goes like $1/((p_1 \cdot k)(p_2 \cdot k)k^2)$. If we take then k collinear to say p_1, we will get $p_1 \cdot k = |\mathbf{p}||\mathbf{k}|(1 - \cos\theta)$ with $\theta \to 1$. Now, if we add the integration measure rewritten in spherical coordinates we get a behaviour $d\cos\theta/(1 - \cos\theta) \sim d\theta/\theta$ which gives a divergent logarithm.

This concludes this brief presentation of the most important and common kinds of divergences that Feynman integrals can exhibit. In the next section we present parametric representations for Feynman integrals, which constitute a powerful tool not only for integration itself but also for the analyses of these divergences.

Parametric representations, Regularization and Properties of Feynman Integrals

Feynman integrals admit parametric representations which allow to transform them into multi-dimensional integrals over a certain number of one-dimensional positive-valued integration variables. The bulk of these representations is given by the so-called *α-representation* (or *Schwinger representation*), which sees a scalar[1] Feynman propagator in momentum space $\tilde{D}_l(p)$ rewritten as an integral over a parameter as follows

$$\tilde{D}_l(p) = \frac{i}{(p^2 - m^2 + i0)^{a_l}} \tag{4.13}$$

$$= i\frac{(-i)^{a_l}}{\Gamma(a_l)} \int_0^\infty d\alpha_l \alpha_l^{a_l - 1} e^{i(p^2 - m^2)a_l} \tag{4.14}$$

which becomes for a $a_l = 1$

$$\tilde{D}(p) = \int_0^\infty d\alpha e^{i(p^2 - m^2)\alpha}. \tag{4.15}$$

Now, to obtain the representation for a generic (scalar) integral, one starts by replacing propagators with their representation

$$F_\Gamma(q_1, ..., q_n) = \int ... \int \frac{d^4k_1 ... d^4k_L}{E_1^{a_1} ... E_N^{a_N}} \tag{4.16}$$

$$= \int d^4k_1 ... \int d^4k_L \frac{i^{-(a_1 + ... a_N)}}{\Gamma(a_1)...\Gamma(a_N)} \times$$

$$\times \int_0^{+\infty} d\alpha_1 ... \int_0^\infty d\alpha_N \alpha_1^{a_1 - 1} ... \alpha_N^{a_N - 1} e^{i(\alpha_1 E_1 + ... + \alpha_N E_N)}. \tag{4.17}$$

Then, one considers the argument of the exponential, which is a function of external momenta q_i and loop momenta k_i and rewrites it as

$$C = (\alpha_1 E_1 + ... + \alpha_N E_N) = k_i M_{ij} k_j - 2Q_j k_j + J \tag{4.18}$$

where M is an $L \times L$ matrix, $Q(\alpha_i, q_j)$ is an L-dimensional vector and $J = J(\alpha_i \alpha_j, m_i^2, q_i, q_j)$ is a scalar function.

[1] We discuss here parametric representation of the only scalar Feynman integrals, since this is what we will actually need in the following for our computation (see Chapter 4).

By performing some manipulations, one manages to rewrite the integrals over the loop momenta k_i in Eq.4.17 as Gaussian integrals, which can be then easily carried out. The final result for the α-parametrization thus reads

$$F_\Gamma(q_1, ..., q_n) = \frac{i^{L-(a_1+...a_N)} \det(M)^{-2} \pi^{L/2}}{\Gamma(a_1)...\Gamma(a_N)} \int \prod_{j=1}^{N} d\alpha_j \alpha_j^{a_j-1} e^{-i\frac{\mathcal{F}(\alpha)}{\mathcal{U}(\alpha)}}, \tag{4.19}$$

with $\mathcal{U}$ and $\mathcal{F}$ being strictly related to the function C as follows

$$\mathcal{U} = \det(M), \quad \mathcal{F} = -\det(M)J + QM^TQ. \tag{4.20}$$

Practically speaking $\mathcal{U}$ and $\mathcal{F}$ are polynomials in the α-parameters of homogeneous degree respectively equal to L and $L+1$ [2].

The representation through α-parameters is interchangeable with another one, which can be obtained with a simple change of variables and goes under the name of *Feynman representation.*

$$\begin{aligned} F_\Gamma(q_1, ..., q_n) = & \frac{(-1)^{a_1+...+a_N} \pi^{L/2} \Gamma(a_1 + ... + a_N - L/2)}{\prod_{i=1}^{N} \Gamma(a_i)} \\ & \times \int \prod_{j=1}^{N} dx_j x_j^{a_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{\mathcal{U}(x)^{a_1+...+a_N-2(L+1)}}{\mathcal{F}(x)^{a_1+...+a_N-2L}} \end{aligned} \tag{4.21}$$

As we already pointed out, Feynman integrals can be ill-defined, since some region of the integration domain can lead to a divergent result. To deal with this, usually such integrals are *regularized* through the introduction of a *regularization parameter* at the integrand level, such that the final result of the integration is a well-defined function of this parameter and the initial divergence is recovered by taking a specific limit of the parameter. Thus the purpose of regularization is that of making the divergence manifest itself as a singularity of the regularization parameter, so that it can be isolated and easily handled.
A number of different types of regularization are available on the market, but we will take into consideration in this context only the one that we used extensively in our computation, which is *dimensional regularization (DR).* This technique consists of promoting the number of dimensions from 4 to $d = 4 - 2\epsilon$. Roughly speaking, what happens is that the final result of the integration will be a well-defined function of the regularization parameter ϵ, which, if expanded around the physical limit $\epsilon \to 0$, will contain poles $1/\epsilon, 1/\epsilon^2$,etc..... representing the divergences of the original integral.

[2]For computation purposes, we point the reader who might want to make use of this useful representation to the `Mathematica` package `UF.m` by A.Smirnov which compute the $\mathcal{U}$ and $\mathcal{F}$ polynomial automatically and which has been successfully used in our project.

In particular, if d is taken to be a complex-valued variable the main feature of dimensional regularization turns out to be the proof that Feynman integrals are analytic functions of ϵ in the complex plane! This gives the possibility to always perform an expansion of the result as a Laurent series around $\epsilon \sim 0$, thus recovering the poles in ϵ mentioned above. When we promote the dimension from 4 to d, the derivation of the α and Feynman representations does not change much: the only difference is that one should now perform d-dimensional Gaussian integrals over the loop momenta, which indeed can be done with no effort. The final results for such representations in d-dimensions read

$$F_\Gamma(q_1,...,q_n) = \frac{i^{L-(a_1+...a_N)}\det(M)^{-d/2}\pi^{Ld/2}}{\Gamma(a_1)...\Gamma(a_N)} \int \prod_{j=1}^{N} d\alpha_j \alpha_j^{a_j-1} e^{-i\frac{\mathcal{F}(\alpha)}{\mathcal{U}(\alpha)}}, \quad (4.22)$$

$$F_\Gamma(q_1,...,q_n) = \frac{(-1)^{a_1+...+a_N}\pi^{Ld/2}\Gamma(a_1+...+a_N-Ld/2)}{\prod_{i=1}^{N}\Gamma(a_i)} \times \int \prod_{j=1}^{N} dx_j x_j^{a_j-1}\delta\left(1-\sum_{i=1}^{N}x_i\right)\frac{\mathcal{U}(x)^{a_1+...+a_N-d(L+1)/2}}{\mathcal{F}(x)^{a_1+...+a_N-dL/2}}. \quad (4.23)$$

The parametric representations Eq.(4.22) and Eq.(4.23) are a useful tool not only for integration itself of Feynman integrals, but also for the analyses of their convergence and for their expansion in given kinematic limits. We will made extensive use in particular of Eq.(4.23) in our computation, as it will be shown in the next chapter.

In the following we briefly state some important properties of Feynman integrals, without providing any proof since this would go beyond the purpose of the present thesis. The first three properties are general to all Feynman integrals, regardless of the adopted regularization scheme, whereas the last two of them are specific to Feynman integrals regularised in DR.

1. Scalar integrals are invariant under the *Poincaré group*. This implies that the integral is a function of only those Lorentz-invariants that can be built starting from external momenta.

2. Integrals transform in a covariant way under *dilatations*. This means that one can always extract the dimension in powers of energy of the integral and thus deal with a dimensionless integral which will be a function of dimensionless ratios of those independent kinematic quantities that can be built stating from independent external momenta.

3. *Lorentz invariance identities* (LI): given a scalar integral $F(q_1,...,q_E)$ de-

pending on E external momenta, the following identities hold

$$\sum_{i=1}^{E} \left(q_i^{\nu} \frac{\partial}{\partial q_{i_\mu}} - q_i^{\mu} \frac{\partial}{\partial q_{i_\nu}} \right) F(q_1, ..., q_E) = 0. \tag{4.24}$$

They follow from the invariance of the integral under Lorentz transformation of the external momenta $q_i^{\mu} \to q_i^{\mu} + \epsilon_{\nu}^{\mu} q_i^{\nu}$.

4. *Integration by parts Identities* (IBPs), namely the possibility of integrating by parts and always neglecting surface terms.

$$\int d^d k_1 ... \int d^d k_L \, \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(k_1^2 - m_1^2)^{a_1} ... (k_L^2 - m_L^2)^{a_L}} \right] = 0$$

The formal argument for which this identities hold relies on the fact that this is the integral in d-dimensions of the total divergence of the vector in curly brackets. It can then be transformed into the flux of this vector over a spherical surface with radius $r \to \infty$ and which goes to zero. A less formal but more intuitive argument is the following: the integral is invariant under translations along the directions of the vector k_i

$$\int d^d k_i f(k_i) = \int d^d (k_i + \epsilon) f(k_i + \epsilon) \tag{4.25}$$

but the partial derivative with respect to vector k_i is the generator of translations along the direction of this vector

$$f(k_i + \epsilon) = f(k_i) + \epsilon \frac{\partial}{\partial k_i} f(k_i) + ... \tag{4.26}$$

so the generator of shift must integrate to zero!
In any case, for the complete proof of IBPs we send the reader to the paper [121] where they were originally discussed.

5. Any scaleless integral in DR is zero.
This property can be proved by use of an auxiliary analytic regularization, using pieces of the integral in α representation considered in different domains of the regularization parameters.
We can consider as an example the massless tadpole diagram, which can be reduced by means of α parameters to a scaleless one-dimensional integral

$$\int \frac{d^d k}{k^2} = -i^{\epsilon} \pi^{d/2} \int_0^{\infty} d\alpha \alpha^{\epsilon - 2}. \tag{4.27}$$

This integral is divided into two parts, from 0 to 1 and from 1 to ∞ and these two parts are integrated. Results are found to be equal except for opposite signs, which lead to zero value. An important subtlety in this procedure should be pointed out. These two pieces converge in different regions of the DR regulator ϵ but DR guarantees that each one of them can be extended through analytic continuation to the entire ϵ axis, so that the result effectively sums up to zero.

For a complete treatment of Dimensional Regularization and its properties we send the reader to the original paper [119] where DR was first introduced by Veltman and 't Hooft and to the paper from Wilson [129], which followed immediately afterwards.

4.1.2 Master Integrals: definition and overview

When computing higher order perturbative corrections to physical quantities, a large number of apparently different integrals appears. In particular, increasing the number of loops and/or external legs, more and more different dimensional scales enter the description of the final result, thus introducing more and more difficulties in the evaluation of the integrals.
In the framework of dimensional regularization, many powerful techniques have been developed in order to make the computation of such corrections feasible. The main goal of such techniques is that of sensibly reducing the number of Feynman integrals that need to be computed explicitly, by relating the original huge amount of integrals needed to describe the process to a much smaller set. Indeed, IBPs identities (plus some other types of identities that will be presented later on in this thesis) can be used to establish a number of relations between the integrals. These relations turn out to be simple linear equations which involve the integrals and only rational functions of the invariants and of the dimensional regularization parameter ϵ. Solving this system of equations finally allows one, as anticipated above, to express most of the integrals in terms of a relatively small subset of irreducible integrals, the so-called *Master Integrals*, which are then to be computed explicitly.
So, given the problem of computing contribution at a given order in perturbation theory to an observable, we can give the following definition of Master Integrals.

Master Integrals form a set of integrals such that any other Feynman integral appearing in the problem can be expressed as a linear combination of the elements of this set. This set thus satisfies the requirements of being closed with respect to IBPs and minimal.

In other words, a set of Master Integrals is a solution of the IBPs system of equations such that

- its elements cannot be further reduced by applying IBPs (i.e. the set is *closed under IBPs operation*),

- the number of elements is the minimal one necessary in order for the previous property to hold.

Consequently, master integrals (MIs) have all the properties of standard Feynman integrals, plus those that derive from the above definition. We would like to stress that, given a system of IBPs identities, one can usually determine an ensemble of equivalent solutions for it. This implies that one always has the freedom to change the set of MIs and this amounts to simply a change of basis in the space of the IBPs system solutions.
Roughly speaking, the computation of a given observable at a given order in perturbation theory, is thus naturally split into two parts

- the reduction of the original set of integrals coming from the matrix elements to a set of MIs as small as possible,
- the explicit computation of the MIs.

A lot of effort has been made in the past twenty years to improve all this machinery.
In non-trivial applications, such as two-loop corrections, the system of equations one has to solve to carry out the reduction can easily grow to include hundreds or thousands of equations, so that one must resort to the use of computer algebra. In the last years many public and private implementations for the automatic reduction to master integrals have become available, many of them relying on the Laporta algorithm [87].
After the problem of reduction is completed, one is left with that of solving the masters. Traditional techniques, such as α- , Feynman or Mellin-Barnes parametric representation are of course available to solve each of the master integrals. But, as the number of loops and external legs increases, also the number of propagators does and these traditional methods are not feasible anymore. Beside these traditional techniques at the beginning of the nineties the method of differential equations has been proposed and over the years has proven to be very powerful in a large number of computations. In this method, differential equations for the integrals under consideration are derived, by differentiating at the integrand level with respect to external scales upon which the integrals depend. The master integrals are then determined by solving such differential equations and matching them to appropriate boundary conditions.
The rest of this chapter will be dedicated to go more into detail in the master integral technique, with particular emphasis on those techniques that we used extensively in our computation and that cannot yet be carried out automatically, namely the solution of masters through differential equations. Among the next three sections, the first will be dedicated to reduction, the second to methods for solving the masters and the third will focus in particular on some recent ideas developed in the field of solution via differential equation, which has been exploited in the present project.

4.2 Reduction of Scalar Matrix Elements to Master Integrals

Let us imagine we want to compute the contribution to a certain order in perturbation theory to the inclusive cross-section for a particular physical process via master integral techniques. Which steps should one take to obtain the set of masters describing such quantity from scratch? This section will be entirely dedicated to describe how a set of master integrals is obtained within the framework of computation of perturbative quantum corrections to an observable.

4.2.1 Topologies and classification of scalar Feynman Integrals

We assume that the reader is already familiar with matrix elements generation for a given process, so we just sketch how it works without entering into details. First Feynman graphs are generated. For processes with a large number of loops/legs, and thus described by a large number of graphs, this happens normally through the help of an automatic generator. Then, a mathematical expression is associated to each graph through Feynman rules, thus giving what is usually called a Feynman diagram. At this stage our process is then described by a sum of Feynman diagrams, usually referred to as Feynman amplitude. For processes involving particles with spin different from zero, (amputated) amplitudes are tensor quantities, namely they have free Lorentz indices. These free Lorentz indices are then saturated when one takes the square modulus of the amplitude and contracts it either with the polarisation tensor or with suitable projectors in order to either compute a cross-section or extract form factors. When such contraction is performed a scalar quantity is obtained, which we will always refer to in the following as *scalar amplitude*.

Also we stress that in the following we will always work in Dimensional Regularization. Given a process $q_1 + q_2 \to q_3 + ... + q_N$ which depends on N external momenta $q_1, ..., q_N$ and L loop momenta $l_1, ..., l_L$, the scalar amplitude $\mathcal{A}$ at this point looks like

$$\mathcal{A}\left(q_1, ..., q_N; l_1, ..., l_L\right) = \int d^d l_1 ... \int d^d l_L \frac{\mathcal{N}\left(q_i \cdot q_j, q_i \cdot l_j, l_i \cdot l_j\right)}{E_1^{a_{E_1}} ... E_M^{a_{E_M}}}, \tag{4.28}$$

where the numerator in the integrand is a polynomial in all the possible scalar products that can be built out of the independent internal and external momenta whereas the denominator is a product of M inverse propagators Eq.(4.3).

In order to obtain quantities with a physical meaning, namely inclusive cross-sections or form factors, one has then to integrate Eq.(4.28) over the Phase Space for the $N-2$ particles $q_3, ..., q_N$ in the final state.

$$\sigma = \int d^d q_3 ... \int d^d q_N \delta_+(q_3^2 - m_3^2) ... \delta_+(q_N^2 - m_N^2) \delta^{(d)}(q_1 + q_2 - (q_3 + ... + q_N)) \times$$
$$\times \ \ \mathcal{A}(q_1, ..., q_N; l_1, ..., l_L). \tag{4.29}$$

Even if it is not necessary for the following discussion, we would like to consider starting from now the most general case in which the technique of master integrals can be used. Indeed, this technique was developed at the beginning with the purpose of dealing with integrations over the only loop momenta, meaning that it was meant to be applied to perform integrations at level of scalar amplitudes Eq.(4.28), thus leaving phase-space integrations to traditional techniques.
Later on, the possibility to treat phase space integrals as loop integrals was introduced, consequently leading to the idea of applying master integrals to both loop and phase-space integrations [**?**]. This idea relies on what is commonly called *reverse unitarity*, which essentially consists in using the Cutkosky rule to replace *delta*-functions by differences of propagators

$$2i\pi\delta(p^2 - m^2) \rightarrow \frac{1}{p^2 - m^2 + i0} - \frac{1}{p^2 - m^2 - i0} \equiv \left(\frac{1}{p^2 - m^2}\right)_c. \tag{4.30}$$

Leaving details for the following subsection, for the moment we can say that this allows us to substitute δ-functions with *cut* propagators in the phase space measure, and, consequently, to treat phase space integrals as loop integrals. Once in Eq.(4.29) we have performed one phase space integral with the momentum conservation δ-function and we have rewritten the remaining on-shell δ-functions using reverse unitarity,

$$\sigma = \int d^d q_3 ... \int d^d q_{N-1} \times$$
$$\times \left(\frac{1}{q_3^2 - m_3^2}\right)_c ... \left(\frac{1}{(q_1 + q_2 - (q_3 + ... q_{N-1}))^2 - m_N^2}\right)_c \ \mathcal{A}(q_1, ..., q_N; l_1, ..., l_L). \tag{4.31}$$

If we now substitute the explicit expression for the scalar amplitude Eq.(4.28), we get

$$\sigma = \int \prod_{i=3}^{i=N-1} d^d q_i \int \prod_{j=1}^{L} d^d l_j \times$$
$$\times \left(\frac{1}{q_3^2 - m_3^2}\right)_c ... \left(\frac{1}{(q_1 + q_2 - (q_3 + ... q_{N-1}))^2 - m_N^2}\right)_c \ \frac{\mathcal{N}(q_i \cdot q_j, q_i \cdot l_j, l_i \cdot l_j)}{E_1^{a_{E_1}} ... E_M^{a_{E_M}}}. \tag{4.32}$$

In Eq.(4.32) it is eventually clear that from this stage on loop and phase space integrations can be treated in the same way, so that we can think of applying master integrals technique not only at level of scalar amplitude (Eq.4.28) but also at level of cross-section (Eq.4.32).
Keeping this in mind, from now on we will address the general mathematical issue of reducing to master integrals a quantity which has the following features:

- it is a function of a number E of 'external' momenta $q_1, ..., q_E$
- it can be generically written as the integral over a number I of 'loop' momenta $k_1, ..., k_I$ of a certain integrand
- the integrand has the generic form of a polynomial in all the possible scalar products that can be built using momenta $q_1, ..., q_E, k_1, ..., k_I$ over a denominator given by products of inverse propagators.

$$A(q_1, ..., q_E) = \int \prod_{p=1}^{I} d^d k_p \frac{\mathcal{N}\left(q_i \cdot q_j, q_i \cdot k_j, k_i \cdot k_j\right)}{E_1^{a_{E_1}} ... E_M^{a_{E_M}}} \tag{4.33}$$

This generic quantity A will have the physical meaning of either a scalar amplitude or a cross-section depending on the precise physical observable one might want to compute.

We start now manipulating Eq.(4.33) as follows. First, we expand A in order to write it as the sum of integrals each one over a ratio having at the numerator a monomial in the scalar products and at the denominator the product of propagators, which are actually scalar Feynman integrals.

$$A(q_1, ..., q_E) = \sum_i c_i\left(q_1, ..., q_E\right) \int \prod_{p=1}^{I} d^d k_p \frac{S_1^{a^i_{S_1}} S_2^{a^i_{S_2}} ... S_M^{a^i_{S_M}}}{E_1^{a_{E_1}} ... E_M^{a_{E_M}}}, \tag{4.34}$$

where

- we pulled the scalar products between external momenta outside the integrals and encoded them in the coefficients $c_i\left(q_1, ..., q_E\right)$ which have then the form

$$c_i\left(q_1, ..., q_E\right) = \prod_{l,m=1}^{E} \left(q_l \cdot q_m\right)^{a^i_{l,m}} \tag{4.35}$$

- we have indicated with $S_1, ..., S_M$ all the possible independent scalar products that one can construct of the type $q_i \cdot k_j$, $k_i \cdot k_j$ using all the independent external and loop momenta.

The number of all possible scalar product of the type $q_i \cdot k_j$, $k_i \cdot k_j$ is fixed by the formula

$$n_S = \frac{I(I+1)}{2} + I\,E. \tag{4.36}$$

The key observation now is that the amplitude has been written as a linear combination of scalar integrals (Eq.4.34) but the ensemble of these integrals is usually not linearly independent, meaning that we still have room to manipulate this expression and reduce the number of integrals appearing in this linear combination. The linear dependence is induced by the fact that factors appearing at denominator are built up themselves from scalar products. This implies two things. First, the number of linearly independent propagators $\{E_1, ..., E_M\}$ that we can build given I loop momenta and E external momenta is fixed by the expression fixing the number for scalar products, namely Eq.(4.36). Second, we can think of expressing each scalar product at numerator as a linear combination of inverse propagators $\{E_1, ..., E_M\}$. More in general, we can think of the sets $\{S_1, ..., S_M\}$ and $\{E_1, ..., E_M\}$ as two bases in the space of all possible linear combinations of scalar products. In this perspective expressing scalar products as linear combinations of a set of selected independent inverse propagators is nothing but a change of basis in this space. So, assuming that our scalar amplitude naturally provides a set of M linearly independent inverse propagators $\{E_1, ..., E_M\}$, we can perform this change of basis

$$S_i = \sum_{j=1}^{M} B_{ij}(E_j - b_j) \quad (j = 1, ..., 7) \tag{4.37}$$

where B and b are respectively an invertible $M \times M$ matrix and an M-dimensional vector whose elements just depend on the scalar products between external momenta $q_i \cdot q_j$ and internal masses m_i^2. We can now use this change of basis to simplify Eq.(4.34): in each term appearing in the sum we substitute scalar products at numerators with their decomposition Eq.(4.37), we expand again, perform all simplifications between numerator and denominator that may occur and finally obtain a scalar amplitude which is a linear combination of Feynman integrals written in terms of only inverse propagator belonging to the set $\{E_1, ..., E_M\}$

$$A(q_1, ..., q_E) = \sum_i c_i'(q_1, ..., q_E) \int \prod_{p=1}^{I} d^d k_p \prod_{j=1}^{M} E_j^{a^i_{E_j}}. \tag{4.38}$$

We can now introduce the concept of 'topology' which will be extensively used in the rest of this thesis and which states in a formal way the idea of sets of independent propagators as basis in a space. We define *topology* T a set $\{E_1, ..., E_M\}$ of inverse propagators E_i which is minimal and complete in the sense that any scalar

product of a loop momentum k_i with either a loop momentum k_j or an external momentum p_j can be uniquely expressed as a linear combination of the E_i and of the kinematic invariants [3].
Given our topology $T = \{E_1, ..., E_M\}$, we can classify all the scalar integrals appearing in the sum in Eq.(4.38) according to it. Indeed, once an order for the propagators in the set T is conveniently fixed, we can associate to each integral a M-dimensional vector whose entries correspond to indices (i.e. powers) of propagators E_i belonging to the topology. We establish this correspondence by adopting the following notation

$$\int \prod_{p=1}^{I} d^d k_p \prod_{j=1}^{M} E_j^{a^i_{E_j}} \rightarrow I[T, \{a^i_{E_1}, a^i_{E_2}, ..., a^i_{E_M}\}]. \tag{4.39}$$

We also define *subtopology* any subset T' of inverse propagator which can be constructed starting from a given original topology T. It is appropriate at this point to discuss an issue which concerns the identification of topologies. In our discussion so far, we took for granted that propagators appearing in the scalar amplitude naturally make up a topology, in a sense that we assumed them to be linearly independent and in a number equal to M with M given by Eq.(4.36). In real computations, this perfect scenario rarely happens. Usually one has to deal with two possible variations of this scenario, which might happen one at a time or also at the same time. First, the propagators might not be a linearly independent set. This problem can be solved with the help of *partial fractioning*. Let us consider the simplest situation, we have a set of M propagators from the scalar amplitude and three of them satisfy the following relation of linear dependence

$$E_i = E_j + E_k. \tag{4.40}$$

This means that this particular set of M propagators is not a topology. But we can create an identity in order to get rid of one of them in favour of the other two. Indeed, we have

$$1 = \frac{E_j + E_k}{E_i}. \tag{4.41}$$

Now, if we take the simplest 'problematic' integrand, namely

$$\frac{1}{E_i E_j E_k} \tag{4.42}$$

and apply to it the identity we get

$$\frac{1}{E_i E_j E_k} = \frac{E_j + E_k}{E_i} \frac{1}{E_i E_j E_k} = \frac{1}{E_i^2 E_k} + \frac{1}{E_i^2 E_j}. \tag{4.43}$$

[3] It must be stressed that a topology is defined only up to shifts of the loop momenta.

We have split our original integral containing the three dependent propagators in the sum of two new integrals which contain each just two of these propagators, namely no more propagators which are linearly dependent. At this point we are left with the task of assigning these integrals to suitable topologies. These are easily obtained by taking the original set M and replacing in turn propagators E_j and E_k with other *auxiliary* propagators constructed *ad hoc* to complete the set in such a way that linear dependence is satisfied. In this way we get from the initial set two new sets of M propagators, linear independent and such that they contain respectively the first and the second integrand that result from partial fractioning. Actually, partial fractioning can be more involved than this, but this example is meant just to convey the general philosophy of partial fractioning:

- exploit the linear dependence between propagators to construct identities that allow to build integrands containing just product of independent propagators
- manipulate the initial set of propagators by eliminating old propagators and adding new auxiliary ones such that in the end they give rise to topologies and such that these topologies actually contain the newly built integrands.

The other problem that might arise concerns the number m of propagators that are naturally contained in a scalar amplitude. It can be a priori either smaller or bigger than the number M given by Eq.(4.36) required to build a topology. In the first case $m < M$, it is sufficient to add one or more auxiliary propagators in order to complete the set in such a way to get a topology out of it. In the second case, $m > M$, the set of m propagators is not linearly dependent, so again we can exploit partial fractioning to reduce it to a smaller set of independent propagators. To conclude we might say that it is often necessary to play with both partial fractioning and auxiliary propagators to get to a form for the scalar amplitude as that described in Eq.(4.38). From now on, keeping in mind the framework of a general computation, we will assume that all the necessary in this sense has been done, so that we have arrived to a form of the amplitude as in Eq.(4.38) and in the next section we can start explaining how such expression is reduced to master integrals.

4.2.2 Identities between Feynman Integrals

In this subsection we will address the problem of further reducing a scalar amplitude in the form Eq.(4.38) by means of *Integration By Parts* (IBPs) identities [4]. Roughly speaking, the idea is to write down, using IBPs, a system of linear equations for scalar Feynman integrals belonging to a certain topology and solve this

[4] IBPs have been introduced in the previous section in the context of properties of dimensionally regularized Feynman integrals.

system with respect to a small set of integrals in terms of which all the others will be expressed through linear relations. Integrals belonging to this set found by solving IBPs systems of equations are what we will call from now on Master Integrals (MIs). MIs are thus characterized by the property of being *irreducible*, in a sense that they cannot be expressed as linear combinations of simpler integrals. To give an idea of how this works, we shall start from the simplest example one can think of, namely the massive tadpole for generic index of its only propagator (in the following we omit the dependence on invariants and dimension for brevity)

$$F(a) = \int \frac{d^d k}{(k^2 - m^2)^a}. \tag{4.44}$$

We apply the IBP identity to this integral

$$\int d^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^2} = 0 \tag{4.45}$$

with $\frac{\partial}{\partial k} \cdot k = \frac{\partial}{\partial k_\mu} \cdot k_\mu$. If we carry out derivatives we get

$$(d - 2a)F(a) - 2am^2 F(a+1) = 0 \tag{4.46}$$

which can be solved to give the following recurrence relation

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1), \tag{4.47}$$

telling us that any Feynman integral with integer $a > 1$ can be expressed recursively in terms of one integral $F(1)$ which we therefore consider as a master integral, namely an integral that cannot be further reduced, as follows [5]

$$F(a) = \frac{(-1)^a (1 - d/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1. \tag{4.48}$$

Let us pick up now a slightly more difficult example, namely the massless 1-loop bubble with generic indices a_1, a_2.

$$F(a_1, a_2) = \int \frac{d^d k}{(k^2)^{a_1} [(q-k)^2]^{a_2}}. \tag{4.49}$$

We apply IBP with respect to the integration momentum

$$\int d^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1} [(q-k)^2]^{a_2}} = 0 \tag{4.50}$$

[5]In the mathematical notation $(m)_n$ is the Pochammer symbol.

and obtain the following relation

$$(d-2a_1-a_2)F(a_1,a_2)-a_2F(a_1-1,a_2+1)+a_2q^2F(a_1,a_2+1)=0, \quad (4.51)$$

which can be rewritten as

$$a_2q^2F(a_1,a_2+1)=a_2F(a_1-1,a_2+1)+(2a_1+a_2-d)F(a_1,a_2). \quad (4.52)$$

If we now shift the index a_2 we can write

$$F(a_1,a_2)=-\frac{1}{(a_2-1)q^2}\left[(d-2a_1-a_2+1)F(a_1,a_2-1)-(a_2-1)F(a_1-1,a_2)\right]. \quad (4.53)$$

This relation relates an integral whose indices sum up to a_1+a_2 on the l.h.s. to integrals whose indices sum up to a_1+a_2-1 on the r.h.s., thus enabling us to reduce the sum of the indices a_1+a_2. From the denominators we see that this relation holds only for $a_2>1$. What if for instance $a_2=1$? Then we can use the symmetry property of this specific integral $F(a_1,a_2)=F(a_2,a_1)$ (which comes from the invariance of the integral under the shift $k\to q-k$) to exchange a_1 and a_2 in this relation and then set $a_2=1$. This gives us

$$F(a_1,1)=-\frac{d-a_1-1}{(a_1-1)q^2}F(a_1-1,1). \quad (4.54)$$

This last relation enables us to reduce the index a_1 to 1. So, Eq.(4.53) and Eq.(4.54) together enable us to express any integral of the given family in terms of the only master integral $F(1,1)$, so that

$$F(a_1,a_2)=c(a_1,a_2,d)F(1,1) \quad (4.55)$$

where C is a rational function of the regulator $d=4-2\epsilon$ and of the only scale of the integrals q^2.

We can extrapolate from this example what happens in a more general situation. By explicitly performing on the integrands derivatives appearing in the IBPs and expressing then scalar products appearing at numerators in terms of the chosen basis of inverse propagators $\{E_1,...,E_M\}$, one obtains identities of the form (we use the notation of Eq.(4.39))

$$\begin{aligned}0=&c\,I[T,\{a_{E_1},a_{E_2},...,a_{E_M}\}]+\sum_{i=1}^{M}a_{E_i}d_iI[T,\{a_{E_1},a_{E_2},...,a_{E_i}+k,...,a_{E_M}\}]\\&+\sum_{i\neq j}^{1,M}a_{E_i}e_{ij}I[T,\{a_{E_1},a_{E_2},...,a_{E_i}+k,...,a_{E_j}-p,...,a_{E_M}\}] \quad (4.56)\end{aligned}$$

involving three kinds of amplitudes

- the original scalar integral itself $I[T, \{a_{E_1}, a_{E_2}, ..., a_{E_M}\}]$;
- scalar integrals $I[T, \{a_{E_1}, a_{E_2}, ..., a_{E_i} + k, ..., a_{E_M}\}]$ with one of the indices increased by some integer number k,$a_{E_i} \to a_{E_i} + k$ ($i = 1, ...M$). These terms originate from the derivation of the factor $E_i^{a_{E_i}}$. Obviously these terms are absent for indices being equal to zero, meaning that it is impossible to generate an absent denominator by differentiation. This implies that IBPs applied to a scalar integral belonging to a certain subtopology T' can never produce scalar integrals belonging to a topology bigger than T';
- scalar integrals $I[T, \{a_{E_1}, a_{E_2}, ..., a_{E_i} + k, ..., a_{E_j} - p, ..., a_{E_M}\}]$ with one index increased by some integer k, $a_{E_i} \to a_{E_i} + k$ and another decreased by some other integer p, $a_{E_j} \to a_{E_j} - p$. These terms originate from the derivation of the factor $E_i^{a_{E_j}}$ together with the cancellation of a power of $E_j^{a_{E_j}}$ with inverse propagators generated at numerators.

The coefficients c,d_i,e_{ij} are functions of ϵ and of the invariants on which the original integral depends.
The IBPs identities constitute a system of linear equations in which the unknowns are the scalar integral themselves. The oldest approach, developed in the original article on the IBPs ([**?**],[**?**]), involves a symbolic solution of the identities, treated as recurrence equations in the indices. In the two examples we gave above one can actually see this kind of method at work, though just in the two easiest situations one can imagine. In the general case, one introduces operators raising or lowering one of the indices

$$\mathbf{i}^{\pm} I[T, \{a_{E_1}, ..., a_{E_i}, ..., a_{E_M}\}] = I[T, \{a_{E_1}, ..., a_{E_i} \pm 1, ..., a_{E_M}\}] \tag{4.57}$$

and tries to combine equations in such a way that a scalar integral is written in terms of amplitudes all containing lowering operators, so that reduction to simpler subtopologies can thus be achieved. The main disadvantage is that a careful, case-by-case, inspection of the equations is required. Also, when the system of equations starts becoming huge, solving for generic values of the indices might become cumbersome and in some cases impossible.
More recently, a new approach radically different from the previous one was proposed in [87]. This method, commonly known as the *Laporta algorithm*, does not attempt to solve systems of IBPs containing an *infinite* number of equations (i.e. for generic values of the indices of propagators) but systems made of a *finite* number of equations generated by specifying some carefully chosen values for the indices. Indeed, it is observed that in practice large values of the indexes a_i do not matter, so that one is able to truncate the initially infinite system of equations at certain, usually small, values of the a_i and thus deal with a finite system of linear equations. The system is solved using the well-known Gauss elimination method. The solution gives the expressions of the integrals as linear combinations of the

master integrals with rational coefficients in the DR parameter ϵ. This approach has the big advantage of being suitable for completely automatic calculations, since it does not require inspections of single identities one-by-one. On the other side, it may require a large CPU time and very long intermediate expressions may be generated, but with some tricks also the entity of these problems can be reduced. For a detailed description of the algorithm and example, we send the reader to the original paper where this was first published ([87]).
In our project we made extensive use of the `Mathematica` implemented version of this algorithm, which is delivered in the `FIRE` package [113]. In particular we used the latest version of the package [114], which allows for performing reduction both with `Mathematica` and with `c++`. However, this is not the only automatic code available for reduction to master integrals. Various implementations exist and the interested reader may find citation of a complete list of them in [76].

One last remark we would like to make is about the compatibility of *reverse unitarity*, mentioned in the previous subsection, with IBPs reduction [8]. Given that we are addressing the computation of a cross-section, once the phase space δ-functions are replaced by the difference of two cut propagators as in Eq.(4.30), we can apply IBPs two cut scalar integrals in the same way we would apply them to standard loop integrals. Indeed, the prescription for the imaginary part of the two propagators in the r.h.s. of Eq.(4.30) is irrelevant for the differentiation. Therefore the IBP relations for the two descendants of these two terms have the same form as the IBP relations for the original integral without the cut. It is then allowed to apply reverse unitarity, thus getting cut scalar integrals, and then apply to them IBP reduction. Reduction will work exactly the same way it works for loop integrals, with the exception that whenever a cut inverse propagator appear at numerator in an integral, that integral is zero (Indeed one has to remember that cut propagators are δ-functions representing on-shell constraints!).

At this stage, we can consider the problem of reduction of a given scalar amplitude to master integrals solved. In the next two sections, we will address the issue of computing the masters themselves.

4.3 Master Integrals Computation - Part 1

Once the reduction to master integrals is performed, one is left with the issue of computing explicitly the masters. In this section we shall address this topic. One has usually at disposal several methods to compute the masters. Among the traditional ones, there are α and Feynman parameters and Mellin-Barnes parametric representation. These methods have been used extensively over the years for one-loop and two-loop computations, but they have some disadvantages. When the number of external and/or internal lines start to grow, the number of possible inverse propagator appearing at denominator in the integrals increase. Since all these methods rely on some way of representing each Feynman propagator as an integral over a certain parameter, integrands involving many propagators give rise to multi-fold parametric integrals with a high number of dimensions (i.e. integration variables), which can easily become pretty cumbersome to be solved. Furthermore, these methods address the computation of each integral separately from the others, so that when the number of master integrals increases, typically in multi-loop/leg computation, the number of integrations to be carried out becomes really challenging.
In this kind of scenario, one might typically want to exploit properties of master integrals, namely the fact of being a closed irreducible set of integrals belonging to one or more topologies, to try to find a more efficient strategy to solve them.
An alternative method, which goes under the name of *Differential Equations*(DE), indeed uses such properties and has proved successful and very powerful since its development in the nineties. Since we heavily relied on this particular strategy for the computation of all master integrals in our project, the rest of chapter 4 will focus specifically on this strategy. In particular, we will review all the ingredients and basics of the method in the present section [6], whereas in the following one we will enter in the details of some quite recent further developments which make the DE method even more efficient and maybe pave the way towards automation of MIs computation.
Since we used very little of the 'traditional' methods in our computation, we decided not to give a full treatment of them.

4.3.1 Generation of Differential Equations for Master Integrals

Once the MIs have been identified, we can derive differential equations for them with respect to the external invariants of which they are functions by using again IBPs relations together with the fact that the MIs are a basis in the vector space of a certain category of Feynman integrals belonging to a given topology. Three steps must essentially be taken.

[6] [2], [125]

- One chooses which set of independent external invariants he/she wants to use to express results for the MIs (which will be then functions of these invariants).
- Then one derives differential operators in the external invariants by observing that derivatives with respect to them can be expressed as derivatives with respect to external momenta through the chain rule.
- The set of differential operators is applied to each master. This allows to express the derivative of a MI as a linear combination of other Feynman integrals belonging to the same topology or to its subtopologies. In order to understand this step, we recall from section 3.2, that derivatives of Feynman integrals, in this case with respect to external momenta, do not produce new propagators in the integrand denominator. In other words, derivatives of Feynman integrals with respect to external momenta will always produce as a result integrals which belong to the same topology or to subtopologies of the original integral.
- After applying derivatives, not all the integrals appearing at r.h.s are masters! IBP reduction is then applied to these integrals, thus reducing those that are not master integrals to linear combinations of these masters.
- As a result we end up with a system of first order, usually coupled, differential equations for the masters.

We shall illustrate how this works in practice with a very basic example, which will then be useful as part of the Single Top NLO computation reported in Chapter 4.

Let us take the 1-loop bubble with an internal mass m^2 and depending on one external momenta q. We take both indices of propagators to be equal to 1.

$$F(q^2, m^2, 1, 1, d) = \int \frac{d^d k}{(k^2 - m^2)(q-k)^2}. \tag{4.58}$$

This integral depends on two scales q^2 and m^2. However, we know from Section 3.1 that it is always possible to pull out from a Feynman integral its dimension in powers of energy and deal with dimensionless integrals. Since it is simpler to deal with dimensionless integrals, this is exactly what we want to do! So we make this integral dimensionless by multiplying it for the inverse power of its dimension in powers of m^2, which corresponds in this case to $m^{2\epsilon}$. The integral we obtain in this way is then a function of the only dimensionless ratios that we can build starting from the external invariants, in this case just $x = \frac{q^2}{m^2}$.

$$F^{ad}(x, 1, 1, \epsilon) = m^{2\epsilon} \int \frac{d^{4-2\epsilon} k}{(k^2 - m^2)(q-k)^2}. \tag{4.59}$$

We now turn to derive the differential operator which gives the derivative with respect to x as a function of the derivative with respect to m^2 by using the chain rule.

$$\frac{d}{dx} = \left(\frac{dx}{dm^2}\right)^{-1} \frac{d}{m^2} = \left((-1)\frac{q^2}{(m^2)^2}\right)^{-1} \frac{d}{dm^2} = \left(-\frac{x}{m^2}\right)^{-1} \frac{d}{dm^2} = -\frac{m^2}{x}\frac{d}{dm^2} \tag{4.60}$$

So, we obtain

$$\begin{aligned}
\frac{d}{dx}F^{ad}(x,1,1,\epsilon) &= -\frac{m^2}{x}\frac{d}{dm^2}\left[(m^2)^{\epsilon}\int\frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2}\right] \\
&= -\frac{m^2}{x}\left[\epsilon(m^2)^{\epsilon-1}\int\frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2} + (m^2)^{\epsilon}\int\frac{d^{4-2\epsilon}k}{(k^2-m^2)^2(q-k)^2}\right] \\
&= -\frac{1}{x}\left[\epsilon(m^2)^{\epsilon}\int\frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2} + (m^2)^{1+\epsilon}\int\frac{d^{4-2\epsilon}k}{(k^2-m^2)^2(q-k)^2}\right] \\
&= -\frac{1}{x}\left[\epsilon F^{ad}(x,1,1,\epsilon) + F^{ad}(x,2,1,\epsilon)\right].
\end{aligned} \tag{4.61}$$

At this stage we inspect integrals appearing on the r.h.s. and we apply IBPs to the new integral $F^{ad}(x,2,1,\epsilon)$ to see if it can be further reduced. This is actually the case since the reduction gives us

$$F(q^2,m^2,2,1,\epsilon) = \frac{1}{m^2-q^2}\left[(1-2\epsilon)F(q^2,m^2,1,1,\epsilon) - F(q^2,m^2,2,0,\epsilon)\right] \tag{4.62}$$

and if we multiply both sides by $(m^2)^{1+\epsilon}$, we obtain the dimensionless relation

$$F^{ad}(x,2,1,\epsilon) = \frac{1}{1-x}\left[(1-2\epsilon)F^{ad}(x,1,1,\epsilon) - F^{ad}(x,2,0,\epsilon)\right]. \tag{4.63}$$

At this point we can decide either to keep $F^{ad}(x,2,0,\epsilon)$ as a master integral or to further reduce it to $F^{ad}(x,1,0,\epsilon)$ through

$$F^{ad}(x,2,0,\epsilon) = (1-\epsilon)F^{ad}(x,1,0,\epsilon). \tag{4.64}$$

In this case we get

$$F^{ad}(x,2,1,\epsilon) = \frac{1}{1-x}\left[(1-2\epsilon)F^{ad}(x,1,1,\epsilon) - (1-\epsilon)F^{ad}(x,1,0,\epsilon)\right] \tag{4.65}$$

which substituted in Eq.4.61, gives

$$\frac{d}{dx}F^{ad}(x,1,1,\epsilon) = -\frac{1}{x}\left[F^{ad}(x,1,1,\epsilon)\left(\epsilon + \frac{1-2\epsilon}{1-x}\right) - \frac{(1-\epsilon)}{1-x}F^{ad}(x,1,0,\epsilon)\right]. \tag{4.66}$$

We have a obtained a differential equation for $F^{ad}(x,1,1,\epsilon)$ which we can solve if we know the inhomogeneous term $F^{ad}(x,1,0,\epsilon)$. The latter has a smaller number of propagators, namely it belongs to a subtopology of the initial one. So, in a bottom-up approach, where we start solving from the smallest subtopologies up to the bigger ones, this term can taken to be known and the differential equation can thus be integrated.

We generalize now the knowledge we have acquired from this example by synthesizing in general formulas for the generation of DE for MIs.
Let us begin by taking a generic scalar integral which depends on a set of N invariants $\mathbf{s} = \{s_1, ..., s_N\}$ built out of E external momenta

$$F(s_1, ..., s_N) = \int d^d k_1 d^d k_L E_1^{a_1} ... E_M^{a_M} \, . \tag{4.67}$$

Let us consider the quantities

$$O_{jk}(\mathbf{s}) = q_{j,\mu} \frac{\partial F(\mathbf{s})}{\partial q_{k,\mu}}. \quad (j,k = 1,2,...,E) \tag{4.68}$$

By the differentiation rules we have

$$O_{jk}(\mathbf{s}) = q_{j,\mu} \sum_{\alpha=1}^{N} \frac{\partial s_\alpha}{\partial q_{k,\mu}} \frac{\partial F(\mathbf{s})}{\partial s_\alpha} = \sum_{\alpha=1}^{N} \left(q_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial q_{k,\mu}} \right) \frac{\partial F(\mathbf{s})}{\partial s_\alpha}. \tag{4.69}$$

According to the available number of the kinematic invariants, the r.h.s. of Eq.(4.68) and the r.h.s. of Eq.(4.69) can be equated to form the following system

$$\sum_{\alpha=1}^{N} \left(q_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial q_{k,\mu}} \right) \frac{F(\mathbf{s})}{\partial s_\alpha} = q_{j,\mu} \frac{\partial F(\mathbf{s})}{\partial q_{k,\mu}} \tag{4.70}$$

which can be solved in order to express $\frac{\partial F(\mathbf{s})}{\partial s_\alpha}$ in terms of $q_{j,\mu} \frac{\partial F(\mathbf{s})}{\partial q_{k,\mu}}$.
Eq.(4.70) holds for any function $F(\mathbf{s})$. Let us now assume that $F = F_i(\mathbf{s})$ is a master integral belonging to a set $\{F_1, F_2, ..., F_p\}$ of MIs describing a certain topology. We apply to it the operator performing derivative with respect to a certain external invariant. This corresponds, through the action of this operator, to a linear combination of derivatives of the master with respect to external momenta and masses. These derivatives are integrals themselves, belonging to the same topology as the MIs $\{F_1, ..., F_p\}$, to which we can apply IBPs reduction. By doing so, we reduce the r.h.s. to a linear combination of MIs, thus expressing the derivative of the master F_i with respect to a certain invariant as a linear combination of the ensemble of MIs themselves. If we apply this procedure to

each master, we get a system of *first-order differential equations* in the ensemble of variables $\mathbf{s}$ for the MIs $\{F_1, ..., F_p\}$

$$\frac{\partial}{\partial s_\alpha} F_j(\mathbf{s}, \epsilon) = A_j(\mathbf{s}, \epsilon) F_j(\mathbf{s}, \epsilon) + \sum_{k \neq j} A_k(\mathbf{s}, \epsilon) F_k(\mathbf{s}, \epsilon) \quad j = 1, ..., p. \tag{4.71}$$

For the sake of precision, we point out that in the set of MIs $\{F_1, ..., F_p\}$ we include master integrals for the topology into consideration plus those for all its possible subtopologies, since the latter appear in the inhomogeneous terms in almost all cases.

Once the system of differential equations Eq.(4.71) is written down for a given set of MIs, the problem of solving the MIs amounts to being able to integrate such equations and matching them with proper boundary conditions. Now it is essential to point out that while the problem of IBP reduction is always solved in closed form in ϵ (and this is usually possible), the same does not happen for the problem concerning the solution of the MIs. There exists a very tiny category of Feynman integrals, mainly 2-point functions, which are solvable in closed form in ϵ. In most cases, given the complexity of the integrals, it is not possible to determine these integrals as exact functions of the regulator ϵ. Also, it happens very often that the results for these integrals will have to be expanded anyway around $d \to 4$, which corresponds to $\epsilon \to 0$. Given these facts, one usually addresses the computation of MIs expanded as Laurent series in ϵ. To do so, one substitutes the ansatz

$$F_j(\mathbf{s}, \epsilon) = \sum_{k=n_0}^{n} \epsilon^k F_j^{(k)}(\mathbf{s}) + \mathcal{O}\left(\epsilon^{n+1}\right) \tag{4.72}$$

where n_0 corresponds to the first power in ϵ that contributes to the series, whereas n is the maximum order we require in the solution and after which the series is truncated. If n_0 is negative, it corresponds to the highest pole of the integral. When expanding systematically in ϵ all the MIs and all the ϵ-dependent coefficient appearing in Eq.(4.71), one obtains a system of chained differential equations for the coefficients $F_j^{(k)}$ of the Laurent expansions of the masters. The first equation, corresponding to the highest pole, involves only the coefficient $F_J^{(n_0)}$ as unknowns. The next equation, corresponding to the next pole in ϵ, involves the $F_j^{(n_0+1)}$ as unknowns and usually $F_j^{(n_0)}$ in the inhomogeneous terms, but, since the equation for the highest pole is considered solved, such a term is considered known. The same approach is adopted for the subsequent equations: if we are solving the equation for the k-th order of the expansion, all coefficients at previous orders will appear in its inhomogeneous term, but in a bottom-up approach these are knowns. Experience shows that typical functions occurring in Feynman integrals are certain classes of iterated integrals, elliptic functions and possibly generalizations thereof.

It is still an open problem that of predicting in general, for a given Feynman graph, what class of functions it is described by. In the next subsection we will introduce the main class of functions in terms of which most results for Feynman integrals are expressed. These functions go under the name of *Goncharov polylogarithms* or *Multiple polylogarithms* or *Generalized Harmonic Polylogarithms*.

4.3.2 Multiple PolyLogarithms: an overview

Many Feynman integrals can be expressed in terms of classical polylogarithms and Nielsen polylogarithms. Starting from the ordinary logarithm

$$\ln z = \int_1^z \frac{dt}{t}, \tag{4.73}$$

we can generalize it as follows

$$Li_n(z) = \int_0^z \frac{dt}{t} Li_{n-1}(t), \quad Li_1(z) = -\ln(1-z). \tag{4.74}$$

Eq.(4.74) defines recursively that class of functions which goes under the name of *classical polylogarithms*. This class of functions can be further generalized to a wider class, called *Nielsen polylogarithms*, defined as

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{(\ln t)^{n-1}[\ln(1-zt)]^p}{t} dt. \tag{4.75}$$

These functions are sufficient to describe large classes of Feynman integrals, but not all of them. Specially some multi-loop and multi-legs integrals can give rise to new classes of functions. Among these new classes, the most common are the so-called *multiple polylogarithms* (MPLs). Since this is indeed the class of functions that we needed in our computation to express results for our master integrals,we will give some more details about it in the following. MPLs are a multi-variable extension of Eq.(4.74) defined recursively via the iterated integral

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, ..., a_n; t) \tag{4.76}$$

with $G(z) = 1$ and $a_i, z \in \mathbb{C}$. The number n of the indices a_i is called *weight* of the MPL, and they can be other constants or variables. In the special case where all the a_i's are zero, we define

$$G(\overbrace{0, ..., 0}^{n}; z) = \frac{1}{n!} \ln^n z. \tag{4.77}$$

Iterated integrals form a *shuffle algebra*, which allows one to express the product of two MPLs of weight n_1 and n_2 as a linear combination with integer coefficients of MPLs of weight $n_1 + n_2$

$$G(a_1, ..., a_{n_1}; z)G(a_{n_1+1}, ..., a_{n_1+n_2}; z) = \sum_{\sigma\in\Sigma(n_1,n_2)} G(a_{\sigma(1)}, ..., a_{\sigma(n_1+n_2)}; z), \tag{4.78}$$

where $\Sigma(n_1, n_2)$ is the set of all shuffles of $n_1 + n_2$ elements, namely the set of all permutations of $n_1 + n_2$ elements with $\sigma(1) < ...\sigma(n_1)$ and $\sigma(n_1 + 1) < ... < \sigma(n_1 + n_2)$. In other words, we consider all possible ways of shuffling the indices $a_1, .., a_{n_1}$ of the first element of the product with the indices $a_{n_1+1}, ..., a_{n_1+n_2}$ of the second element in such a way that each of these shuffles conserves the initial ordering of the indices of both factors in the product.
It is possible to find closed expressions for special classes of MPLs in terms of classical polylogarithms. For instance, for $a \neq 0$, we have

$$\begin{aligned}
&G(\vec{0}_n; z) = \frac{1}{n!}\ln^n z, \quad G(\vec{a}_n; z) = \frac{1}{n!}\ln^n\left(1 - \frac{z}{a}\right), \\
&G(\vec{0}_{n-1}, a; z) = -Li_n\left(\frac{z}{a}\right), \quad G(\vec{0}_n, \vec{a}_p; z) = (-1)^p S_{n,p}(\frac{z}{a}),
\end{aligned} \tag{4.79}$$

where we used the usual vector notation $\vec{a}_n = \overbrace{(a, ..., a)}^{n}$. Given the definition of MPL Eq.(4.76), which we can rewrite conveniently for our purposes as

$$G(a_1, \vec{a}; z) = \int_0^z \frac{dt_1}{t_1 - a_1} G(\vec{a}; t_1), \tag{4.80}$$

the derivative with respect to z is trivial

$$\frac{\partial}{\partial z} G(a_1, \vec{a}; z) = \frac{1}{z - a_1} G(\vec{a}; z). \tag{4.81}$$

For the derivative with respect to the first argument, we get

$$\frac{\partial}{\partial a_1} G(a_1, a_2, \vec{a}; z) = \frac{1}{a_1 - a_2} G(a_1, \vec{a}; z) - \frac{z - a_2}{(z - a_1)(a_1 - a_2)} G(a_2, \vec{a}; z), \tag{4.82}$$

and for the derivative with respect to the second argument

$$\begin{aligned}
\frac{\partial}{\partial a_2} G(a_1, a_2, a_3, \vec{a}; z) = &\frac{1}{a_1 - a_2} G(a_2, a_3, \vec{a}; z) + \frac{1}{a_2 - a_3} G(a_1, a_2, \vec{a}; z) + \\
&- \frac{a_1 - a_3}{(a_1 - a_2)(a_2 - a_3)} G(a_1, a_3, \vec{a}; z).
\end{aligned} \tag{4.83}$$

These identities can be checked by taking double derivatives and verifying that the result does not depend on the order of derivation. In this process, one should use different partial fractioning identities to get to the desired result. Also, these identities hold when the arguments in the weight vector $\vec{a}$ are all different. In case some of them are equal, then some of the denominators in the equations above vanish.

Let's consider now the case of the derivative with respect to the k-th argument, for $k \geq 2$:

$$
\begin{aligned}
\frac{\partial}{\partial a_k} G(\vec{a}; z) &= \int_0^z \frac{dt_1}{t_1 - a_1} ... \int_0^{t_{k-2}} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \frac{\partial}{\partial a_k} G(a_k, a_{k+1}, ...; t_{k-1}) \\
&= \int_0^z \frac{dt_1}{t_1 - a_1} ... \int_0^{t_{k-2}} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \\
&\quad \left(\frac{1}{a_k - a_{k+1}} G(a_k, a_{k+2}, ...; t_{k-1}) \right. \\
&\quad \left. - \frac{t_{k-1} - a_{k+1}}{(t_{k-1} - a_k)(a_k - a_{k+1})} G(a_{k+1}, a_{k+2}, ...; t_{k-1}) \right) . \qquad (4.84)
\end{aligned}
$$

By replacing

$$
\begin{aligned}
&- \frac{t_{k-1} - a_{k+1}}{(t_{k-1} - a_k)(a_k - ak + 1)(t_{k-1} - a_{k-1})} = \\
&= \frac{1}{a_{k-1} - a_k} \frac{1}{t_{k-1} - a_k} - \frac{a_{k-1} - a_{k+1}}{(a_{k-1} - a_k)(a_k - a_{k+1})} \frac{1}{t_{k-1} - a_{k-1}} \qquad (4.85)
\end{aligned}
$$

and integrating the resulting expression we get

$$
\begin{aligned}
\frac{\partial}{\partial a_k} G(\vec{a}; z) = \frac{1}{a_{k-1} - a_k} G(..., \hat{a}_{k-1}, ...; z) + \frac{1}{a_k - a_{k+1}} G(..., \hat{a}_{k+1}, ...; z) \\
- \frac{a_{k-1} - a_{k+1}}{(a_{k-1} - a_k)(a_k - a_{k+1})} G(..., \hat{a}_k, ...; z), \qquad (4.86)
\end{aligned}
$$

where the hat marks the missing arguments. Again, if two consecutive a_j and a_{j+1} are equal, the arguments above should be modified.

There is a last special case for taking derivatives with respect to the last argument

$$
\frac{\partial}{\partial a_n} G(\vec{a}, a_{n-1}, a_n; z) = \frac{1}{a_{n-1} a_n} G(\vec{a}, a_n; z) - \frac{a_{n-1}}{(a_{n-1} - a_n) a_n} G(\vec{a}, a_{n-1}; z). \qquad (4.87)
$$

Using the expressions above for the derivatives of MPLs, we get for the total differential

$$
dG(a_1, a_2, \vec{a}, a_{n-1}, a_n; z) = G(a_2, \vec{a}, a_{n-1}, a_n; z) d \ln \left(\frac{z - a_1}{a_1 - a_2} \right)
$$

$$+ G(a_1, \vec{a}, a_{n-1}, a_n; z) d \ln\left(\frac{a_1 - a_2}{a_2 - a_3}\right) + ...$$
$$+ G(a_1, a_2, \vec{a}, a_{n-1}; z) d \ln\left(\frac{a_{n-1} - a_n}{a_n}\right). \quad (4.88)$$

Eq.(4.88) is the basic result on which we relied in the majority of cases to compute our master integrals. It has been implemented into `Mathematica` routines and widely used to integrate the differential equations for the masters.
In particular, we found out that the category of functions that we need to describe our results is a subset of those defined in Eq.(4.76). Indeed, in physics MPLs usually show up with the entries of the index vector chosen from a limited set, often called the *alphabet* of the problem under consideration. In the simplest case that $a_i \in \{-1, 0, 1\}$, these functions are called *Harmonic Polylogarithms* (HPLs) [105]. In multi-scale integrals the a_i often depend on another variable, in which case one speaks of *two-dimensional Harmonic Polylogarithms* (2dHPLs), which were first introduced in [66]. The latter are actually the restricted category of MPLs in terms of which our results are expressed.

4.3.3 Boundary Conditions: Expansion by regions of Feynman Integrals

Eq.(4.88) constitutes the milestone for the integration of Differential Equations (Eq.(4.71)) written for the masters. Then, the only piece we are missing to complete the picture of the computation of MIs via DE are the boundary conditions. Computing a boundary condition (b.c.) for a Feynman integral means being able to obtain the value of the integral in a given kinematic limit, obviously before knowing the general functional form of the integral itself (which is the result we want to achieve at the end of all this procedure!). This kinematic limit can be freely chosen on the basis of computational simplicity and it corresponds to a particular value of a given ratio of external invariants (dimensional scales) on which the integral depends. In the case of an integral depending on just two dimensional scales, this limit corresponds to a precise kinematic point, so that the resulting b.c. for such an integral will simply be a number. For multi-dimensional integrals, one has at disposal a certain number of ratios that can be built out of the external invariants. When computing boundary conditions, one has the freedom to choose the number of ratios which are going to be constrained to specific values. If the b.c. is computed by fixing only a subset of these ratios, the integral to be evaluated will have a smaller number of scales and the resulting b.c. will be a function of those ratios that have been left free. Otherwise one can also choose to compute a multiple limit and fixing all the possible ratios on which the multi-dimensional integral depends, and in this case the resulting b.c. will be again a pure number.

Given these premises, it is clear that the computation of boundary conditions

reduces to the mathematical problem of compute the value of an integral in a particular limit of the parameters on which it depends, without carrying out the analytic integration in the case of general value of these parameters. We present the problems one encounters when trying to achieve this goal by means of an example.
Let us consider the massless triangle in 4 dimensions

$$F(q^2, p^2, q \cdot p) = \int \frac{d^4k}{k^2(q-k)^2(p+k)^2}. \tag{4.89}$$

The integration is over the four components of the loop momenta $k = (k_0, \mathbf{k})$ and the integral is a scalar quantity, so the result of integration will be a function of the two external momenta p, q through the possible scalar invariants that one can build out of them $(q^2, p^2, q \cdot p)$. Let us imagine we are interested in obtaining the value of this integral in a particular kinematic configuration, identified by the condition $q^2 \gg p^2$. Then we address the question: is it possible to compute $\lim_{q^2 \gg p^2} F(q^2, p^2, q \cdot p)$ without computing F for general values of its arguments? For simplicity we switch to euclidean metric, so that $k, p, q \in \mathbb{R}^4$ and the square modulus of the generic vector l is given by $l^2 = l_0^2 + \mathbf{l}^2$. In euclidean metric we have $q^2, p^2 > 0$ and the condition $q^2 \gg p^2$ is equivalent to $|q| \gg |p|$, which in turn is realized only if there exists a component q_i such that $q_i \gg p_j, \forall j$. Given this, we can classify ratios of invariants as follows

$$\frac{q^2}{p^2} \gg \frac{p \cdot q}{p^2} \gg 1. \tag{4.90}$$

At this point the simplest idea one can come up with to solve this problem is that of exchanging the order of operations by applying first the limit and then the integration operators. This amounts to expanding the integrand around the chosen kinematic limit, namely in our case to send naively p to zero in the integrand denominator, and then trying to integrate the result of the expansion.
Before showing explicitly what happens when we exchange these operations with our example, let us point out the two requirements that we would like our result to have.

- First, we would like the integrals we get on the r.h.s. after the expansion to be Feynman integrals and not other types of functions. This appears rather natural because Feynman integrals are the fundamental objects we are dealing with and it is better to have them in the expansion rather than, say, some artificially introduced parametric integrals.

- The second 'natural' requirement is that we would like our final result to be an expansion in powers and logs of the small parameter around which we are expanding (namely the small ratio(s) of invariants which identify the kinematic limit in which we want to compute our integral).

Going back to our example, when we send p to zero at the integrand level in our toy-integral $F(p^2, q^2, q \cdot p)$ we obtain

$$\int \frac{d^4k}{(k^2)^2(q-k)^2}. \tag{4.91}$$

This integral is clearly ill-defined, since it is logarithmically divergent when k gets small. Thus we expect the result to behave like $\log(p^2)$. Through the introduction for example of a dimensional scale Λ, which allows us to compute divergent integrals obtained from expanding, we can explicitly get the above-mentioned log-type behaviour. Indeed, once we know that this divergent behaviour comes from the small k integration region, what Λ does is cutting the integration domain into two parts, thus separating the small from the large k region

$$f_{small} \simeq f^{(0)}_{small} = F(p^2, q^2, q \cdot p)|_{|p|,|k| \ll |q|} = \frac{1}{q^2} \int_{|k| \leq \Lambda} \frac{d^4k}{(k^2)(p+k)^2}. \tag{4.92}$$

We call the contribution from the small k region f_{small} and we denote with $f^{(0)}_{small}$ the first term of the expansion around the limit $k \ll q$. We would like to stress that in such limit k can be as small as p so that we have to keep both the dependence on k and p in the integrand. This integral can be carried out for example by introducing spherical coordinates, and the result is

$$f_{small} = \frac{4\pi}{q^2} \int_0^{\pi} \sin^2\theta d\theta \int_0^{\Lambda} \frac{rdr}{r^2 + 2|p|r\cos\theta + p^2} \simeq -\frac{\pi^2}{q^2} \ln\left(\frac{p^2}{\Lambda^2}\right). \tag{4.93}$$

Eq.(4.93) exhibits the $\log(p^2)$ behaviour we expected, but we observe that the price we pay for this is that of introducing a spurious dependence of the integral on a non-physical parameter Λ which acts at all effects as a regulator.
It comes natural to wonder if Eq.(4.93) is the only contribution to F in the limit $|q| \gg |p|$... obviously nothing prevents $|p|$ from being much smaller than $|q|$ also when k is not close to zero! What happens in this case? Again we introduce an intermediate scale Λ such that $|p| \ll \Lambda \ll |q|$ to divide the integration domain in two regions, and pick up the large k region this time

$$f_{large} \simeq f^{(0)}_{large} = F(p^2, q^2, q \cdot p)|_{|p| \ll |k|,|q|} = \int_{|k|>\Lambda} \frac{d^4k}{(k^2)^2(q-k)^2}. \tag{4.94}$$

We stress that this time we can safely send p to 0 in the integrand since k is large and thus regulates the divergence $1/(p+k)^2$. On the other hand we cannot touch the terms $1/(q-k)^2$ because now $|k|$ and $|q|$ may happen to be comparable. The result of the integration is

$$f_{large} = -2\pi^2 \frac{\log(\Lambda^2/q^2)}{q^2}. \tag{4.95}$$

We observe that we get a result which is constant in p^2 and exhibits the leading power behaviour in q^2, namely $1/q^2$.
Now we recall that the original integral F in the limit $|p| \ll |q|$ is given by the sum of the two pieces we just computed respectively in the small and large k regions

$$\lim_{|p|\ll|q|} F(q^2,p^2,q\cdot p) = f_{small} + f_{large} \simeq f^{(0)}_{small} + f^{(0)}_{large} = \pi^2 \frac{\ln(p^2/q^2) - 2}{q^2}. \tag{4.96}$$

We observe that, having taken into account all possible integration regions that contribute in the limit $|q| \gg |p|$, when we sum over these contributions the dependence on the regulator cancels between the different pieces and we are left with a result which depends only on physical quantities.
We are pretty happy with this result, but this was quite a simple case where it was still possible to introduce a dimensional scale Λ and carry out integrations of the different pieces with spherical coordinates. In real life, this will almost never be the case because of the complexity of integrals. Thus we address now the issue if it is possible or not to arrive to the same result without splitting the integration domain into regions through an explicit dimensional scale Λ. An even worse problem is represented by the fact that if we are performing computations in the framework of a gauge theory, the introduction of an explicit 'cut-off' breaks gauge-invariance, which we want to avoid, if possible.
A solution to these issues is provided by the use of a different type of regulator, i.e. DR. Let us pick up the same 1-loop triangle but this time we work in DR, so we promote the dimension in which the integration variable k live from 4 to $d = 4 - 2\epsilon$.
We can rewrite F as the sum of f_{large} and f_{small}, then extend the integration domain to the entire space in both these contributions and subtracting at the same time the overlap as follows

$$\begin{aligned}\lim_{|p|\ll|q|} F(q^2,p^2,q\cdot p) =& \frac{1}{q^2}\int_{|k|\leq\Lambda} \frac{d^4k}{(k^2)(p+k)^2} + \int_{|k|>\Lambda} \frac{d^4k}{(k^2)^2(q-k)^2} \\ =& \frac{1}{q^2}\int \frac{d^4k}{(k^2)(p+k)^2} + \int \frac{d^4k}{(k^2)^2(q-k)^2} \\ & - \left[\frac{1}{q^2}\int_{|k|\geq\Lambda} \frac{d^4k}{(k^2)(p+k)^2} + \int_{|k|\leq\Lambda} \frac{d^4k}{(k^2)^2(q-k)^2}\right].\end{aligned} \tag{4.97}$$

We can now manipulate the counter-term we are subtracting by expanding at first order the integrands according to the region of integration

$$-\left[\frac{1}{q^2}\int_{|k|\geq\Lambda} \frac{d^4k}{(k^2)(p+k)^2} + \int_{|k|\leq\Lambda} \frac{d^4k}{(k^2)^2(q-k)^2}\right] \to$$

$$\to -\left[\frac{1}{q^2}\int_{|k|\geq\Lambda}\frac{d^4k}{(k^2)^2}+\int_{|k|\leq\Lambda}\frac{d^4k}{(k^2)^2(q)^2}\right]=$$
$$= -\frac{1}{q^2}\left[\int\frac{d^4k}{(k^2)^2}\right]=0. \tag{4.98}$$

The counter-term amounts to a scaleless integral and thus, since we work in DR, to zero.
So we are left with the following expression for our integral

$$\lim_{|p|\ll|q|} F(q^2,p^2,q\cdot p) \simeq \frac{1}{q^2}\int\frac{d^4k}{(k^2)(p+k)^2}+\int\frac{d^4k}{(k^2)^2(q-k)^2}. \tag{4.99}$$

The two integrals in Eq.(4.99) can be easily solved in DR, giving

$$\lim_{|p|\ll|q|} F(q^2,p^2,q\cdot p) \simeq \pi^{d/2}\left(\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{q^2(p^2)^\epsilon}-\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{(q^2)^{1+\epsilon}}\right)$$
$$\stackrel{\epsilon\to 0}{\to} \pi^2\frac{\ln(p^2/q^2)-2}{q^2}. \tag{4.100}$$

We report result Eq.(4.100) for the sake of completeness, but the interesting notion we learn from this example is contained one step before, namely in Eq.(4.99), which tells us that the initial problem of computing the integral F in a particular kinematic limit without computing it for general values of its parameters is solved! Eq.(4.99) has the following properties:

- the initial integral $F(q^2,p^2,q\cdot p)$, depending on 3 invariants, is replaced by the sum of two integrals depending each on just one scale (p^2 or q^2),
- each of these two integrals is also homogeneous with respect to the expansion parameter p^2,
- they are (simpler) Feynman integrals and not other classes of functions,
- DR is essential, since it allows us to write the r.h.s. integrals contributing in different regions of the integration domain as integrals over the entire domain avoiding at the same time overlapping between these regions,
- the operation of limit and integration in general do not commute and when exchanging their order, namely when taking the operation of limit under integration sign, we have to pay attention to take into account all the possible regions of the integration domain that can contribute non-zero terms in the kinematic limit under study,
- the final result Eq.(4.100) is an expansion in powers and logs of the ratio p^2/q^2 which constitutes the small parameter in which we expand the integral.

If we look back at the requirements we listed above, we clearly see that our result meet them!

From this very basic example, we can thus extrapolate the general guidelines to compute the asymptotic value of a Feynman integral in a given limit of its parameters (internal masses and external invariants).

1. Divide the space of the loop momenta into various *regions* according to the kinematic limit into consideration and *expand the integrand* in a Taylor series with respect to parameters that are considered small there (NB: both loop and external momenta must be taken into consideration when looking for regions and expanding the integrand according to the region!).

2. In every region integrate the expanded integrand over the *whole integration domain* using DR.

3. Sum the resulting contributions from all regions, *setting scaleless integrals to zero*.

This procedures goes under the name of *Expansion by Regions* of Feynman integrals.

Despite the lack of a rigorous mathematical proof stating the correctness of Expansion by Regions (at least for those limits that are typical of Minkowsky space, namely when momenta are located on some singular surface, either on a mass shell or at threshold), there are some 'experimental' general features that still have not been observed to break down in any particular situation. Experience tells us that in all limits the resulting expansion of an integral is always a series in powers and logs of the small parameter of the expansion. If we have an integral that, for simplicity, depends on just two dimensional scales, for instance an internal mass m^2 and an external off-shellness $Q^2 > 0$, its expansion around $x = m^2/Q^2 \ll 1$ will have the general form

$$F(Q^2, m^2) \sim (Q^2)^\omega \sum_{n=n0}^{\infty} \sum_{j=0}^{2h} C_{nj} x^n \ln^j x, \tag{4.101}$$

where we pulled out the overall dimensional factor $(Q^2)^\omega$ and ω is the degree of divergence of the graph associated to F.
The sum over n runs from some minimal value. The index n can generally take, in some limits, not only integer but also half-integer values. The second index j is bounded, for any n, by twice the number of loops.
According to a standard definition of an asymptotic expansion, when we truncate

the series at an arbitrary order N, the remainder is defined as

$$R_N(Q^2, m^2) = F(Q^2, m^2) - (Q^2)^\omega \sum_{n=N_0}^{N} \sum_{j=0}^{2h} C_{nj} x^n \ln^j x \tag{4.102}$$

and it is $\mathcal{O}(x^N)$, namely for Q such that $A < Q < B$, there exist $C > 0$ and $\epsilon > 0$ such that

$$x^{-N} R_N(Q^2, Q^2 x)| \leq C \tag{4.103}$$

for $0 < x < \epsilon$. When we compute multi-dimensional integrals in multiple kinematic limits these formulae get slightly more complicated, but the essence remains the same. Each small parameter is multiplied by a dimensionless 'scaling' variable x and we deal with the resulting function of x in the limit $x \to 0$.
Another 'experimental' fact is the existence of a non-zero radius of convergence of the asymptotic expansion of any Feynman integral in any given limit, which becomes a fundamental property in the moment we want to substitute the value of an integral in a kinematic point with the series obtained through Expansion by Region truncated at a certain order.

At this point we would like to add a couple of remarks before ending our discussion on Expansion by Region.
First, we would like to stress the importance of Dimensional Regularization in this kind of procedure. Indeed, it allows to provide simple prescriptions for the integrals we get on the r.h.s. after the expansion, in a sense that we can keep them as dimensionally regularized integrals over the whole integration domain and these are much easier quantities to deal with and to compute than integrals regularized with a dimensional cut-off. However a remark is in order at this point. The initial integral that we need to expand can be divergent itself. In this case, we need DR to regularize the 'original divergences' intrinsic to the integral itself. On top of that, as we saw, when we expand we can get additional divergences from the regions contributing to the expansion. It can happen then that the singular regions have divergences which are not regulated by the DR parameter, and an additional analytic regulator needs to be introduced in order to take care of these extra divergences.
Finally we want to emphasize that, even once this procedure of Expansion by Regions has been understood and established, the problem of computing Feynman integrals in a precise limit is not trivial. Indeed, when looking at the three main steps listed above to perform the expansion, one will immediately understand that the main issue has been in some sense forced into the first point. Despite the computational complexity that may still be required to compute r.h.s. integrals arising from the expansion, the most difficult part remains that of finding all possible regions contributing to the value of the integral in a particular kinematic

configuration. In the case of multi-loop integrals identification of relevant regions becomes highly non trivial due to the interplay between parameters (i.e. external invariants) and integrations variables (loop momenta or their components). We will not go further into the discussion of these open problems since this goes beyond the purpose of this thesis, but we refer the interested reader to references [116] and [100], the former being a general review of Expansion by Region and the latter a recent approach to automation of this procedure.

4.4 Master Integrals Computation - Part 2: Canonical Basis

We want to conclude this chapter about Master Integrals techniques with a section entirely dedicated to some recently developed ideas which have repeatedly proven successful in improving the method of Differential Equations for Feynman integrals and which have been extensively used in our computation.

In the previous sections we saw that, given a certain topology, we have the freedom to choose the preferred set of Master Integrals, since each set has equivalently the role of a basis for the space of Feynman integrals belonging to that topology. We can exploit this feature of MIs, which is part of the definition of master integral itself, to simplify the system of Differential Equations which is meant to give us the solution for a given set of masters. Indeed, experience shows that DE system can look very different according to the particular choice of MIs basis we make, meaning that a proper choice of basis can considerably simplify the system of differential equations. A clever change of basis is one that is able to diagonalize or at least triangularize the DE system order by order in the DR parameter ϵ, thus guaranteeing the integrability of the system.
The ideas we are going to present in the rest of the section try to approach in a systematic way this issue of finding an optimal basis of master integrals. The material presented is mainly based on the recent papers of J. Henn [75], [76], to which we refer the reader who might want to find out more about this topic.

4.4.1 Feynman integrals singularities and Canonical form of Differential Equations

In order to understand what an optimal choice is for the basis of masters, one should start asking himself what the optimal choice is for the differential equations, which makes their integration as easy as possible. To give an answer to this question, one first has to analyse the singularity structure of Feynman integrals. Actually the kind of singular behaviour that Feynman integrals can exhibit is restricted by the fact that they admit parametric representations such as α and Feynman representations (Eq.4.22, 4.23). Divergent regions can be found by inspecting the divergences in the space of α or Feynman parameters and from such an analysis it follows that in divergent regions an integral has a power-like behaviour $F \sim (x - x_k)^p$, where x_k is the singular point and p is a certain exponent. This means that Feynman integrals [7] can have only *regular singularities* [8]. Fur-

[7] We remind the reader that in the following we will always consider Feynman integrals as complex functions of one or more complex variables, so that the common language will be that of complex analysis.

[8] A *regular singularity* of a function f is an isolated singularity whose growth is bounded by an algebraic function.

thermore parametric representation Eq.4.22, 4.23 exhibit only a linear dependence from the DR parameter ϵ in the exponents. We can summarize in the following two properties [9]

- Feynman integrals only have regular singularities in the kinematic variables
- the scaling exponents near a singularity are linear in ϵ.

These properties are essential to understand not only which is the optimal form of the differential equations for Feynman integrals, namely what we will define in the following as 'canonical form', but also how to reach such form.

We define *canonical form of the differential equation for a given function f* that particular form which makes the singularity structure of f manifest. If we specialize in particular to our subject of interest, i.e. Feynman integrals, given the two above-mentioned properties of Feynman integrals this means that the differential equations can contain only regular singularities [10] in the kinematic variables and that the dependence on ϵ of the coefficients can be only of linear type.

Before discussing in detail how the canonical form for a set of Differential Equations for Feynman integrals can be reached, let us illustrate with a simple example

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To be more precise, suppose U is an open subset of $\mathbb{C}$, the point a is an element of U and f is a complex differentiable function defined on some neighbourhood around a, excluding a, $U\backslash\{a\}$. Complex functions can exhibit four classes of singularities.

- *Isolated singularities*: f is not defined at a, although it does have values defined on $U\backslash\{a\}$.
  - a is a *removable* singularity if there exists a holomorphic function g defined on the whole subset U such that $f(z) = g(z)$ for all z in $U\backslash\{a\}$. The function g is then a continuous replacement for f.
  - a is a *pole* or equivalently a *regular* (or *non-essential*) singularity if there exists a holomorphic function g defined on U with $g(a) \neq 0$ and $n \in \mathbb{N}$ such that $f(z) = g(z)/(z-a)^n$ for all z in $U\backslash\{a\}$.
  - a is an *essential* singularity of f if it is neither a removable singularity nor a pole or alternatively, if and only if the Laurent series of f in a has infinitely many powers of negative degree.
- *Branch cuts*: is a line or a curve excluded from the domain to introduce a technical separation between discontinuous values of the function. The function f will then have distinctly different values on each side of the branch cut.

[9] We stress that a steady mathematical proof of such properties is still lacking. But common knowledge continues confirming their validity.

[10] In the theory of differential equations in the complex plane $\mathbb{C}$, the points of $\mathbb{C}$ are classified into *ordinary points*, at which the equation's coefficient are *analytic functions*, and *singular points*, at which some coefficient has a singularity. Then, among singular points, a distinction is made between a *regular singular points*, where the growth of solutions is bounded by an algebraic function, and an *irregular singular points*, where the full solution set requires functions with higher growth rates.

how it works in practice.
Let us consider the following 2-loop (dimensionless) *cut* integrals describing the phase space of two massless and one massive particle in the final state.

$$m_1 = \int d^dk_1 d^dk_2 \frac{s^{3-d}}{k_1^2 k_2^2[(p+q-k_1-k_2)^2 - m^2]} \tag{4.104}$$

$$m_2 = \int d^dk_1 d^dk_2 \frac{s^{2-d}(k_1+k_2)^2}{k_1^2 k_2^2[(p+q-k_1-k_2)^2 - m^2]}, \tag{4.105}$$

with p, q being the initial state momenta, $s = (p+q)^2$, k_1, k_2 being the outgoing massless particles momenta and $k_m = p+q-k_1-k_2$ being the momentum of the outgoing massive particle with mass $m^2 = k_m^2$.
The dimensional version of these integrals depend on two dimensional scales: the external invariant s and the internal mass m^2. The evolution with respect to s is trivial since it just contains the information about the dimension of the integral in powers of energy. The only non trivial evolution is thus with respect to m^2. When we consider the integrals in their dimensionless version Eq.(4.105), they will depend on the only ratio of invariants we can build out of s and m^2, namely

$$z = \frac{m^2}{s} \tag{4.106}$$

and the differential operator ∂_z will read

$$\partial_z = s\partial_{m^2}. \tag{4.107}$$

If we let this operator acting on the integrals Eq.(4.105), we get

$$\begin{cases} \partial_z m_1 = & m_2 \left(\frac{3(-1+\epsilon)}{2(-1+z)} - \frac{3(-1+\epsilon)}{2z}\right) + m_1 \left(\frac{3-4\epsilon}{-1+z} + \frac{-1+\epsilon}{2z}\right) \\ \partial_z m_2 = & m_1 \left(\frac{1-\epsilon}{2} + \frac{-1+\epsilon}{2z}\right) + m_2 \frac{3(1-\epsilon)}{2z} \end{cases}$$

which we can rewrite in matrix notation as

$$\partial_z \vec{m} = \begin{pmatrix} \frac{3-4\epsilon}{-1+z} + \frac{-1+\epsilon}{2z} & \frac{3(-1+\epsilon)}{2(-1+z)} - \frac{3(-1+\epsilon)}{2z} \\ \frac{1-\epsilon}{2} + \frac{-1+\epsilon}{2z} & \frac{3(1-\epsilon)}{2z} \end{pmatrix} \cdot \vec{m} \tag{4.108}$$

with $\vec{m} = (m_1, m_2)$. The system is coupled and it does not make the singularity structure of the integrals manifest. Indeed it is true that the DE has regular singular points at $z = 0, z = 1$ as expected, but their dependence on ϵ does not tell us anything about the behaviour of functions m_1, m_2 in these singular limits. But we can now choose the following different basis to describe the same topology

$$M_1 = \left(\int d^dk_1 d^dk_2 \frac{s^{5-d}}{(k_1^2)^2 (k_2^2)^2[(p+q-k_1-k_2)^2 - m^2]}\right.$$

$$+2\int d^dk_1 d^dk_2 \frac{s^{5-d}}{(k_1^2)^2 k_2^2[(p+q-k_1-k_2)^2-m^2]^2}\Bigg)(1-z)$$

$$M_2 = \int d^dk_1 d^dk_2 \frac{s^{5-d}}{(k_1^2)^2 k_2^2[(p+q-k_1-k_2)^2-m^2]^2} \tag{4.109}$$

which is connected to the original basis by the transformation (in matrix form)

$$\vec{M} = \left(\frac{a}{z} + \frac{b}{-1+z} + \frac{c}{(-1+z)^2} + \frac{d}{(-1+z)^3} \right) \cdot \vec{m} \tag{4.110}$$

with $\vec{M} = (M_1, M_2)$ and

$$a = \begin{pmatrix} 1-3\epsilon+2\epsilon^2 & -3(1-3\epsilon+2\epsilon^2) \\ \frac{1}{2}(1-3\epsilon+2\epsilon^2) & -\frac{3}{2}(1-3\epsilon+2\epsilon^2) \end{pmatrix}$$
$$b = \begin{pmatrix} -2+6\epsilon-4\epsilon^2 & -3(1-3\epsilon+2\epsilon^2) \\ -\frac{1}{2}+\frac{3\epsilon}{2}-\epsilon^2 & -\frac{3}{2}(1-3\epsilon+2\epsilon^2) \end{pmatrix}$$
$$c = \begin{pmatrix} -88+\frac{12}{\epsilon}+192\epsilon-128\epsilon^2 & -3(1-3\epsilon+2\epsilon^2) \\ -19+\frac{3}{\epsilon}+38\epsilon-24\epsilon^2 & -\frac{3}{2}(1-3\epsilon+2\epsilon^2) \end{pmatrix}$$
$$d = \begin{pmatrix} -176+\frac{24}{\epsilon}+384\epsilon-256\epsilon^2 & -3(1-3\epsilon+2\epsilon^2) \\ -44+\frac{6}{\epsilon}+96\epsilon-64\epsilon^2 & -\frac{3}{2}(1-3\epsilon+2\epsilon^2) \end{pmatrix}. \tag{4.111}$$

The differential equations for the new basis in matrix notation read

$$\partial_z \vec{M} = \begin{pmatrix} \frac{4\epsilon}{1-z} + \frac{2\epsilon}{z} & -\frac{6\epsilon}{z} \\ \frac{\epsilon}{1-z} + \frac{\epsilon}{z} & -\frac{3\epsilon}{z} \end{pmatrix} \cdot \vec{M}. \tag{4.112}$$

Despite the complexity of Eq.(4.110) and Eq.(4.111), this change of basis leads to the new system of differential equations Eq.(4.112), which is not only at a first glance simpler than Eq.(4.108), but also it has the fundamental property of depending linearly on ϵ and having no constant term in ϵ. This property, together with the fact that equations only have regular singular points, guarantees that the singularity structure of the integrals is made manifest at the level of the equations themselves.

Such form of the differential equations, which is indeed what we define as 'canonical', is desirable not only for its beauty and elegance, but specially for two main features which makes integration and matching to boundary conditions as simple as possible.

- Once the solution is written as an expansion in ϵ

$$\vec{M} = \sum_{k\geq n} \epsilon^k \vec{M}^{(k)}(z) \tag{4.113}$$

  and plugged into the system, the system decouples order by order in ϵ and at each order in ϵ the r.h.s. of the equations is known and can thus be integrated.

- The behaviour of the solutions in singular limits can be read directly from the differential equations. Let us consider for instance the limit $z \to 0$. Keeping only the leading term on the r.h.s.

$$\partial_z \vec{M} = \epsilon \frac{Z_0}{z} \cdot \vec{M}. \tag{4.114}$$

we find the solution

$$\lim_{z \to 0} \vec{M}(z, \epsilon) = z^{\epsilon Z_0} \vec{M}_0(\epsilon) \tag{4.115}$$

where $\vec{M}_0(\epsilon)$ is the vector of boundary conditions (which are in this case constants) and Z_0 is the matrix of coefficients which are singular in the $z \to 0$ limit, namely those that are proportional to $1/z$

$$Z_0 = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}. \tag{4.116}$$

The matrix exponential evaluates to

$$z^{\epsilon Z_0} = \begin{pmatrix} 3 - 2z^{-\epsilon} & -6 + 6z^{-\epsilon} \\ 1 - z^{-\epsilon} & -2 + 3z^{-\epsilon} \end{pmatrix} \tag{4.117}$$

so that solutions are different linear combinations of different terms z^α where the exponents α are of the type $\alpha = m\epsilon$, namely linear in ϵ with m being an (semi-)integer number [11]. In particular, the exponents α happen to be the eigenvalues of the matrices of coefficients surviving in the singular limit considered. For instance, in this case the eigenvalues of Z_0 are $\{0, -1\}$ and indeed all terms appearing in Eq. (4.117) are linear combinations of the scalings $\{z^0, z^{-\epsilon}\}$! This is exactly the type of behaviour for the solution that we expect given the analysis of Feynman integrals divergences that we carried out at the beginning of the section.

We would like to emphasize that the singularity structure of the equations does not always reflect so neatly the singularity structure of the solution as it happened in the example above. Indeed the 'matrix' nature of the equations allows for 'spurious' singularities to occur. Let's take the following purely mathematical example

$$\partial_x \vec{f}(x, \epsilon) = \begin{pmatrix} \frac{\epsilon}{x} & 0 \\ -\frac{1}{x^2} & \frac{\epsilon}{1+x} \end{pmatrix} \cdot \vec{f}(x, \epsilon) \tag{4.118}$$

The singularity $-\frac{1}{x^2}$ is not a regular one. Given the simple differential equation

$$\partial_x f(x) = \frac{a}{x^2} f(x) \tag{4.119}$$

[11] Constant terms in Eq.(4.117) are to be read as $z^{m\epsilon}$ with $m = 0$.

its solution is of the type $f(x) = \exp^{-a/x} f_0$. When x approaches 0 this function exhibit a singular behaviour which is not bounded by any algebraic function, so that this singularity in $x = 0$ is classified as an essential one.
Now, we know that if the vector of unknowns f is a vector of Feynman integrals, essential singularities cannot appear. This means that a term like $-1/x^2$ is a spurious singularity which can be removed with an appropriate manipulation. And indeed we can get rid of it by performing a 'change of basis', namely by defining

$$\vec{f} = T\vec{g}, \qquad T = \begin{pmatrix} 1 & 0 \\ \frac{1}{(1-\epsilon)x} & \frac{1}{1-\epsilon} \end{pmatrix} \tag{4.120}$$

which leads to

$$\partial_x \vec{g} = \epsilon \left[\frac{1}{x} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{1+x} \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right] \vec{g}. \tag{4.121}$$

We will discuss in the following how algebraic simplifications can be performed on a basis of integrals in order to reach a canonical form for their differential equations.

4.4.2 Looking for canonical form of DE: an algebraic approach

In order to describe how the expected singularity structure of the integrals can be made manifest, or in other words how the canonical form of the equations can be achieved, we will focus on the case where the integrals depend on just one kinematic variable x, for simplicity. In the case of multi-dimensional integrals depending on more than one dimensionless scales, all the presented algebraic manipulations will hold, but in this case one should apply them to each partial differential equation (PDE) in order to simplify the entire set of PDE describing the evolution of the integral.
We start from equations for some chosen basis of master integrals $\vec{f}$ obtained at the end of an IBP reduction procedure

$$\partial_x \vec{f}(x, \epsilon) = A(x, \epsilon)\vec{f}(x, \epsilon), \tag{4.122}$$

where A is an $N \times N$ matrix where information about the kinematics is encoded. From the structure of the IBP relations it follows that A depends on x and ϵ in a rational way.

Dependence on x
The form of the differential equations we are seeking is such that the singularities of the DE correspond to those of the original Feynman integrals. The latter, we recall, are regular singularities, namely of the type $\sim (x - x_k)^\alpha$, for some values of α and being x_k the singular points. This means that if we inspect the behaviour

of the DE Eq.(4.122) near a singular point, say $x = 0$ without loss of generality, we can expand A as

$$A(x, \epsilon) = \frac{1}{x^p} \sum_{k \geq 0} x^k A_k(\epsilon), \tag{4.123}$$

for some value of p with $p \geq 1$. Because the system of DE is regular singular in $x = 0$ there exists some basis change

$$\vec{f} = T\vec{g}, \quad \partial_x \vec{g}(x, \epsilon) = B(x, \epsilon)\vec{g}(x, \epsilon) \tag{4.124}$$

described by an invertible matrix T,

$$B = T^{-1}AT - T^{-1}\partial_x T \tag{4.125}$$

for which the matrix B describing the DE system in the new basis has $p \leq 1$

$$B(x, \epsilon) = \frac{1}{x} B_0(\epsilon) + \mathcal{O}(x^0). \tag{4.126}$$

Consequently, near each singular point the solution has the desired behaviour $x^{B_0(\epsilon)}$. Without entering the details, we just point the interested reader to the mathematical literature [12] , [14] where the problem of the degree of singularity of a DE system was studied and it was shown that under certain conditions the order of the singular term can be reduced by means of a transformation which is rational in x. It follows that removing spurious singularities at one singular point does not influence the behaviour at other points (except at infinity), so that the following form for the DE system can be algorithmically reached (for simplicity we maintain the name f for the basis, even though a basis change has been performed)

$$\partial_x \vec{f}(x, \epsilon) = \left[\sum_k \frac{a_k(\epsilon)}{x - x_k} + p(x, \epsilon) \right] \vec{f}(x, \epsilon) \tag{4.127}$$

where $p(x, \epsilon)$ is polynomial in x and contains, if non zero, a spurious singularity at infinity. A possible way to remove an eventual such singularity consists in introducing another singular point which has the property to balance the transformation at infinity.
In the end the form

$$\partial_x \vec{f}(x, \epsilon) = \left[\sum_k \frac{a_k(\epsilon)}{x - x_k} \right] \vec{f}(x, \epsilon) \tag{4.128}$$

is reached, where only regular singularities are manifest.

Dependence on ϵ

Given the DE system in the form Eq.(4.128), the solution around a singular point, say again $x = 0$ without loss of generality, takes the form

$$\vec{f} = P(x,\epsilon)x^{a_0(\epsilon)}\vec{f}_0(\epsilon), \tag{4.129}$$

where $\vec{f}_0(\epsilon)$ is a boundary vector, independent of x, at $x \to 0$, and

$$P(x,\epsilon) = \mathbb{I} + \sum_{m\geq 1} x^m P_m(\epsilon) \tag{4.130}$$

is a matrix polynomial whose expansion can be determined recursively from the information in Eq.(4.128). This shows clearly how Eq.(4.128) already contains all the information about the scaling of the integrals in the singular points. To be more specific, the solution around the singular point x_k is a linear combination of terms whose scaling powers are the eigenvalues of the matrix $a_k(\epsilon)$.
From this consideration arises spontaneously the question if the dependence on ϵ can be further simplified. By construction (from IBPs) the dependence on ϵ is rational. Poles in ϵ are spurious and can be removed with a procedure which is similar to the removal of spurious divergences in x [127], [13].
For a polynomial dependence on ϵ, the main cases can be identified. If the r.h.s. of the DE is $\mathcal{O}(\epsilon)$, then the solution at each order in ϵ can be obtained in terms of iterated integrals. If the r.h.s. starts at $\mathcal{O}(\epsilon^0)$, the solution may be more complicated. So the issue boils down to the question if we can construct a transformation that removes the ϵ^0 part of the matrix on the r.h.s. of Eq.(4.128) and what the nature of such transformation is. The answer to this question depends on the nature of the DE. One of three following situations can happen.

1. Removing the ϵ^0 term amounts to choosing a rational normalization factor, so that the needed transformation matrix T is rational.
   Let us give a simple example. Given the $2 \to 2$ process $p + q \to k + k_m$, we consider the dimension-less cut integral representing the phase space for the emission of a massless particle with momentum k and a massive particle with momentum $k_m = p + q - k$

   $$f(z) = \int \frac{s^{2-d/2} d^d k}{k^2 (p+q-k)^2 - m^2} \tag{4.131}$$

   with $s = (p+q)^2$ and $z = m^2/s$. The differential equation for such integral reads

   $$\partial_z f(z) = f(z) \left[\frac{1-2\epsilon}{-1+z} + \frac{1-\epsilon}{z} \right]. \tag{4.132}$$

   We can get rid of terms which are homogeneous in ϵ through the simple rational transformation $T = (1-2\epsilon)(-1+\epsilon)/((1-z)z)$, so that if we define

   $$g(z) = \frac{(1-2\epsilon)(-1+\epsilon)}{(1-z)z} f(z) \tag{4.133}$$

we get for g

$$\partial_z g(z) = g(z)\left[-\frac{2\epsilon}{-1+z} - \frac{\epsilon}{z}\right] \tag{4.134}$$

which is exactly the desired canonical form of the equation.

2. Removing the ϵ^0 term can be done using algebraic functions [12] : the transformation matrix T contains not only rational functions but also algebraic ones, but sometimes a change of variables restores a purely rational dependence in the DE system.
   Let us give the following simple example. We start from the system

$$\partial_x \vec{f}(x,\epsilon) = \begin{pmatrix} 0 & 0 \\ \frac{\epsilon}{4-x} & \frac{2+\epsilon x}{(4-x)x} \end{pmatrix} \vec{f}(x,\epsilon) \tag{4.135}$$

   and apply the transformation $\vec{f} \to T\vec{f}$ with $T = \mathrm{diag}(1, 1/\sqrt{1-4/x})$. We obtain

$$\partial_x \vec{f}(x,\epsilon) = \epsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{x(x-4)}} & \frac{1}{x-4} \end{pmatrix} \vec{f}(x,\epsilon) \tag{4.136}$$

   which makes the r.h.s. $\mathcal{O}(\epsilon)$. Furthermore in this case a rational form of the system can be recovered by applying the variable change $x = -(1-y)^2/y$.

3. Removing the ϵ^0 part is possible through a transformation but this leads to solutions containing elliptic or even more complicated functions. As it is pointed out in [45], a necessary but not sufficient condition to get elliptic functions is that the DE system remains coupled even at $\epsilon = 0$. This comes from the fact that elliptic functions satisfy higher order differential equations and this is equivalent to a system of coupled first order equations.
   Concerning Feynman integrals, the simplest case where this can occur is the two-loop sunrise with equal masses (see [76] for references), which indeed requires elliptic functions to be expressed.

4.4.3 Properties of canonical form and Iterated Integrals

We suppose now that, given a certain system of differential equations, we have managed to simplify both the x and the ϵ dependence, thus arriving at the canonical

[12] An *algebraic function* is a function that can be defined as the root of a polynomial equation. Quite often algebraic functions can be expressed using a finite number of terms, involving only algebraic operations addition, subtraction, multiplication, division, and raising to a fractional power.

form in which the system only contains regular singularities and the ϵ dependence is totally factorized from the kinematic [13]

$$\partial_x \vec{f}(x,\epsilon) = \epsilon A(x)\vec{f}(x,\epsilon) \tag{4.137}$$

with $\vec{f}$ being the vector of unknown functions and A being a $N \times N$ matrix. Such DE system can be solved with Picard's method of successive approximation ([34]) which we briefly recall in the following.

Picard integration. Let's consider the ordinary DE system

$$\frac{d}{dt}X(t) = A(t)X(t), \quad X(t_0) = X_0. \tag{4.138}$$

This is equivalent to the system of integral equations

$$X(t) - X_0 = \int_{t_0}^{t} A(s)X(s)ds. \tag{4.139}$$

The method consists in solving the system recursively. We define

$$X_{n+1}(t) = X_0 + \int_{t_0}^{t} A(s)X_n(s)ds \text{ for } n \geq 0. \tag{4.140}$$

The first couple of terms of the solution will give

$$\begin{aligned} X_1(t) &= X_0 + \int_{t_0}^{t} A(s)dsX_0 \\ X_2(t) &= X_0 + \int_{t_0}^{t} A(s)dsX_0 + \int_{t_0}^{t} A(s)ds \int_{t_0}^{s} ds' A(s')X_0 \\ &\vdots \end{aligned} \tag{4.141}$$

Assuming $t_0 \leq t$, the second term in the expansion can be written as

$$\int_{t_0 \leq s_1 \leq s_2 \leq t} A(s_2)A(s_1)ds_1 ds_2 X_0. \tag{4.142}$$

Continuing in the same way, we can formally write the limit $X(t) = \lim_{n\to\infty} X_n(t)$ as $X(t) = T(t,t_0)X_0$, where $T(t,t_0)$ is given explicitly by

$$T(t,t_0) = 1_n + \sum_{n\geq 1} \int_{t_0 \leq s_1 \leq \ldots \leq s_n \leq t} A(s_n)A(s_{n-1})\ldots A(s_1)ds_1\ldots ds_n. \tag{4.143}$$

[13] Again we suppose to have equations depending on just one dimensionless variable, but all the following argument hold identical also in the case of systems of partial differential equations.

The r.h.s. is an infinite sum of iterated integrals, which we will formally define in the following, and we recognize in $T(t, t_0)$ the expanded form of a path-ordered exponential, so that we can write the solution in the compact form

$$X(t) = P\, e^{\int_{t_0}^{t} A(s)ds} X_0, \tag{4.144}$$

being P the path-ordering operator. We recognize in Eq.(4.144) the well known Dyson series, written in the general case in which matrices $A(s_i)$ evaluated in different points s_i on the path do not commute.

If we now go back to our DE system for a set of Feynman integrals, and rewrite it for convenience in differential form

$$d\vec{f}(x,\epsilon) = \epsilon\, d\tilde{A}(x)\vec{f}(x,\epsilon), \tag{4.145}$$

we see that indeed the solution can be written in compact form as a path-ordered exponential

$$\vec{f}(x,\epsilon) = P\, e^{\epsilon \int_{\mathcal{C}} d\tilde{A}} \vec{f_0}(\epsilon) \tag{4.146}$$

where $\mathcal{C}$ is a path connecting the boundary conditions, for instance $x = 0$, to x and $\vec{f_0}(\epsilon)$ is the vector of boundary conditions. If we now expand the exponential around $\epsilon = 0$, we find a perturbative solution in terms of iterated integrals, where the entries of $d\tilde{A}$ determine the integration kernel. The power of the canonical form Eq.(4.137), and in particular of the factorization of ϵ from the kinematics, is that when the system is solved in such a perturbative approach around $\epsilon = 0$, we are guaranteed that order by order in ϵ the system, which is originally coupled, decouples, thus allowing us to carry out integration.

At this point it is necessary to define more precisely what an iterated integral is, together with its basic properties.

Iterated integrals. Let k be a real or complex number and M a smooth manifold over k. Let $\gamma : [0,1] \to M$ be a piecewise smooth path on M, and let $\omega_1, ..., \omega_n$ be smooth k-valued 1-forms on M. The ordinary line integral is given by

$$\int_{\gamma_1} \omega_1 = \int_0^1 f_1(t_1)dt_1 \tag{4.147}$$

and does not depend on the choice of the parametrization $f(t)$ of γ.
The iterated integral of $\omega_1, ..., \omega_n$ along γ is defined by

$$\int_{\gamma} \omega_1 ... \omega_n = \int_{0 \le t_1 \le ... \le t_n \le 1} f_1(t_1)dt_1 ... f_n(t_n)dt_n. \tag{4.148}$$

More generally, we call iterated integral a k-linear combination of such integrals. Iterated integrals satisfy the following basic properties:

1. The iterated integral $\int_\gamma \omega_1 ... \omega_n$ does not depend on the choice of parametrization of the path γ.
2. If $\gamma^{-1}(t) = \gamma(1-t)$ denotes the reversal of the path γ, then

$$\int_{\gamma^{-1}} \omega_1 ... \omega_n = (-1)^n \int_\gamma \omega_n ... \omega_1 \tag{4.149}$$

3. The shuffle algebra product formula holds

$$\int_\gamma \omega_1 ... \omega_r \int_\gamma \omega_{r+1} ... \omega_{r+s} = \sum_{\sigma \in \Sigma(r,s)} \int_\gamma \omega_{\sigma(1)} ... \omega_{\sigma(r+s)} \tag{4.150}$$

where $\Sigma(r,s)$ is the set of possible shuffles

$$\Sigma(r,s) = \{\sigma \in \Sigma(r+s) : \sigma^{-1}(1) < ... < \sigma^{-1}(r) \text{ and } \sigma^{-1}(r+1) < ... < \sigma^{-1}(r+s)\} \tag{4.151}$$

and $\Sigma(n)$ is the set of permutations on $\{1, ..., n\}$.

Given the definition above of iterated integrals, we recognize immediately that Multiple Polylogarithms constitute a subset of them obtained by requiring the 1-form $\omega_1, ..., \omega_n$ to be logarithmic 1-form. Experience shows that in the majority of cases, MPLs are sufficient to express the solution for an unknown set of Feynman integrals. Whenever this is the case, the canonical form of the DE system for the set of integrals can be further constrained to assume a precise shape. In the next subsection we will analyse this particular case of canonical form, which finds in multi-loop/legs computations broad application and which has been extensively used in this project.

4.4.4 d-log canonical form: Properties and recursive integration

Let us consider the case where the differential equations can be put into the form

$$d\vec{f}(x,\epsilon) = \epsilon (d\tilde{A}) \vec{f}(x,\epsilon) \tag{4.152}$$

with

$$\tilde{A} = \left[\sum_k A_k \log \alpha_k(x) \right] \tag{4.153}$$

and

$$\alpha_k = x - x_k \tag{4.154}$$

with x_k being the locations of the singularities. In other words we assume we have managed to put the DE system in a canonical form where the total differential of the matrix coefficient is a *logarithmic 1-form* whose arguments are rational linear functions of the dimensionless variable x (this can be generalized to the multi-variable case straightforwardly by replacing x with $\vec{x}$ in Eq.(4.152) and (4.153)). We stress that Eq.(4.154) holds whenever this representation can be reached via rational transformations only. In the general case, where the d-log representation is reached through algebraic transformations, the entries α_k will exhibit an algebraic dependence on x. We will assume in the following that a proper remapping can always be found such that, starting from the general case of algebraic dependence of the α_k, a linear dependence can always be restored [14]. Given this, we take Eq.(4.152), (4.153) and (4.154) as starting point and we discuss in the following their properties.

- **Multiple Polylogarithms**. First, we discuss the category of functions appearing in the solution. As it was pointed out, the solution of a DE system in canonical form is given by iterated integrals. But in this particular case, since we are integrating logarithmic 1-forms, it is immediate to see that the iterated integrals we get are by definition Multiple Polylogarithms! Indeed, as usual we can solve Eq.(4.152) pertubatively in ϵ and the contribution of $\mathcal{O}(\epsilon^n)$ to the solution will be an iterated integral of the form

$$\vec{f_n}(t) = f_0^{(n)} + \int_{0\leq t_1\leq \ldots \leq t_n \leq 1} f_1(t_1)dt_1 \ldots f_n(t_n)dt_n \tag{4.155}$$

  where $f_0^{(n)}$ is the initial condition at order $\mathcal{O}(\epsilon^n)$ and the weight functions $f_1(t_1), \ldots, f_n(t_n)$ are the of the type $f_i(t_i) = 1/(x - x_i)$ where x_i belongs to the set $\{x_k\}$ of singular points of the equations. So the first striking feature of a d-log representation is that the solution can be written entirely in terms of MPLs.
  It is important to stress that starting from a general d-log representation (not necessarily of 'rational type') the solution can always be written in terms of MPLs, almost by definition. If the dependence on the dimensionless variables of the d-log form is rational, then MPLs are straightforwardly obtained, as it has just been showed. On the other hand, if we start from a d-log form containing algebraic dependence on x, the solution we get might look very ugly and far from being just a linear combination of MPLs, but in principle it can be rewritten entirely in terms of MPLs via suitable transformations acting on the iterated integrals in terms of which the solution is originally written.

[14] While we can always assume that such a remapping exists, we stress that no algorithm is available to find it in the general case.

- **Uniform transcendentality**. The second property follows directly from the first one, namely from the possibility of writing a solution entirely in terms of MPLs.
It is possible to introduce for MPLs, the concept of *degree of transcendentality* $\mathcal{T}(f)$ of a function f, which is defined as the number of iterated integrals needed to define the function f, e.g. $\mathcal{T}(\log) = 1$, $\mathcal{T}(\mathrm{Li}_n) = n$, etc.... It also holds $\mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$. Constants obtained at special values are also assigned transcendentality, for instance *zeta values* it is set $\mathcal{T}(\zeta_n) = n$[15] . Algebraic factors have degree zero. If a function f is a sum of terms, we say that f has *uniform (degree of) transcendentality* when all its terms have the same degree. In the following we will address the concept of degree of transcendentality also with the word 'weight' since for a MPL these two quantities happen to be the same. On top of that, we define such functions *pure* if their degree of transcendentality is lowered by taking a derivative, i.e. $\mathcal{T}(d\ f) = \mathcal{T}(f) - 1$. This implies that transcendental functions in f cannot be multiplied by algebraic coefficients, which would otherwise be 'seen' by the differential operators. A remarkable property of the solution of a d-log canonical system of equations is that each term in the ϵ-expansion is a $\mathbb{Q}$-linear combination of pure MPLs of the same weight. For a generic integral basis, this is not true and the results looks more complicated, with terms of different weights being mixed and prefactors of MPLs being algebraic functions of the kinematic variables. Finally, if weight -1 is assigned to ϵ, then the solution of a d-log form has uniform weight zero.

These properties make the solution to a canonical basis particularly compact and elegant. But on top of that they have importance on a practical level for two reasons. First they are a good way to check if the solution we have found for the system is correct or not, since this solution must be uniform and pure. Second, they can become a guiding principle for finding an appropriate integral basis. This

[15] We recall that *zeta values* are values taken by the *Riemann zeta function*

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^x - 1}{e^u - 1} du \tag{4.156}$$

for integer arguments $x = n, n \in \mathbb{N}$. The equivalent definitions which can be derived from the previous ones are zeta values as the sum of a series (known as *p-series*)

$$\zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n} \tag{4.157}$$

and zeta values as the result of multiple integration

$$\zeta(n) = \overbrace{\int_0^1 ... \int_0^1}^{n} \frac{\prod_{i=1}^n dx_i}{1 - \prod_{i=1}^n x_i}. \tag{4.158}$$

approach is not based on purely mathematical simplification of the DE system as we showed above, but it deals instead with the original integral representation. Since we did not use this second type of approach in our project, we do not develop it in the present thesis, but we refer the interested reader to [76], where a basic explanation can be found.

Integration of d-log Differential Equations.
To conclude the section, we report on an algorithm to integrate a d-log system of equations recursively in ϵ ([77]). This is actually the method we applied in our project to integrate the master integrals. Since we dealt in our specific case with master integrals depending on two dimension-less variables, i.e. we actually had to integrate system of PDEs, we are going to illustrate how the algorithm works in this specific multi-variable case with an example taken from our own computation, but obviously the generalization to a higher number of variables is straightforward.

The method consists in integrating separately in each variable and order by order in ϵ. We take one of our bases of master integrals

$$\vec{M}_{t_1}(z,y,\epsilon) = \begin{pmatrix} M_1(z,y,\epsilon) \\ M_2(z,y,\epsilon) \\ M_3(z,y,\epsilon) \\ M_4(z,y,\epsilon) \\ M_5(z,y,\epsilon) \\ M_6(z,y,\epsilon) \end{pmatrix} \tag{4.159}$$

related to a certain topology t_1 for which we wrote down differential equations in d-log form as

$$d\vec{M}_{t_1}(z,y,\epsilon) = A_{t_1}(z,y,\epsilon)\cdot \vec{M}_{t_1}(z,y,\epsilon). \tag{4.160}$$

A_{t_1} is the matrix of coefficients of the differential equations (in the following matrices and masters are intended to depend on z, y, ϵ)

$$A_{t_1} = Z1_{t_1} + Z0_{t_1} + ZY_{t_1} + Y1_{t_1} + Y0_{t_1} \tag{4.161}$$

$$Z1_{t_1} = \epsilon \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ -1 & 0 & 0 & 0 & 0 & -4 & 0 \end{pmatrix} d\log(1-z) \tag{4.162}$$

$$Z0_{t_1} = \epsilon \begin{pmatrix} 2 & -6 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} d\log(z) \tag{4.163}$$

$$ZY_{t_1} = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 & -2 & 2 \\ 1 & 0 & 0 & 4 & 0 & 2 & -2 \\ 0 & 0 & 3 & -4 & 0 & -2 & 2 \\ 1 & 0 & 6 & -4 & 0 & -2 & 2 \end{pmatrix} d\log(y+z) \tag{4.164}$$

$$Y1_{t_1} = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix} d\log(1+y) \tag{4.165}$$

$$Y0_{t_1} = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 0 & 2 & -3 \end{pmatrix} d\log(y). \tag{4.166}$$

The vector of solutions will be given by a Laurent series around $\epsilon = 0$

$$\vec{M}_{t_1} = \sum_{i=-1}^{i=3} \vec{M}_{t_1}^{(i)} \epsilon^i + \mathcal{O}(\epsilon^5) \tag{4.167}$$

We establish to integrate first in z and then in y. The partial DE in z and in y at a given order in ϵ reads

$$\partial_z \vec{M}_{t_1}^{(n)}(z,y) = \left(A_{t_1}\right)_z (z,y) \vec{M}_{t_1}^{(n-1)}(z,y) \tag{4.168}$$

$$\partial_y \vec{M}_{t_1}^{(n)}(z,y) = \left(A_{t_1}\right)_y (z,y) \vec{M}_{t_1}^{(n-1)}(z,y) \tag{4.169}$$

with

$$
\begin{aligned}
(A_{t_1})_z &= Z1_{t_1} + Z0_{t_1} + ZY_{t_1} \\
(A_{t_1})_y &= Y1_{t_1} + Y0_{t_1} + ZY_{t_1}
\end{aligned}
\tag{4.170}
$$

We integrate Eq.(4.168) and find a solution up to an arbitrary function of y to be fixed

$$
\vec{M}_{t_1}^{(n)}(z,y) = \vec{h}_{t_1}^{(n)}(y) + \int dz' \, (A_{t_1})_z \, (z',y) \vec{M}_{t_1}^{(n-1)}(z',y). \tag{4.171}
$$

To fix $\vec{h}_{t_1}^{(n)}(y)$, we insert the solution we just found Eq.(4.171) into Eq.(4.169), thus finding a differential equation for $\vec{h}_{t_1}^{(n)}(y)$

$$
\partial_y \vec{h}_{t_1}^{(n)}(y) = (B_{t_1})_y \, (y) \vec{h}_{t_1}^{(n-1)}(y) \tag{4.172}
$$

where the matrix $(B_{t_1})_y \, (y)$ is related in a non trivial way to $(A_{t_1})_y \, (z,y)$. Integrating this equation we get

$$
\vec{h}_{t_1}^{(n)}(y) = \vec{C}^{(n)} + \int dy' \, (B_{t_1})_y \, (y') \vec{h}_{t_1}^{(n-1)}(y') \tag{4.173}
$$

where $\vec{C}^{(n)}$ is a vector of constants which must be determined by imposing the value of the function in a specific point $(\tilde{z}, \tilde{y})$.

We apply all this to our vector of integrals. We start from order $i = -1$, where our ansatz is just a vector of constants

$$
\vec{M}_{t_1}^{(-1)}(z,y) = \begin{pmatrix} c_1^{(0)}/\epsilon \\ c_2^{(0)}/\epsilon \\ c_3^{(0)}/\epsilon \\ c_4^{(0)}/\epsilon \\ c_5^{(0)}/\epsilon \\ c_6^{(0)}/\epsilon \\ c_7^{(0)}/\epsilon \end{pmatrix}. \tag{4.174}
$$

We insert this into Eq.(4.168) and by integrating in terms of Goncharov Polylogarithms, we determine the dependence on z of the ansatz at order $i = 0$

$$
\begin{aligned}
\vec{M}_{t_1}^{(0)}(z,y) = \Big\{ & 2(c_1^{(0)} - 3c_2^{(0)}) G[\{0\}, z] - 4c_1^{(0)} G(\{1\}, z) + h_1(y), \\
& (c_1^{(0)} - 3c_2^{(0)}) G(\{0\}, z) - c_1^{(0)} G(\{1\}, z) + h_2(y), \\
& c_1^{(0)} G(\{1\}, z) + (-c_1^{(0)} - 3c_3^{(0)}) G(\{-y\}, z) + h_3(y),
\end{aligned}
$$

$$
\begin{aligned}
&- c_4^{(0)} G(\{0\}, z) - c_6^{(0)} G(\{1\}, z) \\
&+ (3c_3^{(0)} - 4c_4^{(0)} - 2c_6^{(0)} + 2c_7^{(0)}) G(\{-y\}, z) + h_4(y), \\
&(-4c_4^{(0)} + 2c_7^{(0)}) G(\{0\}, z) - (c_1^{(0)} + 2c_6^{(0)}) G(\{1\}, z) \\
&+ (c_1^{(0)} + 4c_4^{(0)} + 2c_6^{(0)} - 2c_7^{(0)}) G(\{-y\}, z) + h_5(y), \\
&2c_4^{(0)} G(\{0\}, z) - 4c_6^{(0)} G(\{1\}, z) \\
&+ (3c_3^{(0)} - 4c_4^{(0)} - 2c_6^{(0)} + 2c_7^{(0)}) G(\{-y\}, z) + h_6(y), \\
&- c_7^{(0)} G(\{0\}, z) - (c_1^{(0)} + 4c_6^{(0)}) G(\{1\}, z) \\
&+ (c_1^{(0)} + 6c_3^{(0)} - 4c_4^{(0)} - 2c_6^{(0)} + 2c_7^{(0)}) G(\{-y\}, z) + h_7(y) \Big\}
\end{aligned}
\tag{4.175}
$$

In order to determine the vector of functions $h_i(y)$, we insert Eq.(4.175) into the y-partial d.e. and find

$$
\begin{cases}
h_1'(y) = 0 \\
h_2'(y) = 0 \\
d\log(y)(c_1^{(0)} - 2c_2^{(0)} + c_3^{(0)}) + c_3^{(0)} d\log(y+1) + h_3'(y) = 0 \\
c_7^{(0)} d\log(y) + h_4'(y) = c_7^{(0)} d\log(y+1) \\
d\log(y)(c_1^{(0)} + 4c_4^{(0)} + 2c_6^{(0)} - 2c_7^{(0)}) = 2c_5^{(0)} d\log(y+1) + h_5'(y) \\
(d\log(y) - d\log(y+1))(3c_3^{(0)} - 4c_4^{(0)} - 2c_6^{(0)} + 2c_7^{(0)}) = h_6'(y) \\
d\log(y)(c_1^{(0)} - 3c_2^{(0)} - c_7^{(0)}) + 4c_4^{(0)} d\log(y+1) = h_7'(y)
\end{cases}
\tag{4.176}
$$

The solution to this system of differential equations is

$$
\begin{pmatrix}
h_1(y) = 0 + c_1^{(1)} \\[1ex]
h_2(y) = 0 + c_2^{(1)} \\[1ex]
h_3(y) = -c_3^{(0)} G(\{-1\}, y) + (-c_1^{(0)} + 2c_2^{(0)} - c_3^{(0)}) G(\{0\}, y) + c_3^{(1)} \\[1ex]
h_4(y) = c_7^{(0)} G(\{-1\}, y) - c_7^{(0)} G(\{0\}, y) + c_4^{(1)} \\[1ex]
h_5(y) = -2c_5^{(0)} G(\{-1\}, y) + (c_1^{(0)} + 4c_4^{(0)} + 2c_6^{(0)} - 2c_7^{(0)}) G(\{0\}, y) \\
+c_5^{(1)} \\[1ex]
h_6(y) = (-3c_3^{(0)} + 4c_4^{(0)} + 2c_6^{(0)} - 2c_7^{(0)}) G(\{-1\}, y) \\
+(3c_3^{(0)} - 4c_4^{(0)} - 2c_6^{(0)} + 2c_7^{(0)}) G(\{0\}, y) + c_6^{(1)} \\[1ex]
h_7(y) = 4c_4^{(0)} G(\{-1\}, y) + (c_1^{(0)} - 3c_2^{(0)} - c_7^{(0)}) G(\{0\}, y) + c_7^{(1)}
\end{pmatrix}
\tag{4.177}
$$

We have thus determined our solution at order $i = 0$ up to a vector of b.c.

$$\vec{c}^{(1)} = \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_4^{(1)} \\ c_5^{(1)} \\ c_6^{(1)} \\ c_7^{(1)} \end{pmatrix}. \tag{4.178}$$

We proceed by iterating the same procedure in order to construct higher orders in ϵ.
Boundary conditions are then determined typically through expansion by region at every order in ϵ and matched to the solution.
This concludes this section dedicated to the canonical form of differential equations together with the chapter on Master Integrals computation. In the next chapter, the content of this chapter will be applied to our computation, so that more and perhaps clearer examples of the explained tools will be provided.

Chapter 5

Master Integrals for CC-DIS Form Factors

5.1 NLO Form Factors

In this section we present a simple example of computation via Master Integrals. The quantities we choose to calculate are unrenormalized massive coefficient functions for CC-DIS at NLO, namely the Single Top subprocess $b + W^* \to t + X$ at $\mathcal{O}(\alpha_s)$.

5.1.1 From diagrams to scalar amplitudes

Charged-Current Single Top production starts at order $\mathcal{O}(\alpha_s^0)$ (Leading-Order) with the process $b(p_b) + W^*(q) \to t(p_t)$.

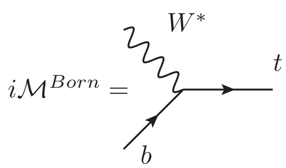


Figure 5.1

We start by drawing diagrams which contribute then at $\mathcal{O}(\alpha_s)$, which we will refer to as NLO in the following. At NLO we have to distinguish between

- **real emission** contributions.
  $\mathbf{b(p_b) + W^*(q) \to t(p_t) + g(k)}$: An extra gluon is radiated and is thus present in the final state as a real particle.

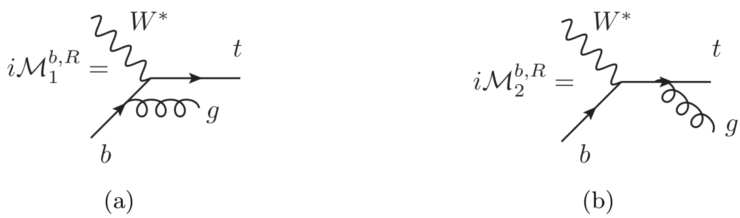


Figure 5.2

$\mathbf{g(p_b) + W^*(q) \to t(p_t) + \bar{b}(k)}$: By applying *crossing symmetry* to the two previous diagrams, we obtain two diagrams for a gluon-initiated process with a top and an anti-bottom in the final state.

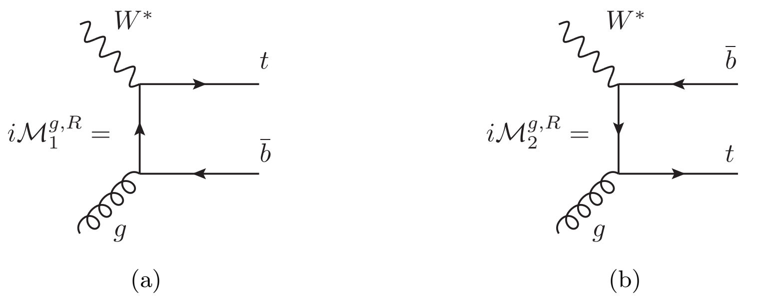


Figure 5.3

- **virtual emission** contributions.
  $\mathbf{b(p_b) + W^*(q) \to t(p_t)}$: Virtual gluons are emitted and reabsorbed giving rise to *vertex* and *self-energy* 1-loop corrections for the bottom and top quarks. Since only connected and amputated Feynman diagrams contribute

to the S-matrix, actually the only contributing diagram is 1-loop correction to the vertex. [1]

(a) (b)

Figure 5.4

With our own `Mathematica` code we translate Feynman diagrams into mathematical expressions, namely amplitudes, and we take the square modulus of the sum of these amplitudes. At this stage, the objects we obtain have all Lorentz indexes saturated except for the ones of the off-shell boson, which will be contracted with the ad hoc projectors in order to extract form factors. Since we have to include only contributions of $\mathcal{O}(\alpha_s)$, the square modulus of the amplitudes will contain the square modulus of the sum of real emission diagrams and the contraction of the virtual diagrams with the Born amplitude. The nature of the contributing sub-processes, namely *bottom*- and *gluon*-initiated naturally divides the computation into two parts, because, having different particles in the initial and final states, these two sub-processes cannot talk to each other when we take the square modulus of the amplitude. We can thus compute separately the contribution coming from the *bottom channel*

$$\left(|i\mathcal{M}^b|^2(\alpha_s)\right)^{\mu\nu} = \left(|i\mathcal{M}_1^{b,R} + i\mathcal{M}_2^{b,R}|^2\right)^{\mu\nu} + 2\Re\left[\left((i\mathcal{M}_2^{b,V}) * i\mathcal{M}^{Born}\right)^{\mu\nu}\right] \quad (5.1)$$

and the one coming from the *gluon-channel*

$$\left(|i\mathcal{M}^g|^2(\alpha_s)\right)^{\mu\nu} = \left(|i\mathcal{M}_1^{g,R} + i\mathcal{M}_2^{g,R}|^2\right)^{\mu\nu}. \quad (5.2)$$

[1] Concerning the self-energy diagrams, the one containing the bottom self-energy evaluates to zero in our 5F scheme ($m_b^2 = 0$) because the 1-loop self-energy of a massless particle is a scaleless 1-loop bubble and thus equal to zero in DR! The diagram containing the top self-energy instead is not zero in our computation because $m_t^2 > 0$. Indeed this diagram, computed off-shell namely by keeping $p_t^2 \neq m_t^2$, provide the renormalization counter-term for the vertex diagram in the on-shell scheme.

In the following, we take C_1 as an example to illustrate how the generic coefficient function C_i is computed.

Example: $\mathbf{C_1}$
The natural thing to do is to divide the computation by channels (bottom and gluon) and, inside each channel to analyse separately real and virtual contributions. In this case, we concentrate on the bottom channel because it involves both real and virtual contributions and can be considered then slightly more complex than the gluon channel, which at NLO involves only real diagrams. On the other hand, it must be said that the gluon real diagrams require the computation of one more master integral with respect to the bottom ones. This additional master has been computed by us with the method of differential equations. We decide nonetheless not to report this MI calculation here, since in the next section the method of differential equations will be widely used and described in all its power and splendour by applying it to 2-loops master integrals.
We proceed to compute the bottom contribution to NLO form factors.

Projectors onto coefficient functions are linear combinations of five fundamental tensor structures

$$\{g_{\mu\nu}, p_{b_\mu} p_{b_\nu}, p_{b_\mu} q_\nu + p_{b_\nu} q_\mu, q_\mu q_\nu, \epsilon_{p_b q \mu\nu}\}. \tag{5.3}$$

In the particular case of C_1, the first three structures contribute. We thus start by contracting all our squared amplitudes with these tensors, to obtain the pieces we will need in the following to construct the desired form factor. In the bottom channel, by using momentum conservation $p_t = p_b + q - k$ to get rid of p_t we obtain for the reals

$$\begin{aligned}
\mathcal{A}_g^{b,R} &= \left(|i\mathcal{M}_1^{b,R} + i\mathcal{M}_2^{b,R}|^2\right)^{\mu\nu} g_{\mu\nu} = \\
&= -\frac{8(-2+d)^2(k\cdot p_b(m_t^2 - q\cdot q) + p_b\cdot q(m_t^2 + 2k\cdot q - 2p_b\cdot q - q\cdot q))}{(k\cdot k - 2k\cdot p_b)^2} \\
&\frac{2\times 8(d-2)(4(k\cdot p_b)^2 + pb\cdot q(2(d-6)(k\cdot q - pb\cdot q) + (d-4)(m_t^2 - q\cdot q))}{(k\cdot k - 2k\cdot p_b)(m_t^2 - 2p_b\cdot q - q\cdot q)} \\
&+ \frac{k\cdot pb((d-2)m_t^2 + 4k\cdot q - 8pb\cdot q + q\cdot q(4-d)))}{(k\cdot k - 2k\cdot p_b)(m_t^2 - 2p_b\cdot q - q\cdot q)} \\
&+ \frac{8(d-2)((d-2)p_b\cdot(q-k)(m_t^2 - q^2) - 2p_b\cdot q(dm_t^2 - (d-2)(-k\cdot q + p_b\cdot q + q^2)))}{(m_t^2 - 2p_b\cdot q - q^2)^2}
\end{aligned} \tag{5.4}$$

$$\mathcal{A}_{p_b p_b}^{b,R} = \left(|i\mathcal{M}_1^{b,R} + i\mathcal{M}_2^{b,R}|^2\right)^{\mu\nu} \left(p_{b_\mu} p_{b_\nu}\right) = \frac{32(d-2)(k\cdot p_b)^2(k\cdot p_b - p_b\cdot q)}{(k\cdot k - 2k\cdot p_b)^2} \tag{5.5}$$

$$\mathcal{A}_{p_b q}^{b,R} = \left(|i\mathcal{M}_1^{b,R} + i\mathcal{M}_2^{b,R}|^2\right)^{\mu\nu} \left(p_{b_\mu} q_\nu + p_{b_\nu} q_\mu\right) =$$

$$= -\frac{(32(d-2)k\cdot p_b(m_t^2 p_b\cdot q + (k\cdot p_b - p_b\cdot q)(-2k\cdot q + 2p_b\cdot q + q\cdot q))}{(k\cdot k - 2k\cdot p_b)^2)}$$
$$+ \frac{2\times 16(d-2)k\cdot p_b(-2(p_b\cdot q)^2 + p_b\cdot q(m_t^2 + 2k\cdot p_b - q\cdot q) + k\cdot p_b q\cdot q)}{(k\cdot k - 2k\cdot p_b)(m_t^2 - 2p_b\cdot q - q\cdot q)} \tag{5.6}$$

and for the virtuals

$$\mathcal{A}_g^{b,V} = \left(\Re\left[\left((i\mathcal{M}_2^{b,V}) * i\mathcal{M}^{Born}\right)^{\mu\nu}\right] g_{\mu\nu}\right) =$$
$$\int d^d l_1 \frac{1}{(l_1\cdot l_1(l_1\cdot l_1 - 2l_1\cdot p_b)(m_t^2 - l_1\cdot l_1 - 2l_1\cdot q - q\cdot q))}\times$$
$$\times \left[8(d-2)(4(l_1\cdot p_b)^2 + ((d-4)l_1\cdot l_1 + 4l_1\cdot q)p_b\cdot q\right.$$
$$\left. + l_1\cdot p_b((2-d)m_t^2 + 4l_1\cdot q + 4p_b\cdot q + (d-4)q\cdot q))\right] \tag{5.7}$$
$$\mathcal{A}_{p_b p_b}^{b,V} = \left(\Re\left[\left((i\mathcal{M}_2^{b,V}) * i\mathcal{M}^{Born}\right)^{\mu\nu}\right](p_{b_\mu} p_{b_\nu})\right) = 0 \tag{5.8}$$
$$\mathcal{A}_{p_b q}^{b,V} = \left(\Re\left[\left((i\mathcal{M}_2^{b,V}) * i\mathcal{M}^{Born}\right)^{\mu\nu}\right](p_{b_\mu} q_\nu)\right) =$$
$$= \int d^d l_1 \frac{16(d-2)l_1\cdot p_b(2(p_b\cdot q)^2 + l_1\cdot p_b\, q\cdot q + p_b\cdot q(-m_t^2 + 2l_1\cdot p_b + q\cdot q))}{(l_1\cdot l_1(l_1\cdot l_1 - 2l_1\cdot p_b)(m_t^2 - l_1\cdot l_1 - 2l_1\cdot q - q\cdot q))}. \tag{5.9}$$

We have thus obtained with scalar objects written in terms of scalar products that constitute the building block for F_1 and also some of the other coefficient functions. Since we need them in more than one coefficient functions, we decide to perform reduction to master integrals directly at the level of these contractions, namely before constructing coefficient functions.

5.1.2 Reduction and MIs computation

We now explain briefly how reduction to MIs is achieved, by taking the $g_{\mu\nu}$ contractions $\mathcal{A}_g^{b,R}$, $\mathcal{A}_g^{b,V}$ as examples.
First, we remind that we are going to use master integral techniques to compute both real and virtual integrations. This implies that the extra-gluon Phase Space measure is converted to cut propagators via reverse unitarity

$$\int d^d k d^d p_t \delta_+(k^2)\delta_+(p_t^2 - m_t^2)\delta^{(4)}(p_b + q - k - p_t) \to$$
$$\to \int d^d k \left(\frac{1}{k^2}\right)_{cut}\left(\frac{1}{(p_b+q-k)^2 - m_t^2}\right)_{cut} \tag{5.10}$$

and cut propagators are then added to the real contributions, so that the object we need to compute is

$$\mathcal{C}_g^{b,R} = \int d^d k \left(\frac{1}{k^2}\right)_{cut}\left(\frac{1}{(p_b+q-k)^2 - m_t^2}\right)_{cut}\mathcal{A}_g^{b,R}. \tag{5.11}$$

For the virtuals instead, we need to integrate just over the top momentum, so that thanks to momentum conservation we will have

$$\int d^d l_1 d^d p_t \delta_+(p_t^2 - m_t^2)\delta^{(4)}(p_b + q - p_t) = \int d^d l_1 \delta((p_b + q)^2 - m_t^2), \tag{5.12}$$

thus obtaining

$$\mathcal{C}_g^{b,V} = \delta(s - m_t^2) \int d^d l_1 \mathcal{A}_g^{b,V}. \tag{5.13}$$

Now, the first step towards reduction is expanding out products and powers appearing at numerator in integrands of Eq.(5.11), (5.13), so that such integrands are written as linear combinations of a certain number of scalar Feynman integrals having at numerator only monomials in the scalar products. The coefficients in front of the integrals will be functions containing scalar products of the only external momenta p_b, q.
Then, the second step is the identification of topologies suitable to express all our scalar integrals. We remind that we call 'topology' a family of propagators which is minimal and complete in a sense that it contains the minimum number of propagators required to express all independent scalar products in terms of them. This implies that the set of propagators in a topology is linearly independent [2].
For more complicated processes than the one under examination there might be a proliferation of topologies coming from different diagrams. One observes that usually some of these topologies happen to be actually the same, up to a shift of integration momenta. In these situations it is usually worth to spend some time to identify among all the possible topologies given by diagrams those that are actually independent. Indeed, this allows to perform reduction to MIs on a smaller number of topologies, thus decreasing since the beginning the number of MIs to be taken into consideration and also the computational time it takes to perform the reductions.
Now, in our NLO example, diagrams contributing are very few and simple, so the situation is simplified with respect to the above-mentioned picture.

- **Real** contribution $\mathcal{C}_g^{b,R}$.
  We have 3 independent scalar products $\{p_b \cdot k, q \cdot k, k^2\}$, so that topologies for these diagrams will contain 3 propagators [3]. From phase space diagrams in this specific case only one topology arise

  $$T^{b,R} = \{k^2, (p_b - k)^2, (p_b + q - k)^2 - m_t^2\}. \tag{5.14}$$

  It is important to mark that the first and third elements of $T^{b,R}$ are the inverse of cut propagators. In other words they have to be understood as

[2] see chapter 4 for more details.
[3] This can be verified by using the formula given in section 4.2 which allows to compute the dimension of a topology given the number of internal and external independent momenta

follows. These quantities are zero when they appear at numerator, whereas they actively contribute to form the integrand of the Phase Space integrals when they appear at denominator.

- **Virtual** contribution $\mathcal{C}_g^{b,V}$.
  Obviously also in this case we have 3 independent scalar products $\{p_b \cdot l_1, q \cdot l_1, l_1^2\}$, so that again topologies for these diagrams will contain 3 propagators. We have one single topology describing the virtuals

$$T^{b,V} = \{l_1^2, (l_1 - p_b)^2, (l_1 + q)^2 - m_t^2\}. \tag{5.15}$$

Now that we have found which topologies describe real and virtual diagrams in the bottom channel [4] , we associate to each scalar integral appearing in the integrands in Eq.(5.11), (5.13) an ordered set of indices where index a_i corresponds to the inverse power with which propagator number i of the topology appears in the integral. This set of indices, together with the topology to which it refers, completely identifies the integral. For instance for the Phase Space we will have

$$\int d^d k \left(\frac{1}{k^2}\right)_{cut} \left(\frac{1}{(p_b + q - k)^2 - m_t^2}\right) = \mathrm{I}[T^{b,R}, \{1, 0, 1\}]. \tag{5.16}$$

We thus rewrite the $g_{\mu\nu}$ contribution to bottom coefficient functions as

$$\begin{aligned}
\mathcal{C}_g^{b,R} =& -4(d-2)^2(m_t^2 - s)\mathrm{I}[T^{b,R}, \{1,1,1\}] \\
&- \frac{2 \times 8(d-2)(\mathrm{I}[T^{b,R}, \{1,0,1\}]((d-4)(m_t^2 - s) - 2Q^2)}{m_t^2 - s} \\
&+ \frac{2(m_t^2 + Q^2)(Q^2 + s)\mathrm{I}[T^{b,R}, \{1,1,1\}])}{m_t^2 - s} \\
&- \frac{4(d-2)\left((d-2)(m_t^2 - s)\mathrm{I}[T^{b,R}, \{1,-1,1\}] + 4m_t^2(Q^2 + s)\mathrm{I}[T^{b,R}, \{1,0,1\}]\right)}{(m_t^2 - s)^2}
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathcal{C}_g^{b,V} =& \delta(s - m_t^2)\Big\{4(d-2)\Big(2\mathrm{I}[T^{b,V}, \{0,0,1\}] - 2\mathrm{I}[T^{b,V}, \{0,1,0\}] - 2Q^2\mathrm{I}[T^{b,V}, \{0,1,1\}] \\
&- 2\mathrm{I}[T^{b,V}, \{1,-1,1\}] + 2\mathrm{I}[T^{b,V}, \{1,0,0\}] - ((d-6)m_t^2 + (d-8)Q^2)\mathrm{I}[T^{b,V}, \{1,0,1\}]
\end{aligned}$$

[4] We stress that in general topologies are independent of the specific contraction we are considering. This means that for instance in the case under inspection topologies Eq.(5.14), (5.15) have been found by analysing the contractions $\mathcal{C}_g^{b,R}$, $\mathcal{C}_g^{b,V}$ with the $g_{\mu\nu}$ tensor, but the same topologies apply to contractions with all the other tensor structures Eq.(5.3). In other words, this just confirms the quite obvious statement that once we decompose the squared amplitude times the Phase Space measure on a basis of tensors, the original singularity structure is conserved untouched in each coefficient of the decomposition. It might happen though that some diagrams gives zero contributions when contracted with some of these structures and in this case the set of topologies coming from that specific contraction is only a subset of the one which is needed to describe all contractions.

$$-2(m_t^2+Q^2)\mathrm{I}[T^{b,V},\{1,1,0\}]-2(m_t^2+Q^2)^2\mathrm{I}[T^{b,V},\{1,1,1\}]\Big)\Big\}\,. \tag{5.18}$$

We collect all scalar integrals appearing in Eq.(5.17), (5.17) and belonging to the given topologies $T^{b,R}$, $T^{b,V}$ and we reduce them to masters. After running the reduction with `Mathematica` package `FIRE`, we see that expressions in Eq.(5.17), (5.18) can actually be expressed in terms of only 3 master integrals

- one MI for the real contribution, namely the Phase Space itself

$$\mathrm{Master}_1=\mathrm{I}[T^{b,R},\{1,0,1\}]=\int d^dk\left(\frac{1}{k^2}\right)_{cut}\left(\frac{1}{(p_b+q-k)^2-m_t^2}\right)_{cut} \tag{5.19}$$

- two for the virtual contribution, namely the massive tadpole

$$\mathrm{Master}_2=\mathrm{I}[T^{b,V},\{0,0,1\}]\overset{l_1\to l_1-q}{\to}\int d^dl_1\frac{1}{l_1^2-m_t^2} \tag{5.20}$$

and the bubble with one massive propagator and external momentum q

$$\mathrm{Master}_3=\mathrm{I}[T^{b,V},\{1,0,1\}]=\int d^dl_1\frac{1}{l_1^2((l_1+q)^2-m_t^2)}. \tag{5.21}$$

By substituting the reduction into Eq.(5.17), (5.18) we obtain the compact expressions

$$\begin{aligned}\mathcal{C}_g^{b,R}=&\mathrm{Master}_1\Bigg\{-\frac{16(1-2\epsilon)(1-\epsilon)^2s}{\epsilon(Q^2+s)}\\&-\frac{16(-1+\epsilon)(Q^2+\epsilon(m_t^2-s)-\frac{(1-2\epsilon)(m_t^2+Q^2)s}{\epsilon(m_t^2-s)})}{m_t^2-s}\\&+\frac{8(1-\epsilon)(-4m_t^2+\frac{(1-\epsilon)(m_t^2-s)^2}{s})(Q^2+s)}{(mt^2-s)^2}\Bigg\}\end{aligned} \tag{5.22}$$

$$\begin{aligned}\mathcal{C}_g^{b,V}=&+\frac{\delta(s-m_t^2)}{(1-2\epsilon)\epsilon m_t^2Q^2}4(-1+\epsilon)\times\\&\times\big(\mathrm{Master}_2((12-7(4-2\epsilon)+(4-2\epsilon)^2)m_t^4-2(1-2\epsilon)m_t^2Q^2+(2-2\epsilon)Q^4)\\&-(1-2\epsilon)\mathrm{Master}_3m_t^2(-2\epsilon m_t^4+4\epsilon^2m_t^2Q^2+(4+6\epsilon-2(4-2\epsilon)\epsilon)Q^4)\big)\,.\end{aligned} \tag{5.23}$$

At this point, we can repeat the same procedure for all contractions with tensor structures in Eq.(5.3) and we can start constructing coefficient functions by using the projectors defined in Eq.(3.41). For instance, we get for C_1 in terms of master integrals

$$C_1^{b,R} = \text{Master}_1 \left\{ \frac{2(s+2\epsilon^2 s - \epsilon(m_t^2+2s)}{\epsilon(Q^2+s)^2} - \frac{2(2\epsilon^2(m_t^2-s)^2 - 2(m_t^2+Q^2)s + \epsilon(m_t^2+s)(m_t^2+2Q^2+s))}{\epsilon(m_t^2-s)^2(Q^2+s)} + \frac{((-1+\epsilon)m_t^4 - 2(-3+\epsilon)m_t^2 s + (-1+\epsilon)s^2)}{(m_t^2-s)^2 s} \right\} \quad (5.24)$$

$$C_1^{b,V} = + \frac{\delta(s-m_t^2)}{\epsilon(-1+2\epsilon)m_t^2(m_t^2+Q^2)} \times \left((-1+\epsilon)\text{Master}_2((-1+2\epsilon)m_t^2+Q^2) - (-1+2\epsilon)\text{Master}_3 m_t^2(2Q^2+\epsilon(m_t^2-Q^2)+2\epsilon^2(m_t^2+Q^2))\right). \quad (5.25)$$

At this stage we just need the expressions for the masters. In this very simple case, we do not need computing none of them, since they are only simple 1-loop integrals that can be found in literature without any effort. In particular the tadpole and the phase space are very basic integrals, so we just report in the following results for them. We want to express final results for NLO coefficient functions in terms of the set of variables $\{s, z, y\}$, with $z = m_t^2/s$ and $y = Q^2/s$. Within this perspective, results for the masters are already expressed in terms of these variables, with s appearing always in front as a prefactor carrying information about the dimension of the integral and multiplying a certain function of z, y which carry instead information about the non trivial dependence of the integral on m_t^2 and Q^2. We also introduce the dependence on the 't Hooft mass of dimensional regularization μ^2, namely a dimensional scale that we need to introduce when working in DR, in order to ensure the coupling g_s to remain dimensionless when switching from 4 to d dimensions.

$$\text{Master}_1 = \left(\frac{s}{\mu^2}\right)^{-\epsilon} (1-z)^{1-2\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \quad (5.26)$$

$$\text{Master}_3 = -s\left(\frac{s}{\mu^2}\right)^{-\epsilon} z^{1-\epsilon}\Gamma(-1+\epsilon)e^{\epsilon\gamma_E} \quad (5.27)$$

The 1-loop bubble, namely Master_2, is also really simple, but we carry out the one-line computation that allows to obtain it to sketch how integration with Feynman parameters works in a very basic case [5].
The integral depends on two dimensional scales, i.e. the internal mass m_t^2 (or

[5] More involved examples can be found in [115].

alternatively s, since in Born kinematic it holds $s = m_t^2$) and the external invariant Q^2. By applying Feynman parametric representation to Eq.(5.21), integrating over one of the two parameters using the δ-function and expressing the result in terms of our set of variables, namely s, y, we get

$$\text{Master}_2 = \left(\frac{s}{\mu^2}\right)^{-\epsilon} \Gamma(\epsilon) \int_0^1 dx_1 \frac{(x_1)^{-\epsilon}}{(1 + y(1 - x_1))^{\epsilon}}. \tag{5.28}$$

The integral can be carried out in closed form in ϵ in terms of Hypergeometric functions, thus giving

$$\text{Master}_2 = \left(\frac{s}{\mu^2}\right)^{-\epsilon} \frac{\pi(1+y)^{-\epsilon} \csc(\epsilon\pi) \text{Hypergeometric}_{2F1}\left(1-\epsilon, \epsilon, 2-\epsilon, \frac{y}{(1+y)}\right)}{\Gamma(2-\epsilon)}. \tag{5.29}$$

We underline that all masters are here expressed in closed form in ϵ because of their simplicity. In more involved computations, this is normally not possible and master integrals can then be computed only as series in ϵ truncated at a certain order. In the next section, where we will compute form factors at the next perturbative order, i.e. $\mathcal{O}(\alpha_s^2)$, this will be actually the case.

Once we have the coefficient function written in terms of masters and also results for the masters , we actually have all the necessary ingredients so that the computation can be considered almost finished, up to a renormalization procedure. By inserting the results for the MIs Eq.(5.26), (5.27), (5.29) into Eq.(5.24), (5.25) and expanding in ϵ, we get

$$\begin{aligned} C_1^{b,R} = \left(\frac{s}{\mu^2}\right)^{-\epsilon} & \left(\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{4}(-8+\pi^2)\right)(1+y)\delta(1-z)\right. \\ & + \frac{1}{2\epsilon}\left(4\left[\frac{1}{1-z}\right]_+ (1+y) - \frac{2(1+2y+z)}{1+y}\right) \\ & + 2\left[\frac{1}{1-z}\right]_+ (1+y) - \left[\frac{\log(1-z)}{1-z}\right]_+ (1+y) \\ & \left.\frac{(2+4y+2z)\log(1-z)}{1+y} + \frac{z^2}{1+y} + \frac{z(1+4y+y^2)}{2(1+y)} - \frac{y+7}{2}\right) \end{aligned} \tag{5.30}$$

Now, concerning the virtual diagrams, we get from the vertex diagram

$$C_1^{b,V,vertex} = \left(\frac{s}{\mu^2}\right)^{-\epsilon} \delta(1-z)(1+y)\left(-\frac{1}{\epsilon^2} - \frac{1 - 2\ln(1+y)}{\epsilon}\right.$$

$$-\frac{48y+\pi^2 y-(12+36y)\ln(1+y)+12y\ln(1+y)^2-24y\mathrm{Li}_2\left(\frac{y}{1+y}\right)}{12y}\Bigg). \quad (5.31)$$

Real and virtual contributions are then to be combined in order to get the unrenormalized coefficient function C_1. When we sum them up all soft poles must cancel and the remaning poles can only represent either UV divergences or IR collinear divergences coming from the initial state. Both these kinds of remaining divergences are then to be removed by means of, respectively, a renormalization and a mass factorization program. We also remind that in our expressions we do not get final state collinear poles because we have a massive top radiating in the final state, but in case we had them they should cancel as well when combining real and virtual contribution. This recaps the strategy that one would adopt in a higher order computation, namely first assembling all the contributing pieces and then performing renormalization and mass factorization. In our case simple NLO computation, since the UV poles just appear in the virtuals, we may also renormalize first the UV divergences in the virtual vertex diagram (thus performing a renormalization at level of single diagrams) and then assembling it with the reals and performing mass factorization (namely renormalization of the initial state collinear poles).
Performing UV renormalization means in general to renormalize running coupling, wave-function and mass. We follow the usual choice, namely to renormalize the running coupling in $\overline{\mathrm{MS}}$ and wave-function and mass in the *on-shell* scheme. At NLO level though, the scenario happens to be quite simplified and it turns out that the only wave-functions need to be renormalized. This can be achieved by means of appropriate counterterms. Following [29], the bare and renormalized coefficient functions satisfy the relation

$$C = Z_{2,b}^{1/2} Z_{2,t}^{1/2} C_{bare}(\alpha_s^{bare}) \quad (5.32)$$

where we dropped for the moment the various indices which identify coefficient functions for simplicity. Indeed we can neglect $Z_{2,b}^{1/2}$ because, being the bottom mass equal to zero in our computation, in an on-shell scheme the expansion of this quantity will read $Z_{2,b}^{1/2} = 1 + \delta Z_{2,b}^{1/2}$, with $\delta Z_{2,b}^{1/2} = 0$. By performing the perturbative expansion of the various left quantities (see again [29], Eq.(3.2)), we get at 1-loop level

$$C^{(1l)} = C_{bare}^{(1l)} + \frac{1}{2}\delta Z_{2,t}^{(1l)} C^{(0l)} \quad (5.33)$$

with $C_{bare}^{(1l)}$ being the sum of real and virtual parts, i.e. in our case

$$C_{1_{bare}}^{(1l)} = C_1^{b,R} + C_1^{b,V,vertex}, \quad (5.34)$$

$C^{(0l)}$ being the LO contribution to the coefficient function which in our case is simply $C_1^{(0l)} = 1$, and finally $\delta Z_{2,t}^{(1l)}$ being the proper counterterm which performs 1-loop wave-function renormalization. The ϵ expansion for this counterterm, truncated at $\mathcal{O}(\epsilon^0)$ reads

$$\delta Z_{2,t}^{(1l)} \overset{\epsilon \to 0}{\to} \left(\frac{s}{\mu^2}\right)^{-\epsilon} \left(-\frac{3}{2\epsilon} - 2\right) + \mathcal{O}(\epsilon). \tag{5.35}$$

By adding Eq.(5.35), (5.30) and (5.31) as prescribed by Eq.(5.33), (5.34), and adding the proper color factor C_F we obtain the renormalized coefficient function

$$\begin{aligned} C_1^{(1l)} = \left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F \Bigg\{ &\frac{1}{\epsilon}\left((1+y)\left(\frac{3}{2} - 2\ln(1+y)\right)\delta(1-z) + 2(1+y)\left[\frac{1}{1-z}\right]_+ \right. \\ &\left. + \frac{(-1-2y-z)}{1+y}\right) \\ &+ \delta(1-z)(1+y)\left(4 + \frac{\pi^2}{3} - 3\ln(1+y) - \frac{\ln(1+y)}{y} + \ln(1+y)^2 - 2\mathrm{Li}_2\left(\frac{y}{1+y}\right)\right) \\ &- 2(1+y)\left(2\left[\frac{\ln(1-z)}{1-z}\right]_+ - \left[\frac{1}{1-z}\right]_+\right) \\ &- \frac{7}{2} - \frac{y}{2} + \frac{z + 4yz + y^2 z + z^2}{2(1+y)} + \frac{2(1+2y+z)\ln(1-z)}{1+y} \Bigg\}. \end{aligned} \tag{5.36}$$

We observe that the double pole, which is a product of a collinear times a soft singularity, gets cancelled between real and virtual contribution, as expected. The UV pole is renormalized, and we are left precisely with just one single collinear pole, due to emission of a real gluon from the massless bottom in the initial state. Mass factorization is then the last step we need to take in order to get a finite result.

At NLO, mass factorization is very simple, since we know that the collinear pole must multiply a coefficient which is exactly the appropriate splitting function, and in this precise case $P_{q/q}(z, y)$.

Now, in order to recognize the splitting function, one should pay particular attention because, due to our variable choice, the splitting will not be manifest. Indeed, the natural DIS-like variables for Single Top would be

$$\tau = \frac{m_t^2 + Q^2}{s + Q^2} = \frac{z+y}{1+y}, \qquad \lambda = \frac{Q^2}{m_t^2 + Q^2} = \frac{y}{1+y} \tag{5.37}$$

in terms of which the product $\frac{1}{\epsilon} P_{q/q}(\tau)$ would reads

$$\frac{1}{\epsilon} P_{q,q}(\tau) = \frac{1}{\epsilon} C_F \left(\frac{m_t^2 + Q^2}{\mu^2}\right)^{-\epsilon} \left[\frac{3}{2}\delta(1-\tau) + \frac{2}{(1-\tau)_+} - (1+\tau)\right]. \tag{5.38}$$

By performing the necessary change of variables, one can easily convince himself that the coefficient of the $1/\epsilon$ pole appearing in Eq.(5.36) is the correct splitting function $P_{q/q}$ reported in Eq.(5.38). After cancelling this last collinear divergence, which is reabsorbed into the b-pdf, we can finally write our final result Eq.(5.39).

$$
\begin{aligned}
C^{(1l)}_{1,ren} = \left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F \{ & \\
& \delta(1-z)(1+y)\left(4+\frac{\pi^2}{3}-3\ln(1+y)-\frac{\ln(1+y)}{y}+\ln(1+y)^2-2\mathrm{Li}_2\left(\frac{y}{1+y}\right)\right) \\
& -2(1+y)\left(2\left[\frac{\ln(1-z)}{1-z}\right]_+-\left[\frac{1}{1-z}\right]_+\right) \\
& -\frac{7}{2}-\frac{y}{2}+\frac{z+4yz+y^2z+z^2}{2(1+y)}+\frac{2(1+2y+z)\ln(1-z)}{1+y}\Bigg\}. \qquad (5.39)
\end{aligned}
$$

In this section we aimed at giving a simple example of how a fixed-order computation works in QCD, via Master Integrals technique. We achieved this goal by showing how one determines and computes Master Integrals for massive CC-DIS at $\mathcal{O}(\alpha_s)$ in QCD, thus getting the renormalized coefficient functions, as for instance the one reported in Eq.(5.39) for the bottom channel.
We stress that renormalized coefficient functions at NLO are fundamental also for the computation of form factors at the next perturbative order, since they constitute the building blocks for the construction of collinear counterterms necessary to renormalize the NNLO coefficient functions.
In the next two sections of chapter 5 we will proceed to illustrate how the techniques of Master Integrals works in the case of $\mathcal{O}(\alpha_s^2)$ corrections, which is more involved because of the larger number of Masters and their increased complexity.

5.2 Master Integrals for CC-DIS at $\mathcal{O}(\alpha_s^2)$

In this section we present the computation of the set of master integrals describing the $\mathcal{O}(\alpha_s^2)$ (NNLO) contribution to form factors.
At this perturbative order, three independent channels are open, namely *bottom, gluon, singlet* channel. It is thus natural to address the computation of NNLO form factors and its explanation channel by channel.
Then, inside each channel the contributing subprocesses will be identified and, for each of them, sets of independent topologies and master integrals will be determined. Results for all master integrals are obtained via the method of Differential Equations (DE). All double real (RR) and real-virtual (RV) MIs are our original results. They have been all computed by using DE in canonical form, which we report in detail in the Appendix for all topologies involved. Since one of the most complicated step into the computation of these masters is the determination of boundary conditions, this part will be addressed in detail throughout the section. The double virtual (VV) masters instead, where already available in [29], [30], [9], [18], [37]. For these MIs, we do not report systematically system of DE, but we take the chance to use them as example to show how non-canonical DE can be integrated. Boundary conditions instead were directly extracted by the above-mentioned references, to which we redirect the reader who might want to know more about this specific topic.

5.2.1 Bottom channel: b-initiated subprocesses at NNLO.

Double Reals $[b + W^* \to t + X_1 + X_2]_{0-loop}$

Topologies and Master Integrals for $b(p_b) + W^*(q) \to t(p_t) + g(k_1) + g(k_2)$.

Diagrams contributing to $b + W^* \to t + g + g$ are drawn in Fig.5.5. We stress that the actual number of diagrams contributing is ten, since for each of the diagrams in Fig.5.5 we have to consider that we have two identical particles (gluons) in the final state, so that the diagram is symmetrical under exchange of the two gluon momenta k_1, k_2. Given this symmetry, in the following we consider matrix elements arising from only one particular assignment of the gluon momenta, since to remaining ones can be obtained be simply exchanging k_1 with k_2 in the matrix elements expressions. By taking the square modulus of diagrams in Fig.5.5 we obtain twenty-one different phase space diagrams. The 3-particle (two massless and one massive) phase space we need for the Double-Reals is given by

$$\mathcal{PS}_3 = \int d^d k_1 \int d^d k_2 \int d^d p_t \left[\frac{1}{k_1^2}\right]_c \left[\frac{1}{k_2^2}\right]_c \left[\frac{1}{p_t^2 - m_t^2}\right]_c \delta^{(4)}\left(p_b + q - k_1 - k_2 - p_t\right)$$

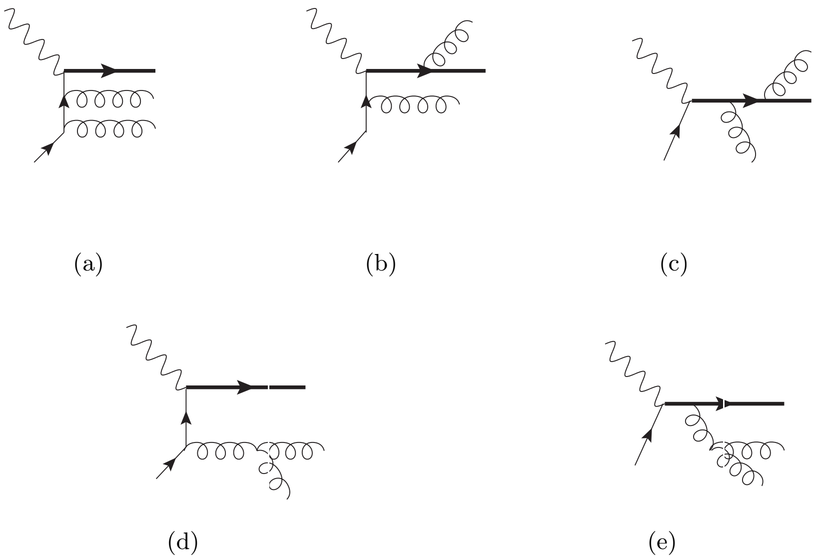


Figure 5.5: Tree-level diagrams for $b + W^* \to t + g + g$

$$= \int d^d k_1 \int d^d k_2 \left[\frac{1}{k_1^2}\right]_c \left[\frac{1}{k_2^2}\right]_c \left[\frac{1}{(p_b + q - k_1 - k_2)^2 - m_t^2}\right]_c . \tag{5.40}$$

If we now multiply the phase-space diagrams by the phase space integration measure [6]

$$d\mathcal{PS}_3 = \frac{1}{k_1^2}\frac{1}{k_2^2}\frac{1}{(p_b + q - k_1 - k_2)^2 - m_t^2}, \tag{5.41}$$

we obtain that our twenty-one phase space diagrams can be all described by just 3 independent topologies

$$\begin{aligned} t_1 =&\{k_1^2, (-k_1 + p_b)^2, k_2^2, (k_1 + k_2 - p_b)^2, (-k_2 + p_b)^2, \\ &- m_t^2 + (-k_2 + p_b + q)^2, -m_t^2 + (k_1 + k_2 - p_b - q)^2\} \\ t_2 =&\{k_1^2, (-k_1 + p_b)^2, k_2^2, (-k_2 + p_b)^2, -m_t^2 + (-k_1 + p_b + q)^2, \\ &- m_t^2 + (-k_2 + p_b + q)^2, -m_t^2 + (k_1 + k_2 - p_b - q)^2\} \\ t_3 =&\{k_1^2, (k_1 + k_2)^2, (-k_1 + p_b)^2, k_2^2, (k_1 + k_2 - p_b)^2, \\ &- m_t^2 + (-k_1 + p_b + q)^2, -m_t^2 + (k_1 + k_2 - p_b - q)^2\}, \end{aligned} \tag{5.42}$$

corresponding to the three diagrams in Fig.5.6.

[6] We omit from now on the symbol $[]_c$ of *cut* propagator for simplicity of notation.

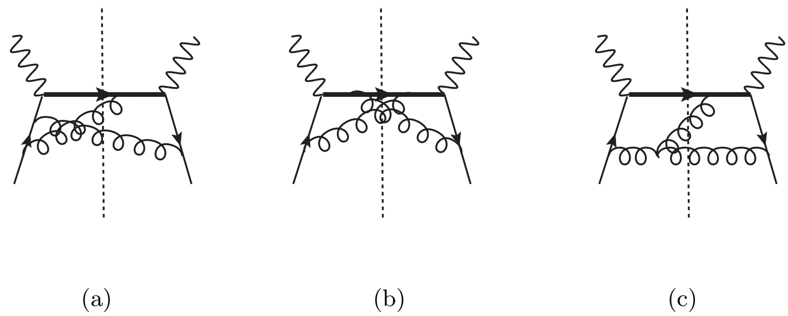


Figure 5.6: Phase-space diagrams for $b+W^* \to t+g+g$ corresponding respectively to the independent topologies t_1, t_2, t_3.

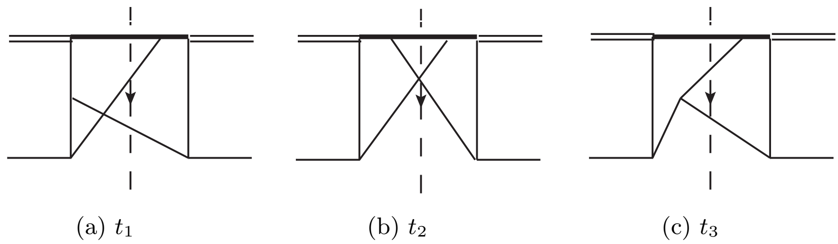


Figure 5.7: Independent topologies to $b + W^* \to t + g + g$.

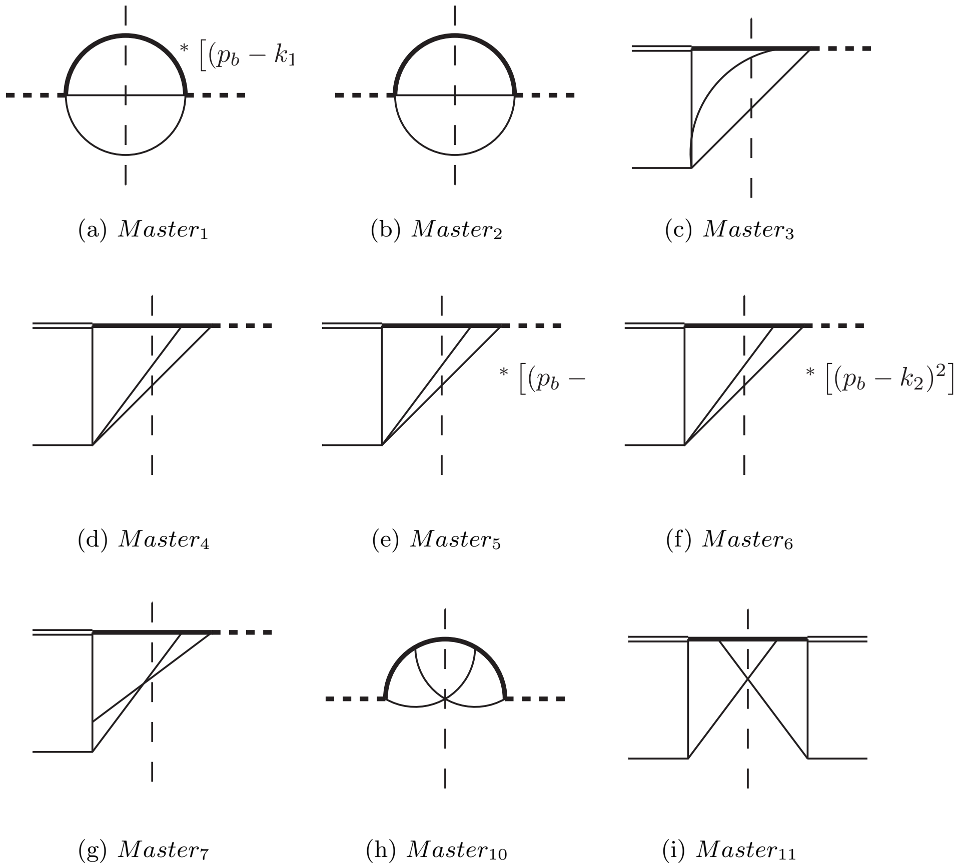
From now on, we will not think anymore in terms of diagrams, but in terms of topologies and subtopologies, meaning that, for the moment being, we will disregard the physical meaning of diagrams and integrals and concentrate only on their mathematical features. We will then adopt a different convention also for drawings, since we will not draw anymore Feynman diagrams, but topologies and integrals. Each (sub-)topology or integral will be represented by a graph. Each internal line correspond to a propagator, whereas external lines represent fixed (external) momenta. Massless lines are represented by normal lines. Massive lines corresponding to m_t^2, Q^2, s are represented respectively by thick lines, double lines and thick dashes lines. Topologies Eq.(5.42) will be represented then by graphs in Fig.5.7. The determination of independent topologies for this subprocess is trivial because set of propagators describing all diagrams other than Fig.5.6 happen to be apparent subtopologies of those in Eq.(5.42), without need to perform any shift on loop momenta. By collecting all scalar integrals given by matrix elements and reducing them to master integrals, we get the following set of master integrals (MIs from $Master_1$ to $Master_7$ belong to t_1, from $Master_8$ to $Master_{11}$ to t_2, from $Master_{12}$ to $Master_{17}$ to t_3).

$$
\begin{aligned}
Master_1 &= G(t_1, \{1, -1, 1, 0, 0, 0, 1\}) \\
Master_2 &= G(t_1, \{1, 0, 1, 0, 0, 0, 1\}) \\
Master_3 &= G(t_1, \{1, 0, 1, 1, 0, 0, 1\}) \\
Master_4 &= G(t_1, \{1, 0, 1, 1, 0, 1, 1\}) \\
Master_5 &= G(t_1, \{1, 1, 1, 1, 0, 1, 1\}) \\
Master_6 &= G(t_1, \{1, -1, 1, 1, 0, 1, 1\}) \\
Master_7 &= G(t_1, \{1, 0, 1, 1, -1, 1, 1\})
\end{aligned}
\tag{5.43}
$$

$$
\begin{aligned}
Master_8 &= G(t_2, \{1, 0, 1, 0, 0, 0, 1\}) \\
Master_9 &= G(t_2, \{1, -1, 1, 0, 0, 0, 1\}) \\
Master_{10} &= G(t_2, \{1, 0, 1, 0, 1, 1, 1\}) \\
Master_{11} &= G(t_2, \{1, 1, 1, 1, 1, 1, 1\})
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
Master_{12} &= G(t_3, \{1, 0, 0, 1, 0, 0, 1\}) \\
Master_{13} &= G(t_3, \{1, -1, 0, 1, 0, 0, 1\}) \\
Master_{14} &= G(t_3, \{1, 0, 0, 1, 1, 0, 1\}) \\
Master_{15} &= G(t_3, \{1, 0, 0, 1, 1, 1, 1\}) \\
Master_{16} &= G(t_3, \{1, -1, 0, 1, 1, 1, 1\}) \\
Master_{17} &= G(t_3, \{1, 0, -1, 1, 1, 1, 1\})
\end{aligned}
\tag{5.45}
$$

The only independent integrals are contained in the basis for the first two topologies t_1, t_2 (Fig.5.8). Masters for the third topology t_3 are not independent since they can be obtained from masters $Master_i$ with $i = 1, ..., 11$ by appropriate shifts of integration momenta, so that from now we will discard this basis, since it does not provide any new independent integral. Inside the basis for t_1, t_2, all master integrals are independent except for $Master_8$ and $Master_9$ which coincide with $Master_1$ and $Master_2$. In order to determine these MIs, according to the approach presented in Chapter 4, we transform each one of these basis into a canonical basis of integrals, listed in Eq.(5.47).



(a) $Master_1$ (b) $Master_2$ (c) $Master_3$

(d) $Master_4$ (e) $Master_5$ (f) $Master_6$

(g) $Master_7$ (h) $Master_{10}$ (i) $Master_{11}$

Figure 5.8: Set of independent MIs for $b + W^* \to t + g + g$.
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

$$
\begin{aligned}
M_1 &= (1-z)(2G(t_1,\{2,0,1,0,0,0,2\}) + G(t_1,\{2,0,2,0,0,0,1\}))\\
M_2 &= G(t_1,\{2,0,1,0,0,0,2\})\\
M_3 &= 2\epsilon(1+y)G(t_1,\{1,0,2,1,0,0,1\})\\
M_4 &= \epsilon(1+y)zG(t_1,\{1,0,1,1,0,1,2\})\\
M_5 &= 2\epsilon^2(1+y)^2G(t_1,\{1,1,1,1,0,1,1\})\\
M_6 &= \epsilon(1+y)(1-z)G(t_1,\{1,0,2,1,0,1,1\})\\
M_7 &= 2\epsilon(1+y)zG(t_1,\{1,0,1,1,0,2,1\})
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
M_8 &= (1-z)(2G(t_2,\{2,0,1,0,0,0,2\}) + G(t_2,\{2,0,2,0,0,0,1\}))\\
M_9 &= G(t_2,\{2,0,1,0,0,0,2\})\\
M_{10} &= 2(1-2\epsilon)G(t_2,\{1,0,1,0,1,1,1\})\\
M_{11} &= \epsilon^2(1+y)^2(1-z)G(t_2,\{1,1,1,1,1,1,1\})
\end{aligned}
\tag{5.47}
$$

For these integrals we are thus able to write a system of equations in canonical form, which can then be integrated using the procedure introduced in Section 4.4. We report here the canonical system for topology t_1 to provide an example of how such a system can look like. Results for canonical basis and canonical DE systems are reported systematically for all topologies in the Appendix. Starting from here and throughout the rest of equations appearing in this Chapter and in the Appendix, we will use the short notation dL to indicate the differential of a logarithm which we would otherwise write as $d\log$.

$$\begin{aligned}
dM_1 =&\epsilon(-6M_2\mathrm{dL}(z) + M_1(-4\mathrm{dL}(1-z) + 2\mathrm{dL}(z)))\\
dM_2 =&\epsilon(-3M_2\mathrm{dL}(z) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))\\
dM_3 =&\epsilon(2M_2\mathrm{dL}(y) + M_3(2\mathrm{dL}(y) - \mathrm{dL}(1+y) - 3\mathrm{dL}(y+z)) + M_1(\mathrm{dL}(1-z) - \mathrm{dL}(y+z)))\\
dM_4 =&\epsilon(M_4(4\mathrm{dL}(y) - \mathrm{dL}(z) - 4\mathrm{dL}(y+z)) + M_6(2\mathrm{dL}(y) - \mathrm{dL}(1-z) - 2\mathrm{dL}(y+z))\\
&+ M_7(-3\mathrm{dL}(y) + \mathrm{dL}(1+y) + 2\mathrm{dL}(y+z)) + M_3(-3\mathrm{dL}(y) + 3\mathrm{dL}(y+z)))\\
dM_5 =&\epsilon(-2M_5\mathrm{dL}(1+y) + M_7(2\mathrm{dL}(z) - 2\mathrm{dL}(y+z)) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(y+z))\\
&+ M_6(-2\mathrm{dL}(1-z) + 2\mathrm{dL}(y+z)) + M_4(-4\mathrm{dL}(z) + 4\mathrm{dL}(y+z)))\\
dM_6 =&\epsilon(M_4(4\mathrm{dL}(1+y) + 2\mathrm{dL}(z) - 4\mathrm{dL}(y+z)) + M_6(2\mathrm{dL}(1+y) - 4\mathrm{dL}(1-z) - 2\mathrm{dL}(y+z))\\
&+ M_7(-2\mathrm{dL}(1+y) + 2\mathrm{dL}(y+z)) + M_3(-3\mathrm{dL}(1+y) + 3\mathrm{dL}(y+z)))\\
dM_7 =&\epsilon(-3M_2\mathrm{dL}(y) + M_4(4\mathrm{dL}(y) + 4\mathrm{dL}(1+y) - 4\mathrm{dL}(y+z))\\
&+ M_6(2\mathrm{dL}(y) - 4\mathrm{dL}(1-z) - 2\mathrm{dL}(y+z)) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(y+z))\\
&+ M_7(-3\mathrm{dL}(y) - \mathrm{dL}(z) + 2\mathrm{dL}(y+z)) + M_3(-6\mathrm{dL}(y) + 6\mathrm{dL}(y+z)))
\end{aligned} \tag{5.48}$$

By integrating such systems of differential equations, we obtain the solution for our canonical integrals (and consequently for our original integral too) up to a boundary condition to be determined in a chosen kinematic point. The following paragraph is dedicated to explain how this can be achieved.

Determination of boundary conditions for RR masters.

We use the threshold region $z \to 1$ to fix the value of our integrals. This implies we need to determine in some way the behaviour of our integrals in this region, without obviously knowing the general solution to the integrals themselves.
Given the kinematic plane described by coordinates z, y, we decide indeed to determine the threshold behaviour in a fixed point , identified by the coordinates $\{z = 1, y = 0\}$. In other words we want to determine the asymptotic behaviour of our integrals in the double limit $\{z \to 1, y \to 0\}$, which corresponds actually to $s \simeq m_t^2$ and $Q^2 \ll s$. In order to do this, we follow the procedure described in [7] and briefly reported in the following.
We rescale the momenta k_1, k_2 of the extra particles emitted by the rescaling factor $\bar{z} = 1 - z$, which encodes information about the behaviour of such momenta in the threshold region.
Then, we expand the integrand and the integration measure around $\bar{z} = 0$, so that we are able to extract the leading (singular) behaviour of the integral in this kinematic point. When we perform such expansion some propagators become linear

in the integration momenta and some others simply do not depend anymore on them, thus not belonging anymore to the topology. In this last case, we have to add to the topology some auxiliary propagators which will not appear in the soft integrals but just serve the purpose of completing the topology and enabling us to pass this topology to the reduction program FIRE.
We observe that in the soft limit all RR masters can be written as linear combinations of just two *soft masters*, which are the Phase Space $Master_2$ and $Master_4$ in Eq.(5.43). These two *soft masters* can then be computed explicitly with traditional techniques, such as Feynman parameters or Mellin-Barnes.
Again we use the integrals of topology t_1 to show explicitly how this procedure works. In particular, let us pick up $Master_6$ as defined in Eq.(5.43), and perform on it the above-mentioned soft expansion. We start by rescaling gluon momenta as $k_1 \to \bar{z}k_1$, $k_2 \to \bar{z}k_2$

$$\begin{aligned}
Master_6 &\overset{k_i \to \bar{z}k_i}{\longrightarrow} \int d^d(\bar{z}k_1) d^d(\bar{z}k_2) \times \\
&\times \frac{(-\bar{z}k_1 + p_b)^2}{(\bar{z}k_1)^2(\bar{z}k_2)^2(\bar{z}k_1 + \bar{z}k_2 - p_b)^2(-m_t^2 + (\bar{z}k_1 + \bar{z}k_2 - p_b - q)^2)(-m_t^2 + (-\bar{z}k_2 + p_b + q)^2)} \\
&\overset{\bar{z}\to 0}{\rightarrow} \int \bar{z}^{2d} d^d k_1 d^d k_2 \times \\
&\times \frac{\bar{z}(-2k_1 \cdot p_b)}{(\bar{z})^4 (k_1)^2 (k_2)^2 (-2\bar{z}(k_1 + k_2) \cdot p_b)[\bar{z}(s - 2k_2 \cdot (p_b + q))][\bar{z}(s - 2(k_1 + k_2) \cdot (p_b + q))]} \\
&= \bar{z}^{2-4\epsilon} \int d^d k_1 d^d k_2 \frac{(-2k_1 \cdot p_b)}{((k_1)^2 (k_2)^2 (-2(k_1 + k_2) \cdot p_b)(s - 2k_2 \cdot (p_b + q))(s - 2(k_1 + k_2) \cdot (p_b + q))} \\
&= \bar{z}^{2-4\epsilon} Master_6^s,
\end{aligned} \tag{5.49}$$

where we defined with $Master_6^s$, the integral over the soft propagators obtained after the expansion in $\bar{z}$. If we apply the same procedure to the rest of the basis for topology t_1, we obtain

$$\begin{aligned}
Master_1 &\to \bar{z}^{(4-4\epsilon)} Master_1^s \\
Master_2 &\to \bar{z}^{(3-4\epsilon)} Master_2^s \\
Master_3 &\to \bar{z}^{(2-4\epsilon)} Master_3^s \\
Master_4 &\to \bar{z}^{(1-4\epsilon)} Master_4^s \\
Master_5 &\to \bar{z}^{(-4\epsilon)} Master_5^s \\
Master_6 &\to \bar{z}^{(4-4\epsilon)} Master_6^s \\
Master_7 &\to \bar{z}^{(2-4\epsilon)} Master_7^s,
\end{aligned} \tag{5.50}$$

where the soft integrals $Master_i^s$ are defined as

$$\begin{aligned}
Master_1^s &= \int d^d k_1 d^d k_2 \frac{(-2k_1 \cdot p_b)}{(k_1^2)\,(k_2^2)\,(-2k_1 \cdot p_b - 2k_1 \cdot q - 2k_2 \cdot_b - 2k_2 \cdot q + s)} \\
Master_2^s &= \int d^d k_1 d^d k_2 \frac{1}{(k_1^2)\,(k_2^2)\,(-2k_1 \cdot p_b - 2k_1 \cdot q - 2k_2 \cdot p_b - 2k_2 \cdot q + s)}
\end{aligned}$$

$$Master_3^s = \int d^dk_1 d^dk_2 \frac{1}{(k_1^2)\,(k_2^2)\,(-2(k_1+k_2)\cdot p_b)(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}$$

$$Master_4^s = \int d^dk_1 d^dk_2 \frac{1}{(k_1^2)\,(k_2^2)\,(-2k_1\cdot p_b - 2k_2\cdot p_b)(-2k_2\cdot p_b - 2k_2\cdot q + s)} \times \\ \times \frac{1}{(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}$$

$$Master_5^s = \int d^dk_1 d^dk_2 \frac{1}{(k_1^2)\,(k_2^2)\,(-2k_1\cdot p_b)(-2k_1\cdot p_b - 2k_2\cdot p_b)(-2k_2\cdot p_b - 2k_2\cdot q + s)} \times \\ \times \frac{1}{(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}$$

$$Master_6^s = \int d^dk_1 d^dk_2 \frac{(-2k_1\cdot p_b)}{(k_1^2)\,(k_2^2)\,(-2k_1\cdot p_b - 2k_2\cdot p_b)(-2k_2\cdot p_b - 2k_2\cdot q + s)} \times \\ \times \frac{1}{(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}$$

$$Master_7^s = \int d^dk_1 d^dk_2 \frac{(-2k_2\cdot p_b)}{(k_1^2)\,(k_2^2)\,(-2k_1\cdot p_b - 2k_2\cdot p_b)(-2k_2\cdot p_b - 2k_2\cdot q + s)} \times \\ \times \frac{1}{(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}. \tag{5.51}$$

We have now to run the reduction program (FIRE in our case) on the $Master_i^s$ to check if they can be further reduced. The soft topology to which these integrals belong would be in principle

$$t_1^s = \{\bar{z}^2 k_1^2, \bar{z}(-2k_1\cdot p_b), \bar{z}^2 k_2^2, \bar{z}(-2k_1\cdot p_b - 2k_2\cdot p_b), \\ \bar{z}(-2k_2\cdot p_b), \bar{z}(-2k_2\cdot p_b - 2k_2\cdot q + s), \bar{z}(-2k_1\cdot p_b - 2k_2\cdot p_b - 2k_1\cdot q - 2k_2\cdot q + s)\}. \tag{5.52}$$

We have no soft propagator which does not depend on the integration momenta k_1, k_2, but we encounter another scenario where again the soft topology must be manipulated in order to obtain an ensemble of 7 independent propagators. Indeed, in this case one of the 7 propagators in t_1^s can be written as linear combination of the others, i.e. $\bar{z}(-2k_2\cdot p_b) = \bar{z}(-2(k_1+k_2)\cdot p_b) - \bar{z}(-2k_1\cdot p_b)$. So we drop this propagator ($\bar{z}(-2k_2\cdot p_b)$) and replace it with another one $-2k_1\cdot k_2$ in order to get a complete topology (namely a minimal closed ensemble of linearly independent propagators) which can be accepted by FIRE. We observe though, that in this case the propagator $\bar{z}(-2k_2\cdot p_b)$ that we are dropping out of the topology actually appears in one of the integrals, namely $Master_7^s$, but this problem can be easily solved by decomposing the integral as

$$Master_7^s = \int d^dk_1 d^dk_2 \frac{(-2(k_1+k_2)\cdot p_b) - (-2k_1\cdot p_b)}{(k_1^2)\,(k_2^2)\,(-2k_1\cdot p_b - 2k_2\cdot p_b)(-2k_2\cdot p_b - 2k_2\cdot q + s)} \times \\ \times \frac{1}{(-2k_1\cdot p_b - 2k_1\cdot q - 2k_2\cdot p_b - 2k_2\cdot q + s)}$$

$$
\begin{aligned}
&= \int d^d k_1 d^d k_2 \frac{1}{\left(k_1^2\right)\left(k_2^2\right)(-2k_2\cdot(p_b+q)+s)(-2(k_1+k_2)\cdot(p_b+q)+s)} \\
&- \int d^d k_1 d^d k_2 \frac{(-2k_1\cdot p_b)}{\left(k_1^2\right)\left(k_2^2\right)(-2k_1\cdot p_b-2k_2\cdot p_b)(-2k_2\cdot p_b-2k_2\cdot q+s)} \times \\
&\qquad \times \frac{1}{(-2k_1\cdot p_b-2k_1\cdot q-2k_2\cdot p_b-2k_2\cdot q+s)}. \qquad (5.53)
\end{aligned}
$$

The original integral has been written as a sum of two other integrals of which the first one is new and the second one is instead nothing but $Master_6^s$. But, regardless of whether these integrals are new or not, the crucial point is that they do not contain anymore the linearly dependent propagator that we dropped. In this particular case, they do not even contain the new propagator $-2k_1\cdot k_2$, but this fact is rather irrelevant for the reduction (and indeed in some other topologies we encountered the situation where the new propagator, added to obtain a complete topology, was indeed present in some soft integrals after decomposition or partial fractioning). By performing reduction on all the soft integrals $Master_i^s$ with $i = 1,...,6$ and on the new integral obtained on the r.h.s. of Eq.(5.53), we see that all these integrals shrink to linear combinations of $Master_2^s, Master_4^s$ as defined in Eq.(5.43).

$$
\begin{aligned}
Master_1^s &= -(1/4)(Q^2+s)Master_2^s \\
Master_2^s &= Master_2^s, \\
Master_3^s &= -(((-3+4\epsilon)Master_2^s)/((-1+2\epsilon)(Q^2+s))) \\
Master_4^s &= Master_4^s \\
Master_5^s &= ((-1+4\epsilon)(2(3-10\epsilon+8\epsilon^2)Master_2^s+\epsilon^2 s(Q^2+s)Master_4^s))/(3\epsilon^3 s(Q^2+s)^2) \\
Master_6^s &= (4(-3+4\epsilon)Master_2^s+(-1+3\epsilon)s(Q^2+s)Master_4^s)/(2(-1+2\epsilon)s) \\
Master_7^s &= \frac{(-3+4\epsilon)Master_2^s}{(-1+2\epsilon)s} \\
&- \frac{4(-3+4\epsilon)Master_2^s+(-1+3\epsilon)s(Q^2+s)Master_4^s}{2(-1+2\epsilon)s}. \qquad (5.54)
\end{aligned}
$$

Computation of the two independent *soft masters*, performed by C. Duhr by means of Mellin-Barnes technique, gives

$$
\begin{aligned}
Master_2^s &= \frac{\Gamma(1-\epsilon)^2}{\Gamma(4-4\epsilon)} \\
Master_4^s &= \frac{\Gamma(1-\epsilon)^2}{\Gamma(4-4\epsilon)}\frac{4(3-4\epsilon)}{\epsilon(1+y)}\frac{(\Gamma(1-3\epsilon)\Gamma(2-2\epsilon)\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}{\epsilon\Gamma(1-\epsilon)^2} \\
&\quad - 2\text{Hypergeometric}_{PFQ}\left(\{1,1-2\epsilon,1-\epsilon\},\{2-2\epsilon,1+\epsilon\},1]\right)). \qquad (5.55)
\end{aligned}
$$

Thus, by plugging these results into Eq.(5.54) and then back into Eq.(5.50), we are able to determine the values of our integrals in the asymptotic limit $\{s \to m_t^2, Q^2 \ll s\}$.
Before concluding this explanation, we would like to do some important remarks.

- The procedure we just reported is actually nothing new. Indeed what we did is Expansion by Region (see Section 4.3) on Double-Real integrals, in order to extract their leading behaviour in the double limit $\{s \to m_t^2, Q^2 \ll s\}$.

- At the beginning of the explanation we pointed out that $\bar{z} = 1 - z$ is the variable which encode the asymptotic behaviour of the extra-radiation momenta in the soft limit, because this limit is identified by all the momentum components of k_1, k_2 scaling as $\bar{z}$. That's why, in order to extract the leading soft behaviour of our integrals, we scale k_1, k_2 by $\bar{z}$, and we replace m_t^2 with $s(1 - \bar{z})$.
  This would be correct without any further remark if we were looking for the only $s \to m_t^2$ limit of our integrals. But we are actually looking for the *double* limit $s \to m_t^2, Q^2 \ll s$! So, given that $\bar{z}$ is the small parameter of our expansion, this means that we should in principle rescale also Q^2 by $\bar{z}$, namely make the substitution $Q^2 \to \bar{z}Q^2$ in the propagators, and then expand.
  But we observe that in our specific case, this is not needed, because, once we expand the scalar products contained in our propagators, and substitute $p_b^2 = 0, p_b \cdot q = (s+Q^2)/2$, no propagator depend anymore on Q^2. This observation holds for all RR masters describing the bottom channel. Obviously, this implies that our integrals will not exhibit singularities in $y = 0$. Thus we can safely conclude that the leading behaviour of MIs expansions in the limit $\{z \to 1, y \to 0\}$ is the same as the leading behaviour in the only limit $z \to 1$ up to y-dependent corrections given by functions which are smooth for $y \to 0$.

- We observe finally that performing Expansion by Region to obtain soft behaviour of integrals which contain only phase space integrations (no real loops!) is indeed trivial, as it can be seen by the above formulas, in a sense that we always get only the hard region contributing to the expansion. This is quite obvious if we think of the nature of these integrals. The soft limit requires all components of extra-radiation momenta k_1, k_2 to be small compared to s. But this same components are in this case the integration variables! The usual scenario of Expansion by Region for loop integrals would see some constraints imposed on some ratio of external invariants to be small and loop momenta totally free to be 'small' or 'big' with respect to this ratio. From the analysis of this interplay between external and integration momenta, one is then able to determine all regions contributing. In this particular case, we do not have such an interplay because integration variables (i.e. components of k_1, k_2) are directly required to be small with the same scaling with respect to s (and this implied the condition on the external invariants $s \simeq m_t^2$ to be satisfied), meaning that only one 'region' contributes to the desired limit!

Topologies and Master Integrals for $b(p_b) + W^*(q) \to t(p_t) + b(k_1) + \bar{b}(k_2)$.

(a) (b)

Figure 5.9: Tree-level diagrams for $b + W^* \to t + b + \bar{b}$

Diagrams contributing to $b + W^* \to t + b + \bar{b}$ are drawn in Fig.5.9. By taking the square modulus of these diagrams and adding the Phase Space integration measure Eq.(5.41), we obtain ten different phase space diagrams which happened to be described by just one independent topology.

$$\begin{aligned} t_4 =& \{k_1^2, (k_1+k_2)^2, k_2^2, (k_1+k_2-p_b)^2, (-k_2+p_b)^2, \\ & - m_t^2 + (k_1-q)^2, -m_t^2 + (k_1+k_2-p_b-q)^2\}, \end{aligned} \tag{5.56}$$

which is represented according to our convention in Fig.5.10.

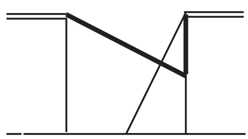


Figure 5.10: Independent topologies to $b + W^* \to t + b + \bar{b}$.

When reducing all scalar integrals to masters, we get the following set of MIs, which are all new with respect to the sets of masters obtained for $b + W^* \to t + g + g$, except for the first three, namely from $Master_1$ to $Master_3$.

$$\begin{aligned} Master_1 &= G(t_4, \{1, -1, 1, 0, 0, 0, 1\}) \\ Master_2 &= G(t_4, \{1, 0, 1, 0, 0, 0, 1\}) \\ Master_3 &= G(t_4, \{1, 0, 1, 1, 0, 0, 1\}) \end{aligned}$$

$$\begin{aligned}
&Master_4 = G(t_4, \{1, -1, 1, 0, 0, 1, 1\})\\
&Master_5 = G(t_4, \{1, 0, 1, -1, 0, 1, 1\})\\
&Master_6 = G(t_4, \{1, 0, 1, 0, 0, 1, 1\})\\
&Master_7 = G(t_4, \{1, -1, 1, 1, 0, 1, 1\})\\
&Master_8 = G(t_4, \{1, 0, 1, 1, 0, 1, 1\})\\
&Master_9 = G(t_4, \{1, 1, 1, 0, 0, 1, 1\})\\
&Master_{10} = G(t_4, \{1, 1, 1, 0, 1, 1, 1\})
\end{aligned} \tag{5.57}$$

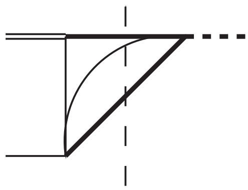
Along the same path followed to solve integrals $b + W^* \to t + g + g$, we perform a change of basis towards a canonical one. This canonical basis and the correspondent system of canonical DEs are reported in the Appendix for the sake of brevity (see Eq.(6.8) and (6.9)). We underline that the alphabet describing $b+W^* \to t+g+g$ happens to be a subset of the alphabet for $b+W^* \to t+b+\bar{b}$, which contains in addition the two new letters $1 + y + z, 1 + 2y + z$.

$$\mathcal{A}_{bb}^{RR} = \{z, z - 1, y, 1 + y, z + y, 1 + y + z, 1 + 2y + z\}. \tag{5.58}$$

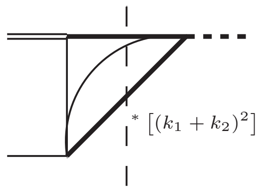
The double limit $y \to 0, z \to 1$ for $Master_i$ as defined in Eq.(5.57) reads

$$\begin{aligned}
&Master_1 \to \bar{z}^{(3-4\epsilon)} Master_2^s\\
&Master_2 \to \bar{z}^{(5-4\epsilon)}((-1 + \epsilon)sMaster_2^s)/(-10 + 8\epsilon)\\
&Master_3 \to \bar{z}^{(2-4\epsilon)}(-(((-3 + 4\epsilon)Master_2^s)/((-1 + 2\epsilon)(Q^2 + s))))\\
&Master_4 \to \bar{z}^{(3-4\epsilon)} Master_2^s\\
&Master_5 \to \bar{z}^{(5-4\epsilon)}((-1 + \epsilon)sMaster_2^s)/(-10 + 8\epsilon)\\
&Master_6 \to \bar{z}^{(4-4\epsilon)}(-(1/2)(Q^2 + s)Master_2^s)\\
&Master_7 \to \bar{z}^{(2-4\epsilon)}(-(((-3 + 4\epsilon)Master_2^s)/((-1 + 2\epsilon)(Q^2 + s))))\\
&Master_8 \to \bar{z}^{(4-4\epsilon)}(-((sMaster_2^s)/(4(Q^2 + s))))\\
&Master_9 \to \bar{z}^{(1-4\epsilon)}(2(-3 + 4\epsilon)Master_2^s)/(\epsilon s)\\
&Master_{10} \to \bar{z}^{(-4\epsilon)}(-(((-1 + 2\epsilon)(-3 + 4\epsilon)(-1 + 4\epsilon)Master_2^s)/(\epsilon^3 s(Q^2 + s)))).
\end{aligned} \tag{5.59}$$

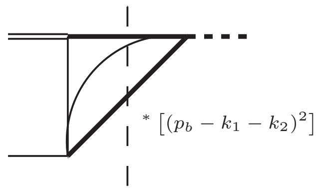
Being that $Master_2$ in Eq.(5.57) is the same integral as $Master_2$ in Eq.(5.43), all the information needed to compute masters for $b + W^* \to t + b + \bar{b}$ are provided. This concludes our discussion of the Double-Real master integrals and we address in the next subsection the Real-Virtual contribution.



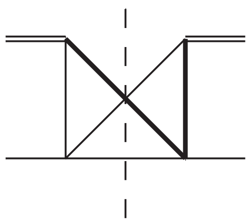
(a) $Master_4$



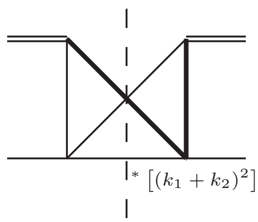
(b) $Master_5$



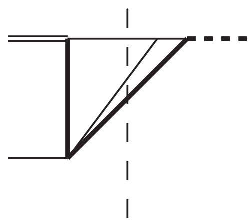
(c) $Master_6$



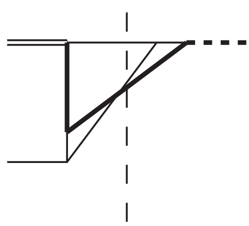
(d) $Master_7$



(e) $Master_8$



(f) $Master_9$



(g) $Master_{10}$

Figure 5.11: Set of independent MIs for $b + W^* \to t + b + \bar{b}$.
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

Real-Virtuals $[b + W^* \to t + X_1]_{1-loop}$

Diagrams contributing to $[b + W^* \to t + g]_{1-loop}$ are drawn in Fig.5.12. Phase space for the Real-Virtual corrections is a simple 2-particle (one massive and one massless) one, given by

$$\begin{aligned}\mathcal{PS}_2 &= \int d^d k \int d^d p_t \left[\frac{1}{k^2}\right]_c \left[\frac{1}{p_t^2 - m_t^2}\right]_c \delta^{(4)}\left(p_b + q - k - p_t\right) \\ &= \int d^d k \left[\frac{1}{k^2}\right]_c \left[\frac{1}{(p_b + q - k)^2 - m_t^2}\right]_c . \end{aligned} \tag{5.60}$$

The corresponding measure is thus[7]

$$d\mathcal{PS}_2 = \frac{1}{k^2}\frac{1}{(p_b + q - k)^2 - m_t^2}. \tag{5.61}$$

By contracting amplitudes for diagrams in Fig.5.12 with the one for the tree-level process $b + W^* \to t + g$, adding the 2-particle (one massive and one massless) phase space measure Eq.(5.61) and searching for independent topologies, we find that three independent topologies in Eq.(5.62) (drawn in Fig.5.13) are sufficient to describe all the Phase Space diagrams.

$$\begin{aligned} t_1 =& \{k^2, (k+p_b)^2, l_1^2 - m_t^2, (l_1 - q)^2, (k + l_1 - q)^2, (k + l_1 - p_b - q)^2, -m_t^2 + (-k + p_b + q)^2\} \\ t_2 =& \{k^2, (k+p_b)^2, l_1^2, (l_1 + p_b)^2, -m_t^2 + (l_1 - q)^2, -m_t^2 + (k + l_1 - q)^2, -m_t^2 + (-k + p_b + q)^2\} \\ t_3 =& \{k^2, (k+p_b)^2, l_1^2, (l_1 + p_b)^2, (-k + l_1 + p_b)^2, -m_t^2 + (l_1 - q)^2, -m_t^2 + (-k + p_b + q)^2\}. \end{aligned} \tag{5.62}$$

Reduction to master integrals result in the set of MIs Eq.(5.63), where we decided this time to gather all master integrals coming from the three topologies Eq.(5.62) into just one set of masters for which we will write and integrate just one system of DE.

$$\begin{aligned} Master_1 &= G(t_1, \{1, 1, 0, 1, 0, 2, 0\}) \\ Master_2 &= G(t_1, \{1, 1, 0, 2, 0, 1, 0\}) \\ Master_3 &= G(t_1, \{1, 1, 0, 1, 0, 1, 0\}) \\ Master_4 &= G(t_1, \{1, 1, 0, 1, 0, 0, 0\}) \\ Master_5 &= G(t_1, \{1, 1, 0, 1, 1, 1, 0\}) \\ Master_6 &= G(t_1, \{1, 1, 0, 1, 1, 0, 0\}) \\ Master_7 &= G(t_1, \{1, 1, 0, 0, 1, 0, 1\}) \end{aligned}$$

[7]We omit from now on the symbol $[]_c$ of *cut* propagator for simplicity of notation.

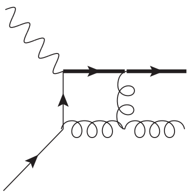
$$
\begin{aligned}
Master_8 &= G(t_1, \{1,1,0,1,1,0,2\}) \\
Master_9 &= G(t_1, \{1,1,0,2,1,0,1\}) \\
Master_{10} &= G(t_1, \{1,1,0,1,1,0,1\}) \\
Master_{11} &= G(t_1, \{1,1,0,1,1,1,1\}) \\
Master_{12} &= G(t_2, \{1,1,0,0,1,1,0\}) \\
Master_{13} &= G(t_2, \{1,1,0,1,0,0,1\}) \\
Master_{14} &= G(t_2, \{2,1,0,1,0,0,1\}) \\
Master_{15} &= G(t_2, \{1,2,0,1,0,0,1\}) \\
Master_{16} &= G(t_2, \{1,1,0,1,0,1,1\}) \\
Master_{17} &= G(t_2, \{1,1,0,2,0,1,1\}) \\
Master_{18} &= G(t_2, \{1,2,0,1,0,1,1\}) \\
Master_{19} &= G(t_2, \{1,1,0,0,1,1,1\}) \\
Master_{20} &= G(t_3, \{1,1,0,1,0,1,0\}\}) \\
Master_{21} &= G(t_3, \{1,1,0,1,0,1,1\}) \\
Master_{22} &= G(t_3, \{2,1,0,1,0,1,1\}) \\
Master_{23} &= G(t_3, \{1,1,0,2,0,1,1\}) \\
Master_{24} &= G(t_2, \{1,1,1,1,1,1,1\})
\end{aligned} \tag{5.63}
$$

Once again, we transform the original masters Eq.(5.63) into the canonical set identified by Eq.(6.10), which satisfies in turn the system of DEs in canonical form Eq.(6.11), reported in the Appendix. [8]. The alphabet which describes the system is found to be

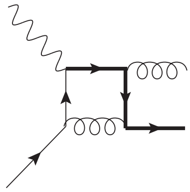
$$
\mathcal{A}^{RV} = \{z, 1-z, y, 1+y, z+y, 1+y+z\}. \tag{5.64}
$$

Canonical DE are integrated with the same method explained in Chapter 4, as for the Double-Reals masters.
The computation of the boundary condition is instead quite different and definitely more tricky than that of the RR case, so we will focus on it for the rest of the RV subsection.

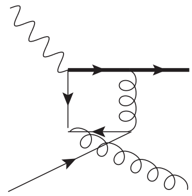
[8]We thank Dr. B. Mistlberger for kindly providing cross-check to Eq.(6.10) and (6.11)



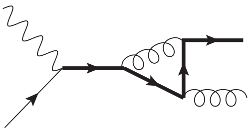
(a) Box n.1



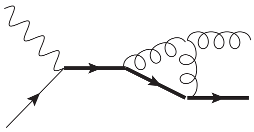
(b) Box n.2



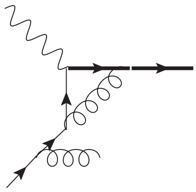
(c) Box n.3



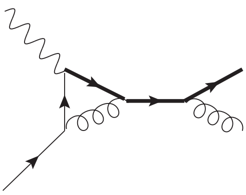
(d) Triangle n.1



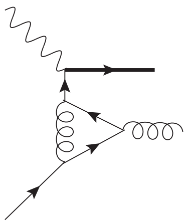
(e) Triangle n.2



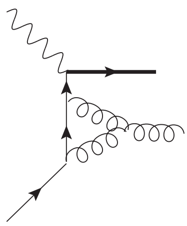
(f) Triangle n.3



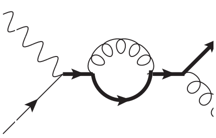
(g) Triangle n.4



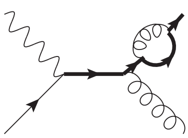
(h) Triangle n.5



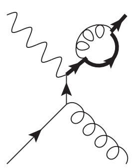
(i) Triangle n.6



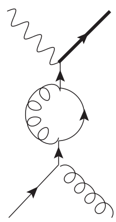
(j) Bubble n.1



(k) Bubble n.2

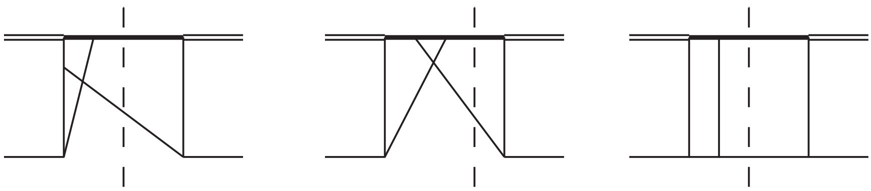


(l) Bubble n.3



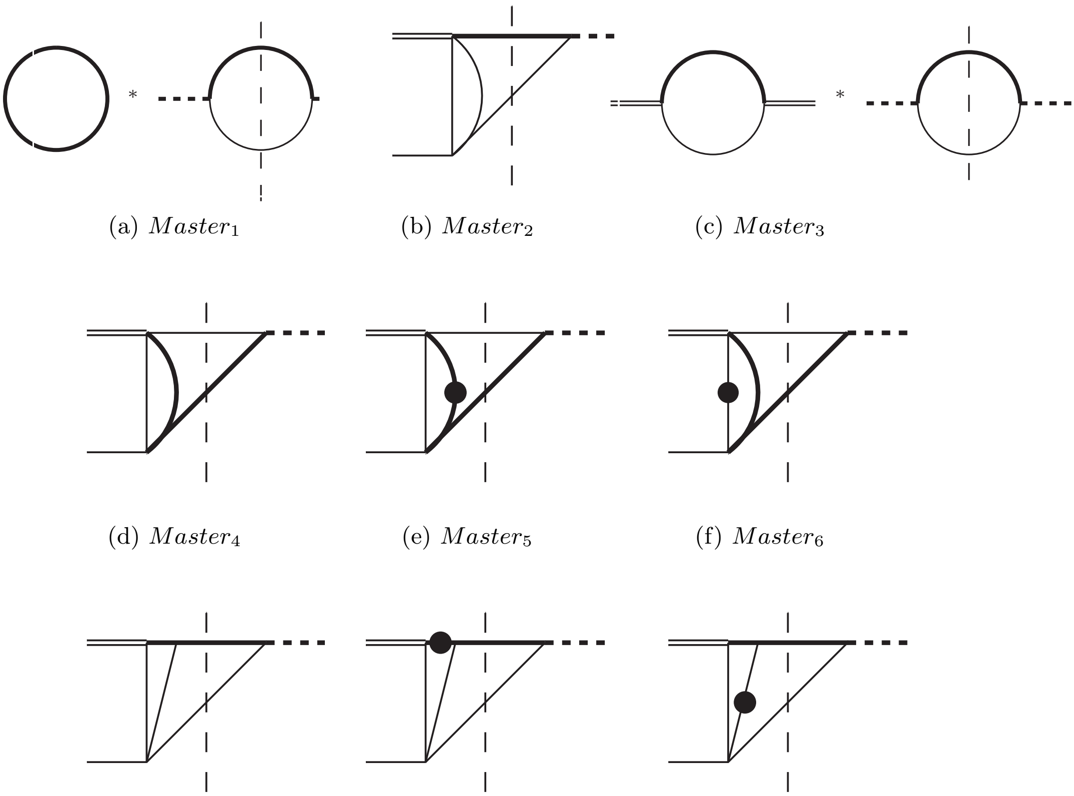
(m) Bubble n.4

Figure 5.12: 1-loop corrections to $b + W^* \to t + g$



(a) t_1 (b) t_2 (c) t_3

Figure 5.13: Independent topologies to $[b + W^* \to t + g]_{1-loop}$.

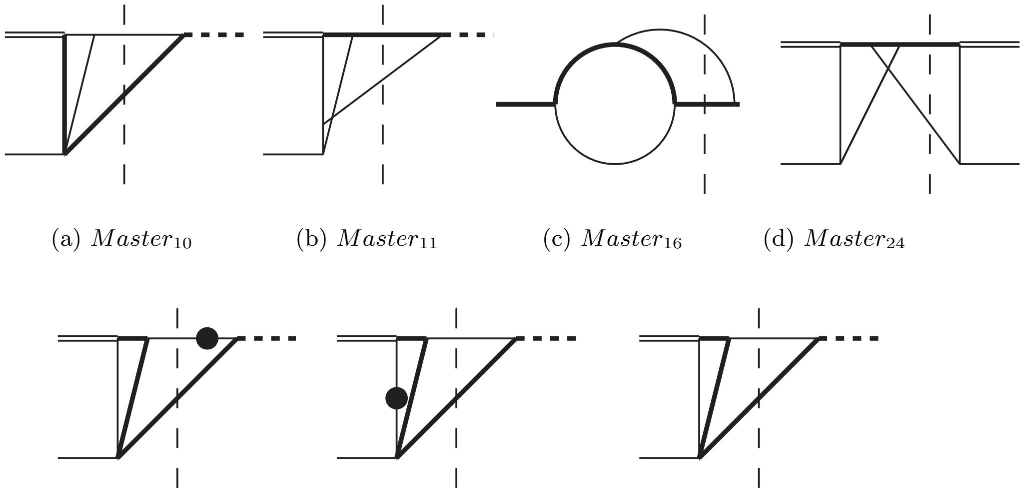


(a) $Master_1$ (b) $Master_2$ (c) $Master_3$

(d) $Master_4$ (e) $Master_5$ (f) $Master_6$

(g) $Master_7$ (h) $Master_8$ (i) $Master_9$

Figure 5.14: Set of independent MIs for $b + W^* \to t + g$ at 1-loop ($Master$ from 1 to 9).
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).



(a) $Master_{10}$ (b) $Master_{11}$ (c) $Master_{16}$ (d) $Master_{24}$

(e) $Master_{17}$ (f) $Master_{18}$ (g) $Master_{19}$

Figure 5.15: Set of independent MIs for $b + W^* \to t + g$ at 1-loop ($Master$ from 10 to 24).
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

Determination of boundary conditions for the b-channel RV masters.

We remind that our RV integrals depend on 3 dimensional scales $\{s, m_t^2, Q^2\}$, which we recast into the set $\{s, z = m_t^2/s, y = Q^2/s\}$.
Being the only dimensional scale, s parametrizes the dimension of the integral in powers of energy. The evolution of our integrals with respect to s is then trivial. On the other hand, we do not have a priori any information about the evolution with respect to z and y. This information is extracted by integrating the system of DEs for the MIs.
For the rest of this subsection, we assume that DEs systems (Eq.(6.11)) for the masters have already been integrated.
We are left only with the determination of initial conditions. We choose to compute them in the same double limit $z \to 1 \wedge y \to 0$, which we used also for the RR (we remind that it corresponds to the limit in which the real emitted gluon is soft ($s \simeq m_t^2$) and the virtuality of the W^* is much smaller than s ($Q^2 \ll s$).

The way we compute these initial conditions is the following.
We are dealing with integrals where one integration variable l_1 is a 'real' loop momentum (belonging to a virtual particle) and the other one k is the momentum of a real emitted particle whose delta of on-shellness has been replaced by a *cut* propagator.
In order to get the asymptotic behaviour of our integrals in the chosen limit $\{z \to 1, y \to 0\}$, we would like to use Expansion by Region (see 4). But this method is designed for pure loop momenta, whose components and off-shellness can assume any value.
We cannot apply this method to the integration over k, because k describes a real particle ($k^2 = 0$) and also it must be softer than any other external momentum in order to realize the threshold limit $s \simeq m_t^2$.
The procedure we decide to adopt can be then resumed in the following steps.

- We 'cut' again our integrals, namely transforming them back into real 1-loop integrals integrated over a 2-particles Phase Space.
- We apply Expansion by Region only to the pure 1-loop integral.
- We finally integrate the result over the *soft* Phase Space (the integration being thus over only the angular variable).

<u>2-particle Phase Space and invariants parametrization</u>
We imagine now that we have cut our diagrams Phase Space diagrams as we said above, so that for the moment we forget about the phase space integration over the real momentum k and we are thus left with 1-loop corrections to the process

$$b(p_b) + W^*(q) \to t(p_b + q - k) + g(k). \tag{5.65}$$

We remind that for this process at any number of loops, the kinematic is parametrized by two masses

$$Q^2 = -q^2, \quad m_t^2 = (p_b + q - k)^2 \tag{5.66}$$

and two Mandelstam invariants among [9]

$$s = (p_b + q)^2, \quad t = (p_b - k)^2, \quad u = (q - k)^2. \tag{5.67}$$

Since in our process the off-shell W-boson acts only as external particle, its off-shellness Q^2 will never appear as an internal mass in the pure 1-loop integrals. Obviously the same holds for the Mandelstam invariants s, t, u. Consequently, the pure 1-loop integrals will be functions of the kinematic invariants classified as follows

- internal mass: m_t^2
- external scales: $m_t^2, \; Q^2, \; s, \; t, \; u$.

We choose to use as independent variables to describe our process $\mathcal{S}_1 = \{s, Q^2, m_t^2, \lambda\}$, where $\lambda = 1/2(1 + \cos\theta)$, being θ the angle between p_b and k. In terms of this set $\mathcal{S}_1$, the remaining variables are expressed as

$$\begin{aligned} t &= -\frac{s + Q^2}{s}(s - m_t^2)(1 - \lambda) \\ u &= m_t^2 - Q^2 - s - t = -\frac{m_t^2 Q^2}{s} + \frac{\lambda(m_t^2 - s)(Q^2 + s)}{s}, \end{aligned} \tag{5.68}$$

$$d\phi_2 = \frac{1}{8\pi}\frac{s - m_t^2}{s}\frac{1}{\Gamma(1-\epsilon)}\left[\frac{(s - m_t^2)^2}{4\pi s}\right]^{\epsilon}[\lambda(1-\lambda)]^{\epsilon}\, d\lambda. \tag{5.69}$$

The physical region is given by the volume in the parameter space given by

$$s > 0 \wedge Q^2 > 0 \wedge s > m_t^2 > 0 \wedge 0 < \lambda < 1. \tag{5.70}$$

which defines also the sign of the remaining invariants $t < 0, \;\; u < 0$.
We perform a last change of variables going from set $\mathcal{S}_1$ to set $\mathcal{S}_2 = \{z, y, s, \lambda\}$, where we introduce the usual dimensionless variables z, y. As for the Double-Real masters, also in this case we define the variable $\bar{z} = 1 - z$, which captures the scaling of the extra radiation momenta in the threshold limit ($\bar{z} \to 0$). Our 1-loop integrals will be then computed as expansions around the 'small' parameter $\bar{z}$.

[9] The on-shellness and momentum conservation constraints yields the relation $s + t + u = m_t^2 - Q^2$.

To be able to expand at integrand level around $\bar{z} \to 0, y \to 0$, we need to know the scaling of the all invariants in this limit:

$$\begin{aligned}
m_t^2 &\to (1-\bar{z})s \\
Q^2 &\to ys \\
t &\to -\bar{z}\,(1+y)\,(1-\lambda)s \\
u &\to (-sy) - \bar{z}\,(\lambda + (\lambda-1)y)\,s.
\end{aligned} \tag{5.71}$$

Finally, in order to integrate the result of Expansion by Region at 1-loop over the Phase Space, we need the expression this latter in terms of $\bar{z}$ (Eq.(5.72)).

$$d\phi_2 = \frac{2^{(-3+2\epsilon)}\pi^{(-1+\epsilon)}\left((1-\lambda)^{-\epsilon}\lambda^{-\epsilon}\bar{z}^{1-2\epsilon}\right)}{\Gamma[1-\epsilon]}. \tag{5.72}$$

1-loop topologies in the double limit $y \to 0, z \to 1$

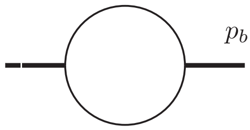
When we cut our RV integrals, we get a certain number of pure 1-loop integrals, to be perform in the desired kinematic limit. Among these integrals, we have 2-,3-,4-point functions.
We describe in the following the computation via Expansion by Region of each one of these categories of functions. The method is explained in 4. We briefly recall here what the method consists of. Suppose we want to compute a Feynman integral which depend on a_1, a_2,.. dimensional scales in a particular kinematic limit which is identified by the ratio between two of these scales being small, for instance $a = a_1/a_2 \to 0$. Obviously we want to do this without needing to the integral in full kinematic. Then, 'expansion by regions' consists in

- revealing 'regions' where the parameters of the Feynman representation of the integral scale as the expansion parameter a elevated to such integer power that the $\mathcal{U}$ and $\mathcal{F}$ polynomial result to scale homogeneously under a global rescaling of all the parameters (these are the regions which are supposed to contribute to the integral non-zero contributions in DR),
- computing via traditional techniques (Feynman Parameters or Mellin-Barnes) these regions,
- summing up the contributions from these regions. This is the result for our integral in the limit $a \to 0$.

The most difficult part is revealing all the regions contributing in a given limit. In our case fortunately this is quite, because we are at 1-loop and we can quite safely rely on the results of `Asy2.m` ([80]) which reveals these regions automatically for us. Then computation of regions is carried out via Feynman parameters. In the following we analyse in detail topology by topology how this procedure works.

- **BUBBLES**

1. **Massless bubble**



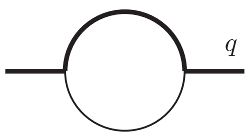
$$b_1(p_b \cdot k) = \int d^d l_1 \frac{1}{(l_1 - q)^2 (k + l_1 - p_b - q)^2}$$
$$= \Gamma(\epsilon) \int dx_1 dx_2 \delta(1 - x_1 - x_2)(-t\ x_1\ x_2)^{-\epsilon}(x_1 + x_2)^{-2+2\epsilon} \tag{5.73}$$

This integral depends on just one dimensional scale $t = -2p_b \cdot k$ and the $\mathcal{F}$ polynomial is positive-definite for $t < 0$, namely for physical values of t. So, the dependence on t of the result can be just one: t elevated to the dimension (in powers of energy) of the integral, which means in this case, since $t < 0$, $(-t)^{(4-2\epsilon-4)/2} = (-t)^{-\epsilon}$. If we recall Eq.(5.71), then we see immediately that this integral present a double branch-cut for $z \leq 1, \lambda \leq 1$. The result is simply (without needing to compute any region)

$$b_1(z, y, \lambda, s) = \frac{(1-\lambda)^{-\epsilon} s^{-\epsilon} (y+1)^{-\epsilon} (\bar{z})^{-\epsilon} \Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)}. \tag{5.74}$$

If we expand in ϵ we get logs of $\bar{z}$, which represent nothing but the above-mentioned branch-cut.

2. **Massive bubble with internal mass $\mathbf{m_t^2}$ and external mass $\mathbf{Q^2}$**

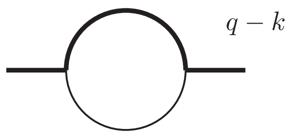


$$\begin{aligned}
b_2(Q^2, m_t^2) =& \int d^d l_1 \frac{1}{(l1^2 - m_t^2)(l_1 - q)^2} \\
=& \Gamma(\epsilon) \int dx_1 dx_2 \delta(1 - x_1 - x_2)(x_1 + x_2)^{-2+2\epsilon} \\
& \times (x_1(Q^2 x_2 + m_t^2(x_1 + x_2)))^{-\epsilon} \qquad (5.75)
\end{aligned}$$

This integral depend on 2 dimensional scales: the internal mass m_t^2 and the external mass Q^2, which we can re-parametrize by saying that it depends on one dimensional scale,say m_t^2 which gives its dimension in power of energy and then on the dimensionless ratio Q^2/m_t^2. It can be seen by analysing the behaviour of either the $\mathcal{F}$ polynomial or the denominators that no singularities occur when the double condition $\{s \to m_t^2, Q^2 \ll s\}$ is realized. Indeed, `Asy2.m` gives just the hard region $\{0, 0\}$ in this limit. We also mark that the F polynomial is positive-definite in the physical region. By evaluating the hard region, we get

$$b_2(Q^2 \ll s, m_t^2 \simeq s) = -s^{-\epsilon}\Gamma(-1 + \epsilon). \qquad (5.76)$$

3. **Massive bubble with internal mass $\mathbf{m_t^2}$ and external mass u**

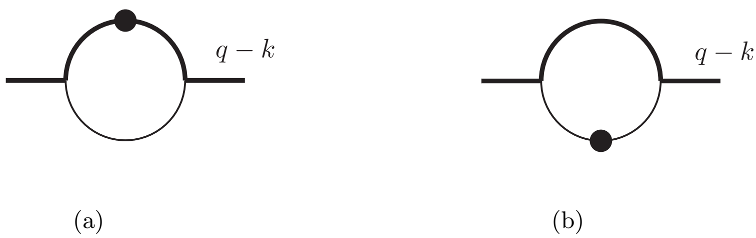


$$\begin{aligned}
b_3(-u, m_t^2) =& b_6(-u, m_t^2) = \int d^d l_1 \frac{1}{(l_1^2 - m_t^2)(k + l_1 - q)^2} \\
=& \Gamma(\epsilon) \int dx_1 dx_2 \delta(1 - x_1 - x_2)(x_1 + x_2)^{-2+2\epsilon} \\
& \times (x_1(-u x_2 + m_t^2(x_1 + x_2)))^{-\epsilon} \qquad (5.77)
\end{aligned}$$

Exactly the same consideration holds for this integral as for the previous massive bubble. The F polynomial is positive-definite in the physical region $\{u < 0, m_t^2 > 0\}$ and, since the integral depend on $-u/m_t^2$, the limit $s \to m_t^2$ is clearly smooth. Moreover, if we also ask for $Q^2 \ll s$, again the external scale goes to zero (it would remain $q \cdot k$ but k is soft!), but the integral, as in the previous case, does not develop any new singularity thanks to the presence of the internal scale m_t^2. We conclude that the double limit $\{s \to m_t^2, Q^2 \ll s\}$ does not produce singularities and thus we just need to compute the hard region, which reads again

$$b_3(Q^2 \ll s, m_t^2 \simeq s) = -s^{-\epsilon}\Gamma(-1+\epsilon). \tag{5.78}$$

For this particular topology and external mass, we also have to evaluate the two integrals with dotted propagators in Fig.5.16a,5.16b.



For both these integrals the double limit is smooth, so that we only have to compute the hard region, as confirmed by `Asy2.m`. (NB: The only pathological case may be that of integral with a massless dotted propagator, but also in this case the integral is convergent in the IR region, as it can be seen from power counting). For these integrals we get

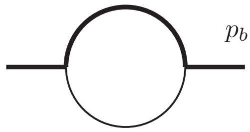
$$\begin{aligned} b_4(-u, m_t^2) = &\int d^d l_1 \frac{1}{[(l1^2 - m_t^2)]^2 (k + l_1 - q)^2} \\ =& s^{-1-\epsilon}\Gamma[1+\epsilon] x_1^{-\epsilon}(x_1+x_2)^{-1+2\epsilon} \\ &\times (\lambda(1+y)x_2 + (1-\bar{z})(x_1 - (-1+\lambda)(1+y)x_2))^{-1-\epsilon} \end{aligned} \tag{5.79}$$

$$b_4(Q^2 \ll s, m_t^2 \simeq s) = (s^{-1-\epsilon}\Gamma[1+\epsilon])/(1-\epsilon). \tag{5.80}$$

$$\begin{aligned} b_5(-u, m_t^2) = &\int d^d l_1 \frac{1}{(l1^2 - m_t^2)[(k + l_1 - q)^2]^2} \\ =& s^{-1-\epsilon}\Gamma[1+\epsilon] x_1^{-1-\epsilon} x_2 (x_1+x_2)^{-1+2\epsilon} \\ &\times (\lambda(1+y)x_2 + (1-\bar{z})(x_1 - (-1+\lambda)(1+y)x_2))^{-1-\epsilon} \end{aligned} \tag{5.81}$$

$$b_5(Q^2 \ll s, m_t^2 \simeq s) = (s^{-1-\epsilon}\Gamma[1+\epsilon])/((-1+\epsilon)\epsilon). \tag{5.82}$$

4. **Massive bubble with internal mass $\mathbf{m_t^2}$ and external mass s**



$$
\begin{aligned}
b_7(s, m_t^2) =& \int d^d l_1 \frac{1}{(l_1 + p_b)^2(-m_t^2 + (l_1 - q)^2)} \\
=& \Gamma(\epsilon) \int dx_1 dx_2 \delta(1 - x_1 - x_2)(x_1 + x_2)^{-2+2\epsilon} \\
& \times (x_2(-s x_1 + m_t^2(x_1 + x_2)))^{-\epsilon}
\end{aligned}
\tag{5.83}
$$

This integral depends only on 2 dimensional scales, s (external) and m_t^2 (internal). This implies that the F polynomial is positive-definite in the region $s < 0, m_t^2 > 0$. Because of the condition $s < 0$, this region is non-physical and we will refer to it in the following as 'euclidean' or 'non-physical' region. This tells us that the result for our integral will be a function which takes complex values for physical values of s ($s > 0$) and this consideration holds in general, regardless of the various limits we can take for this function.

Now, let us consider also the fact that we are interested in computing not for general values of s, but for $s \to m_t^2$, namely when the external scale approaches the internal one. This kind of limit, typical of Minkowsky space, is usually known as *on-shell* limit of a Feynman integral, and experience warns us that this limit might not be smooth. And this is indeed the case! If we ask `Asy2.m` to give us the expansions in the double limit $\{Q^2 \ll s, s \to m_t^2\}$ (which in this case reduces to only $s \to m_t^2$), it gives us two regions $\{0, 0\}, \{0, 1\}$ and from the second region we actually get a branch cut singularity.

We dedicate the following paragraph to explaining the procedure we use in order to determine exactly the imaginary part acquired by the integral in the physical region.

Analytical continuation: Whenever we have to deal with integrals whose result is a complex-valued function, it is always good to compute them in a region where they are real (which is identified by the condition that the $\mathcal{F}$ polynomial be positive-definite) and then analytically continue them back to a region of physical interest, where they develop then a non-zero imaginary part. Analytical continuation is performed starting from the Feynman prescription on the invariants, which gives positive imaginary part to external invariants (e.g. $s \to s + i0$) and negative

imaginary part to the external invariants (e.g. $m_t^2 \to m_t^2 - i0$). This procedure guarantees, if correctly carried out, the imaginary part of the result to have the correct sign.
Now, if we were to compute the integral Eq.(5.83) for general value of s, m_t^2, we would simply compute for negative s and then continue back with the just-mentioned prescription $s \to s + i0$. But since we are interested in the value of the integral around the point $s \simeq m_t^2$, things are slightly more subtle. In order to combine expansion around $\bar{z} \simeq 0$ with analytical continuation, the following strategy revealed itself more efficient, specially for the computation of more complicated integrals (e.g. see the Boxes computation below) which depend on other scales on top of s and m_t^2 [10].
In order to expand around $\bar{z} \to 0$, we need to parametrize m_t^2 as $s(1-\bar{z})$. Once we insert this parametrization in the $\mathcal{F}$ polynomial and we simplify it, we assist to a cancellation of the terms having $-s$ as a prefactor, and we are left with a polynomial whose sign is not definite because of the only presence of terms having $-s\bar{z}$ as prefactors. Integral b_7 is the easiest example where we can see this.

$$\begin{aligned}\mathcal{F}_{b_7} &= x_2(-sx_1 + m_t^2(x_1 + x_2)) \\ &= x_2(-sx_1 + s(1-\bar{z})(x_1 + x_2)) \\ &= x_2(sx_2 - s\bar{z}(x_1 + x_2))\end{aligned} \tag{5.84}$$

It is clear then that the easiest thing to do is to compute for $\bar{z} < 0$, which means we are in the unphysical region 'above threshold' $m_t^2 > s$, and then continue back to physical values $0 < \bar{z} < 1$. In order to do so, we define for simplicity

$$\bar{x} = -\bar{z}, \quad -1 < \bar{x} < 0 \text{ for } 0 < m_t^2 < s, \quad 0 < \bar{x} \text{ for } m_t^2 > s. \tag{5.85}$$

The last ingredient we need to work out is the prescription to analytically continue $\bar{x}$, which we derive from the prescription on s as follows.

$$1 - \bar{z} = z = \frac{m_t^2}{s} \to \frac{m_t^2}{s + i0} = \frac{m_t^2}{s} - i0 = 1 - \bar{z} - i0 \tag{5.86}$$

This implies the following prescription

$$\bar{z} \to \bar{z} + i0, \tag{5.87}$$

[10] the consideration we are going to explain in the following hold not only for this specific integral, but also for all other integrals (triangles and boxes) whose $\mathcal{F}$ polynomials depend on both s and m_t^2 are characterized by the same kind of problems concerning threshold expansion and analytical continuation.

thus implying

$$\bar{x} = -\bar{z} \to -\bar{z} - i0 = \bar{z}(-1 - i0) = \bar{z}e^{-i\pi}. \tag{5.88}$$

We can now start computing our integral.

HARD REGION $\{0, 0\}$

For the hard region we do not even need analytical continuation because in this region we simply set $\bar{z} = 0$ in the $\mathcal{F}$ polynomial, thus eliminating since the beginning any ambiguity in its sign.

$$\begin{aligned} b_7(\bar{z}, s)_{\{0,0\}} &= \int dx_2 s^{-\epsilon}\Gamma(\epsilon)x_2^{-2\epsilon}(1 + x_2)^{-2+2\epsilon} \\ &= \frac{s^{-\epsilon}\Gamma(\epsilon)}{1 - 2\epsilon} \end{aligned} \tag{5.89}$$

REGION $\{0, 1\}$

$$\begin{aligned} b_7(\bar{z}, s)_{\{0,1\}} &= a^{1-2\epsilon}s^{-\epsilon}\Gamma(\epsilon) \int dx_2(\bar{x} + x_2)^{-\epsilon}x_2^{-\epsilon} \\ &= s^{-\epsilon}a^{1-2\epsilon}\bar{x}^{1-2\epsilon}\Gamma(1 - \epsilon)\Gamma(-1 + 2\epsilon) \\ &= -e^{2i\pi\epsilon}s^{-\epsilon}a^{1-2\epsilon}\bar{z}^{1-2\epsilon}\Gamma(1 - \epsilon)\Gamma(-1 + 2\epsilon) \end{aligned} \tag{5.90}$$

We see that the branch cut start actually at $\mathcal{O}(\bar{z})$, so we need to compute the complete contribution to this order, which includes actually the second term (Next-to-Leading-Order) in the expansion of the hard region.
HARD REGION $\{0, 0\}$ *at NLO-*$\bar{z}$

$$b_7(\bar{z}, s)^{NLO}_{\{0,0\}} = -\frac{1}{2}s^{-\epsilon}\bar{z}\Gamma(\epsilon). \tag{5.91}$$

Thus our final result is

$$b_7(z \to 1, s) = \frac{s^{-\epsilon}\Gamma(\epsilon)}{1 - 2\epsilon} + -\frac{1}{2}s^{-\epsilon}\bar{z}\Gamma(\epsilon) - e^{2i\pi\epsilon}s^{-\epsilon}\bar{z}^{1-2\epsilon}\Gamma(1 - \epsilon)\Gamma(-1 + 2\epsilon). \tag{5.92}$$

- **TRIANGLES**

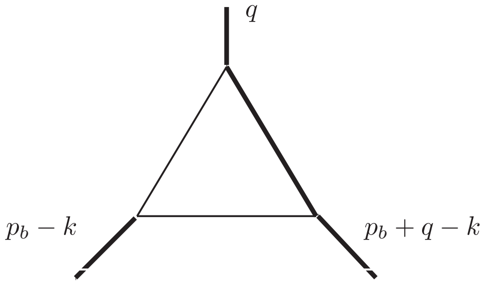


Figure 5.16: Triangle n.1: t_1

1. **Triangle with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2, t, m_t^2}\}$**

$$\begin{aligned} t_1(Q^2, -t, m_t^2) =& \int d^d l_1 \frac{1}{(l_1^2 - m_t^2)(l_1 - q)^2 (k + l_1 - p_b - q)^2} \\ =& \Gamma(\epsilon) \int_0^\infty dx_2 dx_3 \Gamma(1+\epsilon)(1 + x_2 + x_3)^{-1+2\epsilon} \\ & (m_t^2(1 + x_2) + x_2(Q^2 - t\ x_3))^{-1-\epsilon} \end{aligned} \tag{5.93}$$

This integral depends on 3 dimensional scales and, among these scales, m_t^2 is both internal and external. So we are not guaranteed a priori that the $\mathcal{F}$ polynomial respects the rules of being positive-definite for negative external masses and positive internal masses. But, in the case under study, this is true if m_t^2 is considered as internal mass. $\mathcal{F}$ is positive-definite for the physical values of parameters $\{Q^2 > 0, m_t^2 > 0, t < 0\}$. The regions given by Asy2.m for this integral are 3: $\{\{0,0,0\}$, $\{0,-1,-1\}$, $\{0,0,-1\}\}$. Since the computation of regions does not present any problems in this case and we do not need any analytical continuation, we just report results, without any intermediate passage.

$$\begin{aligned} t_1(\bar{z}, y, s)_{\{0,0,0\}} =& \frac{s^{-1-\epsilon}\Gamma(\epsilon)}{2\epsilon} \\ t_1(\bar{z}, y, s)_{\{0,-1,-1\}} =& -\frac{4^\epsilon \pi^{3/2} s^{-1-\epsilon}(\ (1-\lambda)\ \bar{z})^{-\epsilon} \csc(\epsilon\pi)}{\epsilon\Gamma(1/2 - \epsilon)} \\ t_1(\bar{z}, y, s)_{\{0,0,-1\}} =& \frac{\pi s^{-1-\epsilon}((1-\lambda)\ \bar{z})^{-2\epsilon} \csc(2\epsilon\pi)\Gamma(1+\epsilon)}{\epsilon} \end{aligned} \tag{5.94}$$

We mark that different regions may require the regulator ϵ to be positive or negative, depending on which kind of divergence (IR or UV) it is supposed to regulate in that region. But, since we are working in DR, we are guaranteed that we can always make an analytic continuation

in ϵ of our results. This means that we can perform, for instance, an analytic continuation in all regions which are initially defined for negative ϵ. In general, the analytic continuation we have to do are trivial (actually simple substitutions) because we never get ϵ elevated to an non-integer exponent (which would acquire a complex phase). Indeed, ϵ only appears in Γ functions, inverse integer powers (namely explicit poles $1/\epsilon, 1/\epsilon^2, ...$) and in csc, sec. These functions are defined in the whole complex plane (except for well-known isolated poles in ϵ), so as a matter of fact, we do not need to do concretely anything. Once we integrate in terms of these functions, analytic continuation is automatically done for us.
That's why we list our results without worrying about the values of ϵ for which these integrals are obtained. So for t_1 we get by summing up regions

$$
\begin{aligned}
t_1(z \to 1, y \to 0, s) =& \frac{s^{-1-\epsilon}\Gamma(\epsilon)}{2\epsilon} \\
&- \frac{4^{\epsilon}\pi^{3/2}s^{-1-\epsilon}\left(\,(1-\lambda)\;\bar{z}\right)^{-\epsilon}\csc(\epsilon\pi)}{\epsilon\Gamma(1/2-\epsilon)} \\
&\frac{\pi s^{-1-\epsilon}((1-\lambda)\;\bar{z})^{-2\epsilon}\csc(2\epsilon\pi)\Gamma(1+\epsilon)}{\epsilon}
\end{aligned}
$$

In the same topology we also find other three integrals (t_2, t_3, t_9, drawn in Fig.5.17) with dotted propagators for which we just report final results.

$$
\begin{aligned}
t_2(z \to 1, y \to 0, s) &= t_2(\bar{z}, y, s)_{\{0,0,0\}} + t_2(\bar{z}, y, s)_{\{0,-1,-1\}} + t_2(\bar{z}, y, s)_{\{0,0,-1\}} \\
t_2(\bar{z}, y, s)_{\{0,0,0\}} &= -\frac{s^{-2-\epsilon}\Gamma(2+\epsilon)}{\epsilon(1+2\epsilon)} \\
t_2(\bar{z}, y, s)_{\{0,-1,-1\}} &= +\frac{\pi s^{-2-\epsilon}((1-\lambda)\bar{z})^{-\epsilon}\csc(\epsilon\pi)\Gamma(-1-\epsilon)}{\epsilon(1+\epsilon)\Gamma(-2\epsilon)} \\
t_2(\bar{z}, y, s)_{\{0,0,-1\}} &= -\pi s^{-2-\epsilon}((1-\lambda)\bar{z})^{-1-2\epsilon}\csc(2\epsilon\pi)\Gamma(1+\epsilon),
\end{aligned} \tag{5.95}
$$

$$
\begin{aligned}
t_3(z \to 1, y \to 0, s) &= t_3(\bar{z}, y, s)_{\{0,0,0\}} + t_3(\bar{z}, y, s)_{\{0,-1,-1\}} + t_3(\bar{z}, y, s)_{\{0,0,-1\}} \\
t_3(\bar{z}, y, s)_{\{0,0,0\}} &= -\frac{s^{-2-\epsilon}\Gamma(1-\epsilon)}{(1-\epsilon)(2-4\epsilon)} \\
t_3(\bar{z}, y, s)_{\{0,-1,-1\}} &= +\frac{\pi s^{-2-\epsilon}(1-\lambda)^{-1-\epsilon}\bar{z}^{-1-\epsilon}\csc(\epsilon\pi)\Gamma(1-\epsilon)}{(1+\epsilon)\Gamma(-2\epsilon)} \\
t_3(\bar{z}, y, s)_{\{0,0,-1\}} &= +\pi s^{-2-\epsilon}((1-\lambda)\bar{z})^{-2(1+\epsilon)}\csc(2\epsilon\pi)\Gamma(1+\epsilon),
\end{aligned} \tag{5.96}
$$

$$t_9(z \to 1, y \to 0, s) = t_9(\bar{z}, y, s)_{\{0,0,0\}} + t_9(\bar{z}, y, s)_{\{0,-1,0\}} + t_9(\bar{z}, y, s)_{\{0,0,1\}}$$
$$t_9(\bar{z}, y, s)_{\{0,0,0\}} = -\frac{s^{-2-\epsilon}\Gamma(2+\epsilon)}{(1+2\epsilon)\epsilon(1+\epsilon)},$$
$$t_9(\bar{z}, y, s)_{\{0,-1,0\}} = \frac{4^{-\epsilon}s^{-2-\epsilon}((1-\lambda)\bar{z})^{-1-2\epsilon}\Gamma(1/2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(1+2\epsilon)}{\sqrt{\pi}},$$
$$t_9(\bar{z}, y, s)_{\{0,0,1\}} = \frac{s^{-2-\epsilon}(1-\lambda)^{-1-\epsilon}\bar{z}^{-1-\epsilon}\Gamma(-\epsilon)^2\Gamma(2+\epsilon)}{(1+\epsilon)\Gamma(-2\epsilon)}. \quad (5.97)$$

2. **Triangle with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2}, \mathbf{u}\}$**

$$t_4(Q^2, -u, m_t^2) = \int d^d l_1 \frac{1}{(l_1^2 - m_t^2)(l_1 - q)^2(k + l_1 - q)^2}$$
$$= \Gamma(1+\epsilon)\int_0^\infty (1 + x_2 + x_3)^{-1+2\epsilon}$$
$$\times (Qx^2 x_2 - ux_3 + m_t^2(1 + x_2 + x_3))^{-1-\epsilon} \quad (5.98)$$

Since we do not have overlapping between internal and external invariants (m_t^2 just appears as internal mass), the $\mathcal{F}$ polynomial follows very simply the rule which assigns it positive sign for negative external and positive internal invariants. Furthermore, since $\mathcal{F}$ does not depend on s, it results to be positive-definite in the physical region $\{Q^2 > 0, t < 0, m_t^2 > 0\}$ and we thus do not need any analytical continuation. Last but not least, when we send $\mathcal{F}$ for $Q^2 \to 0$ and $m_t^2 \simeq s$, the $\mathcal{F}$ polynomial does not goes to zero, meaning that the integral will exhibit a smooth behaviour in this double limit. And indeed `Asy2.m` gives us just the hard region for this topology, which reads

$$t_4(z \to 1, y \to 0, s) = t_4(\bar{z}, y, s)_{\{0,0,0\}} = \frac{s^{-1-\epsilon}\Gamma(1+\epsilon)}{(-1+\epsilon)\epsilon}. \quad (5.99)$$

3. **Triangle with 2 internal masses $\mathbf{m_t^2}$ and external masses $\{\mathbf{m_t^2}, \mathbf{s}\}$**

$$t_5(m_t^2, s) = \int d^d l_1 \frac{1}{(l_1 + p_b)^2((l_1 - q)^2 - m_t^2)((k + l_1 - q)^2 - m_t^2)}$$
$$= \Gamma(1+\epsilon)\int_0^\infty (1 + x_2 + x_3)^{-1+2\epsilon}(-sx_2 + mt^2(x_2 + (x_2 + x_3)^2))^{-1-\epsilon}. \quad (5.100)$$

The same considerations we did for bubble b_7 (Eq.(5.83)) hold also for triangle t_5. The topology depends on just 2 dimensional scales s, m_t^2,

and m_t^2 is both internal and external mass. This means that, given positive internal negative and external invariants, the $\mathcal{F}$ polynomial is not guaranteed to have positive sign. But once again $\mathcal{F}$ happens to be positive-definite for negative s and positive m_t^2 (namely if we consider m_t^2 as internal mass). The integral will be thus real for $s < 0$ but complex for $s > 0$, so that we will need to perform analytical continuation in order to get the correct result. Finally, since we want to compute an *on shell* limit of this integral, we know that it might develop singularities in this limit, and this is indeed the case, as confirmed by `Asy2.m`, which gives us two regions $\{\{0,0,0\},\{0,1,1\}\}$, of which the second one develops a branch cut. We report explicitly the computation of the second region, since it requires some manipulations.
HARD REGION $\{0,0,0\}$

$$t_5(\bar{z}, y, s)_{\{0,0,0\}} = -\frac{s^{-1-\epsilon}\Gamma(1+\epsilon)}{2\epsilon} \tag{5.101}$$

REGION $\{0,1,1\}$
We perform the same kind of analytical continuation that we performed for bubble b_7. Namely we compute 'above threshold', i.e. for $\bar{x} = -\bar{z} > 0$, and then we continue back to the physical region 'below threshold', i.e. $\bar{z} > 0$, by using the prescription Eq.(5.88). We start then by making the substitution $\bar{z} \to -\bar{x}$ and computing for $\bar{x} > 0$.

$$t_5(\bar{z}, y, s)_{\{0,1,1\}} = a^{-2\epsilon} s^{-1-\epsilon} \int_0^\infty \Gamma(1+\epsilon)(\bar{x}x_2 + (x_2+x_3)^2)^{-1-\epsilon}. \tag{5.102}$$

This integral is tricky because of the presence of both linear and quadratic dependence of the integrand on the integration variables. We can bypass this obstacle by performing the following change of variables

$$x_2 = \xi\eta/\bar{x}, \;\; x_3 = (1-\xi)\eta/\bar{x}. \tag{5.103}$$

The Jacobian of the transformation is

$$J = \begin{pmatrix} \frac{\eta}{\bar{x}} & \frac{\xi}{\bar{x}} \\ -\frac{\eta}{\bar{x}} & \frac{1-\xi}{\bar{x}} \end{pmatrix} \tag{5.104}$$

with determinant

$$det(J) = \frac{\eta}{\bar{x}^2}. \tag{5.105}$$

The inverse transformation

$$\xi = \frac{x_2}{(x_2+x_3)}, \;\; \eta = \bar{z}(x_2+x_3) \tag{5.106}$$

provides the integration domain expressed in terms of boundaries on the ξ, η variables

$$0 < \xi < 1, \quad 0 < \eta < \infty. \tag{5.107}$$

So, our integral now reads

$$\begin{aligned} t_5(\bar{z}, y, s)_{\{0,1,1\}} &= \Gamma(1+\epsilon) a^{-2\epsilon} s^{-1-\epsilon} (\bar{x})^{2\epsilon} \int_0^1 d\xi \int_0^\infty d\eta (\eta + \xi \bar{x}^2)^{-1-\epsilon} \eta^{-\epsilon} \\ &= a^{-2\epsilon} s^{-1-\epsilon} \bar{x}^{-2\epsilon} \Gamma(1-\epsilon)\Gamma(2\epsilon) \int_0^1 d\xi \xi^{-2\epsilon} \\ &= s^{-1-\epsilon} \bar{z}^{-2\epsilon} e^{2i\pi\epsilon} \Gamma(1-\epsilon)\Gamma(-1+2\epsilon). \end{aligned} \tag{5.108}$$

Analytical continuation is performed in the last line of Eq.(5.108). Our final result reads

$$\begin{aligned} t_5(z \to 1, y \to 0, s) = &- \frac{s^{-1-\epsilon}\Gamma(1+\epsilon)}{2\epsilon} \\ &+ s^{-1-\epsilon} \bar{z}^{-2\epsilon} e^{2i\pi\epsilon} \Gamma(1-\epsilon)\Gamma(-1+2\epsilon). \end{aligned} \tag{5.109}$$

4. **Triangle with 2 internal masses $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2}, \mathbf{u}\}$**
In this topology we have two integrals.

$$\begin{aligned} t_6(Q^2, -u, m_t^2) &= \int d^d l_1 \frac{1}{l1^2(-m_t^2 + (l1-q)^2)(-m_t^2 + (k + l1 - q)^2)} \\ &= \Gamma(1+\epsilon) \int_0^\infty (1 + x_2 + x_3)^{-1+2\epsilon} \\ &\times (m_t^2(x_2 + x_3)(1 + x_2 + x_3) + (Q^2 x_2 - u x_3))^{-1-\epsilon}, \end{aligned} \tag{5.110}$$

$$\begin{aligned} t_7(Q^2, -u, m_t^2) &= \int d^d l_1 \frac{1}{[l1^2]^2(-m_t^2 + (l1-q)^2)(-m_t^2 + (k + l1 - q)^2)} \\ &= \Gamma(2+\epsilon) \int_0^\infty (1 + x_2 + x_3)^{2\epsilon} \\ &\times (m_t^2(x_2 + x_3)(1 + x_2 + x_3) + (Q^2 x_2 - u x_3))^{-2-\epsilon}. \end{aligned} \tag{5.111}$$

The same considerations we have done for topology t_4 also hold for this topology, so that results for this integral in the double limit $\{z \to 1, y \to 0\}$ correspond to the only hard region and read

$$t_6(z \to 1, y \to 0, s) = t_6(\bar{z}, y, s)_{\{0,0,0\}} = \frac{s^{-1-\epsilon}\Gamma(1+\epsilon)}{1-\epsilon}, \tag{5.112}$$

$$t_7(z \to 1, y \to 0, s) = t_7(\bar{z}, y, s)_{\{0,0,0\}} = \frac{s^{-2-\epsilon}\Gamma(2+\epsilon)}{(-1+\epsilon)\epsilon}. \tag{5.113}$$

- **BOX**

1. **Box with 1 internal mass m_t^2 (1 massive propagator) and external masses $\{\mathbf{u}, \mathbf{t}, \mathbf{Q^2}\}$**

$$
\begin{aligned}
box_1(Q^2, u, t, p_t^2, m_t^2) = & \int d^d l_1 \frac{1}{(l_1^2 - m_t^2)(l_1 - q)^2 (k + l_1 - q)^2 (k + l_1 - p_b - q)^2} \\
= & \Gamma(2+\epsilon) \int_0^\infty (1 + x_2 + x_3 + x_4)^{2\epsilon} \\
& \times (-u x_1 x_3 - p_t^2 x_1 x_4 - t x_2 x_4 + Q^2 x_1 x_2 \\
& + m_t^2 x_1 (x_1 + x_2 + x_3 + x_4))^{-2-\epsilon}. \qquad (5.114)
\end{aligned}
$$

The genuine $\mathcal{F}$, before any manipulations, depends on the external invariants $\{u, t, p_t^2, Q^2 = -q^2\}$. Being its sign positive in the region identified by negative external and positive internal invariants, $\mathcal{F}$ is positive-definite in the physical region $\{u < 0, t < 0, p_t^2 < 0, Q^2, m_t^2\}$. We observe that, thanks to a cancellation occurring in the polynomial, we can safely set $p_t^2 = m_t^2$ without spoiling the sign of the polynomial.

$$
\begin{aligned}
box_1(Q^2, u, t, m_t^2) = & \int d^d l_1 \frac{1}{(l_1^2 - m_t^2)(l_1 - q)^2 (k + l_1 - q)^2 (k + l_1 - p_b - q)^2} \\
= & \Gamma(2+\epsilon) \int_0^\infty (1 + x_2 + x_3 + x_4)^{2\epsilon} \\
& \times (-u x_1 x_3 - t x_2 x_4 + Q^2 x_1 x_2 \\
& + m_t^2 x_1 (x_1 + x_2 + x_3))^{-2-\epsilon}. \qquad (5.115)
\end{aligned}
$$

We obtain thus an integral which is real in the region of physical interest and thus do not need analytical continuation. `Asy2.m` gives three regions: $\{\{0,0,0,0\}, \{0,-1,-1,-1\}, \{0,0,0,-1\}\}$. Since integrations are quite simple and hand real-valued output, we report directly results for the various regions.

$$
\begin{aligned}
box_1(z \to 1, y \to 0, s) = & box_1(\bar{z}, y, s)_{\{0,0,0,0\}} + box_1(\bar{z}, y, s)_{\{0,-1,-1,-1\}} \\
& + box_1(\bar{z}, y, s)_{\{0,0,0,-1\}} \qquad (5.116)
\end{aligned}
$$

HARD REGION $\{0,0,0,0\}$

$$
box_1(\bar{z}, y, s)_{\{0,0,0,0\}} = -\frac{s^{-2-\epsilon}\Gamma(2+\epsilon)}{(1+2\epsilon)\epsilon(1+\epsilon)}. \qquad (5.117)
$$

REGION $\{0,-1,-1,-1\}$

$$
box_1(\bar{z}, y, s)_{\{0,-1,-1,-1\}} = a^{-1-\epsilon} s^{-2-\epsilon} (1-\lambda)^{-1-\epsilon} \bar{z}^{-1-\epsilon} \frac{2^{1+\epsilon}\Gamma(-\epsilon)^3 \Gamma(1+\epsilon)}{\Gamma(1-\epsilon)\Gamma(-2\epsilon)}. \qquad (5.118)
$$

REGION $\{0, 0, 0, -1\}$

$$box_1(\bar{z}, y, s)_{\{0,0,0,-1\}} = -a^{-1-2\epsilon}\pi s^{-2-\epsilon}(1-\lambda)^{-1-2\epsilon}{z_b}^{-1-2\epsilon}\csc(2\epsilon\pi)\Gamma(\epsilon). \tag{5.119}$$

2. **Box with 1 internal mass m_t^2 (2 massive propagators) and external masses $\{\mathbf{s}, \mathbf{Q^2}, \mathbf{u}\}$**

$$\begin{aligned} box_2(Q^2, s, u, p_t^2, m_t^2) &= \int d^d l_1 \frac{1}{(l_1^2)(l_1+p_b)^2((k+l_1-q)^2-m_t^2)((l_1-q)^2-m_t^2)} \\ &= \Gamma(2+\epsilon)\int_0^\infty (1+x_2+x_3+x_4)^{2\epsilon} \\ &\quad (-sx_2x_3 - ux_1x_4 - p_t^2 x_2x_4 + Q^2 x_1x_3 \\ &\quad + m_t^2(x_3+x_4)(x_1+x_2+x_3+x_4))^{(-2-\epsilon)}. \end{aligned} \tag{5.120}$$

Following the usual rule, $\mathcal{F}$ is positive-definite for negative external invariants $\{s < 0, p_t^2 < 0, u < 0, Q^2 < 0\}$ and positive internal invariant m_t^2. First we observe that we can safely set $p_t^2 = m_t^2$ thanks to a cancellation in the $\mathcal{F}$ polynomial which guarantees the sign of the polynomial to remain definite (as it happened for box_1). So we can rewrite

$$\begin{aligned} box_2(Q^2, s, u, m_t^2) &= \int d^d l_1 \frac{1}{(l_1^2)(l_1+p_b)^2((k+l_1-q)^2-m_t^2)((l_1-q)^2-m_t^2)} \\ &= \Gamma(2+\epsilon)\int_0^\infty (1+x_2+x_3+x_4)^{2\epsilon} \\ &\quad (-sx_2x_3 - ux_1x_4 + Q^2 x_1x_3 \\ &\quad + m_t^2((x_3+x_4)^2 x_1x_4 + x_3(x_1+x_2)))^{(-2-\epsilon)}. \end{aligned} \tag{5.121}$$

We also clearly see that for this box, we have to deal with analytical continuation, since the result is expected to be complex for $s > 0$. `Asy2.m` gives us 2 regions: $\{\{0, 0, 0, 0\}, \{0, -1, 0, 0\}\}$ and indeed the result for the second region happens to develop a complex phase. Since integrations are quite trivial to perform and analytical continuation is done using the usual prescriptions Eq.(5.88), we just report the final results for regions.

$$\begin{aligned} box_2(\bar{z} \to 0, y \to 1, s) &= box_2(\bar{z}, y, s)_{\{0,0,0,0\}} + box_2(\bar{z}, y, s)_{\{0,-1,0,0\}} \\ box_2(\bar{z}, y, s)_{\{0,0,0,0\}} &= -\frac{s^{-2-\epsilon}\Gamma(2+\epsilon)}{(1+2\epsilon)\epsilon} \\ box_2(\bar{z}, y, s)_{\{0,-1,0,0\}} &= a^{-1-2\epsilon}e^{2i\pi\epsilon}s^{-2-\epsilon}\bar{z}^{-1-2\epsilon}\Gamma(-\epsilon)\Gamma(2\epsilon). \end{aligned} \tag{5.122}$$

Now that all 1-loop integrals are computed, we can integrate them over the Phase Space Eq.(5.72), namely over the angular variable λ. There is just one last subtlety that separates us from obtaining the initial condition and we would like to spend a few words about it. Up to here we computed 1-loop topologies with only denominators and with dotted propagators. This is because these 1-loop integrals where obtained by taking the basis in Eq.(5.63) and cutting away the Phase Space. Now, the point is that when we want to get the initial condition for instance for this specific choice of basis (no numerators but dotted propagators) we will have to integrate our 1-loop results not only over the standard Phase Space, but also over 'dotted Phase Spaces', namely combinations of one or more dotted cut propagators. Whenever we get a dotted cut propagator, this is not physical and a 'dotted Phase Space' cannot be parametrized like an ordinary Phase Space (Eq.(5.72). So, we adopt the following strategy. We express the 'dotted basis' Eq.(5.63) in terms of a basis with no dotted propagators but with numerators (which is actually the original bases given by the `Mathematica` version of the reduction program `FIRE`, [113]). When we cut away the Phase Space in the new basis, those masters which have a numerator can be distinguished into two categories

- the numerator does not contain the loop momentum l_1, ex.

$$\begin{aligned}
&\int d^dk d^dl_1 \frac{(k+p_b)^2}{k^2 l_1^2(-k+l_1+p_b)^2(-mt^2+(l_1-q)^2)(-mt^2+(-k+p_b+q)^2)} \\
&\to \int d^dk \frac{(k+p_b)^2}{k^2(-mt^2+(-k+p_b+q)^2)} T(k) \\
&\text{with } T(k) = \int d^dl_1 \frac{1}{l_1^2(-k+l_1+p_b)^2(-mt^2+(l_1-q)^2)}
\end{aligned} \tag{5.123}$$

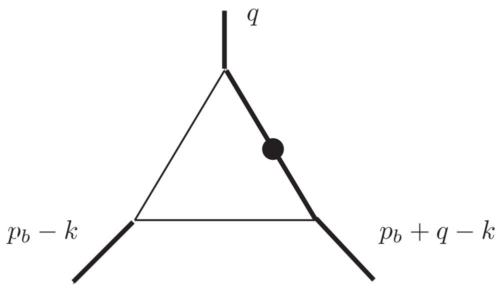
  In this case the 1-loop integral is one of the results obtained above and and then we parametrize the numerator depending on k in terms of the Phase Space variables z, λ and integrate over it together with the Phase Space.

- the numerator contains the loop momentum l_1, ex.

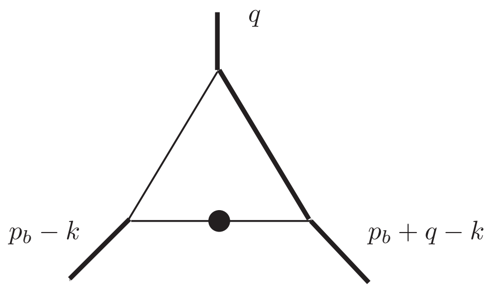
$$\begin{aligned}
&\int d^dk d^dl_1 \frac{(l_1+p_b)^2}{k^2 l_1^2(-k+l_1+p_b)^2(-mt^2+(l_1-q)^2)(-mt^2+(-k+p_b+q)^2)} \\
&\to \int d^dk \frac{1}{k^2(-mt^2+(-k+p_b+q)^2)} T'(k) \\
&\text{with } T'(k) = \int d^dl_1 \frac{(l_1+p_b)^2}{l_1^2(-k+l_1+p_b)^2(-mt^2+(l_1-q)^2)}
\end{aligned} \tag{5.124}$$

  In this latter case the 1-loop integral has a numerator. In principle we would need to compute a new 1-loop integral, but practically speaking this is not necessary because this new integral $T'(k)$ can be expressed as a linear combination of the above computed 1-loop 'master' integrals. Once this reduction is done we just need to carry out the integration over k of a standard Phase Space Eq.(5.72) times the 1-loop $T'(k)$.

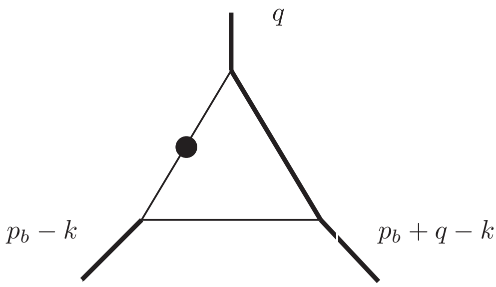
With this last paragraph, we have explained in detail all the elements needed to compute the MIs describing $[b + W^* \to t + g]_{1-loop}$. We can then conclude here the RV section for the bottom channel. Complete results for the MIs are provided in the Appendix.



(a) Triangle n.2: t_2



(b) Triangle n.3: t_3



(c) Triangle n.9: t_9

Figure 5.17

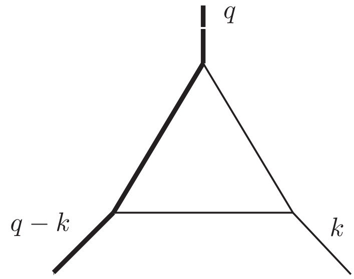


Figure 5.18: Triangle n.4: t_4

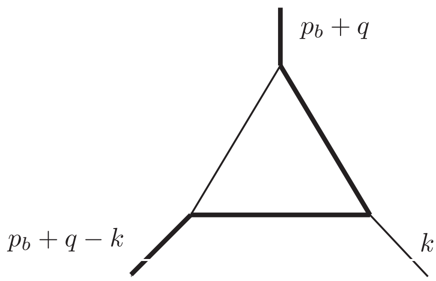
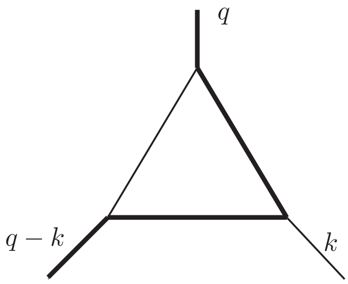
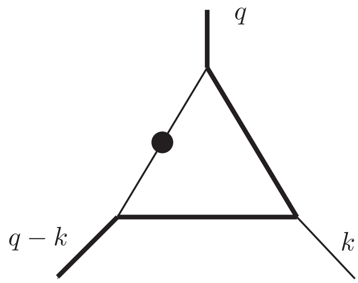


Figure 5.19: Triangle n.5: t_5



(a) Triangle n.6: t_6



(b) Triangle n.7: t_7

Figure 5.20

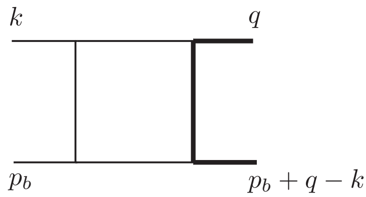


Figure 5.21: Box n.1: box_1

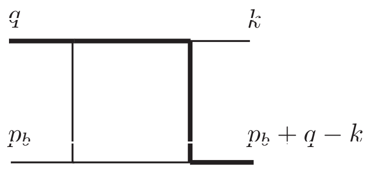


Figure 5.22: Box n.2: box_2

Double-Virtuals $[b + W^* \to t]_{2-loop}$

Diagrams contributing to the 2-loop corrections to the vertex $b + W^* \to t$ are drawn in Fig.(5.23). They give birth to four independent topologies (Eq.(5.125) and Fig.(5.24)).

$$\begin{aligned}
t_1^{VV} &= \{l_1^2, l_2^2, (l_1 - p_b)^2, (l_2 + p_b)^2, (-l_1 + l_2 + p_b)^2, -m_t^2 + (l_2 - q)^2, -m_t^2 + (-l_1 + p_b + q)^2\} \\
t_2^{VV} &= \{l_1^2, l_2^2, (l_1 + l_2)^2, (l_1 + l_2 - p_b)^2, (l_2 - p_b)^2, -m_t^2 + (l_1 + l_2 - p_b - q)^2, -m_t^2 + (l_1 - q)^2\} \\
t_3^{VV} &= \{l_1^2, l_2^2, (l_1 + p_b)^2, (l_2 - p_b)^2, -m_t^2 + (l_1 - q)^2, -m_t^2 + (-l_1 + l_2 + q)^2, -m_t^2 + (l_2 + p_b + q)^2\} \\
t_4^{VV} &= \{l_1^2, l_2^2 - m_t^2, (l_1 + p_b)^2, -m_t^2 + (l_1 - q)^2, (-l_1 + l_2 + q)^2, (l_2 + q)^2, (l_2 + p_b + q)^2\}.
\end{aligned} \quad (5.125)$$

The master integrals obtained from the reduction of these diagrams are listed in Eq.(5.126). In Fig.(5.25) we draw the subtopologies to which master integrals with denominator equal to 1 correspond (the remaining masters are obtained by multiplying these subtopologies by the appropriate numerators).

$$\begin{aligned}
Master^{VV}_{t_1,1} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 l_2^2 (-l_1 + l_2 + p_b)^2 [-mt^2 + (l_2 - q)^2]} \\
Master^{VV}_{t_1,2} &= \int d^d l_1 d^d l_2 \frac{(l_1 - p_b)^2}{l_1^2 l_2^2 (-l_1 + l_2 + p_b)^2 [-mt^2 + (l_2 - q)^2][-mt^2 + (-l_1 + p_b + q)^2]} \\
Master^{VV}_{t_2,1} &= \int d^d l_1 d^d l_2 \frac{l_1^2}{l_2^2 (l_1 + l_2)^2 [-mt^2 + (l_1 - q)^2]} \\
Master^{VV}_{t_2,2} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 l_2^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,3} &= \int d^d l_1 d^d l_2 \frac{l_1^2}{l_2^2 (l_1 + l_2)^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,4} &= \int d^d l_1 d^d l_2 \frac{(l_1 + l_2)^2}{l_1^2 l_2^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,5} &= \int d^d l_1 d^d l_2 \frac{(l_1 + l_2 - p_b)^2}{l_1^2 l_2^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,6} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 (l_1 + l_2)^2 (l_2 - p_b)^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,7} &= \int d^d l_1 d^d l_2 \frac{l_2^2}{l_1^2 (l_1 + l_2)^2 (l_2 - p_b)^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_2,8} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 l_2^2 (l_1 + l_2)^2 (l_2 - p_b)^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (l_1 + l_2 - p_b - q)^2]} \\
Master^{VV}_{t_3,1} &= \int d^d l_1 d^d l_2 \frac{1}{[-mt^2 + (l_1 - q)^2][-mt^2 + (-l_1 + l_2 + q)^2][-mt^2 + (l_2 + p_b + q)^2]} \\
Master^{VV}_{t_3,2} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 [-mt^2 + (l_1 - q)^2][-mt^2 + (-l_1 + l_2 + q)^2][-mt^2 + (l_2 + p_b + q)^2]}
\end{aligned}$$

$$
\begin{aligned}
Master^{VV}_{t_3,3} &= \int d^d l_1 d^d l_2 \frac{l_2^2}{l_1^2[-mt^2+(l_1-q)^2][-mt^2+(-l_1+l_2+q)^2][-mt^2+(l_2+p_b+q)^2]}\\
Master^{VV}_{t_3,4} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2 l_2^2[-mt^2+(-l_1+l_2+q)^2][-mt^2+(l_2+p_b+q)^2]}\\
Master^{VV}_{t_3,5} &= \int d^d l_1 d^d l_2 \frac{(l_1+p_b)^2}{l_1^2 l_2^2[-mt^2+(-l_1+l_2+q)^2][-mt^2+(l_2+p_b+q)^2]}\\
Master^{VV}_{t_4,1} &= \int d^d l_1 d^d l_2 \frac{1}{l_1^2[l_2^2-mt^2][(-l_1+l_2+q)^2]}\\
Master^{VV}_{t_4,2} &= \int d^d l_1 d^d l_2 \frac{1}{[l_2^2-mt^2][(l_1+p_b)^2][(-l_1+l_2+q)^2]}\\
Master^{VV}_{t_4,3} &= \int d^d l_1 d^d l_2 \frac{(l_1+p_b)^2}{l_1^2[l_2^2-mt^2][(-l_1+l_2+q)^2]}\\
Master^{VV}_{t_4,4} &= \int d^d l_1 d^d l_2 \frac{(l_1+p_b)^2}{l_1^2[l_2^2-mt^2][-mt^2+(l_1-q)^2][(-l_1+l_2+q)^2][(l_2+p_b+q)^2]}.
\end{aligned}
\tag{5.126}
$$

Results for all master integrals in Eq.(5.126) have been provided in [29], [30], [9], [18], [37] and were cross-checked by us. For a complete list of them, we send the interested reader to the above-mentioned references. We limit ourselves to provide for pedagogical purposes an example of how one of these integrals is computed through standard Differential Equations, where by 'standard' we mean not in canonical form. Let us pick up for instance $Master^{VV}_{t_1,1}$ and define its dimensionless equivalent

$$
\begin{aligned}
Master^{VV,ad}_{t_1,1}(y) &= (m_t^2)^{d-4} Master^{VV,ad}_{t_1,1}(y)\\
&= (m_t^2)^{-2\epsilon} Master^{VV,ad}_{t_1,1}(y).
\end{aligned}
\tag{5.127}
$$

The D.E. for $Master^{VV,ad}_{t_1,1}(y)$ reads

$$
\begin{aligned}
\frac{d}{dy} Master^{VV,ad}_{t_1,1}(y) =& \frac{4(3-(4-2\epsilon)(1-y)-4y) Master^{VV,ad}_{t_1,1}(y)}{4y(1+y)}\\
&- \frac{(8-3(4-2\epsilon)) Master^{VV,ad}_{t_4,2}(y)}{4y(1+y)}.
\end{aligned}
\tag{5.128}
$$

A first comment that needs to be made is that all VV masters satisfy ordinary DEs. This represents quite a big simplification with respect to the RR and RV masters, for which we had to integrate partial DEs. The reason for this lies in the kinematic of the double-virtual amplitudes, which, given the absence of real emissions, coincides with the Born kinematic. At tree-level the process is governed by only two dimensional scales, i.e. Q^2, m_t^2, because the energy in the center of mass frame of the b, W^* pair s is exactly the needed amount to produce a top in the final state, so that $s = m_t^2$ holds. This implies that at two loops the dimensionless ratio $z = m_t^2/s$ is constant and equal to 1 , so that the only variable on which the

masters can depend is actually $y = Q^2/m_t^2$.
Now, the r.h.s of Eq.(5.128) has a usual structure: it is the sum of a *homogeneous* term, proportional to the integral we want to solve, namely $Master^{VV}_{t_1,1}$, and an *inhomogeneous term* given by the linear combinations of other integrals of the basis Eq.(5.126) with an equal or smaller number of propagators with respect to $Master^{VV}_{t_1,1}$. In this specific case, the structure is particularly simple, since the inhomogeneous term contains the only integral $Master^{VV}_{t_4,2}$, which constitutes indeed a subtopology of the original integral $Master^{VV}_{t_1,1}$. Now, the standard method to solve differential equations in one variable, such as Eq.(5.128), is the so-called *variation of constants method*, which can be found in any standard text-book of elementary calculus. A prerequisite for applying such method is the knowledge of the inhomogeneous term, which implies we have to know the result for $Master^{VV}_{t_4,2}$ in order to be able to compute $Master^{VV}_{t_1,1}$.
In other words, we have to adopt a bottom-up approach to solve the system of (coupled) differential equations that we obtain for the entire basis. We start by solving DEs for the masters of the basis with the lowest number number of propagators, and proceed then solving the masters with an increasing number of propagators, till we get to solve the biggest integrals appearing in the basis. At each order m in the number of propagators, masters which have been previously solved (namely with a number of propagators $m' < m$) enter the DEs as part of the inhomogeneous terms, which is then totally known. So, assuming we know the result for $Master^{VV}_{t_4,2}$, we show how Eq.(5.128) is solved order by order in ϵ, taking as an example the first three non-trivial orders. We start from the observation that, being at two loops, the most singular behaviour both integrals $Master^{VV}_{t_1,1}$ and $Master^{VV}_{t_4,2}$ can exhibit is of the type $1/\epsilon^4$. From the result for $Master^{VV}_{t_4,2}$ we know that this integral starts at order $1/\epsilon^2$, so that it admits the expansion

$$Master^{VV}_{t_4,2}(y) = \frac{m^{(-2)}_{t_4,2}(y)}{\epsilon^2} + \frac{m^{(1)}_{t_4,2}(y)}{\epsilon} + m^0_{t_4,2}(y) + m^{(1)}_{t_4,2}\epsilon(y) + m^{(2)}_{t_4,2}\epsilon^2(y) + \mathcal{O}(\epsilon^3), \quad (5.129)$$

where the coefficients $m^{(i)}_{t_4,2}(y)$ are known. Along the same line, we can write an ansatz for $Master^{VV}_{t_1,1}$, in which we have to include a priori also terms of order $1/\epsilon^3$ and $1/\epsilon^4$.

$$\begin{aligned} Master^{VV}_{t_1,1}(y) =& \frac{m^{(-4)}_{t_1,1}(y)}{\epsilon^4} + \frac{m^{(-3)}_{t_1,1}(y)}{\epsilon^3} + \frac{m^{(-2)}_{t_1,1}(y)}{\epsilon^2} + \frac{m^{(1)}_{t_1,1}(y)}{\epsilon} \\ &+ m^0_{t_1,1}(y) + m^{(1)}_{t_1,1}\epsilon(y) + m^{(2)}_{t_1,1}\epsilon^2(y) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (5.130)$$

Let us substitute in Eq.(5.128) both the ϵ-expansions for the inhomogeneous and homogeneous terms (Eq.(5.129) and Eq.(5.130)) and expand again in ϵ both sides of Eq.(5.128). We get for the first five coefficient in the expansion of $Master^{VV}_{t_1,1}(y)$ the following differential equations.

$$\begin{aligned}
&\frac{d}{dy} m_{t_1,1}^{(-4)}(y) = 0 \\
&\frac{d}{dy} m_{t_1,1}^{(-3)}(y) = 0 \\
&\frac{d}{dy} m_{t_1,1}^{(-2)}(y) = \frac{m_{t_4,2}^{(-2)}(y) - m_{t_1,1}^{(-2)}(y)}{\epsilon^2 y(1+y)} \\
&\frac{d}{dy} m_{t_1,1}^{(-1)}(y) = \frac{-6m_{t_4,2}^{(-2)}(y) + 4m_{t_4,2}^{(-1)}(y) + 8(1-y)m_{t_1,1}^{(-2)}(y) - 4m_{t_1,1}^{(-1)}(y)}{4\epsilon y(1+y)} \\
&\frac{d}{dy} m_{t_1,1}^{(0)}(y) = \frac{-6m_{t_4,2}^{(-1)}(y) + 4m_{t_4,2}^{(0)}(y) + 8(1-y)m_{t_4,2}^{(-1)}(y) - 4m_{t_4,2}^{(0)}(y)}{4y(1+y)},
\end{aligned} \tag{5.131}$$

For brevity, we omit the next orders ($\mathcal{O}(\epsilon)$, $\mathcal{O}(\epsilon^2)$, and so on) in Eq.(5.131) [11].

Let us start integrating the first two orders, which are trivial and give simply constants as a result of the integration

$$m_{t_1,1}^{(-4)}(y) = c^{(-4)}, \quad m_{t_1,1}^{(-3)}(y) = c^{(-3)}. \tag{5.132}$$

Such constants, which appear as a result of the fact that we are performing indefinite integrations on both sides of the DE, have to be fixed by computing the integral in a fixed kinematic point $y = \tilde{y}$. The vector of boundary conditions $\vec{c} = (c^{(-4)}, ..., c^{(0)})$ can be obtained from the explicit computation of the integral in the chosen kinematic limit, by applying for instance the strategy of Expansion by Regions (previously illustrated in the case of the RR and RV integrals). Alternatively one can study the singular behaviour of the differential equation and check whether there are or not spurious singularities, namely singularities which do not have any precise physical meaning (we remind the reader that the singular behaviour of the differential equation reflect that of its solution!). In case a spurious singularity is present say for $y = y'$, one can use it to extract the boundary conditions by imposing the integral to be finite in y'. This is indeed the case for the case study we have chosen. In order to study the behaviour, we let the regulator ϵ going to zero in the differential equations, thus getting

$$\frac{d}{dy} Master_{t_1,1}^{VV}(y) = \left(-\frac{1}{y} + \frac{1}{1+y}\right) Master_{t_1,1}^{VV}(y) + \left(\frac{1}{y} - \frac{1}{1+y}\right) Master_{t_4,2}^{VV}(y). \tag{5.133}$$

Given the physical boundary $y \geq 0$ and given that no physical divergence is expected at $y = 0$, this equation exhibits a clear spurious singularity in $y = 0$.

[11] They have a structure similar to the one of the first five orders, reported in Eq.(5.131) and the strategy to integrate them is the same as the one that we illustrate in the following for the lowest order.

Starting from the first non trivial order in ϵ (here ϵ^{-2}) and by requiring order by order in Eq.(5.131) the r.h.s to be finite in $y = 0$, namely the coefficient of the $1/y$ pole to be zero, the boundary conditions $\vec{c}$ can be extracted[12]. Without entering further in the discussion of boundary conditions, which were not indeed computed directly by us, let us assume the vector of boundary condition $\vec{c}$ to be known and show then how integration order by order works.
The vector $\vec{c}$ reads up to $\mathcal{O}(\epsilon^0)$

$$\vec{c} = \left(0, 0, \frac{1}{32}, -\frac{5}{32}, \frac{1}{96}(-57 - 2\pi^2)\right), \tag{5.134}$$

so that the two first order in Eq.(5.131) simply amount to zero contribution. For the first non trivial order, namely $\mathcal{O}(\epsilon^{-2})$ we need to substitute in Eq.(5.131) the inhomogeneous term $m^{(-2)}_{t_4,2}(y) = 1/32$ and we immediately observe that given the structure of the r.h.s, we do not even need to integrate. It is sufficient to impose the r.h.s. to be finite in the limit $y \to 0$ in order to extract the desired coefficient, namely $m^{(-2)}_{t_1,1}(y) = 1/32$. Moving on to $\mathcal{O}(\epsilon^{-1})$, and substituting $m^{(-1)}_{t_4,2}(y) = 5/64$ plus the results from the previous orders, we get the DE

$$\frac{d}{dy} m^{(-1)}_{t_1,1}(y) = \frac{3}{32y} - \frac{5}{32(1+y)} + \left(-\frac{1}{y} + \frac{1}{1+y}\right) m^{(-1)}_{t_1,1}(y). \tag{5.135}$$

The homogeneous solution reads then

$$m^{(-1),hom}_{t_1,1}(y) = \frac{(1+y)C[-1]}{y} \tag{5.136}$$

with $C[-1]$ being an integration constant. By applying the variation of constant method, we get as a solution to the full equation

$$\begin{aligned} m^{(-1)}_{t_1,1}(y) &= \frac{(1+y)C[-1]}{y} \int dy \left[\frac{3}{32y} - \frac{5}{32(1+y)}\right] \left[\frac{(1+y)C[-1]}{y}\right]^{-1} + c^{(-1)} \\ &= -\frac{5}{32y} + \frac{(1+y)\ln(1/(1+y))}{16y} + c^{(-1)} \\ &= \frac{1}{32}(5 - 2\ln(1+y)) - \frac{\ln(1+y)}{16y}. \end{aligned} \tag{5.137}$$

At this point we are ready this result for $m^{(-1)}_{t_1,1}(y)$ together with the known inhomogeneous coefficient $m^{(0)}_{t_4,2}(y)$ in the DE for the next order, namely $\mathcal{O}(\epsilon^0)$ and solve it in the same way we did at $\mathcal{O}(\epsilon^{-1})$. By repeating this procedure order by order we can obtain the desired integral up to the needed precision in ϵ.

[12]This method cannot be applied to the first two orders, for which the boundary conditions have to explicitly computed.

Our discussion of Double-Virtual contribution to CC-DIS Form Factors ends here. Further details and complete results for master integrals and 2-loop Form Factors can be found in the above-cited references.

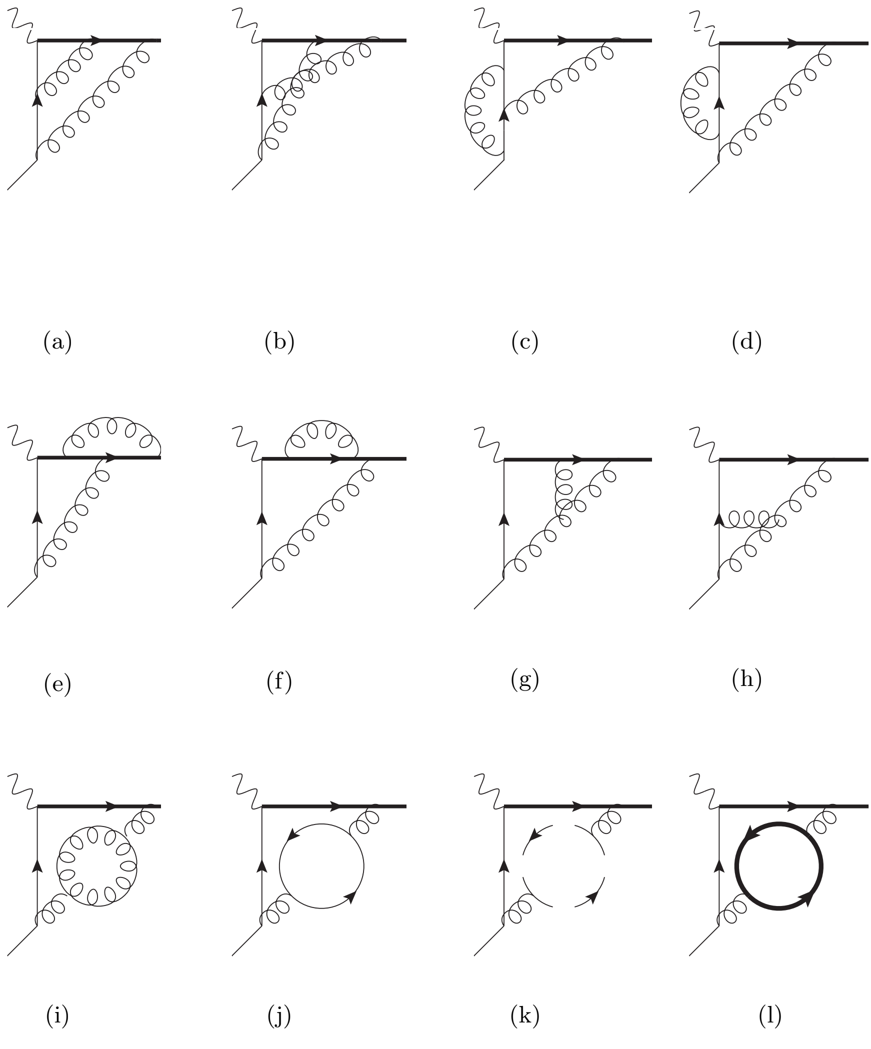
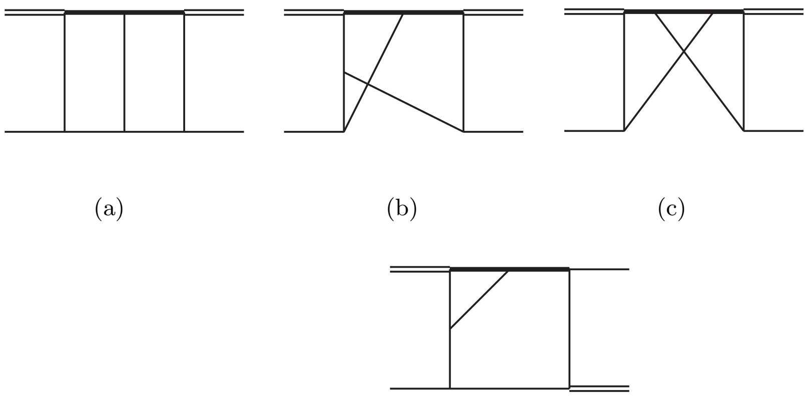


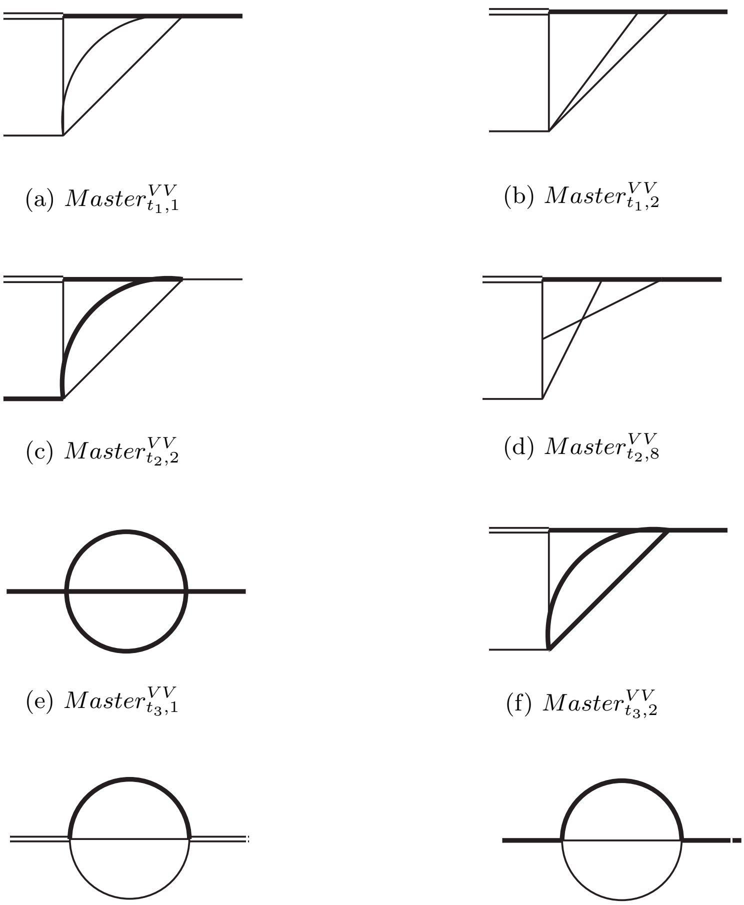
Figure 5.23: 2-loop corrections to $b + W^* \to t$



(a) (b) (c)

(d)

Figure 5.24: Independent topologies for $[b + W^* \to t]_{2l}$.



(a) $Master^{VV}_{t_1,1}$ (b) $Master^{VV}_{t_1,2}$

(c) $Master^{VV}_{t_2,2}$ (d) $Master^{VV}_{t_2,8}$

(e) $Master^{VV}_{t_3,1}$ (f) $Master^{VV}_{t_3,2}$

(g) $Master^{VV}_{t_4,1}$ (h) $Master^{VV}_{t_4,2}$

Figure 5.25: Set of independent subtopologies for $b + W^* \to t$ at 2-loop. Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

5.2.2 Gluon channel: g-initiated subprocesses at NNLO.

Double Reals $\left[g + W^* \to t + \bar{b} + g\right]_{0-loop}$

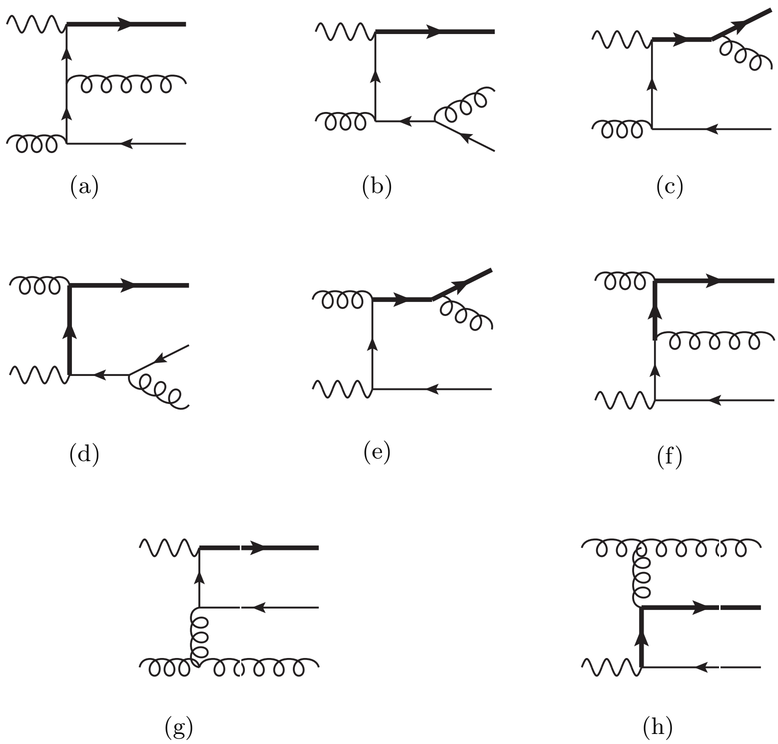


Figure 5.26: Tree-level diagrams for $g + W^* \to t + \bar{b} + g$

The Feynman diagrams for the Double Real (RR) contribution (Fig.5.26) to the gluon channel can be generated from scratch or obtained by applying *crossing symmetry* to the Double Real diagrams for the bottom-initiated subprocess $b + W^* \to t + g + g$. We remind the reader that crossing symmetry is a property of amplitudes which states that the S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k = -p$ in the final state. In other words, given an amplitude $\mathcal{M}$, it holds

$$\mathcal{M}\left(\phi(p) + ... \to ...\right) = \mathcal{M}\left(... \to ... + \bar{\phi}(k)\right) \tag{5.138}$$

where $\bar{\phi}$ is the antiparticle of ϕ and $k = -p$ (see [102]). In our case, given the starting amplitudes for $b(p_b) + W^*(q) \to t(p_t) + g(k_1) + g(k_2)$, we can obtain amplitudes for $g(k_1)+W^*(q) \to t(p_t)+g(k_2)+\bar{b}(p_b)$ by applying crossing symmetry defined as $\mathcal{CS}(p_b \to -p_b, k_1 \to -k_1)$. Once crossing symmetry is applied, it is convenient to rename particle momenta as

$$g(k_1) + W^*(q) \to t(p_t) + g(k_2) + \bar{b}(p_b) \overset{(-k_1 \to p_b, -p_b \to k_1)}{\to} g(p_b) + W^*(q) \to t(p_t) + g(k_2) + \bar{b}(k_1), \tag{5.139}$$

in order to maintain definitions of Mandelstam invariants unchanged. By inspection of Fig.(5.26), we see that because of the appearance of tree-level diagrams with a new structure, already at this primary stage we can intuitively expect to get new topologies and master integrals with respect to those already found in the bottom channel.
By taking the square modulus of diagrams Fig.5.26 and adding the 3-particle Phase Space Eq.(5.40), we get at a first inspection 9 independent set of propagators Eq.(5.140).

$$\begin{aligned} t_{in} = \{&\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1+k_2-q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1-q)^2 - m_t^2, (k_1+k_2-q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1-q)^2 - m_t^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-q)^2 - m_t^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1-q)^2 - m_t^2, (k_1+k_2-q)^2 - m_t^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (k_1-q)^2 - m_t^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (k_1-q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}, \\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (p_b-k_2)^2, (k_1-q)^2 - m_t^2, (-k_1+p_b+q)^2 - m_t^2, \\ &(k_1+k_2-p_b-q)^2 - m_t^2\}\} \end{aligned} \tag{5.140}$$

Only five among the sets of propagators in Eq.(5.140) are topologies. The remaining four sets, listed in Eq.(5.141), contain linearly dependent propagators.

$$t_{dep} = \{\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1+k_2-q)^2 - m_t^2,$$

$$
\begin{aligned}
&(k_1+k_2-p_b-q)^2-m_t^2\},\\
&\{k_1^2,(p_b-k_1)^2,k_2^2,(k_1+k_2-p_b)^2,(k_1-q)^2-m_t^2,(-k_1+p_b+q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\},\\
&\{k_1^2,(p_b-k_1)^2,k_2^2,(k_1-q)^2-m_t^2,(k_1+k_2-q)^2-m_t^2,(-k_1+p_b+q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\},\\
&\{k_1^2,(p_b-k_1)^2,k_2^2,(p_b-k_2)^2,(k_1-q)^2-m_t^2,(-k_1+p_b+q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\}\}
\end{aligned}
\tag{5.141}
$$

This implies that we have to perform partial fractioning on those scalar integrals containing products of such linearly dependent propagators. Let us explain with an example. For instance, we consider the first set in Eq.(5.141), which contains the set of linearly dependent propagators Eq.(5.142).

$$
[(k_1+k_2)^2]-[(k_1+k_2-p_b)^2]-[(k_1+k_2-q)^2-m_t^2]+[(k_1+k_2-p_b-q)^2-m_t^2]=0.
\tag{5.142}
$$

Among the scalar integrals described by this set of propagators we also encounter integrals containing at denominator products of these four dependent propagators. In order to perform the reduction to masters integrals on this kind of integrals, we need to correct for this linear dependence and this is achieved by designing a proper *partial fractioning rule*. For instance, given the relation Eq.(5.142), we design the partial fractioning rule Eq.(5.143).

$$
\begin{aligned}
\frac{1}{[(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2]} \rightarrow \frac{1}{s+Q^2}\Bigg[&\frac{1}{[(k_1+k_2-q)^2-m_t^2]}-\frac{1}{[(k_1+k_2-p_b)^2]}\\
&-\frac{[(k_1+k_2)^2]}{[(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2]}\\
&+\frac{[(p_b+q-k_1-k_2)^2-m_t^2]}{[(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2]}\Bigg]
\end{aligned}
\tag{5.143}
$$

If we apply Eq.(5.143) to the simplest integral we can imagine, namely the one containing the phase space and a product of the only dependent propagators, we can rewrite it as the sum of integrals each one containing only a subset of these propagators, so that the problem of linear dependence is solved.

$$
\begin{aligned}
&\int d^dk_1 d^dk_2 \frac{1}{[k_1^2][k_2^2][(k_1+k_2)^2][(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2][(k_1+k_2-p_b-q)^2-m_t^2]}\\
&=\frac{1}{s+Q^2}\Bigg[\int d^dk_1k_2 \frac{1}{[k_1^2][k_2^2][(k_1+k_2)^2][(k_1+k_2-q)^2-m_t^2][(k_1+k_2-p_b-q)^2-m_t^2]}\\
&-\int d^dk_1k_2 \frac{1}{[k_1^2][k_2^2][(k_1+k_2)^2][(k_1+k_2-p_b)^2][(k_1+k_2-p_b-q)^2-m_t^2]}\\
&-\int d^dk_1d^dk_2 \frac{1}{[k_1^2][k_2^2][(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2][(k_1+k_2-p_b-q)^2-m_t^2]}
\end{aligned}
$$

$$+ \int d^dk_1 d^dk_2 \frac{1}{[k_1^2][k_2^2][(k_1+k_2)^2][(k_1+k_2-p_b)^2][(k_1+k_2-q)^2-m_t^2]}\Bigg] \tag{5.144}$$

We see that none of the integrals appearing on the right hand side of Eq.(5.144) contains now a set of linearly dependent propagators, so that we achieved our goal. In addition, we observe that the term on the r.h.s. coloured in red is zero, since it happens not to contain anymore the full phase space measure. The last step we need to take to definitely get rid of the problems of linearly dependent propagators is to make sure that every scalar integral we get after performing partial fractioning is actually contained in a topology, or, in other words, that the propagators contained in each of these scalar integrals actually represent a subset of a proper topology. We explain using again the same example. Let us write down the set of topology of which we dispose up to now, given by Eq.(5.145).

$$\begin{aligned} t_{indep} =&\{\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1-q)^2-m_t^2, (k_1+k_2-q)^2-m_t^2,\\ &(k_1+k_2-p_b-q)^2-m_t^2\},\\ &\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-q)^2-m_t^2, (-k_1+p_b+q)^2-m_t^2,\\ &(k_1+k_2-p_b-q)^2-m_t^2\},\\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (-k_1+p_b+q)^2-m_t^2,\\ &(k_1+k_2-p_b-q)^2-m_t^2\},\\ &\{k_1^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (k_1-q)^2-m_t^2, (-k_1+p_b+q)^2-m_t^2,\\ &(k_1+k_2-p_b-q)^2-m_t^2\},\\ &\{k_1^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (p_b-k_2)^2, (k_1-q)^2-m_t^2,\\ &(k_1+k_2-p_b-q)^2-m_t^2\}\} \end{aligned} \tag{5.145}$$

Now, if we look at the scalar integrals on the r.h.s. of Eq.(5.144), we see that the first and third integrals are indeed subtopologies respectively of the second and the first topologies listed in Eq.(5.145). But the second integral happens to be a subtopology of none of the topologies in Eq.(5.145). To solve this problem we simply need to create an *ad hoc* topology that contains such integral. This can be done by adjusting the original set of dependent propagators $\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1+k_2-q)^2-m_t^2, (k_1+k_2-p_b-q)^2-m_t^2\}$ (the first in Eq.(5.141)) in order to get out of it a proper topologies that contain the three propagators $[(k_1+k_2)^2], [(k_1+k_2-p_b)^2], [(k_1+k_2-p_b-q)^2-m_t^2]$. This is easily achieved by properly replacing one of the linearly dependent propagators in the set with a new propagator chosen such that the two above-mentioned requirements are satisfied. For instance, the first set in Eq.(5.141) can be modified by replacing $[(k_1+k_2-q)^2-m_t^2]$ with $[(k_1-q)^2-m_t^2]$. In this way we obtain the new topology

$$\{k_1^2, (k_1+k_2)^2, (p_b-k_1)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1-q)^2-m_t^2, (k_1+k_2-p_b-q)^2-m_t^2\}, \tag{5.146}$$

which actually contains all propagators of the second integrand in Eq.(5.144) as a subtopology.

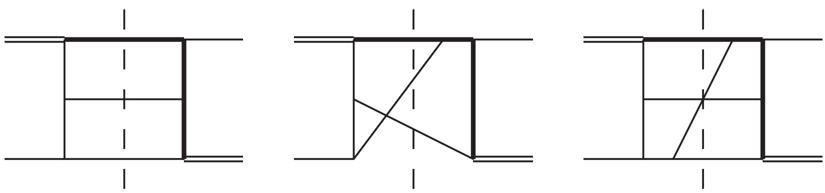
For completeness we report in the following all partial fractioning rules we created and used for the other three topologies containing linearly dependent propagators, namely the last three sets of propagators in Eq.(5.141). The relation of linear dependence is the same for all these three sets, so that we can design just one partial fractioning rule (Eq.(5.147)) valid for the three of them.

$$\frac{1}{[(k_1-p_b)^2][(q-k_1)^2-m_t^2]} \to \frac{1}{s+Q^2}\left[\frac{[(p_b+q-k_1)^2-m_t^2]}{[(k_1-p_b)^2][(q-k_1)^2-m_t^2]} - \frac{1}{[(k_1-p_b)^2]} - \frac{1}{[(q-k_1)^2-m_t^2]}\right] \tag{5.147}$$

By proceeding in this way, namely performing partial fractioning and manipulating topologies where needed, we end up with a certain number of independent topologies describing our process, but, after running reduction to MIs, it turns out that the entire set of independent master integrals is actually contained in only three of such topologies, reported in Eq.(5.148) and drawn in Fig.5.27.

$$\begin{aligned}
t_1^G =&\{k_1^2, (-k_1+p_b)^2, k_2^2, (k_1+k_2-p_b)^2, (k_1-q)^2-m_t^2, (k_1+k_2-q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\}\\
t_2^G =&\{k_1^2, k_2^2, (k_1+k_2-p_b)^2, (-k_2+p_b)^2, (k_1-q)^2-m_t^2, (-k_1+p_b+q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\}\\
t_3^G =&\{k_1^2, (k_1+k_2)^2, (-k_1+p_b)^2, k_2^2, (k_1+k_2-q)^2-m_t^2, (-k_1+p_b+q)^2-m_t^2,\\
&(k_1+k_2-p_b-q)^2-m_t^2\}.
\end{aligned} \tag{5.148}$$

The sets of Master Integrals found in each of these topologies are listed in



(a) (b) (c)

Figure 5.27: Independent topologies to $g+W^* \to t+\bar{b}+g$.

Eq.(5.149), 5.150, 5.151. We underline that not all of these masters are independent. Indeed we can find some of these integrals repeated in more than one topology, but for a matter of order in the computation we decide to keep the masters separated according to the topology to which they belong instead of merging them into one big set. The set of independent MIs describing double real contribution to the gluon channel contains twenty-one integrals.

$$\begin{aligned}
Master_1^G &= G[t_1^g, \{1,0,1,0,0,0,1\}]\\
Master_2^G &= G[t_1^g, \{1,0,1,0,0,0,2\}]\\
Master_3^G &= G[t_1^g, \{1,0,1,1,0,0,1\}]\\
Master_4^G &= G[t_1^g, \{1,0,1,0,1,0,1\}]\\
Master_5^G &= G[t_1^g, \{1,0,1,0,1,0,2\}]\\
Master_6^G &= G[t_1^g, \{1,0,1,0,2,0,1\}]\\
Master_7^G &= G[t_1^g, \{1,0,1,1,1,0,1\}]\\
Master_8^G &= G[t_1^g, \{1,0,1,1,1,0,2\}]\\
Master_9^G &= G[t_1^g, \{1,0,1,1,0,1,1\}]\\
Master_{10}^G &= G[t_1^g, \{1,0,1,0,1,1,1\}]\\
Master_{11}^G &= G[t_1^g, \{1,0,1,0,1,1,2\}]\\
Master_{12}^G &= G[t_1^g, \{1,0,1,1,1,1,1\}]\\
Master_{13}^G &= G[t_1^g, \{1,1,1,1,1,1,1\}]
\end{aligned} \tag{5.149}$$

$$\begin{aligned}
Master_{14}^G &= G[t_2^G, \{1,1,0,0,0,0,1\}]\\
Master_{15}^G &= G[t_2^G, \{1,1,0,0,0,0,2\}]\\
Master_{16}^G &= G[t_2^G, \{1,1,1,0,0,0,1\}]\\
Master_{17}^G &= G[t_2^G, \{1,1,0,0,1,0,1\}]\\
Master_{18}^G &= G[t_2^G, \{1,1,0,0,1,0,2\}]\\
Master_{19}^G &= G[t_2^G, \{1,1,0,0,2,0,1\}]\\
Master_{20}^G &= G[t_2^G, \{1,1,1,0,1,0,1\}]\\
Master_{21}^G &= G[t_2^G, \{1,1,1,0,1,0,2\}]\\
Master_{22}^G &= G[t_2^G, \{1,1,1,0,0,1,1\}]\\
Master_{23}^G &= G[t_2^G, \{1,1,1,0,0,1,2\}]\\
Master_{24}^G &= G[t_2^G, \{1,1,1,0,0,2,1\}]\\
Master_{25}^G &= G[t_2^G, \{1,1,1,0,1,1,1\}]\\
Master_{26}^G &= G[t_2^G, \{1,1,1,1,0,1,1\}]
\end{aligned} \tag{5.150}$$

$$\begin{aligned}
Master_{27}^G &= G[t_3^G, \{1,0,0,1,0,0,1\}]\\
Master_{28}^G &= G[t_3^G, \{1,0,0,1,0,0,2\}]\\
Master_{29}^G &= G[t_3^G, \{1,0,0,1,1,1,1\}]
\end{aligned}$$

$$Master_{30}^{G} = G[t_3^G, \{1,0,0,1,1,1,2\}]$$
$$Master_{31}^{G} = G[t_3^G, \{1,1,1,1,1,1,1\}] \quad (5.151)$$

We also would like to stress that the majority of integrals in Eq.(5.149), (5.150), (5.151) was already encountered in the reduction of the bottom channel double real diagrams. But indeed, as expected, nine new master integrals, distributed in the three topologies Eq.(5.148), are found, corresponding to $Master_9^G$, $Master_{10}^G$, $Master_{11}^G$, $Master_{12}^G$, $Master_{13}^G$, $Master_{25}^G$, $Master_{29}^G$, $Master_{30}^G$, $Master_{31}^G$. These new masters are drawn in Fig.(5.28).
Once again we turn these basis of masters into canonical ones, given in Eq.(6.12), (6.13), (6.14) of the Appendix.

From here on the computation, at least in principles, develops as usual, namely we generate systems of canonical D.E. we integrate them in terms of GPLs and we match them with appropriate boundary conditions generated in the same way we already did for the RR masters integrals in the bottom channel (D.E.and b.c. are systematically reported in Eqs. from (6.15) to (6.20) of the Appendix). The procedure to compute such boundary conditions is the same used for the bottom channel RR masters, thus for a detailed description we send the reader back to the part of this section dedicated to the computation of b-initiated subprocesses. Nonetheless, we would like to stress that all gluon masters in the soft limit $z \to 1$ shrink to linear combination of only two soft masters. These two soft masters happen to be the same which also describes the soft limit of the Double-Real MIs for the b-channel (see Eq.(5.55)). It is quite remarkable that we only need two soft integrals to describe the soft behaviour of double real radiation in both the bottom and gluon channel!

Unfortunately, we find in topologies t_1^g and t_2^G an additional difficulty that needs to be discussed and sorted out before being able to actually compute integrals belonging to this topologies. We address the discussion of this topic in the next paragraph.

Master Integrals generating Quadratic Letters.

Let us start by having a look at the type of transformation which give rise to canonical basis for topologies t_1^G, t_2^G. By inspecting the correspondent Eq.(6.12), (6.13) (see Appendix), it can be observed that in general canonical masters are written as linear combination of the original masters with coefficients which depends rationally on the dimensionless variable z, y. In other words, the majority of canonical masters is obtained via a *rational transformation*. But this is not true in a couple of cases. Indeed, M_{12}^G and M_{25}^G are obtained by multiplying the original integrals by algebraic functions of z, y (in this case square roots having as argument quadratic polynomials in z, y). For this two masters we need to introduce an *algebraic transformation* and the source of such transformation simply lies in the

structure of the D.E. for the original masters.
Given the general structure for such D.E (Eq.(5.152))

$$\partial_z Master_j(z,y,\epsilon) = A_z^{j,j}(z,y,\epsilon)Master_j(z,y,\epsilon) + \sum_{i\neq j} A_z^{j,i}(z,y,\epsilon)Master_i(z,y,\epsilon)$$
$$\partial_y Master_j(z,y,\epsilon) = A_y^{j,j}(z,y,\epsilon)Master_j(z,y,\epsilon) + \sum_{i\neq j} A_y^{j,i}(z,y,\epsilon)Master_i(z,y,\epsilon), \quad (5.152)$$

the homogeneous terms for $Master_{12}$,$Master_{25}$ equations read (Eq.(5.153), (5.154))

$$\begin{aligned} A_z^{12,12}(z,y,\epsilon) =& \frac{2z\epsilon - z - 2\epsilon + 1}{4y^2 + 4y + z^2 - 2z + 1} - \frac{2\epsilon}{z} \\ A_y^{12,12}(z,y,\epsilon) =& \frac{2(2y+1)(2\epsilon - 1)}{4y^2 + 4y + z^2 - 2\ z + 1} + \frac{\epsilon - 2}{y+1} \\ & - \frac{2\epsilon}{y+1} - \frac{y\ \epsilon}{(y+1)^2} - \frac{\epsilon}{(y+1)^2} + \frac{1}{y+1}, \end{aligned} \quad (5.153)$$

$$\begin{aligned} A_z^{25,25}(z,y,\epsilon) =& \frac{4y\epsilon - 2y + 2z\epsilon - z + 2\epsilon - 1}{4y\ z + z^2 + 2z + 1} - \frac{2\epsilon}{z} \\ A_y^{25,25}(z,y,\epsilon) =& \frac{2z(2\epsilon - 1)}{4yz + z^2 + 2z + 1} - \frac{2\epsilon\ + 1}{y+1}. \end{aligned} \quad (5.154)$$

By inspection of such homogeneous terms, we see that the only kind of transformation which can make terms containing quadratic polynomials $4y^2 + 4y + z^2 - 2z + 1, 4yz + z^2 + 2z + 1$ at denominators linear in ϵ is algebraic. In other words, the presence of quadratic denominators in the original equations imply the use of algebraic transformation such as the ones in Eq.(6.12), (6.13) in order to obtain equations which are linear in ϵ and thus in canonical form.
Indeed, the differential equations for M_{12}, M_{25} can directly be written as a total differential of a logarithmic 1-form as

$$\begin{aligned} dM_{12} =& \epsilon\, \big(M_{12}(-2\mathrm{dL}[1+y] + \mathrm{dL}[4y + 4y^2 + (-1+z)^2] - 2\mathrm{dL}[z]) \\ & + 1/2 M_1(4(\mathrm{dL}[y] + \mathrm{dL}[1+y]) - 7\mathrm{dL}[1 - \sqrt{4y + 4y^2 + (-1+z)^2} - z] \\ & - \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y + 4y^2 + (-1+z)^2} - z] + \mathrm{dL}[z]) \\ & + 1/2 M_{10}(2(\mathrm{dL}[y] + \mathrm{dL}[1+y]) - 3\mathrm{dL}[1 - \sqrt{4y + 4y^2 + (-1+z)^2}] - z \\ & - \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y + 4y^2 + (-1+z)^2} - z + \mathrm{dL}[z]) \\ & + 1/2 M_{11}(2(\mathrm{dL}[y] + \mathrm{dL}[1+y]) - 3\mathrm{dL}[1 - \sqrt{4y + 4y^2 + (-1+z)^2} - z] \\ & - \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y + 4y^2 + (-1+z)^2} - z] + \mathrm{dL}[z]) \end{aligned}$$

$$
\begin{aligned}
&+ 2M_3(2(\mathrm{dL}[y] + \mathrm{dL}[1+y]) - 3\mathrm{dL}[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y+4y^2+(-1+z)^2} - z] + \mathrm{dL}[z]) \\
&- 2M_8(\mathrm{dL}[y] + \mathrm{dL}[1+y] - \mathrm{dL}[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y+4y^2+(-1+z)^2} - z] + \mathrm{dL}[z]) \\
&- M_9(\mathrm{dL}[y] + \mathrm{dL}[1+y] - \mathrm{dL}[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y+4y^2+(-1+z)^2} - z] + \mathrm{dL}[z]) \\
&+ M_4(\mathrm{dL}[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y+4y^2+(-1+z)^2} - z] + \mathrm{dL}[z]) \\
&- 2M_5(\mathrm{dL}[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y + (2y+1)\sqrt{4y+4y^2+(-1+z)^2} - z] + \mathrm{dL}[z]) \\
&- 3M_6(textdL[1 - \sqrt{4y+4y^2+(-1+z)^2} - z] \\
&- \mathrm{dL}[1 + 4y^2 + 4y(2y+1)\sqrt{4y+4y^2+(-1+z)^2]} - z] + \mathrm{dL}[z]) \\
&- 4M_2(\mathrm{dL}[y] + \mathrm{dL}[1+y] - 2\mathrm{dL}[-1 + \sqrt{4y+4y^2+(-1+z)^2} + z]) \\
&-2M_7(\mathrm{dL}[y] + \mathrm{dL}[1+y] - 2\mathrm{dL}[-1 + \sqrt{4y+4y^2+(-1+z)^2} + z])\Big)
\end{aligned}
\tag{5.155}
$$

$$
\begin{aligned}
dM_{25} =& \epsilon \big(M_{25}(-2\mathrm{dL}[1+y] - 2\mathrm{dL}[z] + \mathrm{dL}[1+2z+4yz+z^2]) \\
&+ 4M_{24}(\mathrm{dL}[y] + \mathrm{dL}[1+y] - 2\mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}]) \\
&- 8M_{16}(\mathrm{dL}[y] + \mathrm{dL}[1+y] - 2\mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}]) \\
&+ 4M_{21}(\mathrm{dL}[y] + \mathrm{dL}[1+y] - 2\mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}]) \\
&- 2M_{23}(\mathrm{dL}[y] + \mathrm{dL}[1+y] + 3\mathrm{dL}[z] + \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] \\
&- 3\mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&- 6M_{15}(\mathrm{dL}[y] + \mathrm{dL}[1+y] + \mathrm{dL}[z] - \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] \\
&- \mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&- 2M_{17}(\mathrm{dL}[y] + \mathrm{dL}[1+y] + \mathrm{dL}[z] - \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] \\
&- \mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&+ 2M_{14}(\mathrm{dL}[z] + \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] - \mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&+ 2M_{18}(\mathrm{dL}[z] + \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] - \mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&- 4M_{20}(\mathrm{dL}[z] + \mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] - \mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}]) \\
&+ 4M_{22}(\mathrm{dL}[y] + \mathrm{dL}[1+y] - \mathrm{dL}[z] - 3\mathrm{dL}[1+2y+z+\sqrt{1+2z+4yz+z^2}] \\
&+\mathrm{dL}[1+z+2yz+\sqrt{1+2z+4yz+z^2}])\Big) .
\end{aligned}
\tag{5.156}
$$

It was observed in Section 4.4 that, given the definition of Multiple Polylogarithm (MPL) Eq.(4.76) and the definition of Iterated Integrals Eq.(4.148), if a canonical system of D.E. can be written as the total differential of a logarithmic 1-form with all the arguments of the d-Logs being linear function of the dimensionless variables describing the problem, then the system can be integrated in terms of MPLs.
It was also observed that this automatically happens when a canonical basis is reached via rational transformations.
On the other hand, if a canonical basis is reached via an algebraic transformation, normally an algebraic dependence on the dimensionless variables is introduced in the arguments of the d-Logs. This is exactly what happens in our case, as it can be clearly seen from Eq.(5.155), (5.156).
Given the presence of such algebraic dependence in the d-Logs arguments, we cannot integrate the system as it is in terms of MPLs. We need first to find a suitable remapping to eliminate the presence of square roots in the arguments and eventually to confine the quadratic dependence to only one variable at choice between z and y.
As explained in [44], it is indeed possible to integrate in terms of MPLs whenever the system exhibits quadratic dependence of the *letters* on just one variable, provided that integration over the quadratic variable is postponed until the very end of the calculation. We will explain more in detail how this works in the following. For the moment we start providing the necessary remappings for t_1^G, t_2^G and describing what their exact purpose is.

1. We start from the original *alphabets* for t_1^G, t_2^G which contain square roots and quadratic dependence on both z and y.

$$\begin{aligned}\mathcal{A}_{t_1^G} = \Big\{ & 1-z, z, y+z, y, y+1, y+z+1, 2y+z+1, -\sqrt{4y^2+4\,y+(z-1)^2}-z+1, \\ & (2y+1)\left(\sqrt{4y^2+4\,y+(z-1)^2}+2\right)+4y^2-z+1, \\ & \sqrt{4\,y^2+4y+(z-1)^2}+z-1, 4y^2+4y+(z-1)^2\Big\} \end{aligned} \tag{5.157}$$

$$\begin{aligned}\mathcal{A}_{t_2^G} = \Big\{ & 1-z, z, y+z, y, y+1, y+z+1, 2y+z+1, \sqrt{4yz+z^2+2z+1}+2\,y+z+1, \\ & \sqrt{4yz+z^2+2z+1}+2yz+z+1, 4yz+z^2+2z+1\Big\} \end{aligned} \tag{5.158}$$

2. We observe in general that pure loop-integrals, namely integrals for which we can write down a Feynman parameters representation, transforms in a not only well-defined but indeed *even* way under sign-change of the square roots possibly appearing in the alphabet describing the related canonical basis. Indeed, every loop-integral is a purely rational object, since integrations

over Feynman parameters do not contain any square root and, as such, it is always *even* under a change in the choice of any square root branch. On the other hand, integrals belonging to the canonical basis can be obtained by multiplying the original loop-integrals either by rational functions, either by algebraic functions which are nothing but the square roots that we recover then in the alphabet. Consequently, we understand that canonical masters M_i do conserve the property of transforming in a well-defined way under sign-change of this square root function, but they have the freedom to transform either evenly or oddly. In order to make manifest this property of the integrals, it is then desirable to write down the solutions to the canonical basis in terms of functions which are even or odd under sign change of the square root functions. This corresponds to having coefficients of the canonical D.E., namely entries of the d-Log matrices, with the same property, namely that of transforming in a manifestly either even or odd way under sign-change of the square root. In case coefficients do not automatically satisfy this property, the goal can be achieved by manipulating these coefficients in order to (anti-)symmetrize them with respect to the above-mentioned sign transformation.

Now, in the present situation, the scenario is a bit different, since the original basis are not made up of pure loop-integrals but of cut integrals. We cannot write down a Feynman parameters representation for cut integrals and, as a consequence these kind of integrals and the related canonical ones do not satisfy a priori any property of well-defined transformation under sign change of possible square root functions.
Still, it is useful to write the solution to these integrals as linear combinations of functions being either even or odd under such transformation, so that if one of the integrals is even or odd, this becomes immediately manifest.
So, before proceeding with remappings, we anti-symmetrize those d-Logs entries that do not transform in a well-defined way when we change the sign of square root functions appearing in alphabets for canonical basis of t_1^G, t_2^G (Eq.(5.159), (5.161)). The (anti-)symmetrized alphabets are then reported in Eq.(5.160), (5.162).

t_1^G:

$$\Big\{\mathrm{dL}\left(1-\sqrt{4y+4y^2+(-1+z)^2}-z\right) \to \frac{1}{2}\left(\mathrm{dL}\left(4y(1+y)\right)\right.$$
$$\left.+\ \mathrm{dL}\left(\frac{1-\sqrt{4y+4y^2+(-1+z)^2}-z}{1+\sqrt{4y+4y^2+(-1+z)^2}-z}\right)\right),$$
$$\mathrm{dL}\left(1+4y^2+4y+(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z\right) \to \frac{1}{2}\left(\mathrm{dL}\left(4y(1+y)\right)\right.$$

$$
\begin{aligned}
. \qquad & +\mathrm{dL}\left(\frac{1+4y^2+4y+(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z}{1+4y^2+4y-(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z}\right)+2\mathrm{dL}(z)\Bigg),\\
&\mathrm{dL}\left(-1+\sqrt{4y+4y^2+(-1+z)^2}+z\right)\to\frac{1}{2}\left(\mathrm{dL}(4y(1+y))\right.\\
&\left.+\mathrm{dL}\left(\frac{-1+\sqrt{4y+4y^2+(-1+z)^2}+z}{-1-\sqrt{4y+4y^2+(-1+z)^2}+z}\right)\right)\Bigg\}, \qquad (5.159)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{SYMM}_{t_1^G} = & \{1-z,z,y+z,y,y+1,y+z+1,2y+z+1,\\
& \frac{1-\sqrt{4y+4y^2+(-1+z)^2}-z}{1+\sqrt{4y+4y^2+(-1+z)^2}-z},\\
& \frac{1+4y^2+4y+(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z}{1+4y^2+4y-(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z},\\
& \left.\frac{-1+\sqrt{4y+4y^2+(-1+z)^2}+z}{-1-\sqrt{4y+4y^2+(-1+z)^2}+z}\right\}. \qquad (5.160)
\end{aligned}
$$

t_2^G :

$$
\begin{aligned}
&\Bigg\{\mathrm{dL}\left(1+2y+z+\sqrt{1+2z+4yz+z^2}\right)\to\frac{1}{2}\left(\mathrm{dL}(4y(1+y))\right.\\
&\quad\left.+\mathrm{dL}\left(\frac{1+2y+z+\sqrt{1+2z+4yz+z^2}}{1+2y+z-\sqrt{1+2z+4yz+z^2}}\right)\right),\\
&\mathrm{dL}\left(1+z+2yz+\sqrt{1+2z+4yz+z^2}\right)\to\frac{1}{2}\left(\mathrm{dL}(4y(1+y))+2\mathrm{dL}(z)\right.\\
&\quad\left.+\mathrm{dL}\left(\frac{1+z+2yz+\sqrt{1+2z+4yz+z^2}}{1+z+2yz-\sqrt{1+2z+4yz+z^2}}\right)\right)\Bigg\}, \qquad (5.161)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{SYMM}_{t_2^G} = & \{1-z,z,y+z,y,y+1,y+z+1,2y+z+1,\\
& \frac{1+2y+z+\sqrt{1+2z+4yz+z^2}}{1+2y+z-\sqrt{1+2z+4yz+z^2}},\\
& \left.\frac{1+z+2yz+\sqrt{1+2z+4yz+z^2}}{1+z+2yz-\sqrt{1+2z+4yz+z^2}}\right\}. \qquad (5.162)
\end{aligned}
$$

3. On both topologies we perform a first remapping which has the unique purpose of eliminate square roots.

   t_1^G :

$$\left\{z \to \frac{x+w(2+x)}{-1+w}, y \to \frac{w}{x}\frac{(1+x)^2}{(1-w)^2}\right\} \tag{5.163}$$

t_2^G :

$$\left\{z \to z, y \to \frac{x}{z}\frac{(1+z)^2}{(1-x)^2}\right\} \tag{5.164}$$

4. By applying remappings Eq.(5.163), (5.164) to the alphabets Eq.(5.160), (5.162), we see that square roots are indeed eliminated, but still the letters conserve quadratic dependence on both the newly introduced dimensionless variables. In order to reduce such quadratic dependence to just one of the two variables, we introduce other remappings. These consist actually of just one transformation for t_2^G and two transformations in a row for t_1^G, as listed in the following.

   t_1^G :

$$\left\{w \to -\frac{1}{a}, x \to -\frac{b}{1+b}\right\}, \qquad \left\{a \to 1+c-2d, b \to \frac{-1+d}{2+c-2d}\right\} \tag{5.165}$$

   t_2^G :

$$\left\{x \to \frac{a}{1+a}, z \to -\frac{a}{a-b}\right\} \tag{5.166}$$

5. By applying in a row Eq.(5.163), (5.164) and Eq.(5.165), (5.166) to the initial alphabets, we obtain in output the new alphabets Eq.(5.167), (5.168).

$$\begin{aligned}\mathcal{A}_{t_1^G}^{SYMM,remapped} = \{&c-2d, 1+c-d, 1+d, -1+d, c+(-2+d)d, 1+c-2d,\\ &c-d, d, -1+cd, c-2d+cd, -c+d, c+cd-2d^2\}\end{aligned} \tag{5.167}$$

$$\begin{aligned}\mathcal{A}_{t_2^G}^{SYMM,remapped} = \{&-2a+b, -a+b, a, a+(1+a)b^2, 1+a, b, 1+b,\\ &a(-1+b)+b, 1+b+ab, 1+2(1+a)b, a-b, 1+2a\}\end{aligned} \tag{5.168}$$

We clearly see that our goal is achieved! We have got rid of all square roots and we have managed to confine the quadratic dependence to only one dimensionless variable! We can finally write down the following list of rules for the d-Log entries

t_1^G :

$$
\begin{aligned}
&\{\mathrm{dL}(1-z) \to \mathrm{dL}(c-2d) - \mathrm{dL}(1+c-d)\,,\\
&\mathrm{dL}(z) \to -\mathrm{dL}(1+c-d) + \mathrm{dL}(1+d)\,,\\
&\mathrm{dL}(y+z) \to -\mathrm{dL}(1+c-d) - \mathrm{dL}(-1+d) + \mathrm{dL}(c+(-2+d)d)\,,\\
&\mathrm{dL}(y) \to \mathrm{dL}(1+c-2d) - \mathrm{dL}(1+c-d) - \mathrm{dL}(-1+d)\,,\\
&\mathrm{dL}(1+y) \to \mathrm{dL}(c-d) - \mathrm{dL}(1+c-d) - \mathrm{dL}(-1+d) + \mathrm{dL}(d)\,,\\
&\mathrm{dL}(1+y+z) \to -\mathrm{dL}(1+c-d) - \mathrm{dL}(-1+d) + \mathrm{dL}(-1+cd)\,,\\
&\mathrm{dL}(1+2y+z) \to -\mathrm{dL}(1+c-d) - \mathrm{dL}(-1+d) + \mathrm{dL}(c-2d+cd)\,,\\
&\mathrm{dL}(4y(1+y)) \to \mathrm{dL}(1+c-2d) + \mathrm{dL}(c-d) - 2\mathrm{dL}(1+c-d) - 2\mathrm{dL}(-1+d) + \mathrm{dL}(d)\,,\\
&\mathrm{dL}\left(\frac{1-\sqrt{4y+4y^2+(-1+z)^2}-z}{1+\sqrt{4y+4y^2+(-1+z)^2}-z}\right) \to -\mathrm{dL}(1+c-2d) - \mathrm{dL}(d) + \mathrm{dL}(-c+d)\,,\\
&\mathrm{dL}\left(\frac{1+4y^2+4y+(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z}{1+4y^2+4y-(2y+1)\sqrt{4y+4y^2+(-1+z)^2}-z}\right) \to \mathrm{dL}(1+c-2d) + \mathrm{dL}(c-d)\\
&-2\mathrm{dL}(-1+d) - \mathrm{dL}(d) + 2\mathrm{dL}(1+d)\,,\\
&\mathrm{dL}\left(\frac{-1+\sqrt{4y+4y^2+(-1+z)^2}+z}{-1-\sqrt{4y+4y^2+(-1+z)^2}+z}\right) \to -\mathrm{dL}(1+c-2d) - \mathrm{dL}(d) + \mathrm{dL}(-c+d)\,,\\
&\mathrm{dL}\left(4y+4y^2+(-1+z)^2\right) \to -2\mathrm{dL}(1+c-d) - 2\mathrm{dL}(-1+d) + 2\mathrm{dL}\left(c+cd-2d^2\right)\}
\end{aligned}
\tag{5.169}
$$

t_2^G :

$$
\begin{aligned}
&\{\mathrm{dL}(1-z) \to \mathrm{dL}(-2a+b) - \mathrm{dL}(-a+b)\,,\\
&\mathrm{dL}(z) \to \mathrm{dL}(a) - \mathrm{dL}(-a+b)\,,\\
&\mathrm{dL}(y+z) \to -\mathrm{dL}(-a+b) + \mathrm{dL}\left(a+(1+a)b^2\right)\,,\\
&\mathrm{dL}(y) \to \mathrm{dL}(1+a) + 2\mathrm{dL}(b) - \mathrm{dL}(-a+b)\,,\\
&\mathrm{dL}(1+y) \to \mathrm{dL}(1+b) - \mathrm{dL}(-a+b) + \mathrm{dL}(a(-1+b)+b)\,,\\
&\mathrm{dL}(1+y+z) \to \mathrm{dL}(b) - \mathrm{dL}(-a+b) + \mathrm{dL}(1+b+ab)\,,\\
&\mathrm{dL}(1+2y+z) \to \mathrm{dL}(b) - \mathrm{dL}(-a+b) + \mathrm{dL}(1+2(1+a)b)\,,\\
&\mathrm{dL}(4y(1+y)) \to \mathrm{dL}(1+a) + 2\mathrm{dL}(b) + \mathrm{dL}(1+b) - 2\mathrm{dL}(-a+b) + \mathrm{dL}(a(-1+b)+b)\,,\\
&\mathrm{dL}\left(\frac{1+2y+z+\sqrt{1+2z+4yz+z^2}}{1+2y+z-\sqrt{1+2z+4yz+z^2}}\right) \to \mathrm{dL}(1+a) + \mathrm{dL}(1+b) - \mathrm{dL}(a(-1+b)+b)\,,\\
&\mathrm{dL}\left(\frac{1+z+2yz+\sqrt{1+2z+4yz+z^2}}{1+z+2yz-\sqrt{1+2z+4yz+z^2}}\right) \to -2\mathrm{dL}(a) + \mathrm{dL}(1+a) - \mathrm{dL}(1+b)
\end{aligned}
$$

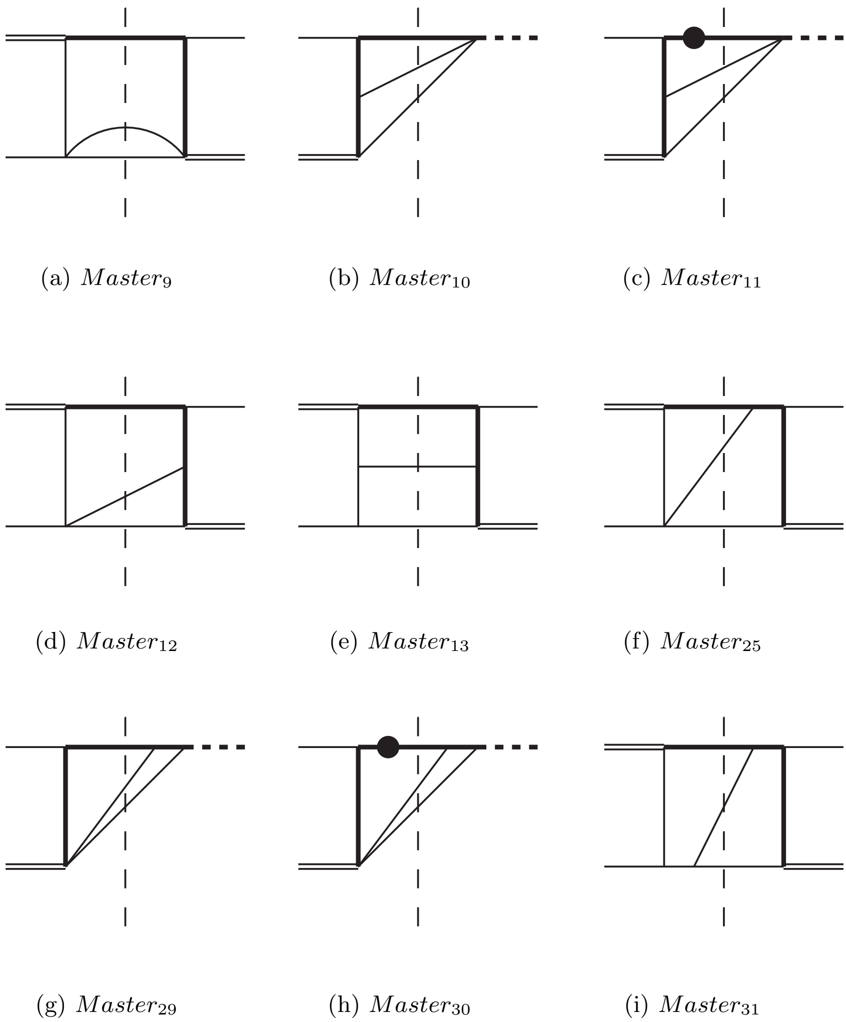
$$+\,\mathrm{dL}\left(a(-1+b)+b\right),$$
$$\mathrm{dL}\left(1+2z+4yz+z^2\right)\to 2\mathrm{dL}\left(1+2a\right)+2\mathrm{dL}\left(b\right)-2\ \mathrm{dL}\left(-a+b\right)\big\} \tag{5.170}$$

Eventually, we use these list of rules to remap our systems of canonical D.E. into new ones (reported in Eq.(6.15), (6.16) of the Appendix).

6. We finally explain with an example why it is possible to integrate a canonical system of D.E. whose alphabet exhibits quadratic dependence on just one of the variables, following reference [35], [44]. We consider for instance the alphabet for t_1^G in equation 5.167, whose only quadratic letter reads $c(1+d)-2d^2$ and we imagine to integrate first in c. When we perform integration in c, we get from this quadratic letter MPLs of the form $G\left(\{2d^2/(1+d),...\},c\right)$. Following the integration procedure explained in Section 3.4, this solution is then used to derive the differential equations for the functions of the left-over variable d and we know that at this stage all functions depending on c should disappear (this is actually one of the main consistency check one has to do when integrating such systems of differential equations!).
   Since there are no letters in the alphabet which are quadratic in d and are independent of c, we conclude that only letters that are linear in d will appear in this final step of the integration.

   This concludes our discussion about how to deal with Master Integrals generating quadratic letters.

The discussion about the Double-Real master integrals for the gluon channel ends here. All intermediate and final results for the independent masters can be found in the Appendix.



(a) $Master_9$ (b) $Master_{10}$ (c) $Master_{11}$

(d) $Master_{12}$ (e) $Master_{13}$ (f) $Master_{25}$

(g) $Master_{29}$ (h) $Master_{30}$ (i) $Master_{31}$

Figure 5.28: Set of new MIs appearing in $g + W^* \to t + \bar{b} + g$.
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

Real-Virtuals $\left[g + W^* \to t + \bar{b}\right]_{1-loop}$

The Feynman diagrams for the Real-Virtual (RV) contribution to the gluon channel are obtained by applying *crossing symmetry* to the Real-Virtual diagrams for the bottom-initiated subprocess $[b + W^* \to t + g]_{1-loop}$. Given the identification of particles momenta $b(p_b) + W^*(q) \to t(p_t) + g(k)$, the crossing transformation is identified by the rules $(p_b \to -p_b, k \to -k)$. The resulting diagrams are reported in Fig.(5.29). As we did for RR diagrams, in order to maintain the same definition of Mandelstam invariants in terms of particle momenta we used up to now, after performing crossing symmetry we rename particles momenta as $(-k \to p_b, -p_b \to k)$.

By contracting amplitudes for diagrams in Fig.(5.29) with the one for the corresponding tree-level process $g + W^* \to t + \bar{b}$, adding the 2-particle phase space measure Eq.(5.61), we find ten sets of propagators, among which four are not topologies, since they contain linearly dependent propagators. This means we need again to perform partial fractioning as we did for the Double Real matrix elements.
By doing so and by modifying such dependent sets of propagators in order to build topologies out of them as previously described in the case of the RR matrix elements reduction, we find that gluon RV matrix elements can be described by an ensemble of 7 independent topologies. Reduction to master integrals then tells us that independent master integrals are distributed in just 5 of these topologies (listed in Eq.(5.171) and drawn in Fig.(5.30)), which are then the ones that we really need to describe our process.

$$
\begin{aligned}
t_1^G =&\{k^2, l_1^2 - m_t^2, (k+p_b)^2, (l_1-q)^2, (k+l_1-p_b-q)^2, (-l_1+p_b+q)^2, -m_t^2 + (-k+p_b+q)^2\} \\
t_2^G =&\{k^2, l_1^2 - m_t^2, (k+p_b)^2, (l_1-q)^2, (k+l_1-p_b-q)^2, (-l_1+p_b+q)^2, -m_t^2 + (-k+p_b+q)^2\} \\
t_3^G =&\{k^2, (k-l_1)^2, l_1^2, (k+p_b)^2, (-k+l_1+p_b)^2, -m_t^2 + (l_1-q)^2, -m_t^2 + (-k+p_b+q)^2\} \\
t_4^G =&\{k^2, l_1^2 - m_t^2, -m_t^2 + (k-q)^2, (l_1-q)^2, (k+l_1-p_b-q)^2, (-l_1+p_b+q)^2, \\
&- m_t^2 + (-k+p_b+q)^2\} \\
t_5^G =&\{k^2, (k-l_1)^2, l_1^2, -m_t^2 + (k-q)^2, -m_t^2 + (l_1-q)^2, -m_t^2 + (-k+p_b+q)^2, \\
&- m_t^2 + (-l_1+pb+q)^2\}
\end{aligned}
\tag{5.171}
$$

We report in Eq.(5.172), (5.173), (5.174), (5.175), (5.176) the basis of masters we find for each topology. We stress that, as for the Double Real set of MIs, some integrals appear in more than one topology, but for practical reasons, we prefer to keep the masters separated according to the topology they belong to. The independent MIs are in total 25, out of which 12 are actually new with respect to the ones already encountered in the bottom channel. These new masters are drawn in Fig.(5.31), (5.32).

$$
\begin{aligned}
Master_1^G &\to G[t_1^G, \{1,1,0,0,0,0,1\}]\\
Master_2^G &\to G[t_1^G, \{1,1,0,1,0,0,1\}]\\
Master_3^G &\to G[t_1^G, \{1,1,0,0,0,1,1\}]\\
Master_4^G &\to G[t_1^G, \{1,0,0,1,1,0,1\}]\\
Master_5^G &\to G[t_1^G, \{1,1,0,1,1,0,1\}]\\
Master_6^G &\to G[t_1^G, \{1,1,0,1,1,0,2\}]\\
Master_7^G &\to G[t_1^G, \{1,1,0,1,2,0,1\}]
\end{aligned} \tag{5.172}
$$

$$
\begin{aligned}
Master_8^G &\to G[t_2^G, \{1,0,0,0,1,1,0\}]\\
Master_9^G &\to G[t_2^G, \{1,0,0,0,0,1,1\}]\\
Master_{10}^G &\to G[t_2^G, \{1,0,1,0,1,1,0\}]\\
Master_{11}^G &\to G[t_2^G, \{1,1,0,0,1,2,0\}]\\
Master_{12}^G &\to G[t_2^G, \{1,1,0,0,2,1,0\}]\\
Master_{13}^G &\to G[t_2^G, \{1,1,0,0,1,1,0\}]\\
Master_{14}^G &\to G[t_2^G, \{1,0,1,0,0,1,1\}]\\
Master_{15}^G &\to G[t_2^G, \{1,1,1,0,1,1,0\}]\\
Master_{16}^G &\to G[t_2^G, \{1,0,1,0,1,1,1\}]\\
Master_{17}^G &\to G[t_2^G, \{1,1,0,0,1,1,1\}]\\
Master_{18}^G &\to G[t_2^G, \{1,1,0,0,1,1,2\}]\\
Master_{19}^G &\to G[t_2^G, \{1,1,1,1,1,1,1\}]
\end{aligned} \tag{5.173}
$$

$$
\begin{aligned}
Master_{20}^G &\to G[t_3^G, \{1,0,0,0,0,1,1\}]\\
Master_{21}^G &\to G[t_3^G, \{1,0,1,0,1,0,1\}]\\
Master_{22}^G &\to G[t_3^G, \{1,0,1,0,0,1,1\}]\\
Master_{23}^G &\to G[t_3^G, \{1,1,0,0,0,1,2\}]\\
Master_{24}^G &\to G[t_3^G, \{1,1,0,0,0,2,1\}]\\
Master_{25}^G &\to G[t_3^G, \{1,1,0,0,0,1,1\}]\\
Master_{26}^G &\to G[t_3^G, \{1,0,1,0,1,1,2\}]\\
Master_{27}^G &\to G[t_3^G, \{1,0,1,0,1,2,1\}]\\
Master_{28}^G &\to G[t_3^G, \{1,0,1,0,1,1,1\}]\\
Master_{29}^G &\to G[t_3^G, \{1,1,1,0,0,1,1\}]\\
Master_{30}^G &\to G[t_3^G, \{1,1,1,0,1,1,1\}]
\end{aligned} \tag{5.174}
$$

$$
\begin{aligned}
Master_{31}^G &\to G[t_4^G, \{1,1,0,0,0,0,1\}]\\
Master_{32}^G &\to G[t_4^G, \{1,1,1,0,0,0,1\}]\\
Master_{33}^G &\to G[t_4^G, \{1,0,0,1,1,0,1\}]
\end{aligned}
$$

$$\begin{aligned}
&Master^G_{34} \to G[t^G_4, \{1,1,0,0,0,1,1\}]\\
&Master^G_{35} \to G[t^G_4, \{1,1,0,1,0,0,1\}]\\
&Master^G_{36} \to G[t^G_4, \{1,1,1,1,0,0,1\}]\\
&Master^G_{37} \to G[t^G_4, \{1,1,1,0,0,1,1\}]\\
&Master^G_{38} \to G[t^G_4, \{1,0,1,1,1,0,1\}]\\
&Master^G_{39} \to G[t^G_4, \{1,1,0,1,1,0,1\}]\\
&Master^G_{40} \to G[t^G_4, \{1,1,0,1,1,0,2\}]\\
&Master^G_{41} \to G[t^G_4, \{1,1,0,1,2,0,1\}]\\
&Master^G_{42} \to G[t^G_4, \{1,1,1,1,1,0,1\}]\\
&Master^G_{43} \to G[t^G_4, \{1,1,1,1,1,1,1\}]
\end{aligned} \tag{5.175}$$

$$\begin{aligned}
&Master^G_{44} \to G[t^G_5, \{1,0,0,0,1,1,0\}]\\
&Master^G_{45} \to G[t^G_5, \{1,0,0,0,0,1,1\}]\\
&Master^G_{46} \to G[t^G_5, \{1,0,0,1,1,1,0\}]\\
&Master^G_{47} \to G[t^G_5, \{1,1,0,0,1,2,0\}]\\
&Master^G_{48} \to G[t^G_5, \{1,1,0,0,2,1,0\}]\\
&Master^G_{49} \to G[t^G_5, \{1,1,0,0,1,1,0\}]\\
&Master^G_{50} \to G[t^G_5, \{1,0,1,0,1,1,0\}]\\
&Master^G_{51} \to G[t^G_5, \{1,0,0,1,0,1,1\}]\\
&Master^G_{52} \to G[t^G_5, \{1,0,1,0,0,1,1\}]\\
&Master^G_{53} \to G[t^G_5, \{1,0,1,1,1,1,0\}]\\
&Master^G_{54} \to G[t^G_5, \{1,1,1,0,1,1,0\}]\\
&Master^G_{55} \to G[t^G_5, \{1,0,1,1,0,1,1\}]\\
&Master^G_{56} \to G[t^G_5, \{1,1,0,0,1,1,1\}]\\
&Master^G_{57} \to G[t^G_5, \{1,1,0,0,1,1,2\}]\\
&Master^G_{58} \to G[t^G_5, \{1,0,1,0,1,1,1\}]\\
&Master^G_{59} \to G[t^G_5, \{1,0,1,1,1,1,1\}]
\end{aligned} \tag{5.176}$$

As usual, we transform then each of these original integral basis into canonical ones (reported in Eq.(6.21), (6.22), (6.23), (6.24), (6.25) of the Appendix). We report the corresponding system of canonical D.E. in Eq.(6.26), (6.27), (6.28), (6.29), (6.30) of the Appendix.
We point out that in topology t^G_4, we encountered the same quadratic denominator found in the RR topology t^G_2. The alphabets for these topologies present then the same quadratic letters, which implies we need to perform remapping Eq.(5.164), (5.166) on the canonical D.E. for the RV topology t^G_4. The system of canonical D.E., rewritten in terms of the new dimensionless variables (a, b), is free of letters containing square roots and has quadratic dependence on just one variable, meaning that this system is now integrable in terms of MPLs.

Boundary conditions for the g-channel RV masters.

For a complete explanation of how boundary conditions are computed in general for Real-Virtual master integrals, we send the reader back to the subsection dedicated to the bottom channel, since the same technique applies also to gluon channel RV masters.
Following this technique, we observe that the ensemble of pure 1-loop integrals that we obtain when we cut the gluon RV original basis of masters (Eq.(5.172), (5.173), (5.174), (5.175), (5.176)) in correspondence of the Phase Space, contains mostly 1-loop integrals that we already encountered in the bottom channel. We obtain only five new 1-loop integrals to compute in the double limit $\{z \to 1, y \to 0\}$, which we draw in Fig.. In the rest of this subsection we address the computation of these 1-loop integrals.

- **TRIANGLES**

1. **Triangle with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2}, \mathbf{s}\}$**

$$
\begin{aligned}
t_1^G\left(Q^2, s, m_t^2\right) &= \int d^d l_1 \frac{1}{l_1^2(-m_t^2 + (l_1 - q)^2)(-m_t^2 + (-l_1 + p_b + q)^2)} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \Gamma(1+\epsilon) \times \\
&\quad \times \delta(1 - x_1 - x_2 - x_3)(x_1 + x_2 + x_3)^{-1+2\epsilon} \\
&\quad \times \left(m_t^2(x_2 + x_3)(x_1 1 + x_2 + x_3) + x_1(Q^2 x_2 - s x_3)\right)^{-1-\epsilon}
\end{aligned} \tag{5.177}
$$

Triangle t_1^G corresponds to Fig.5.33a. In this integral, m_t^2 appears only as internal invariant, and Q^2, s as external ones. We can then expect the $\mathcal{F}$ polynomial to be, with no ambiguity, positive-definite in the region $\{m_t^2 > 0, Q^2 > 0, s < 0\}$, and, by inspection of the $\mathcal{F}$ polynomial in Eq.(5.177), we can easily conclude that this is indeed the case.
Consequently, the integration over Feynman parameters in Eq.(5.177), will be carried out in the 'over-threshold' region identified by the condition $\bar{z} = 1 - m_t^2/s < 0$, where the integral is real, and then the result will be continued back to physical values $0 < \bar{z} < 1$ as explained in the paragraph 'Analytical continuation' (see from Eq.(5.86) to Eq.(5.88)).
Two regions are revealed by `Asy2.m` in the limit $\{z \to 1, y \to 0\}$, identified by the scalings of Feynman parameters respectively by $\{0, 0, 0\}$ and $\{0, 2, 1\}$.

HARD REGION $\{0,0,0\}$

$$
\begin{aligned}
t_1^G(\bar{z},y,s)_{0,0,0} =&\Gamma(1+\epsilon)\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\times\\
&\times\delta(1-x_1-x_2-x_3)(x_1+x_2+x_3)^{-1+2\epsilon}\\
&(s(x_2+x_3)(x_1+x_2+x_3)+(-sx_1x_3))^{-1-\epsilon}\\
=&\Gamma(1+\epsilon)\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\delta(1-x_1-x_2-x_3)\\
&(x_1+x_2+x_3)^{-1+2\epsilon}(s)^{-1-\epsilon}\left(x_1x_2+(x_2+x_3)^2\right)^{-1-\epsilon}\\
=&\Gamma(1+\epsilon)(s)^{-1-\epsilon}\int_0^\infty dx_1\int_0^1 dx_2(x_1+1)^{-1+2\epsilon}(x_1x_2+1)^{-1-\epsilon}\\
=&-s^{-1-\epsilon}\Gamma(\epsilon)(\psi(-2\epsilon+1)-\psi(-\epsilon+1)) \qquad (5.178)
\end{aligned}
$$

REGION $\{0,2,1\}$
By making the necessary rescalings of parameters Eq.(5.179)

$$
\{m_t^2\to(1+a\bar{x})s, Q^2\to ays, x_2\to a^2x_2, x_3\to a^1x_3\},\quad \bar{x}=-\bar{z}>0 \qquad (5.179)
$$

inside Eq.(5.177), setting $x_1=1$ and keeping the first term in the expansion around $a\to 0$, we get the integral in Eq.(5.180) to compute.

$$
t_1^G(\bar{z},y,s)_{\{0,2,1\}}=a^{1-2\epsilon}s^{-1-\epsilon}\int_0^\infty x_2\int_0^\infty x_3\Gamma(1+\epsilon)\left(x_2+\bar{x}x_3+x_3^2\right)^{-1-\epsilon} \qquad (5.180)
$$

We carry out the integration first over x_2 and then over x_3. Then, by applying the prescription for analytic continuation on $\bar{x}$, $\bar{x}\to\bar{z}e^{-i\pi}$, and setting $a=1$, we get as final result

$$
t_1^G(\bar{z},y,s)_{\{0,2,1\}}=e^{2i\epsilon\pi}s^{-1-\epsilon}\bar{z}^{1-2\epsilon}\Gamma(1-\epsilon)\Gamma(-1+2\epsilon). \qquad (5.181)
$$

The two regions just computed can be finally summed up to give

$$
\begin{aligned}
t_1^G(z\to 1,y\to 0,s)=&-s^{-1-\epsilon}\Gamma(\epsilon)(\psi(-2\epsilon+1)-\psi(-\epsilon+1))\\
&+e^{2i\epsilon\pi}s^{-1-\epsilon}\bar{z}^{1-2\epsilon}\Gamma(1-\epsilon)\Gamma(-1+2\epsilon) \qquad (5.182)
\end{aligned}
$$

2. **Triangle with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{p_t^2}, \mathbf{u}\}$**

$$\begin{aligned} t_2^G(m_t^2, u, p_t^2) =&\Gamma(1+\epsilon) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \\ &\delta(x_1+x_2+x_3)(x_1+x_2+x_3)^{-1+2\epsilon} \\ &\left(m_t^2(x_2+x_3)(x_1+x_2+x_3) + (-u)x_1x_2 + (-p_t)^2 x_1x_3\right)^{-1-\epsilon}. \end{aligned} \tag{5.183}$$

Triangle t_2^G corresponds to Fig.5.33b. By inspection of Eq.(5.183), we see that, as expected, the $\mathcal{F}$ polynomial is positive-definite for negative values of the external invariants $Q^2 > 0, p_t^2 < 0$ (we remind that $Q^2 = -q^2$!) and positive values of the internal ones $m_t^2 > 0$. We observe that we can safely set $p_t^2 = m_t^2$ without spoiling the sign of the $\mathcal{F}$ polynomial, thus obtaining

$$\begin{aligned} t_2^G(m_t^2, u) =&\Gamma(1+\epsilon) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \\ &\delta(x_1+x_2+x_3)(x_1+x_2+x_3)^{-1+2\epsilon} \\ &\left(m_t^2(x_1x_2 + (x_2+x_3)^2) + (-u)x_1x_2\right)^{-1-\epsilon}. \end{aligned} \tag{5.184}$$

By looking at Eq.(5.184), it is quite clear that the $\mathcal{F}$ polynomial does not develop problematic behaviour, namely does not goes to zero, when we take $s \to m_t^2$ and $Q^2 \simeq 0$. In this limit, u is also $\mathcal{O}(0)$, so that the $\mathcal{F}$ polynomial simply reduces to $\left(s(x_1x_2 + (x_2+x_3)^2)\right)$. The smooth behaviour of the integral in this limit is confirmed by the presence of the only hard region, according to the output of `Asy2.m`. Since the computation of such region is straightforward for both integrals belonging to this subtopology (Fig.**??**, 5.33c), we directly quote results.

$$t_2^G(z \to 1, y \to 0, s) = -s^{-1-\epsilon}\Gamma(\epsilon)(\psi(-2\epsilon+1) - \psi(-\epsilon+1)) \tag{5.185}$$
$$t_3^G(z \to 1, y \to 0, s) = -s^{-1-\epsilon}\Gamma(\epsilon)\left(1 + \epsilon + \epsilon(\psi_0(1-\epsilon) - \psi_0(-2\epsilon))\right) \tag{5.186}$$

- **BOX**
Box with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2}, \mathbf{p_t^2}, \mathbf{t}, \mathbf{s}\}$.

$$\begin{aligned} Box_1^G(m_t^2, Q^2, p_t^2, t, s) = &\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \delta(1-x_1-x_2-x_3-x_4) \times \\ &\times \Gamma(2+\epsilon)(x_1+x_2+x_3+x_4)^{2\epsilon} \times \\ &\times (Q^2 x_1x_2 - p_t^2 x_1x_3 - t x_2x_3 - s x_1x_4 + \end{aligned} \tag{5.187}$$

$$+ m_t^2 x_1(x_1 + x_2 + x_3 + x_4))^{-2-\epsilon} \tag{5.188}$$

Box_1^g corresponds to Fig.5.33d. The $\mathcal{F}$ polynomial follows the general rule and is positive-definite for positive internal invariants and negative external ones. Once again, we can safely set $p_t^2 = m_t^2$ without altering the sign of the polynomial. On the other side, it is clear that the integrals requires analytical continuation, since it is real for $s < 0$. Analytical continuation is performed once again on the variable $\bar{z}$, as already mentioned above for the triangle t_1^G (Eq.(5.177)).
`Asy2.m` individuates the presence of 3 regions, identified respectively by $\{0,0,0,0\}$, $\{0,-1,-1,-1\}$, $\{0,0,-1,-1\}$.

REGION $\{0,0,0,0\}$

$$Box_1^G(\bar{z}, y, s)_{\{0,0,0,0\}} = -\frac{\epsilon s^{-2-\epsilon}\Gamma(\epsilon)}{2(1+\epsilon)(1+2\epsilon)} \tag{5.189}$$

REGION $\{0,-1,-1,-1\}$
After having expanded at integrand level with the Feynman parameters scaling dictated by this region, we end up with the integrand contained in Eq.(5.190)

$$box_1^G(Q^2, s, t, m_t^2)_{\{0-1-1-1\}} = \int_0^\infty dx_2 dx_3 dx_4 s^{-2-\epsilon}\Gamma(2+\epsilon)(x_1 x_2 + \theta\bar{z}x_2 x_3)^{-2-\epsilon} (x_2 + x_3 + x_4)^{2\epsilon}, \tag{5.190}$$

where we already used the Cheng-Wu theorem to set $x_1 = 1$ and extending the integration over the other parameters to ∞. For simplicity of notation, we have introduced the variable $\theta = 1 - \lambda$, so that the t invariant results to be parametrized by $t = -s(1+y)\bar{z}\theta$. The integrand in Eq.(5.190) depends on the two dimensionless variables $\bar{z}, \theta$. The final result for this region will be then a function of these two parameters.
We observe that the $\mathcal{F}$-polynomial is positive-definite in the physical region $\{0 < \theta < 1, 0 < \bar{z} < 1\}$, so that we do not need to perform any kind of analytical continuation in this region. The integrations over x_2, x_3, x_4 are carried without any difficulty, thus giving as a result

$$box_1^G(Q^2, s, t, m_t^2)_{\{0-1-1-1\}} = -\frac{\pi s^{-2-\epsilon}(\theta\bar{z})^{-1-\epsilon}\csc(\epsilon\pi)\Gamma(-1-\epsilon)}{\Gamma(-2\epsilon)}. \tag{5.191}$$

REGION $\{0,0,-1,-1\}$

We are now left with the last region to compute, identified by the scalings $\{0,0,-1,-1\}$ of the Feynman parameters. After expanding at integrand level, we end up with

$$box_1^G(Q^2,s,t,m_t^2)_{\{00-1-1\}} = \int_0^\infty dx_2 dx_3 dx_4 s^{-2-\epsilon}\Gamma(2+\epsilon)(x_3+x_4)^{2\epsilon} \\ (1+x_2+\theta\bar{z}x_2x_3 - z_b x_4)^{-2-\epsilon}. \quad (5.192)$$

As for the previous region, we see that also in this case the integrand depends on $\bar{z},\theta$, implying that the result for this region will be again a function of these variables. But, with respect to the previous region, we have to deal in this case with analytical continuation. In fact, the $\mathcal{F}$ polynomial $1+x_2+\theta\bar{z}x_2x_3 - z_b x_4$ is clearly positive-definite for $\{-1<\bar{z}<0, -1<\theta<0\}$. So, the procedure we adopt is the usual one which allows to get the correct complex phases in this cases, namely

- we define two new variables

$$\bar{x} = -\bar{z}, \quad \bar{\theta} = -\theta, \quad (5.193)$$

- we perform integrations in Eq.(5.192) in the unphysical region $0<\bar{x}<1, 0<\bar{\theta}<1$,
- we work out the analytical continuation prescriptions for the dimensionless variables $\bar{x},\bar{\theta}$ starting from the ones from the invariants,
- we apply such prescriptions to get our final result in the physical region.

Let us go through this procedure step by step. We start substituting Eq.(5.193) into Eq.(5.192), thus getting

$$box_1^G(Q^2,s,t,m_t^2)_{\{0,0,-1,-1\}} = \int_0^\infty dx_2 dx_3 dx_4 s^{-2-\epsilon}\Gamma(2+\epsilon)(x_3+x_4)^{2\epsilon} \\ (1+x_2+\bar{\theta}\bar{x}x_2x_3 + x_b x_4)^{-2-\epsilon}. \quad (5.194)$$

We carry out integration in x_2, which hands

$$box_1^G(Q^2,s,t,m_t^2)_{\{0,0,-1,-1\}} = \int_0^\infty dx_3 dx_4 \frac{\epsilon s^{-2-\epsilon}\Gamma(\epsilon)(x_3+x_4)^{2\epsilon}(1+\bar{x}x_4)^{-1-\epsilon}}{1+\bar{\theta}x_b x_3}. \quad (5.195)$$

In order to be able to integrate in x_3, x_4, we first rescale both these parameters by $\bar{x}$, namely by doing the replacements $x_i \to x_i/\bar{x}$, $i=3,4$. After this manipulation, we get

$$box_1^G(Q^2,s,t,m_t^2)_{\{0,0,-1,-1\}} = \int_0^\infty dx_3 dx_4 \frac{s^{-2-\epsilon}\bar{x}^{-2-2\epsilon}\Gamma(1+\epsilon)(1+x_4)^{-1-\epsilon}(x_3+x_4)^{2\epsilon}}{1+\bar{\theta}x_3}. \tag{5.196}$$

At this point we can integrate Eq.(5.196) over x_3, x_4. The result obtained in closed form in ϵ is Eq.(5.197).

$$\begin{aligned} box_1^G(Q^2,s,t,m_t^2)_{\{00-1-1\}} = &- \frac{\pi s^{-2-\epsilon}\bar{x}^{-2-2\epsilon}\csc(\epsilon\pi)\Gamma(1+2\epsilon)}{\bar{\theta}}\left((-\bar{\theta})^{-\epsilon}(1+\bar{\theta})^{\epsilon}\Gamma(-\epsilon)\right. \\ &\left.+\bar{\theta}\text{Hypergeometric}_{2F1}\text{Regularized}(1,1,2+\epsilon,-\bar{\theta})\right) \\ &- \pi s^{-2-\epsilon}\bar{\theta}^{-1-2\epsilon}\bar{x}^{-2-2\epsilon}\csc(2\epsilon\pi)\Gamma(1+\epsilon) \\ &\times\left(-\frac{2^{1+2\epsilon}\pi^{3/2}(-\bar{\theta})^{2\epsilon}\bar{\theta}^{-\epsilon}(1+\bar{\theta})^{\epsilon}\csc(2\epsilon\pi)}{\Gamma(1/2-\epsilon)\Gamma(1+\epsilon)}\right. \\ &\left.+\frac{\text{Hypergeometric}_{2F1}\left(1,1+\epsilon,2(1+\epsilon),-\frac{1}{\bar{\theta}}\right)}{\bar{\theta}+2\epsilon\bar{\theta}}\right). \end{aligned} \tag{5.197}$$

Result in Eq.(5.197) is valid in the region $\{0<\bar{x}<1, 0<\bar{\theta}<1\}$. We need then to perform analytical continuation on such result in order to go back to the physical region $\{-1<\bar{x}<0, -1<\bar{\theta}<0\}$, which corresponds to $\{0<\bar{z}<1, 0<\theta<1\}$. We already have the prescription for $\bar{x}$ (Eq.(5.88)). We then need to work out the rule for $\bar{\theta}$. We start from the representation for the t invariant $t=-s(1+y)\bar{z}\theta$. For physical values of $s, y, \bar{z}, \theta$ ($s>0, y>0, 1>\bar{z}>0, 1>\theta>0$), t is negative, and indeed, as already pointed out at the beginning of our calculation, the physical region is identified by $t<0$. Now, what happens is that we have re-parametrized $\bar{z}$ with $\bar{x}$, so that for t we get now $t=s(1+y)x_b\theta$ and if we compute for $x_b>0$, then it means that we are computing for $t>0$. But the original $\mathcal{F}$ polynomial $Q^2x_1x_2 - tx_2x_3 - sx_1x_4 + m_t^2x_1(x_1+x_2+x_4)$ (Eq.(**??**))is positive-definite for negative t and the same happens after we expand it in the region we are considering, as it can be seen from Eq.(5.192) (the term coming from t is $\theta\bar{z}$). From this consideration we draw the conclusion that if we compute above threshold ($-1<\bar{z}<0$), and we want to keep $t<0$, we need to flip the sign of the only independent variable on which t depends, which is the angular variable θ. This explains why we introduced $\bar{\theta}=-\theta$ and also why in this framework a change in the sign of θ directly corresponds to a change in the sign of t.

We are now ready to work out the rule. We start from the rule for t, which is nothing but the usual Feynman prescription external invariants $t \to t+i0$. This give then

$$-t = -t - i0 = t(-1-i0) = te^{-i\pi}. \tag{5.198}$$

We can work out the rule for θ as follows.

$$\bar{\theta} = -\theta = \frac{t}{s+Q^2}\frac{s}{s-m_t^2} = \frac{-t}{s+Q^2}\frac{s}{s-m_t^2}e^{i\pi} = \theta e^{i\pi} \tag{5.199}$$

We can finally substitute the prescription Eq.(5.88),(5.199) into Eq.(5.197), thus obtaining our final result [13]

$$\begin{aligned} box_1^G(Q^2,s,t,m_t^2)_{\{00-1-1\}} =& \frac{\pi s^{-2-\epsilon}\bar{z}^{-2-2\epsilon}e^{2i\pi\epsilon}\csc(\epsilon\pi)\Gamma(1+2\epsilon)}{\theta}\left((\theta)^{-\epsilon}(1-\theta)^{\epsilon}\Gamma(-\epsilon)\right.\\ &\left.-\theta\text{Hypergeometric}_{2F1}\text{Regularized}\left(1,1,2+\epsilon,\theta\right)\right)\\ &\pi s^{-2-\epsilon}\theta^{-1-2\epsilon}\bar{z}^{-2-2\epsilon}\csc(2\epsilon\pi)\Gamma(1+\epsilon)\\ &\times\left(-\frac{2^{1+2\epsilon}e^{-i\pi\epsilon}\pi^{3/2}(\theta)^{\epsilon}(1-\theta)^{\epsilon}\csc(2\epsilon\pi)}{\Gamma(1/2-\epsilon)\Gamma(1+\epsilon)}\right.\\ &+\frac{\frac{e^{i(1+\epsilon)\pi}(1-\theta)^{\epsilon}\theta^{1+\epsilon}\Gamma(-\epsilon)\Gamma(2(1+\epsilon))}{\Gamma(-1-\epsilon+2(1+\epsilon))}}{-\theta-2\epsilon\theta}\\ &\left.-\frac{\frac{\theta\Gamma(\epsilon)\Gamma(2(1+\epsilon))\text{Hypergeometric}_{2F1}(1,2-2(1+\epsilon),1-\epsilon,\theta)}{\Gamma(1+\epsilon)\Gamma(-1+2(1+\epsilon))}}{-\theta-2\epsilon\theta}\right). \end{aligned} \tag{5.201}$$

The final result for box_1^G is the sum of the three regions

[13] We remark that before performing the analytical continuation on $\bar{\theta}, \bar{x}$, which gives as a result Eq.(5.201), we perform a manipulation on the function Hypergeometric2F1 $\left(1, 1+\epsilon, 2(1+\epsilon), -\frac{1}{\theta}\right)$ appearing in the previous equation (Eq.(5.197)), by means of the formula

$$\begin{aligned} \text{Hypergeometric}_{2F1}(a,b,c,z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\text{Hypergeometric}_{2F1}\left(a,1-c+a,1-b+a,\frac{1}{z}\right)\\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\text{Hypergeometric}_{2F1}\left(b,1-c+b,1-a+b,\frac{1}{z}\right). \end{aligned} \tag{5.200}$$

Eq.(5.200) applied to Hypergeometric2F1 $\left(1, 1+\epsilon, 2(1+\epsilon), -\frac{1}{\theta}\right)$ allows us to rewrite it in terms of Hypergeometric$_{2F1}$ of argument $\bar{\theta}$, instead of $\frac{1}{\theta}$. In other words, first we analytically continue Hypergeometric2F1 $\left(1, 1+\epsilon, 2(1+\epsilon), -\frac{1}{\theta}\right)$ from the region where the argument is $\frac{1}{-\theta} < -1$ to the region where it is $(-1 < -\bar{\theta} < 0)$. And only after that we analytically continue to the region where the argument is $0 < \theta < 1$ (and in this case we do not even really need analytical continuation, since the Hypergeometric$_{2F1}$ is already convergent in the circle where its argument z is $|z| < 1$).
It is necessary to adopt such order in the analytical continuation, otherwise if we use rule Eq.(5.199) directly on Hypergeometric2F1 $\left(1, 1+\epsilon, 2(1+\epsilon), -\frac{1}{\theta}\right)$, we would get Hypergeometric2F1 $\left(1, 1+\epsilon, 2(1+\epsilon), \frac{1}{\theta}\right)$, which expanded in ϵ would give logarithms of the type $\ln(1-1/\theta)$. These logs are clearly well defined for $\theta > 1$, but not for $0 < \theta < 1$. In this latter case, such logs develop an imaginary part, whose sign would not be properly defined.

$$
\begin{aligned}
box_1^G(z_b \to 0, y \to 0, s, \theta) = &- \frac{\epsilon s^{-2-\epsilon}\Gamma(\epsilon)}{2(1+\epsilon)(1+2\epsilon)} \\
&- \frac{\pi s^{-2-\epsilon}(\theta\bar{z})^{-1-\epsilon}\csc(\epsilon\pi)\Gamma(-1-\epsilon)}{\Gamma(-2\epsilon)} \\
&\frac{\pi s^{-2-\epsilon}\bar{z}^{-2-2\epsilon}e^{2i\pi\epsilon}\csc(\epsilon\pi)\Gamma(1+2\epsilon)}{\theta}\left((\theta)^{-\epsilon}(1-\theta)^{\epsilon}\Gamma(-\epsilon)\right. \\
&\left.-\theta\text{Hypergeometric}_{2F1}\text{Regularized}\left(1,1,2+\epsilon,\theta\right)\right) \\
&\pi s^{-2-\epsilon}\theta^{-1-2\epsilon}\bar{z}^{-2-2\epsilon}\csc(2\epsilon\pi)\Gamma(1+\epsilon) \\
&\times\left(-\frac{2^{1+2\epsilon}e^{-i\pi\epsilon}\pi^{3/2}(\theta)^{\epsilon}(1-\theta)^{\epsilon}\csc(2\epsilon\pi)}{\Gamma(1/2-\epsilon)\Gamma(1+\epsilon)}\right. \\
&+\frac{\frac{e^{i(1+\epsilon)\pi}(1-\theta)^{\epsilon}\theta^{1+\epsilon}\Gamma(-\epsilon)\Gamma(2(1+\epsilon))}{\Gamma(-1-\epsilon+2(1+\epsilon))}}{-\theta-2\epsilon\theta} \\
&\left.-\frac{\frac{\theta\Gamma(\epsilon)\Gamma(2(1+\epsilon))\text{Hypergeometric}_{2F1}(1,2-2(1+\epsilon),1-\epsilon,\theta)}{\Gamma(1+\epsilon)\Gamma(-1+2(1+\epsilon))}}{-\theta-2\epsilon\theta}\right).
\end{aligned}
\tag{5.202}
$$

Box with 1 internal mass $\mathbf{m_t^2}$ and external masses $\{\mathbf{Q^2}, \mathbf{p_t^2}, \mathbf{u}, \mathbf{s}\}$.

$$
\begin{aligned}
Box_2^G(m_t^2, Q^2, s, u) &= \int d^d l \frac{1}{(k-l)^2 l^2[(l-q)^2 - m_t^2][(l-p_b-q)^2 - m_t^2]} \\
&= \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1)\Gamma(2+\epsilon)(x_1+x_2+x_3+x_4)^{2\epsilon} \\
&(-ux_1x_3 + x_2(Q^2x_3 - sx_4) + mt^2(x_1x_3 + (x_3+x_4)(x_2+x_3+x_4)))^{-2-\epsilon}.
\end{aligned}
\tag{5.203}
$$

Box_2^G corresponds to Fig.5.33e. The initial $\mathcal{F}$-polynomial is positive-definite for positive internal invariants and negative external ones. As for all previous cases in which the $\mathcal{F}$-polynomial depends on p_t^2, also for Box_2^G we can safely set $p_t^2 = m_t^2$ without spoiling the sign of the polynomial. By doing so, we end up with the representation Eq.(5.203). The integral is real in the euclidean region $s < 0$. In order to obtain its value in the region $s > 0$, we thus need to perform analytical continuation. By running `Asy2.m` on the Feynman representation of this integral, we get two regions identified by the Feynman parameters scalings respectively as $\{0,0,0,0\}, \{0,0,2,1\}$.

HARD REGION $\{0,0,0,0\}$

The hard region $\{0,0,0,0\}$ requires to perform the integral in Eq.(5.204), where we already used the Cheng-Wu theorem to get rid of the integration over x_1 by restricting the $\delta(x_1+x_2+x_3+x_4-1)$ to $\delta(x_1+x_2-1)$ and extending then the integration over x_3, x_4 to $[0,\infty]$.

$$box_2^G(Q^2,s,u,m_t^2)_{HR} = \int_0^1 dx_2 \int_0^\infty dx_3 \int_0^\infty dx_4 s^{-2-\epsilon}\Gamma(2+\epsilon)(1+x_3+x_4)^{2\epsilon} \\ ((1-x_2)x_3+x_2x_3+(x_3+x_4)^2)^{-2-\epsilon} \tag{5.204}$$

Integral in Eq.(5.204) is easily solved via the change of variable

$$x_3 = (\eta-\xi)\xi,\ x_4 = \xi-(\eta-\xi)\xi, \quad \{0<\xi<\infty, 1<\eta<\infty\}, \tag{5.205}$$

and the obtained expression for this region is then

$$box_2^G(Q^2,s,u,m_t^2)_{HR} = \frac{\pi s^{-2-\epsilon}\csc(\epsilon\pi)\Gamma(-\epsilon)}{\Gamma(-2\epsilon)}. \tag{5.206}$$

REGION $\{0,0,2,1\}$

We now turn to region $\{0,0,2,1\}$, which amounts to compute, after having expanded at integrand level, the integral reported in Eq.(5.207),

$$box_2^G(Q^2,s,u,m_t^2)_{\{0,0,2,1\}} = s^{-2-\epsilon}\Gamma(2+\epsilon)\int_0^1 dx_1dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4-1)(x_1+x_2)^{2\epsilon} \\ (x_1x_3+x_2x_3-\bar{z}x_2x_4+x_4^2)^{-2-\epsilon}. \tag{5.207}$$

with $\bar{z}$ being as usual the threshold variable defined as $\bar{z} = \frac{s-m_t^2}{s} \in [0,1]$. We observe that the only term spoiling the sign of the $\mathcal{U}$-polynomial is $-\bar{z}x_2x_4$. Once more it the convenient to perform analytical continuation on $\bar{z}$. Following the usual procedure, we introduce then the variable $\bar{x} = -\bar{z}$ and compute the integral 'above threshold', namely for $0<\bar{x}<1$. By performing the integrations in Eq.(5.207), we get

$$box_2^G(Q^2,s,u,m_t^2)_{\{0,0,2,1\}} = s^{-2-\epsilon}\bar{x}^{-1-2\epsilon}\Gamma(-\epsilon)\Gamma(2\epsilon). \tag{5.208}$$

The analytical continuation prescription Eq.(5.88) applied to the previous result yields

$$box_2^G(Q^2,s,u,m_t^2)_{\{0,0,2,1\}} = s^{-2-\epsilon}\exp(2i\pi\epsilon)\bar{z}^{-1-2\epsilon}\Gamma(-\epsilon)\Gamma(2\epsilon), \tag{5.209}$$

so that our final result for this box in the desired limit reads

$$box_2^G(Q^2\to 0,s,u,m_t^2\to s) = \frac{\pi s^{-2-\epsilon}\csc(\epsilon\pi)\Gamma(-\epsilon)}{\Gamma(-2\epsilon)} \\ + s^{-2-\epsilon}\exp(2i\pi\epsilon)\bar{z}^{-1-2\epsilon}\Gamma(-\epsilon)\Gamma(2\epsilon). \tag{5.210}$$

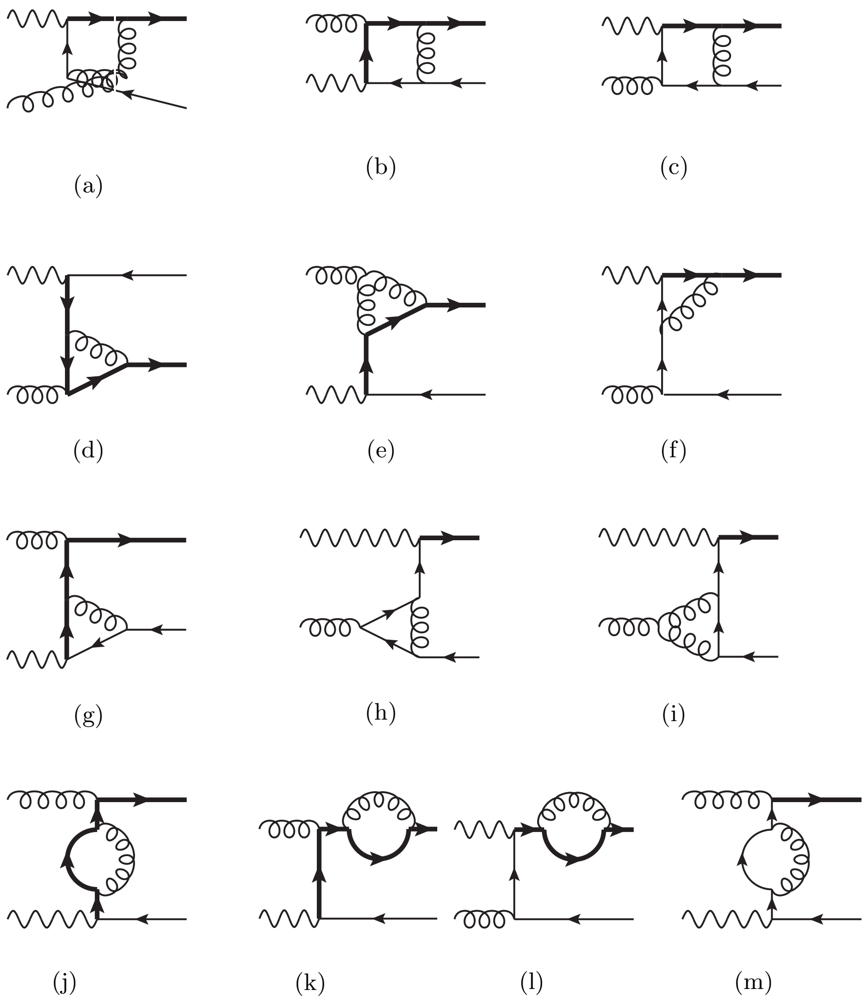


Figure 5.29: 1-loop corrections to $g + W^* \to t + \bar{b}$

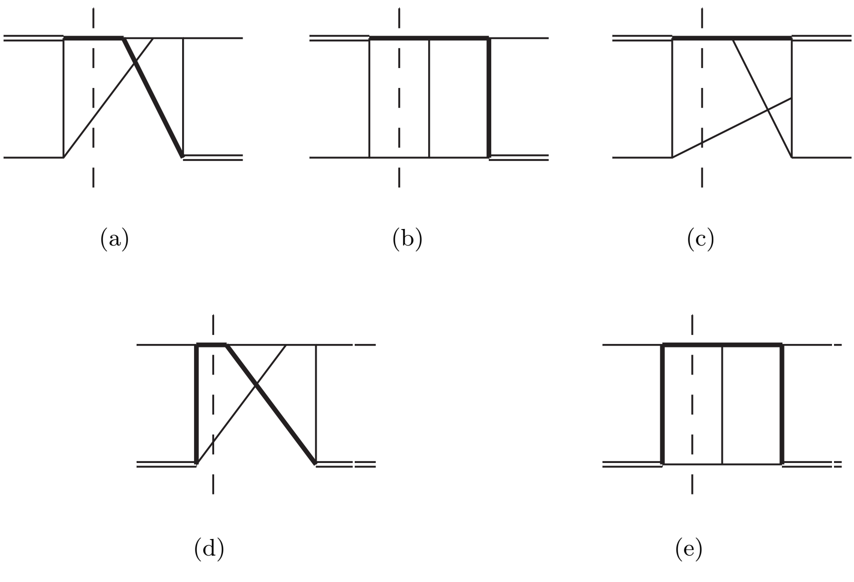
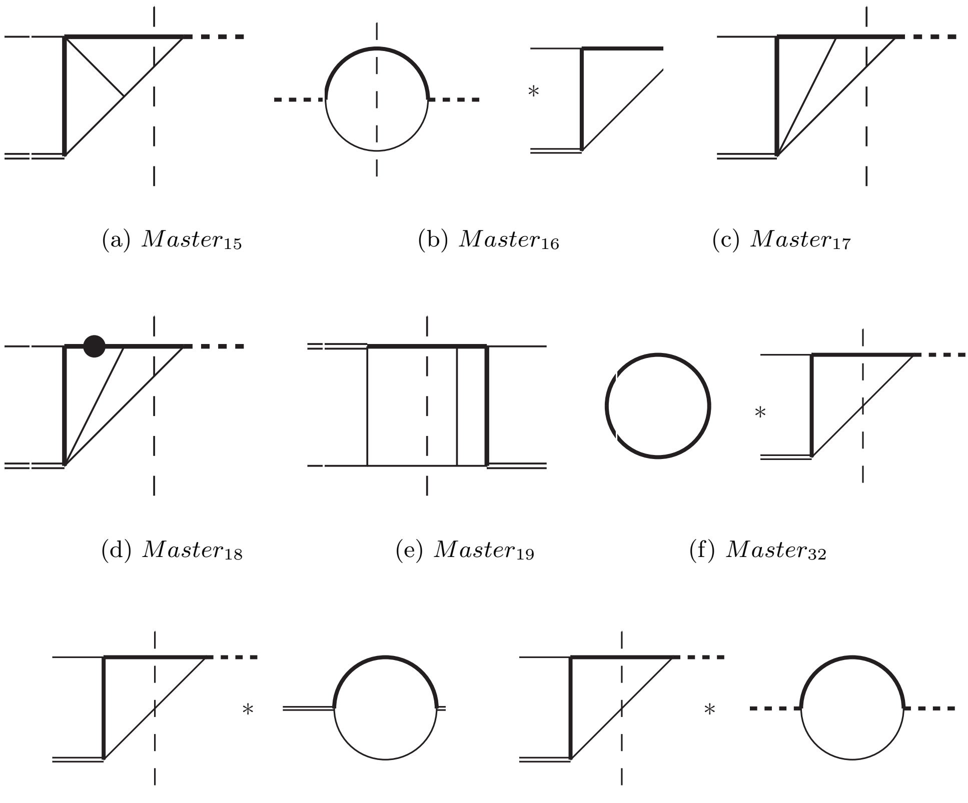


Figure 5.30: Independent topologies to $[g + W^* \to t + \bar{b}]_{1-loop}$.

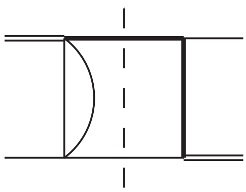


(a) $Master_{15}$ (b) $Master_{16}$ (c) $Master_{17}$

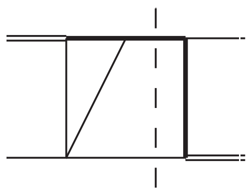
(d) $Master_{18}$ (e) $Master_{19}$ (f) $Master_{32}$

(g) $Master_{36}$ (h) $Master_{37}$

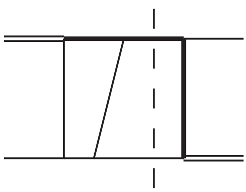
Figure 5.31: Set of new MIs appearing in $g + W^* \to t + \bar{b}$ at 1-loop ($Master$ from 15 to 37).
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).



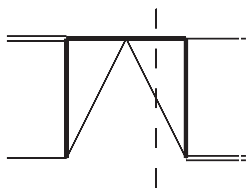
(a) $Master_{38}$



(b) $Master_{42}$



(c) $Master_{43}$



(d) $Master_{59}$

Figure 5.32: Set of new MIs appearing in $g + W^* \to t + \bar{b}$ at 1-loop ($Master$ from 38 to 59).
Simple thin lines are massless, simple thick lines, double lines and thick dashed lines are massive and correspond respectively to m_t^2 (either internal or external) , Q^2 and s (only external).

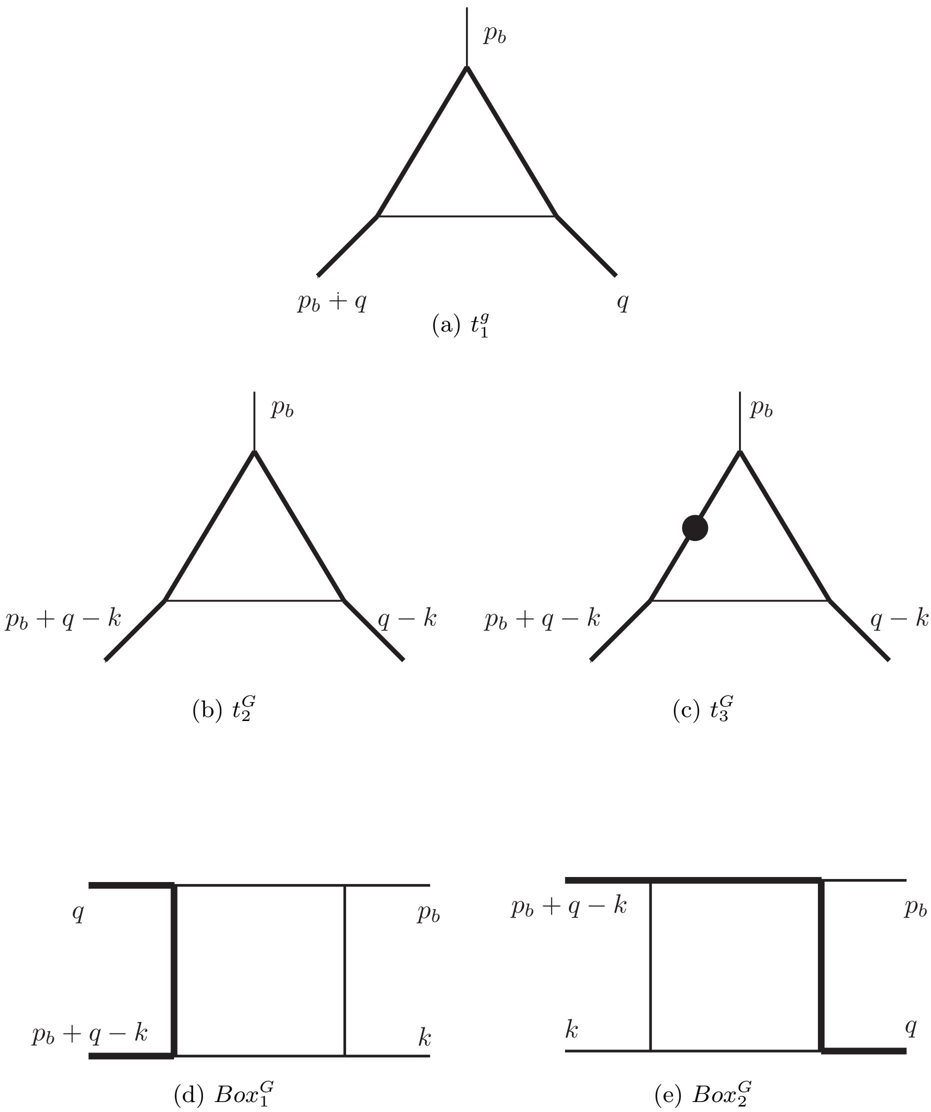


Figure 5.33: Independent 1-loop boundary conditions for the gluon RV Master Integrals (new with respect to the ones encountered in the bottom channel).

Chapter 6

Conclusion and Outlook

In this thesis, we presented partial results achieved in the analytical computation of the QCD corrections at $\mathcal{O}(\alpha_s^2)$ (NNLO) to the inclusive cross-section for Single Top in t-channel. In particular we presented the determination and computation of the complete set of Master Integrals which describes the NNLO QCD corrections to massive Form Factors for CC-DIS. This set of Master Integrals is a crucial ingredient in the computation of NNLO QCD corrections to inclusive Single Top in t-channel.

We started by reviewing (chapter 2) the importance of the Single Top production mechanism and the need for accurate predictions of its cross-section for SM and BSM searches, in the more general context of Top physics at the LHC.
After a brief recap of the basics of QCD, we described the computation of QCD corrections to Single Top in t-channel in the framework of a DIS-like approach (chapter 3). The original hadron-initiated process $p + p \to t + q' + X$ is 'cut' into the two weak currents $p \to W^* + q' + X_1$ and $p + W^* \to t + X_2$ which constitute the building blocks of Single Top in t-channel. In other words, we neglect all the non-factorizable corrections, namely all gluon-exchanges between these currents. As a result, all the information about higher-order QCD corrections is encoded in the two Form Factors, namely the cross-sections for the two subprocesses $p \to W^* + q' + X_1$ and $p + W^* \to t + X_2$. This DIS-like picture holds exactly for this process at LO and NLO in QCD. At NNLO instead it becomes an approximation because at this perturbative order the non-factorizable corrections begin to give non-zero contribution. Some qualitative arguments for the reliability of such approximation are provided at the end of the chapter.
In order to construct the cross-section for $[p + p \to t + q' + X]_{\mathcal{O}(\alpha_s^2)}$, the knowledge of Form Factors up $\mathcal{O}(\alpha_s^2)$ is required. Form Factors can be obtained through the 'master formula' of perturbative QCD as convolutions between the appropriate

Parton Distribution Functions and the cross-sections for the underlying partonic processes. In the process under consideration, what we need are thus the cross-sections for the partonic weak currents $q \to W^* + q' + X_1$ and $b + W^* \to t + X_2$ up to $\mathcal{O}(\alpha_s^2)$. These can be computed fully analytically order by order in perturbation theory. The first of these subprocesses only involves massless quarks and its cross-section is already available in literature in analytical form up to NNLO QCD. Instead, the partonic current containing the top quark is yet unknown. It depends on one additional scale because it contains a heavy quark in the final state, whose mass m_t cannot be neglected. The computation of $\sigma(b + W^* \to t + X_2)$ therefore ends up depending on three dimensional scales (energy in the c.o.m. frame s, top mass m_t, and virtuality of the W-boson Q^2) and its computation beyond leading order turns out to be quite involved. In the literature results already exist for this cross-section up to NLO in QCD, but no analytical result is available up to NNLO. Thus, the only piece we need to complete our picture is the analytical result for $\sigma(b + W^* \to t + X)_{\mathcal{O}(\alpha_s^2)}$. The rest of this thesis, namely Chapter 3 and 4, are thus dedicated to the analytical computation of QCD corrections at $\mathcal{O}(\alpha_s^2)$ to $\sigma(b + W^* \to t + X)$, which constitutes then the bulk of our original work.
Chapter 4 is dedicated to explaining in detail the technique of Master Integrals, which we adopt in order to carry out analytically all the needed integrations over momenta of both real and virtual particles.
Finally, in chapter 5, we explain in detail all the steps which lead to our original result, which consists in the determination and explicit computation of the complete set of Master Integrals for $b + W^* \to t$, or more generally for Charged-Current DIS Form Factors.
Results for all the independent Master Integrals are then reported systematically in the Appendix, together with some other useful intermediate results.

We would like to conclude this dissertation with an outlook on the possible future developments of the results obtained up to now.

The first, natural development will be the completion of the computation of massive CC-DIS Form Factors, describing the $b + W^* \to t + X$ current up to NNLO-QCD. At this stage, all partonic matrix elements written in terms of Master Integrals are ready and all Master Integrals have been explicitly computed. Thus, what needs to be done is the following (in order).

- Plug the results for the masters into the partonic matrix elements. The expressions we get are at this stage full of every possible type of divergences and needs to be further manipulated.

- Check that all final state IR divergences and all soft-IR initial state divergences cancel when we sum up the expressions corresponding to the different pieces that build up our final result, namely Double-Real, Real-Virtual and Double-Virtual diagrams.

- Perform UV renormalization, which amounts in this case to renormalize the coupling constant, the bottom and top wave-functions and the top mass.
- Accomplish mass factorization, in order to reabsorb into the PDFs the initial-state collinear divergences which are still left in our expressions.

At this point we will end up with a finite expression for the NNLO partonic cross-section $\sigma(b + W^* \to t + X)$, and by convolving it with the appropriate PDFs, we will finally get the desired CC-DIS massive Form Factors.
Since this result is obtained in the above-mentioned DIS-like approximation, a quantitative estimate of the error introduced by this approximation must be provided. This may be accomplished by means of an analysis like the one carried out in [**?**].

Now, our formal result for CC-DIS massive Form Factors can be used for some major phenomenological applications.
In the first place, we would obviously like to use it to construct the desired prediction for Single Top in t-channel at NNLO-QCD. This can be achieved by implementing our analytical results for partonic Form Factors into an already available Fortran code, which will then perform numerically the convolution with PDFs and the integration over the remaining variables which describes the global $2 \to 2$ process, as explained in Chapter 2. This will provide a cross-check to the fully numerical NNLO computation carried out in [**?**] and also a fast and efficient way to evaluate the inclusive cross-section for the process of interest. In fact, an analytical result has two main advantages. On one hand it is more stable and faster under numerical evaluation. On the other hand it is 'safer' for what concerns the cancellation of singularities, which happens exactly, thus providing a stringent test of the correctness of the computation.
A more ambitious program would involve the construction of a *theoretical precision benchmark* for Single Top in t-channel. This could be achieved by combining all kinds of available corrections to this process. On top of the previously discussed *QCD corrections up to NNLO*, which would constitute the bulk of the precision benchmark, the following corrections, available in the literature, may be taken into consideration.

- *EW corrections* : The other type of quantum corrections that one can consider are electro-weak corrections (SM). Such corrections have been computed for the Single Top in t-channel inclusive cross-section at NLO-EW (α^3) both in SM and MSSM scenarios (see [**?**]). The overall SM one-loop effect is however pretty small, again of the size of a few percent with respect to the tree-level cross-section, due to a compensation of weak and QED contributions, which are of opposite sign.

- *Soft resummation* : The effect of resummation of soft-gluon contributions to all orders are computed at NNLL in [?]. These results need to be carefully matched with the NNLO-QCD computation, in order to avoid double-counting of those logarithms which are already included in the fixed-order computation.

- *m_b corrections* : As discussed at the end of Chapter 2, our computation of the analytical NNLO-QCD contribution to t-channel Single top is carried out in a five-flavour (5F) scheme, where the b-quark is considered massless and the only massive quark happens to be the top quark. Given the physical value of 4.5GeV of the bottom mass, this can be considered a realistic description only up to a certain extent. It is thus natural to wonder what happens if we perform our computation in a four-flavour (4F) scheme, namely if we let the bottom be massive. In this case the level of complexity of the computation, already at NLO, significantly increases, since the final result will depend upon a new additional dimensional scale m_b. Since the full NNLO computation in 4F is then beyond our possibilities, the exact contribution due to m_b cannot be achieved in exact form. Nonetheless, we can think of estimating such contribution in an approximated manner, by means of the procedure explained at the end of Chapter 3. Once our NNLO analytical result in 5F will be available, such procedure could be used to evaluate the impact of m_b corrections at $\mathcal{O}(\alpha_s^2)$. The estimate thus obtained could be then added, together with the other corrections above-mentioned, to the 5F NNLO inclusive cross-section, and it will represent a corrective factor which takes into account, though only in an approximate way, the presence of m_b.

Beside this theoretical precision benchmark for Single Top, massive CC-DIS Form Factors can find other phenomenological application in the extraction of PDFs from global parton fits and in the computation of other processes where this kind of Form Factors enter as building blocks.
On a more formal level instead, the performed computation itself exhibits some interesting features. In the first place the newly computed Master Integrals increase the general knowledge we have of such basis of integrals at 2-loop. Secondly, our masters were computed by following the idea ([?]) of finding a canonical form for the differential equations which describe them, thus confirming once more the success of this recent approach. Last but not least, suitable remappings were found in order to linearise systems of differential equations whose coefficients contained non-rational functions of the kinematic invariants. The existence of such remappings is not guaranteed a priori, but once more they have been successfully found, as it happened for some other even more complicated processes in the very recent literature.

To conclude, we have achieved our objective, namely the computation of the set of

Masters describing the partonic process $[b+W^* \to t+X]_{\mathcal{O}(\alpha_s^2)}$. Starting from this achieved milestone, analytical results for massive CC-DIS Form Factors and inclusive Single Top in t-channel will become available, hopefully in the next future, as tools to further increase our knowledge of this tricky, puzzling and fascinating world which is Particle Physics.

Appendix A: Canonical Basis and PDEs

Canonical Differential Equations and Boundary Conditions for $\mathbf{b+W^* \to t+g+g}$.

We now report systematically for the 3 topologies the system of differential equations in canonical form, and the expansion in terms of soft masters $Master_2^2, Master_4^s$, which determines the boundary conditions. The alphabet involved in the DE for the subprocess $b+W^* \to t+g+g$ is

$$\mathcal{A}_{gg}^{RR} = \{z, 1-z, y, 1-y, y+z\}. \tag{6.1}$$

Systems of differential equations in canonical form:

- t_1

$$\begin{aligned}
dM_1 =&\epsilon(-6M_2\mathrm{dL}(z) + M_1(-4\mathrm{dL}(1-z) + 2\mathrm{dL}(z)))\\
dM_2 =&\epsilon(-3M_2\mathrm{dL}(z) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))\\
dM_3 =&\epsilon(2M_2\mathrm{dL}(y) + M_3(2\mathrm{dL}(y) - \mathrm{dL}(1+y) - 3\mathrm{dL}(y+z)) + M_1(\mathrm{dL}(1-z) - \mathrm{dL}(y+z)))\\
dM_4 =&\epsilon(M_4(4\mathrm{dL}(y) - \mathrm{dL}(z) - 4\mathrm{dL}(y+z)) + M_6(2\mathrm{dL}(y) - \mathrm{dL}(1-z) - 2\mathrm{dL}(y+z))\\
&+ M_7(-3\mathrm{dL}(y) + \mathrm{dL}(1+y) + 2\mathrm{dL}(y+z)) + M_3(-3\mathrm{dL}(y) + 3\mathrm{dL}(y+z)))\\
dM_5 =&\epsilon(-2M_5\mathrm{dL}(1+y) + M_7(2\mathrm{dL}(z) - 2\mathrm{dL}(y+z)) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(y+z))\\
&+ M_6(-2\mathrm{dL}(1-z) + 2\mathrm{dL}(y+z)) + M_4(-4\mathrm{dL}(z) + 4\mathrm{dL}(y+z)))\\
dM_6 =&\epsilon(M_4(4\mathrm{dL}(1+y) + 2\mathrm{dL}(z) - 4\mathrm{dL}(y+z)) + M_6(2\mathrm{dL}(1+y) - 4\mathrm{dL}(1-z) - 2\mathrm{dL}(y+z))\\
&+ M_7(-2\mathrm{dL}(1+y) + 2\mathrm{dL}(y+z)) + M_3(-3\mathrm{dL}(1+y) + 3\mathrm{dL}(y+z)))\\
dM_7 =&\epsilon(-3M_2\mathrm{dL}(y) + M_4(4\mathrm{dL}(y) + 4\mathrm{dL}(1+y) - 4\mathrm{dL}(y+z))\\
&+ M_6(2\mathrm{dL}(y) - 4\mathrm{dL}(1-z) - 2\mathrm{dL}(y+z)) + M_1(-\mathrm{dL}(1-z) + \mathrm{dL}(y+z))\\
&+ M_7(-3\mathrm{dL}(y) - \mathrm{dL}(z) + 2\mathrm{dL}(y+z)) + M_3(-6\mathrm{dL}(y) + 6\mathrm{dL}(y+z)))
\end{aligned} \tag{6.2}$$

- t_2

$$\begin{aligned}
dM_8 =&\epsilon(-6M_9\mathrm{dL}(z) + M_8(-4\mathrm{dL}(1-z) + 2\mathrm{dL}(z)))\\
dM_9 =&\epsilon(-3M_9\mathrm{dL}(z) + M_8(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))\\
dM_{10} =&\epsilon(M_9(4\mathrm{dL}(1-z) - 4\mathrm{dL}(z)) + M_{10}(-2\mathrm{dL}(1-z) - \mathrm{dL}(z)) + M_8(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))\\
dM_{11} =&\epsilon(-2M_{11}\mathrm{dL}(1-z) + M_9(2\mathrm{dL}(1-z) - 2\mathrm{dL}(z))\\
&+ M_8(-\mathrm{dL}(1-z) + \mathrm{dL}(z)) + M_{10}(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))
\end{aligned} \tag{6.3}$$

- t_3

$$\begin{aligned}
dM_{12} =&\epsilon(-6M_{13}\mathrm{dL}(z) + M_{12}(-4\mathrm{dL}(1-z) + 2\mathrm{dL}(z)))\\
dM_{13} =&\epsilon(-3M_{13}\mathrm{dL}(z) + M_{12}(-\mathrm{dL}(1-z) + \mathrm{dL}(z)))\\
dM_{14} =&\epsilon(2M_{13}\mathrm{dL}(y) + M_{14}(2\mathrm{dL}(y) - \mathrm{dL}(1+y) - 3\mathrm{dL}(y+z))\\
&+ M_{12}(\mathrm{dL}(1-z) - \mathrm{dL}(y+z)))\\
dM_{15} =&\epsilon(2M_{13}\mathrm{dL}(y) + M_{16}(2\mathrm{dL}(y) - \mathrm{dL}(z)) + M_{15}(2\mathrm{dL}(1+y) - \mathrm{dL}(z))\\
&+ M_{14}(2\mathrm{dL}(y) + \mathrm{dL}(z)) + M_{17}(2\mathrm{dL}(y) + 2\mathrm{dL}(z)))\\
dM_{16} =&\epsilon(M_{15}(-2\mathrm{dL}(1+y) + 2\mathrm{dL}(1-z))\\
&+ M_{17}(2\mathrm{dL}(y) - 4\mathrm{dL}(1+y) + 4\mathrm{dL}(1-z) - 2\mathrm{dL}(z))\\
&+ M_{16}(2\mathrm{dL}(y) - 4\mathrm{dL}(y+z))\\
&+ M_{14}(-\mathrm{dL}(1+y) + 4\mathrm{dL}(1-z) - 3\mathrm{dL}(y+z)) + M_{12}(\mathrm{dL}(1-z) - \mathrm{dL}(y+z)))\\
dM_{17} =&\epsilon(-3M_{13}\mathrm{dL}(y) + M_{15}(-\mathrm{dL}(1+y) - 2\mathrm{dL}(1-z))\\
&+ M_{17}(-\mathrm{dL}(y) - 4\mathrm{dL}(1-z) - \mathrm{dL}(z))\\
&+ M_{16}(-\mathrm{dL}(y) - \mathrm{dL}(1+y) + \mathrm{dL}(y+z)) + M_{12}(-\mathrm{dL}(1-z) + \mathrm{dL}(y+z))\\
&+ M_{14}(-3\mathrm{dL}(y) + \mathrm{dL}(1+y) - 4\mathrm{dL}(1-z) + 3\mathrm{dL}(y+z)))
\end{aligned} \tag{6.4}$$

Soft limit :

- t_1

$$\begin{aligned}
&Master_1 \to \bar{z}^{(4-4\epsilon)}(-(1/4)(Q^2+s)Master_2^s)\\
&Master_2 \to \bar{z}^{(3-4\epsilon)}Master_2^s,\\
&Master_3 \to \bar{z}^{(2-4\epsilon)}(-(((-3+4\epsilon)Master_2^s)/((-1+2\epsilon)(Q^2+s))))\\
&Master_4 \to \bar{z}^{(1-4\epsilon)}Master_4^s\\
&Master_5 \to \bar{z}^{(-4\epsilon)}((-1+4\epsilon)(2(3-10\epsilon+8\epsilon^2)Master_2^s + \epsilon^2 s(Q^2+s)Master_4^s))/(3\epsilon^3 s(Q^2+s)^2)\\
&Master_6 \to \bar{z}^{(4-4\epsilon)}(4(-3+4\epsilon)Master_2^s + (-1+3\epsilon)s(Q^2+s)Master_4^s)/(2(-1+2\epsilon)s)\\
&Master_7 \to \bar{z}^{(2-4\epsilon)}(-(Master_2^s/(Q^2+s)) + ((1-3\epsilon)sMaster_4^s)/(-6+8\epsilon)).
\end{aligned} \tag{6.5}$$

- t_2

$$\begin{aligned}
&Master_8 \to \bar{z}^{(3-4\epsilon)}Master_2^s\\
&Master_9 \to \bar{z}^{(4-4\epsilon)}(-(1/4)(Q^2+s)Master_2^s)\\
&Master_{10} \to \bar{z}^{(1-4\epsilon)}((-3+4\epsilon)Master_2^s)/((-1+2\epsilon)s)\\
&Master_{11} \to \bar{z}^{(-1-4\epsilon)}((-1+2\epsilon)(-3+4\epsilon)(-1+4\epsilon)Master_2^s)/(\epsilon^3 s(Q^2+s)^2)
\end{aligned} \tag{6.6}$$

- t_3

$$\begin{aligned}
&Master_{12} \to \bar{z}^{(3-4\epsilon)}Master_2^s\\
&Master_{13} \to \bar{z}^{(5-4\epsilon)}((-1+\epsilon)sMaster_2^s)/(-10+8\epsilon)\\
&Master_{14} \to \bar{z}^{(2-4\epsilon)}(-(((-3+4\epsilon)Master_2^s)/((-1+2\epsilon)(Q^2+s))))\\
&Master_{15} \to \bar{z}^{(1-4\epsilon)}Master_4^s\\
&Master_{16} \to \bar{z}^{(3-4\epsilon)}(Master_2^s/(Q^2+s) + ((-1+3\epsilon)sMaster_4^s)/(-6+8\epsilon))\\
&Master_{17} \to \bar{z}^{(2-4\epsilon)}((6-8\epsilon)Master_2^s - (-1+3\epsilon)s(Q^2+s)Master_4^s)/(2(-1+2\epsilon)s)
\end{aligned} \tag{6.7}$$

Canonical Differential Equations and Boundary Conditions for $\mathbf{b}+\mathbf{W}^{*}\to\mathbf{t}+\mathbf{b}+\bar{\mathbf{b}}$.

The canonical basis is defined as

$$
\begin{aligned}
&M_1 = (1-z)*(2*G(t_4,\{2,0,1,0,0,0,2\})+G(t_4,\{2,0,2,0,0,0,1\}))\\
&M2 = G(t_4,\{2,0,1,0,0,0,2\})\\
&M_3 = 2*\epsilon*(1+y)*G(t_4,\{1,0,2,1,0,0,1\})\\
&M_4 = 2*(z*(1+y+z)*G(t_4,\{1,0,1,0,0,2,2\})\\
&\qquad +2*\epsilon*(1+y+z)*(G(t_4,\{1,0,1,0,0,1,2\})-G(t_4,\{1,0,2,0,0,1,1\}))\\
&\qquad +3*\epsilon*(1+y+z)*G(t_4,\{1,0,2,0,0,1,1\}))\\
&M_5 = 2*\epsilon*(1+y)*G(t_4,\{1,0,2,0,0,1,1\})\\
&M_6 = \epsilon*(1+y)*(G(t_4,\{1,0,1,0,0,1,2\})-G(t_4,\{1,0,2,0,0,1,1\}))\\
&M_7 = 2*\epsilon*y*(1+y)*G(t_4,\{1,0,1,1,0,1,2\})\\
&M_8 = 2*(\epsilon*y*(1+y)*G(t_4,\{1,0,1,1,0,2,1\})\\
&\qquad +(\epsilon*(1+y)*(G(t_4,\{1,0,1,0,0,1,2\})-G(t_4,\{1,0,2,0,0,1,1\})))/2\\
&\qquad +\epsilon*(1+y)*G(t_4,\{1,0,2,0,0,1,1\})+2*\epsilon*(1+y)*G(t_4,\{1,0,2,1,0,0,1\})\\
&\qquad +((1-z)*(G(t_4,\{2,0,1,0,0,0,2\})+G(t_4,\{2,0,2,0,0,0,1\})))/4)\\
&M_9 = 2*\epsilon^2*(1+y)*G(t_4,\{1,1,1,0,0,1,1\})\\
&M_{10} = 2*\epsilon^2*(1+y)^2*G(t_4,\{1,1,1,0,1,1,1\}).
\end{aligned}
\tag{6.8}
$$

Differential equations read in this basis (in the form of total differential)

$$
\begin{aligned}
dM_1 =&\epsilon(-6M_2\mathrm{dL}(z)+M_1(-4\mathrm{dL}(1-z)+2\mathrm{dL}(z)))\\
dM_2 =&\epsilon(-3M_2\mathrm{dL}(z)+M_1(-\mathrm{dL}(1-z)+\mathrm{dL}(z)))\\
dM_3 =&\epsilon(2M_2\mathrm{dL}(y)+M_3(2\mathrm{dL}(y)-\mathrm{dL}(1+y)-3\mathrm{dL}(y+z))+M_1(\mathrm{dL}(1-z)-\mathrm{dL}(y+z)))\\
dM_4 =&\epsilon(M_2(-6\mathrm{dL}(1+y)+6\mathrm{dL}(z)-6\mathrm{dL}(y+z))+M_4(-2\mathrm{dL}(1+y)+\mathrm{dL}(z)-2\mathrm{dL}(y+z))\\
&+M_1(2\mathrm{dL}(1+y)-2\mathrm{dL}(z)+\mathrm{dL}(y+z))+M_5(3\mathrm{dL}(1+y)-3\mathrm{dL}(z)+3\mathrm{dL}(y+z))\\
&+M_6(6\mathrm{dL}(1+y)-6\mathrm{dL}(z)+4\mathrm{dL}(y+z)))\\
dM_5 =&\epsilon(M_5(2\mathrm{dL}(y)-4\mathrm{dL}(1+y)-3\mathrm{dL}(z))+M_1(\mathrm{dL}(y)-2\mathrm{dL}(z))\\
&+M_6(2\mathrm{dL}(y)-4\mathrm{dL}(1+y)-2\mathrm{dL}(z))\\
&+M_2(-4\mathrm{dL}(y)+6\mathrm{dL}(z))+M_4(-\mathrm{dL}(y)+\mathrm{dL}(z)+\mathrm{dL}(1+y+z)))\\
dM_6 =&\epsilon(3M_5\mathrm{dL}(1+y)+M_1\mathrm{dL}(z)-3M_2\mathrm{dL}(z)\\
&+M_6(4\mathrm{dL}(1+y)-2\mathrm{dL}(z))-M_4\mathrm{dL}(1+y+z))\\
dM_7 =&\epsilon(M_6(-2\mathrm{dL}(1+y)+4\mathrm{dL}(1-z)+2\mathrm{dL}(z)-4\mathrm{dL}(y+z))+M_8(2\mathrm{dL}(1+y)-2\mathrm{dL}(y+z))\\
&+M_7(2\mathrm{dL}(y)-2\mathrm{dL}(1-z)-2\mathrm{dL}(y+z))+M_5(-2\mathrm{dL}(1+y)+4\mathrm{dL}(1-z)-2\mathrm{dL}(y+z))\\
&+M_1(-\mathrm{dL}(1+y)+2\mathrm{dL}(1-z)-\mathrm{dL}(y+z))+M_4(-2\mathrm{dL}(1-z)+2\mathrm{dL}(y+z))\\
&+M_3(-4\mathrm{dL}(1+y)+2\mathrm{dL}(1-z)+2\mathrm{dL}(y+z))+M_2(-6\mathrm{dL}(1-z)+6\mathrm{dL}(y+z)))\\
dM_8 =&\epsilon(M_7(2\mathrm{dL}(1+y)-2\mathrm{dL}(y+z))+M_5(-\mathrm{dL}(1+y)+\mathrm{dL}(y+z))\\
&+M_3(-2\mathrm{dL}(1+y)+2\mathrm{dL}(y+z))+M_4(-\mathrm{dL}(y)+\mathrm{dL}(1+y+z))\\
&+M_8(2\mathrm{dL}(y)-2\mathrm{dL}(y+z)-2\mathrm{dL}(1+2y+z)))\\
dM_9 =&\epsilon(-M_1\mathrm{dL}(y)+4M_2\mathrm{dL}(y)+M_4\mathrm{dL}(y)+M_6(-2\mathrm{dL}(y)-2\mathrm{dL}(1-z))\\
&+M_9(2\mathrm{dL}(1+y)-2\mathrm{dL}(1-z)-2\mathrm{dL}(y+z))+M_5(-2\mathrm{dL}(y)-2\mathrm{dL}(1-z)+\mathrm{dL}(y+z)))\\
dM_{10} =&\epsilon(-2M_{10}\mathrm{dL}(1+y)+M_5(-2\mathrm{dL}(1-z)+6\mathrm{dL}(z)-4\mathrm{dL}(y+z))\\
&+M_6(-2\mathrm{dL}(1-z)+6\mathrm{dL}(z)-4\mathrm{dL}(y+z))+M_1(-\mathrm{dL}(1-z)+2\mathrm{dL}(z)-\mathrm{dL}(y+z))\\
&+M_9(-2\mathrm{dL}(1-z)+2\mathrm{dL}(y+z))+M_4(-2\mathrm{dL}(z)+2\mathrm{dL}(y+z))+M_2(-6\mathrm{dL}(z)+6\mathrm{dL}(y+z)))
\end{aligned}
\tag{6.9}
$$

Canonical Differential Equations and Boundary Conditions for $\mathbf{b}+\mathbf{W}^{*}\to\mathbf{t}+\mathbf{g}$ at 1-loop.

The canonical basis is defined as

$$M_1 = Master_1(1+y)\epsilon$$
$$M_2 = Master_2(1+y)\epsilon$$
$$M_3 = \frac{1}{(-1+z)z}\Big[Master_4(-1+3\epsilon-2\epsilon^2)+z(Master_1(1+y)z\epsilon$$
$$+Master_3(1-5\epsilon+6\epsilon^2)+Master_2(-(1+y)\epsilon+z(-1+(5+y)\epsilon)))\Big]$$
$$M_4 = -\frac{Master_4(1-z)(1-3\epsilon+2\epsilon^2)}{(-1+z)^2z}$$
$$M_5 = Master_5(1+y)\epsilon^2$$
$$M_6 = \frac{y(-1+2\epsilon)(Master_4-Master_4\epsilon+Master_6z(-1+2\epsilon))}{(-1+z)z(y+z)}$$
$$M_7 = \frac{Master_7(1-5\epsilon+6\epsilon^2)}{2(-1+z)}$$
$$M_8 = -\frac{1}{2(-1+z)^2z(y+z)^2(1+yz)(1+2\epsilon)}(1-z)$$
$$\times\Big[2Master_4(1+y)(1-3\epsilon+2\epsilon^2)(-y(-1+z)z+z^2(-1+(-3+z)\epsilon)+y^2(z-\epsilon+3z\epsilon))$$
$$+z(1+2\epsilon)(Master_7(3y^3z+yz^2(8+z)+z^2(1+2z)+y^2(-1+6z+4z^2))(1-5\epsilon+6\epsilon^2)$$
$$+2(1+y)(-2Master_6y(1+y)z(1-2\epsilon)^2$$
$$+(-1+z)^2(y+z)(Master_8(1+y)z(y+z)(1+3\epsilon)$$
$$-2\epsilon(Master_9z(-y^2+z)+Master_{10}(y+2z+yz)\epsilon))))\Big]$$
$$M_9 = \frac{1}{(-1+z)z(y+z)}\Big[Master_4(y-z)(1-3\epsilon+2\epsilon^2)+z(-2Master_6y(1-2\epsilon)^2$$
$$+(y+z)(2(1+y)(-1+z)\epsilon(Master_9z+Master_{10}\epsilon)+Master_7(1-5\epsilon+6\epsilon^2)))\Big]$$
$$M_{10} = 2Master_{10}(1+y)\epsilon^2$$
$$M_{11} = Master_{11}(1+y)^2\epsilon^2$$
$$M_{12} = \frac{(1-z)(-((Master_{12}(1+z)(1-2\epsilon)^2)/(-1+z)^3)+(Master_4(1+z)(1-3\epsilon+2\epsilon^2)}{(-1+z)^3z(1+z)}$$
$$M_{13} = (1+y)\left[\frac{Master_{14}(-1-2y+z)}{y-z}+\frac{Master_{15}(y-yz)}{yz-z^2}+\frac{Master_{13}(1-3\epsilon)}{z}\right]\epsilon$$
$$M_{14} = (1+y)(1+y-z)\epsilon\left[\frac{Master_{13}(1-3\epsilon)}{z+yz}-\frac{Master_{15}y(-1+z)(z+\epsilon+y\epsilon-2z\epsilon)}{(1+y)^2(y-z)z\epsilon}\right.$$
$$\left.-\frac{Master_{14}(z+\epsilon+2y\epsilon-3z\epsilon)}{y\epsilon+y^2\epsilon-z\epsilon-yz\epsilon}+\frac{Master_4(-1+3\epsilon-2\epsilon^2)}{(1+y)^2(-1+z)z\epsilon}\right]$$
$$M_{15} = (1+y)\epsilon\left[\frac{Master_{13}(1+z)(-1+3\epsilon)}{(1+y)z}+\frac{Master_4(1-3\epsilon+2\epsilon^2)}{(1+y)^2z\epsilon}\right.$$
$$+\frac{Master_{15}y(-1+z)(z^2(1-2\epsilon)+(1+y)\epsilon+z(-2+5\epsilon+y(-1+3\epsilon)))}{(1+y)^2(y-z)z\epsilon}$$
$$\left.+\frac{Master_{14}(y(1+z)(-1+4\epsilon)-(-1+z)(\epsilon+z(-1+3\epsilon)))}{(1+y)(y-z)\epsilon}\right]$$
$$M_{16} = 2Master_{16}(1+y)\epsilon^2$$
$$M_{17} = Master_{17}(1+y)(y+z)\epsilon$$

$$M_{18} = 2(1+y)(1-z)\epsilon^2 \left[\frac{Master_{19}(-1+2\epsilon)}{(1+y)(-1+z)\epsilon} \right.$$

$$+ \frac{2Master_{18}y^2(-1+z)z}{(1+y)\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$+ \frac{Master_{17}(y^3-yz+y^2z-z^2)(y^2(1+\epsilon)+z(1+\epsilon)-y(-2+\epsilon+z(2+\epsilon)))}{(1+y)(-1+z)\epsilon(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$- \frac{Master_6 y(1-2\epsilon)^2(y^2(1+\epsilon)+z(1+\epsilon)-y(-1+\epsilon+z(3+\epsilon))}{(1+y)(-1+z)^2\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$+ Master_4(1-3\epsilon+2\epsilon^2) \times ($$

$$\frac{(-3z^2(1+\epsilon)+y^4(3+\epsilon)+2yz(-4+3\epsilon+z(3+\epsilon))+y^2(1+4z(-2+\epsilon)-3\epsilon+z^2(3+\epsilon))}{2(1+y)^3(-1+z)z\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$\left. + \frac{-2y^3(3(-1+\epsilon)+z(4+\epsilon)))}{2(1+y)^3(-1+z)z\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))} \right)$$

$$+ Master_{16} ($$

$$\frac{y^4\epsilon(1+\epsilon)+z^2\epsilon(1+\epsilon)-2y^3\epsilon(-1+z+\epsilon+z\epsilon)}{(1+y)(-1+z)\epsilon(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$\left. + \frac{-2yz\epsilon(-1+z+\epsilon+z\epsilon)+y^2(-1+\epsilon)(\epsilon+4z\epsilon+z^2(4+\epsilon)))}{(1+y)(-1+z)\epsilon(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))} \right)$$

$$- \frac{Master_{13}(-1+3\epsilon)}{2(1+y)^2(-1+z)z\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$\times \left[y^5\epsilon(1+\epsilon)+z^2(2+3z)\epsilon(1+\epsilon)-yz\epsilon(z+4(-1+\epsilon)+5z\epsilon+2z^2(3+\epsilon)) \right.$$

$$+ y^2(2(-1+\epsilon)\epsilon+4z^2(1+\epsilon)+z\epsilon(3+\epsilon)-z^3\epsilon(3+\epsilon))$$

$$\left. -y^4\epsilon(4\epsilon+z(7+3\epsilon))+y^3((-5+\epsilon)\epsilon+2z\epsilon(-1+3\epsilon)+z^2(4+9\epsilon+3\epsilon^2)) \right]$$

$$- \frac{Master_{15}y}{2(1+y)^3(y-z)z\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$\times \left[y^6\epsilon(1+\epsilon)-z^2(1+\epsilon)(z(8-19\epsilon)-2\epsilon+z^2(-3+6\epsilon))+y^5(\epsilon-3\epsilon^2+z(5-12\epsilon-13\epsilon^2)) \right.$$

$$+ y^2(2(-1+\epsilon)\epsilon+z^2(-17+54\epsilon-23\epsilon^2)+z^3(22-47\epsilon-15\epsilon^2)+z^4(-3+5\epsilon+2\epsilon^2)+z(4-5\epsilon+5\epsilon^2))$$

$$+ yz(-4(-1+\epsilon)\epsilon+z(-12+37\epsilon-27\epsilon^2)+2z^3(-3+5\epsilon+2\epsilon^2)+z^2(15-38\epsilon+11\epsilon^2))$$

$$+ y^3(\epsilon(-7+3\epsilon)+z^3(13-22\epsilon-15\epsilon^2)+z(19-44\epsilon+21\epsilon^2)+z^2(-26+71\epsilon+23\epsilon^2))$$

$$\left. +y^4(-\epsilon(5+3\epsilon)-3z(-6+13\epsilon+3\epsilon^2)+z^2(-11+28\epsilon+25\epsilon^2)) \right]$$

$$+ \frac{Master_{14}}{2(1+y)^2(y-z)(-1+z)\epsilon^2(y^3(1+\epsilon)-z^2(1+\epsilon)-y^2(-1+\epsilon+2z(1+\epsilon))+yz(2(-1+\epsilon)+z(3+\epsilon)))}$$

$$\times \left[-2y^6\epsilon(1+\epsilon)+(-1+z)z^2(1+\epsilon)(2\epsilon+z(-3+9\epsilon))+y^5(-3+2\epsilon+13\epsilon^2+z(-5+20\epsilon+17\epsilon^2)) \right.$$

$$+ y^3(-1+12\epsilon-7\epsilon^2+z(-9+14\epsilon-17\epsilon^2)+z^2(11-64\epsilon+\epsilon^2)+z^3(-9+34\epsilon+19\epsilon^2))$$

$$- yz(-4(-1+\epsilon)\epsilon+z(-11+22\epsilon+\epsilon^2)+2z^3(-3+8\epsilon+3\epsilon^2)+z^2(9-24\epsilon+23\epsilon^2))$$

$$- y^4(2(3-12\epsilon+\epsilon^2)+z(9-32\epsilon+19\epsilon^2)+z^2(-7+44\epsilon+31\epsilon^2))+y^2(-2(-1+\epsilon)\epsilon$$

$$\left. \left. +z^4(3-8\epsilon-3\epsilon^2)+z^3(-13+50\epsilon+11\epsilon^2)+z(3-22\epsilon+15\epsilon^2)+z^2(7-32\epsilon+33\epsilon^2)) \right] \right] + M_{17}$$

$$M_{19} = Master_{19}(1-2\epsilon)\epsilon$$

$$M_{20} = \frac{Master_{20}(1-5\epsilon+6\epsilon^2)}{2(-1+z)}$$

$$M_{21} = 2Master_{21}(1+y)\epsilon^2$$

$$
\begin{aligned}
M_{22} &= Master_{23}(1+y)(y+z)\epsilon \\
M_{23} &= 2(1+y)(1-z)\epsilon^2 \left[\frac{Master_{21}}{1-z} - \frac{Master_{12}(1-2\epsilon)^2}{(1+y)(-1+z)^3\epsilon^2} - \frac{Master_6 y(1-2\epsilon)^2}{(1+y)(-1+z)^2(y+z)\epsilon^2} + \frac{Master_{22} z}{\epsilon - z\epsilon} \right. \\
&\left. + \frac{Master_4(1-3\epsilon+2\epsilon^2)}{(-1+z)^3(y+z)\epsilon^2} + \frac{Master_{20}(1-5\epsilon+6\epsilon^2)}{(1+y)(-1+z)^2\epsilon^2} \right] \\
M_{24} &= Master_{24}(1+y)^2(1-z)\epsilon^2
\end{aligned}
\tag{6.10}
$$

The canonical DE system reads then

$$
\begin{aligned}
dM_1 =&(\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(z))M_1 + (2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - \epsilon \mathrm{dL}(z))M_2 \\
&+ (-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(1+y-z) + \epsilon \mathrm{dL}(z))M_3 + \epsilon \mathrm{dL}(z)M_4 \\
dM_2 =& -\epsilon textdL(y)M_1 + (-2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 3\epsilon \mathrm{dL}(z))M_2 + (\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(1+y-z) + \epsilon \mathrm{dL}(z))M_3 \\
dM_3 =&(-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(1-z))M_1 + (-2\epsilon \mathrm{dL}(y) + 3\epsilon \mathrm{dL}(1+y) + 2\epsilon \mathrm{dL}(1-z) - 3\epsilon \mathrm{dL}(z))M_2 \\
&+ (\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z))M_3 + (-\epsilon \mathrm{dL}(1+y) + \epsilon \mathrm{dL}(z))M_4 \\
dM_4 =&(-2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_4 \\
dM_5 =&(\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(y+z))M_1 + (2\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(y+z))M_2 - \epsilon \mathrm{dL}(y)M_3 \\
&+ (2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(y+z))M_5 + \epsilon \mathrm{dL}(y)M_6 \\
dM_6 =&(\epsilon \mathrm{dL}(z) - \epsilon \mathrm{dL}(y+z))M_4 + (\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(y+z))M_6 \\
dM_7 =&(-\epsilon \mathrm{dL}(1+y) - 3\epsilon \mathrm{dL}(1-z))M_7 \\
dM_8 =&(2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z))M_4 + (-2\epsilon \mathrm{dL}(y) + 4\epsilon \mathrm{dL}(1+y) - 4\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_6 \\
&+ (-3\epsilon \mathrm{dL}(1+y) + 3\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(z))M_7 - 4\epsilon \mathrm{dL}(1-z)M_8 \\
&+ (2\epsilon \mathrm{dL}(1+y) + \epsilon \mathrm{dL}(z) - 2\epsilon \mathrm{dL}(y+z))M_9 - \epsilon \mathrm{dL}(z)M_{10} \\
dM_9 =&(\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z))M_4 + (-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_7 + (-\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z))M_8 \\
&+ (2\epsilon \mathrm{dL}(y) - 4\epsilon \mathrm{dL}(y+z))M_9 + (\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z))M_{10} \\
dM_{10} =&(-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_4 + (-2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(z))M_6 + (\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z))M_7 + (\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_8 \\
&+ (-2\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_9 + (-\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(z))M_{10} \\
dM_{11} =&(2\epsilon \mathrm{dL}(1-z) - 3\epsilon \mathrm{dL}(z) + \epsilon \mathrm{dL}(y+z))M_1 + (2\epsilon \mathrm{dL}(1-z) - 3\epsilon \mathrm{dL}(z) + \epsilon \mathrm{dL}(y+z))M_2 \\
&+ (-2\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_3 + (-\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(y+z))M_4 + (-2\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(y+z))M_5 \\
&+ (-2\epsilon \mathrm{dL}(z) + 2\epsilon \mathrm{dL}(y+z))M_6 + (-3\epsilon \mathrm{dL}(1-z) + 3\epsilon \mathrm{dL}(z))M_7 + (-\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z))M_8 \\
&+ (-\epsilon \mathrm{dL}(z) + \epsilon \mathrm{dL}(y+z))M_9 - 2\epsilon \mathrm{dL}(1+y)M_{11} \\
dM_{12} =&(\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_4 - 4\epsilon \mathrm{dL}(1-z)M_{12} \\
dM_{13} =&(-2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 3\epsilon \mathrm{dL}(z))M_{13} + (\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(1+y-z) + \epsilon \mathrm{dL}(z))M_{14} - \epsilon \mathrm{dL}(y)M_{15} \\
dM_{14} =&(-\epsilon \mathrm{dL}(1+y) + \epsilon \mathrm{dL}(z))M_4 + (-2\epsilon \mathrm{dL}(y) + 3\epsilon \mathrm{dL}(1+y) + 2\epsilon \mathrm{dL}(1-z) - 3\epsilon \mathrm{dL}(z))M_{13} \\
&+ (\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z))M_{14} + (-\epsilon \mathrm{dL}(y) + \epsilon textdL(1-z))M_{15} \\
dM_{15} =&\epsilon \mathrm{dL}(z)M_4 + (2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - \epsilon \mathrm{dL}(z))M_{13} + (-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(1+y-z) + \epsilon \mathrm{dL}(z))M_{14} \\
&+ (\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(z))M_{15} \\
dM_{16} =&(\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_{13} + (3\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z))M_{15} \\
&+ (-\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_{16} + (\epsilon \mathrm{dL}(y) \\
&- \epsilon \mathrm{dL}(z))M_{17} + \epsilon \mathrm{dL}(y)M_{18} - 2\epsilon \mathrm{dL}(y)M_{19} \\
dM_{17} =& -\epsilon \mathrm{dL}(z)M_4 - 2\epsilon \mathrm{dL}(y)M_6 + (-4\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + 4\epsilon \mathrm{dL}(z))M_{13} \\
&+ (2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(z))M_{14} + (-2\epsilon \mathrm{dL}(y) + 4\epsilon \mathrm{dL}(1+y) - 4\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_{15} \\
&+ (2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(y+z))M_{17} \\
dM_{18} =& -2\epsilon \mathrm{dL}(z)M_4 + (-2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(z))M_6 + 2\epsilon \mathrm{dL}(z)M_{12} \\
&+ (-4\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(1-z) + 4\epsilon \mathrm{dL}(z))M_{13} \\
&+ (2\epsilon \mathrm{dL}(y) - 4\epsilon \mathrm{dL}(z))M_{14} + (-2\epsilon \mathrm{dL}(y) + 4\epsilon \mathrm{dL}(1+y) \\
&- 4\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_{15} + (-2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_{18}
\end{aligned}
$$

$$
\begin{aligned}
dM_{19} =& \epsilon \mathrm{dL}(z)M_{12} + (-2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_{19} \\
dM_{24} =& (\epsilon \mathrm{dL}(1+y) + \epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z) - \epsilon \mathrm{dL}(y+z))M_4 + (2\epsilon \mathrm{dL}(z) - 2\epsilon \mathrm{dL}(y+z))M_6 \\
&+ (-2\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_{12} + (-\epsilon \mathrm{dL}(1+y) + \epsilon \mathrm{dL}(z))M_{13} + (2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(z))M_{14} \\
&+ (\epsilon \mathrm{dL}(1+y) - \epsilon \mathrm{dL}(z))M_{15} + (-\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(y+z))M_{17} + (\epsilon \mathrm{dL}(1-z) \\
&- \epsilon \mathrm{dL}(z))M_{18} - 2\epsilon \mathrm{dL}(1-z)M_{24} \\
dM_{20} =& (-\epsilon \mathrm{dL}(1+y) - 3\epsilon \mathrm{dL}(1-z))M_{20} \\
dM_{21} =& \epsilon \mathrm{dL}(y)M_4 - 2\epsilon \mathrm{dL}(z)M_6 + (-2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(z))M_{12} + (4\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(z))M_{20} \\
&+ (-\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1+y) - 2\epsilon \mathrm{dL}(z))M_{21} + (2\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z))M_{22} + (\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_{23} \\
dM_{22} =& (\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(y+z))M_4 + (4\epsilon \mathrm{dL}(1+y) + 2\epsilon \mathrm{dL}(z) - 4\epsilon \mathrm{dL}(y+z))M_6 \\
&+ (-2\epsilon \mathrm{dL}(y) + 4\epsilon \mathrm{dL}(1+y) - 4\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z))M_{12} \\
&+ (4\epsilon \mathrm{dL}(y) - 6\epsilon \mathrm{dL}(1+y) + 6\epsilon \mathrm{dL}(1-z) - 4\epsilon \mathrm{dL}(z))M_{20} \\
&+ (-\epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z))M_{21} + (2\epsilon \mathrm{dL}(y) - 4\epsilon \mathrm{dL}(y+z))M_{22} \\
&+ (\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1+y) + 2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z))M_{23} \\
dM_{23} =& -2\epsilon \mathrm{dL}(z)M_{20} - \epsilon \mathrm{dL}(z)M_{21} + (-2\epsilon \mathrm{dL}(1+y) - \epsilon \mathrm{dL}(z) + 2\epsilon \mathrm{dL}(y+z))M_{22} - 4\epsilon \mathrm{dL}(1-z)M_{23}.
\end{aligned}
\tag{6.11}
$$

Canonical Differential Equations and Boundary Conditions for $g + W^* \to t + \bar{b} + g$.

We report in the following, for the all three topologies t_1^G, t_2^G, t_3^G, the systems of differential equations in canonical form after remappings, and the expansions in terms of soft masters, which determines the boundary conditions.

$$
\begin{aligned}
M_1^G =& (1-z)(2\ G(t_1^G, \{2,0,1,0,0,0,2\}) + G(t_1^G, \{2,0,2,0,0,0,1\})) \\
M_2^G =& G(t_1^G, \{2,0,1,0,0,0,2\}) \\
M_3^G =& 2\ (y+1)\epsilon G(t_1^G, \{1,0,2,1,0,0,1\}) \\
M_4^G =& 2(2\epsilon(y+z+1)\ (G(t_1^G, \{1,0,1,0,1,0,2\}) - G(t_1^G, \{1,0,2,0,1,0,1\})) \\
&+ 3\epsilon(y+z+1)\ G(t_1^G, \{1,0,2,0,1,0,1\}) + z(y+z+1)G(t_1^G, \{1,0,1,0,2,0,2\})) \\
M_5^G =& 2(y+1)\epsilon\ G(t_1^G, \{1,0,2,0,1,0,1\}) \\
M_6^G =& (y+1)\epsilon\ (G(t_1^G, \{1,0,1,0,1,0,2\}) - G(t_1^G, \{1,0,2,0,1,0,1\})) \\
M_7^G =& 2y(y+1)\epsilon\ G(t_1^G, \{1,0,1,1,1,0,2\}) \\
M_8^G =& 2\left(y(y+1)\epsilon\ G(t_1^G, \{1,0,1,1,2,0,1\}) + \frac{1}{2}(y+1)\epsilon\right. \\
&(G(t_1^G, \{1,0,1,0,1,0,2\}) - G(t_1^G, \{1,0,2,0,1,0,1\})) + (y+1)\epsilon \\
&G(t_1^G, \{1,0,2,0,1,0,1\}) + 2(y+1)\epsilon\ G(t_1^G, \{1,0,2,1,0,0,1\}) + \frac{1}{4}(1-z) \\
&\left.(G(t_1^G, \{2,0,1,0,0,0,2\}) + G(t_1^G, \{2,0,2,0,0,0,1\}))\right) \\
M_9^G =& 2(y+1)(1-2\ \epsilon)\epsilon G(t_1^G, \{1,0,1,1,0,1,1\}) \\
M_{10}^G =& 4(y+1)\epsilon^2\ G(t_1^G, \{1,0,1,0,1,1,1\}) \\
M_{11}^G =& 4(y+1)z\epsilon G(t_1^G, \{1,0,1,0,2,1,1\}) \\
M_{12}^G =& -2\ (y+1)\epsilon^2 \sqrt{4y^2+4y+z^2-2z+1}\ G(t_1^G, , \{1,0,1,1,1,1,1\})
\end{aligned}
$$

$$
M_{13}^G = (y+1)^2\epsilon^2(2y+z+1)\ G(t_1^G,,\{1,1,1,1,1,1,1\}) \tag{6.12}
$$

$$
\begin{aligned}
M_{14}^G =& (1-z)(2\ G(t_2^G,\{2,1,0,0,0,0,2\}) + G(t_2^G,\{2,2,0,0,0,0,1\}))\\
M_{15}^G =& G(t_2^G,\{2,1,0,0,0,0,2\})\\
M_{16}^G =& 2\ (y+1)\epsilon G(t_2^G,\{1,2,1,0,0,0,1\})\\
M_{17}^G =& 2(2\epsilon(y+z+1)\ (G(t_2^G,\{1,1,0,0,1,0,2\}) - G(t_2^G,\{1,2,0,0,1,0,1\}))\\
&+ 3\epsilon(y+z+1)\ G(t_2^G,\{1,2,0,0,1,0,1\}) + z(y+z+1)G(t_2^G,\{1,1,0,0,2,0,2\}))\\
M_{18}^G =& 2(y+1)\epsilon\ G(t_2^G,\{1,2,0,0,1,0,1\})\\
M_{19}^G =& (y+1)\epsilon\ (G(t_2^G,\{1,1,0,0,1,0,2\}) - G(t_2^G,\{1,2,0,0,1,0,1\}))\\
M_{20}^G =& 2y(y+1)\epsilon\ G(t_2^G,\{1,1,1,0,1,0,2\})\\
M_{21}^G =& 2\left(y(y+1)\epsilon\ G(t_2^G,\{1,1,1,0,2,0,1\}) + \frac{1}{2}(y+1)\epsilon\ (G(t_2^G,\{1,1,0,0,1,0,2\})\right.\\
&- G(t_2^G,\{1,2,0,0,1,0,1\})) + (y+1)\epsilon\ G(t_2^G,\{1,2,0,0,1,0,1\}) + 2(y+1)\epsilon\\
&\left. G(t_2^G,\{1,2,1,0,0,0,1\}) + \frac{1}{4}(1-z)(2\ G(t_2^G,\{2,1,0,0,0,0,2\}) + G(t_2^G,\{2,2,0,0,0,0,1\}))\right)\\
M_{22}^G =& 2(y+1)\epsilon^2\ G(t_2^G,\{1,1,1,0,0,1,1\})\\
M_{23}^G =& 2(y+1)z\epsilon G(t_2^G,\{1,1,1,0,0,2,1\})\\
M_{24}^G =& 2(y+1)\ z\epsilon G(t_2^G,\{1,1,1,0,0,1,2\})\\
M_{25}^G =& \frac{4(y+1)\epsilon^2\sqrt{4y\ z+z^2+2z+1}G(t_2^G,\{1,1,1,0,1,1,1\})}{1-z}\\
M_{26}^G =& 2(y+1)^2\epsilon^2\ G(t_2^G,\{1,1,1,1,0,1,1\})
\end{aligned} \tag{6.13}
$$

$$
\begin{aligned}
M_{27}^G =& (1-z)(2\ G(t_3^G,\{2,0,0,1,0,0,2\}) + G(t_3^G,\{2,0,0,2,0,0,1\}))\\
M_{28}^G =& G(t_3^G,\{2,0,0,1,0,0,2\})\\
M_{29}^G =& 4\ (y+1)\epsilon^2 G(t_3^G,\{1,0,0,1,1,1,1\})\\
M_{30}^G =& 2(y+1)z\epsilon\ G(t_3^G,\{1,0,0,1,1,2,1\})\\
M_{31}^G =& 2(y+1)^2(1-z)\epsilon^2 G(t_3^G,\{1,1,1,1,1,1,1\})
\end{aligned} \tag{6.14}
$$

Systems of differential equations in canonical form:

- t_1^G

$$
\begin{aligned}
dM_1 =& \epsilon\,(2M_1(-2\mathrm{dL}(c-2d) + \mathrm{dL}(c-d+1) + \mathrm{dL}(d+1)) + 6M_2(\mathrm{dL}(c-d+1) - \mathrm{dL}(d+1)))\\
dM_2 =& \epsilon\,(M_1(\mathrm{dL}(d+1) - \mathrm{dL}(c-2d)) + 3M_2(\mathrm{dL}(c-d+1) - \mathrm{dL}(d+1)))\\
dM_3 =& \epsilon\,(M_1(\mathrm{dL}(c-2d) - \mathrm{dL}(c+(d-2)d) + \mathrm{dL}(d-1))\\
&+ 2M_2(\mathrm{dL}(c-2d+1) - \mathrm{dL}(c-d+1) - \mathrm{dL}(d-1))\\
&- M_3(-2\mathrm{dL}(c-2d+1) + \mathrm{dL}(c-d) - 2\mathrm{dL}(c-d+1) + 3\mathrm{dL}(c+(d-2)d) - 2\mathrm{dL}(d-1) + \mathrm{dL}(d)))
\end{aligned}
$$

$$
\begin{aligned}
dM_4 =&\epsilon\,(M_1(2\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)+\mathrm{dL}(c+(d-2)d)-3\mathrm{dL}(d-1)+2\mathrm{dL}(d)-2\mathrm{dL}(d+1))\\
&-6M_2(\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)+\mathrm{dL}(c+(d-2)d)-2\mathrm{dL}(d-1)+\mathrm{dL}(d)-\mathrm{dL}(d+1))\\
&+M_4(-2\mathrm{dL}(c-d)+3\mathrm{dL}(c-d+1)-2\mathrm{dL}(c+(d-2)d)+4\mathrm{dL}(d-1)-2\mathrm{dL}(d)+\mathrm{dL}(d+1))\\
&+3M_5(\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)+\mathrm{dL}(c+(d-2)d)-2\mathrm{dL}(d-1)+\mathrm{dL}(d)-\mathrm{dL}(d+1))\\
&+2M_6(3\mathrm{dL}(c-d)-2\mathrm{dL}(c-d+1)+2\mathrm{dL}(c+(d-2)d)-5\mathrm{dL}(d-1)+3\mathrm{dL}(d)-3\mathrm{dL}(d+1)))\\
dM_5 =&\epsilon\,(M_1(\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d+1)-\mathrm{dL}(d-1)-2\mathrm{dL}(d+1))\\
&-2M_2(2\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d+1)-2\mathrm{dL}(d-1)-3\mathrm{dL}(d+1))\\
&+M_4(-\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d+1)+\mathrm{dL}(cd-1)+\mathrm{dL}(d+1))\\
&+M_5(2\mathrm{dL}(c-2d+1)-4\mathrm{dL}(c-d)+5\mathrm{dL}(c-d+1)+2\mathrm{dL}(d-1)-4\mathrm{dL}(d)-3\mathrm{dL}(d+1))\\
&+2M_6(\mathrm{dL}(c-2d+1)-2\mathrm{dL}(c-d)+2\mathrm{dL}(c-d+1)+\mathrm{dL}(d-1)-2\mathrm{dL}(d)-\mathrm{dL}(d+1)))\\
dM_6 =&\epsilon\,(M_1(\mathrm{dL}(d+1)-\mathrm{dL}(c-d+1))+3M_2(\mathrm{dL}(c-d+1)-\mathrm{dL}(d+1))\\
&+M_4(\mathrm{dL}(c-d+1)-\mathrm{dL}(cd-1)+\mathrm{dL}(d-1))\\
&+3M_5(\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)-\mathrm{dL}(d-1)+\mathrm{dL}(d))\\
&+2M_6(2\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)-2\mathrm{dL}(d-1)+2\mathrm{dL}(d)-\mathrm{dL}(d+1)))\\
dM_7 =&\epsilon\,(-M_1(-2\mathrm{dL}(c-2d)+\mathrm{dL}(c-d)+\mathrm{dL}(c+(d-2)d)-2\mathrm{dL}(d-1)+\mathrm{dL}(d))\\
&-6M_2(\mathrm{dL}(c-2d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d-1))\\
&+2M_3(\mathrm{dL}(c-2d)-2\mathrm{dL}(c-d)+\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d-1)-2\mathrm{dL}(d))\\
&-2M_4(\mathrm{dL}(c-2d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d-1))\\
&-2M_5(-2\mathrm{dL}(c-2d)+\mathrm{dL}(c-d)+\mathrm{dL}(c+(d-2)d)-2\mathrm{dL}(d-1)+\mathrm{dL}(d))\\
&-2M_6(-2\mathrm{dL}(c-2d)+\mathrm{dL}(c-d)+2\mathrm{dL}(c+(d-2)d)-3\mathrm{dL}(d-1)+\mathrm{dL}(d)-\mathrm{dL}(d+1))\\
&+2M_7(-\mathrm{dL}(c-2d)+\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d+1)-\mathrm{dL}(c+(d-2)d))\\
&+2M_8(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d)))\\
dM_8 =&\epsilon\,(-2M_3(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+M_4(\mathrm{dL}(cd-1)-\mathrm{dL}(c-2d+1))-M_5(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+2M_7(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+2M_8(\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d+1)-\mathrm{dL}(cd+c-2d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d-1)))\\
dM_9 =&\epsilon\,(M_1(-\mathrm{dL}(c-d)+\mathrm{dL}(d-1)-\mathrm{dL}(d)+\mathrm{dL}(d+1))\\
&-2M_2(\mathrm{dL}(c-2d+1)-2\mathrm{dL}(c-d)+\mathrm{dL}(d-1)-2\mathrm{dL}(d)+\mathrm{dL}(d+1))\\
&+2M_3(-\mathrm{dL}(c-2d+1)+\mathrm{dL}(d-1)+\mathrm{dL}(d+1))\\
&-M_9(\mathrm{dL}(c-d)-2\mathrm{dL}(c-d+1)-\mathrm{dL}(d-1)+\mathrm{dL}(d)+\mathrm{dL}(d+1)))\\
dM_{10} =&\epsilon\,(M_1(-\mathrm{dL}(c-2d+1)+\mathrm{dL}(d-1)+\mathrm{dL}(d+1))\\
&-M_{10}(\mathrm{dL}(c-2d+1)-2\mathrm{dL}(c-d+1)-\mathrm{dL}(d-1)+\mathrm{dL}(d+1))\\
&+M_{11}(-\mathrm{dL}(c-2d+1)+\mathrm{dL}(d-1)+\mathrm{dL}(d+1))\\
&+4M_2(\mathrm{dL}(c-2d+1)-\mathrm{dL}(d-1)-\mathrm{dL}(d+1))\\
&+2M_5(\mathrm{dL}(c-d)-\mathrm{dL}(d-1)+\mathrm{dL}(d)-\mathrm{dL}(d+1))\\
&-2M_6(\mathrm{dL}(c-2d+1)-2\mathrm{dL}(c-d)+\mathrm{dL}(d-1)-2\mathrm{dL}(d)+\mathrm{dL}(d+1)))\\
dM_{11} =&\epsilon\,(4M_1(\mathrm{dL}(c-2d+1)-\mathrm{dL}(c+(d-2)d))\\
&+2M_{10}(\mathrm{dL}(c-2d+1)-\mathrm{dL}(c+(d-2)d))\\
&+2M_{11}(\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d+1)-\mathrm{dL}(c+(d-2)d)-\mathrm{dL}(d+1))\\
&+16M_2(\mathrm{dL}(c+(d-2)d)-\mathrm{dL}(c-2d+1))+2M_4(\mathrm{dL}(cd-1)-\mathrm{dL}(c-2d+1))\\
&+2M_5(2\mathrm{dL}(c-2d+1)-3\mathrm{dL}(c-d)+\mathrm{dL}(c+(d-2)d)-3\mathrm{dL}(d))\\
&+8M_6(\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d)-\mathrm{dL}(d)))\\
dM_{12} =&\epsilon\left(-2M_{12}\left(-\mathrm{dL}\left(cd+c-2d^2\right)+\mathrm{dL}(c-d)-\mathrm{dL}(c-d+1)+\mathrm{dL}(d)+\mathrm{dL}(d+1)\right)\right.\\
&+\frac{1}{4}M_1(6\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d)-7\mathrm{dL}(d-c)+2\mathrm{dL}(d-1)+8\mathrm{dL}(d)-2\mathrm{dL}(d+1))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}M_{10}(2\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d)-3\mathrm{dL}(d-c)+2\mathrm{dL}(d-1)+4\mathrm{dL}(d)-2\mathrm{dL}(d+1))\\
&+\frac{1}{4}M_{11}(2\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d)-3\mathrm{dL}(d-c)+2\mathrm{dL}(d-1)+4\mathrm{dL}(d)-2\mathrm{dL}(d+1))\\
&-4M_2(\mathrm{dL}(c-2d+1)-\mathrm{dL}(d-c)+\mathrm{dL}(d))\\
&+M_3(2\mathrm{dL}(c-2d+1)-\mathrm{dL}(c-d)-3\mathrm{dL}(d-c)+2\mathrm{dL}(d-1)+4\mathrm{dL}(d)-2\mathrm{dL}(d+1))\\
&-\frac{1}{2}M_4(2\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d)-\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)+2\mathrm{dL}(d+1))\\
&+M_5(2\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d)-\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)+2\mathrm{dL}(d+1))\\
&+\frac{3}{2}M_6(2\mathrm{dL}(c-2d+1)+\mathrm{dL}(c-d)-\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)+2\mathrm{dL}(d+1))\\
&-2M_7(\mathrm{dL}(c-2d+1)-\mathrm{dL}(d-c)+\mathrm{dL}(d))\\
&+M_8(\mathrm{dL}(c-d)+\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)-2\mathrm{dL}(d)+2\mathrm{dL}(d+1))\\
&+\frac{1}{2}M_9(\mathrm{dL}(c-d)+\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)-2\mathrm{dL}(d)+2\mathrm{dL}(d+1))\Big)\\
dM_{13}=&\epsilon\,(2M_1(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+M_{10}(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+M_{11}(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+M_{12}(\mathrm{dL}(c-d)+\mathrm{dL}(d-c)-2\mathrm{dL}(d-1)-2\mathrm{dL}(d)+2\mathrm{dL}(d+1))\\
&+2M_{13}(\mathrm{dL}(c-d+1)-\mathrm{dL}(cd+c-2d)+\mathrm{dL}(d-1))\\
&-8M_2(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&+2M_3(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&-2M_7(\mathrm{dL}(c-d)-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d))\\
&-2M_8(-\mathrm{dL}(c+(d-2)d)+\mathrm{dL}(d-1)+\mathrm{dL}(d+1)))
\end{aligned}
\tag{6.15}
$$

- t_2^G

$$
\begin{aligned}
dM_{14}=&\epsilon\,(2M_{14}(-2\mathrm{dL}(b-2a)+\mathrm{dL}(b-a)+\mathrm{dL}(a))-6M_{15}(\mathrm{dL}(a)-\mathrm{dL}(b-a)))\\
dM_{15}=&\epsilon\,(M_{14}(\mathrm{dL}(a)-\mathrm{dL}(b-2a))-3M_{15}(\mathrm{dL}(a)-\mathrm{dL}(b-a)))\\
dM_{16}=&\epsilon\left(M_{14}\left(\mathrm{dL}(b-2a)-\mathrm{dL}\left((a+1)b^2+a\right)\right)\right.\\
&-M_{16}\left(3\mathrm{dL}\left((a+1)b^2+a\right)-2\mathrm{dL}(b-a)+\mathrm{dL}(a(b-1)+b)-2\mathrm{dL}(a+1)-4\mathrm{dL}(b)+\mathrm{dL}(b+1)\right)\\
&+2M_{15}(-\mathrm{dL}(b-a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b)))\\
dM_{17}=&\epsilon\left(-M_{14}\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(b-a)-2\mathrm{dL}(a(b-1)+b)+2\mathrm{dL}(a)-2\mathrm{dL}(b+1)\right)\right.\\
&-6M_{15}\left(\mathrm{dL}\left((a+1)b^2+a\right)-\mathrm{dL}(b-a)+\mathrm{dL}(a(b-1)+b)-\mathrm{dL}(a)+\mathrm{dL}(b+1)\right)\\
&+M_{17}\left(-2\mathrm{dL}\left((a+1)b^2+a\right)+3\mathrm{dL}(b-a)-2\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)-2\mathrm{dL}(b+1)\right)\\
&+3M_{18}\left(\mathrm{dL}\left((a+1)b^2+a\right)-\mathrm{dL}(b-a)+\mathrm{dL}(a(b-1)+b)-\mathrm{dL}(a)+\mathrm{dL}(b+1)\right)\\
&\left.-2M_{19}\left(-2\mathrm{dL}\left((a+1)b^2+a\right)+2\mathrm{dL}(b-a)-3\mathrm{dL}(a(b-1)+b)+3\mathrm{dL}(a)-3\mathrm{dL}(b+1)\right)\right)\\
dM_{18}=&\epsilon\,(M_{14}(\mathrm{dL}(b-a)-2\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b))\\
&+2M_{15}(-\mathrm{dL}(b-a)+3\mathrm{dL}(a)-2\mathrm{dL}(a+1)-4\mathrm{dL}(b))\\
&+M_{17}(-\mathrm{dL}(b-a)+\mathrm{dL}(ab+b+1)+\mathrm{dL}(a)-\mathrm{dL}(a+1)-\mathrm{dL}(b))\\
&-M_{18}(-5\mathrm{dL}(b-a)+4\mathrm{dL}(a(b-1)+b)+3\mathrm{dL}(a)-2\mathrm{dL}(a+1)-4\mathrm{dL}(b)+4\mathrm{dL}(b+1))\\
&+2M_{19}(2(\mathrm{dL}(b-a)-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b)-\mathrm{dL}(b+1))-\mathrm{dL}(a)+\mathrm{dL}(a+1)))
\end{aligned}
$$

$$
\begin{aligned}
dM_{19} =& \epsilon \left(M_{14}(\mathrm{dL}(a) - \mathrm{dL}(b-a)) - 3M_{15}(\mathrm{dL}(a) - \mathrm{dL}(b-a)) \right. \\
& - M_{17}(-\mathrm{dL}(b-a) + \mathrm{dL}(ab+b+1) + \mathrm{dL}(b)) \\
& + 3M_{18}(-\mathrm{dL}(b-a) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1)) \\
& \left. -2M_{19}(\mathrm{dL}(b-a) - 2\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) - 2\mathrm{dL}(b+1)) \right) \\
dM_{20} =& \epsilon \Big(-M_{14} \Big(\mathrm{dL}\left((a+1)b^2+a\right) - 2\mathrm{dL}(b-2a) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \\
& - 6M_{15} \Big(\mathrm{dL}(b-2a) - \mathrm{dL}\left((a+1)b^2+a\right) \Big) \\
& + 2M_{16} \Big(\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(b-2a) - 2\mathrm{dL}(a(b-1)+b) - 2\mathrm{dL}(b+1) \Big) \\
& - 2M_{17} \Big(\mathrm{dL}(b-2a) - \mathrm{dL}\left((a+1)b^2+a\right) \Big) \\
& - 2M_{18} \Big(\mathrm{dL}\left((a+1)b^2+a\right) - 2\mathrm{dL}(b-2a) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \\
& + 2M_{19} \Big(-2\mathrm{dL}\left((a+1)b^2+a\right) + 2\mathrm{dL}(b-2a) - \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) - \mathrm{dL}(b+1) \Big) \\
& + 2M_{20} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) - \mathrm{dL}(b-2a) + \mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) \Big) \\
& +2M_{21} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \Big) \\
dM_{21} =& \epsilon \Big(-2M_{16} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \\
& - M_{18} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \\
& + 2M_{20} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1) \Big) \\
& + 2M_{21} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(b-a) - \mathrm{dL}(2(a+1)b+1) + \mathrm{dL}(a+1) + \mathrm{dL}(b) \Big) \\
& -M_{17}(-\mathrm{dL}(ab+b+1) + \mathrm{dL}(a+1) + \mathrm{dL}(b)) \Big) \\
dM_{22} =& \epsilon \left(M_{15}(-\mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b)) \right. \\
& + 2M_{16}(-\mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b)) \\
& - 2M_{22}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) - \mathrm{dL}(b+1)) \\
& + M_{23}(-2\mathrm{dL}(b-a) + \mathrm{dL}(a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b)) \\
& \left. +M_{24}(\mathrm{dL}(b-a) + \mathrm{dL}(a) - 2\mathrm{dL}(a+1) - 4\mathrm{dL}(b)) \right) \\
dM_{23} =& \epsilon \Big(M_{14} \Big(\mathrm{dL}\left((a+1)b^2+a\right) - \mathrm{dL}(b-2a) \Big) \\
& - 4M_{16} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(b-2a) - \mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) \Big) \\
& + 2M_{22} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) - 2\mathrm{dL}(b-2a) + 2\mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) \Big) \\
& + 2M_{24} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) - \mathrm{dL}(b-a) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) + \mathrm{dL}(b+1) \Big) \\
& - 3M_{15}(-\mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b)) \\
& -M_{23}(4\mathrm{dL}(b-2a) - 6\mathrm{dL}(b-a) + \mathrm{dL}(a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b)) \Big) \\
dM_{24} =& \epsilon \Big(-2M_{16} \Big(-\mathrm{dL}\left((a+1)b^2+a\right) + \mathrm{dL}(b-2a) - \mathrm{dL}(b-a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) \Big) \\
& + 2M_{22} \Big(-2\mathrm{dL}\left((a+1)b^2+a\right) - \mathrm{dL}(b-2a) + \mathrm{dL}(b-a) + 2\mathrm{dL}(a+1) + 4\mathrm{dL}(b) \Big) \\
& - M_{24} \Big(4\mathrm{dL}\left((a+1)b^2+a\right) - \mathrm{dL}(b-a) + \mathrm{dL}(a) - 4\mathrm{dL}(a+1) - 8\mathrm{dL}(b) \Big) \\
& -2M_{23}(\mathrm{dL}(b-2a) - \mathrm{dL}(b-a) - \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b) - \mathrm{dL}(b+1)) \Big) \\
dM_{25} =& \epsilon \left(2M_{14}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) + \mathrm{dL}(b+1)) \right. \\
& + 8M_{16}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + \mathrm{dL}(b+1)) \\
& + 2M_{18}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) + \mathrm{dL}(b+1)) \\
& - 4M_{20}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) + \mathrm{dL}(b+1))
\end{aligned}
$$

$$
\begin{aligned}
&- 4M_{21}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + \mathrm{dL}(b+1)) \\
&- 4M_{22}(-2\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a) + \mathrm{dL}(a+1) + 2\mathrm{dL}(b+1)) \\
&+ 2M_{23}(2\mathrm{dL}(a(b-1)+b) - 3\mathrm{dL}(a) + \mathrm{dL}(a+1) - 2\mathrm{dL}(b+1)) \\
&- 4M_{24}(-\mathrm{dL}(a(b-1)+b) + \mathrm{dL}(a+1) + \mathrm{dL}(b+1)) \\
&+ 2M_{25}(\mathrm{dL}(b-a) - \mathrm{dL}(a(b-1)+b) - \mathrm{dL}(a) + \mathrm{dL}(2a+1) + \mathrm{dL}(b) - \mathrm{dL}(b+1)) \\
&-6M_{15}(\mathrm{dL}(a) - \mathrm{dL}(a+1)) - 2M_{17}(\mathrm{dL}(a) - \mathrm{dL}(a+1))) \\
dM_{26} =& \epsilon \left(M_{14} \left(\mathrm{dL}\left((a+1)b^2 + a \right) - \mathrm{dL}(b-2a) \right) \right. \\
&- 2M_{16} \left(\mathrm{dL}(b-2a) - \mathrm{dL}\left((a+1)b^2 + a \right) \right) \\
&- 2M_{22} \left(\mathrm{dL}(b-2a) - \mathrm{dL}\left((a+1)b^2 + a \right) \right) \\
&- 2M_{24} \left(\mathrm{dL}(a) - \mathrm{dL}\left((a+1)b^2 + a \right) \right) \\
&\left. +2M_{23}(\mathrm{dL}(a) - \mathrm{dL}(b-2a)) - 2M_{26}(-\mathrm{dL}(b-a) + \mathrm{dL}(a(b-1)+b) + \mathrm{dL}(b+1)) \right)
\end{aligned} \tag{6.16}
$$

- t_3^G

$$
\begin{aligned}
dM_{27} =& \epsilon \left(M_{27}(2\mathrm{dL}(z) - 4\mathrm{dL}(1-z)) - 6M_{28}\mathrm{dL}(z) \right) \\
dM_{28} =& \epsilon \left(M_{27}(\mathrm{dL}(z) - \mathrm{dL}(1-z)) - 3M_{28}\mathrm{dL}(z) \right) \\
dM_{29} =& \epsilon \left(M_{27}\mathrm{dL}(z) - 4M_{28}\mathrm{dL}(z) - M_{29}\mathrm{dL}(z) + 2M_{30}\mathrm{dL}(z) \right) \\
dM_{30} =& \epsilon \left(M_{27}(\mathrm{dL}(z) - 2\mathrm{dL}(1-z)) + M_{28}(8\mathrm{dL}(1-z) - 3\mathrm{dL}(z)) \right. \\
&\left. -M_{29}\mathrm{dL}(1-z) + M_{30}(-2\mathrm{dL}(1-z) - 2\mathrm{dL}(z)) \right) \\
dM_{31} =& \epsilon \left(M_{27}(3\mathrm{dL}(z) - 3\mathrm{dL}(1-z)) + M_{28}(8\mathrm{dL}(1-z) - 8\mathrm{dL}(z)) \right. \\
&\left. +M_{29}(\mathrm{dL}(z) - \mathrm{dL}(1-z)) + M_{30}(2\mathrm{dL}(z) - 2\mathrm{dL}(1-z)) - 2M_{31}\mathrm{dL}(1-z) \right)
\end{aligned} \tag{6.17}
$$

Soft limit:

- t_1^G:

$$
\begin{aligned}
Master_1^G &\to \bar{z}^{3-4\epsilon} Master_1^s \\
Master_2^G &\to \bar{z}^{2-4\epsilon} \frac{Master_1^s(4\epsilon-3)}{s} \\
Master_3^G &\to \bar{z}^{2-4\epsilon} \left(-\frac{Master_1^s(4\epsilon-3)}{(2\epsilon-1)(Q^2+s)} \right) \\
Master_4^G &\to \bar{z}^{3-4\epsilon} Master_1^s \\
Master_5^G &\to \bar{z}^{2-4\epsilon} \frac{Master_1^s(4\epsilon-3)}{s} \\
Master_6^G &\to \bar{z}^{3-4\epsilon} Master_1^s \\
Master_7^G &\to \bar{z}^{2-4\epsilon} \left(-\frac{Master_1^s(4\epsilon-3)}{(2\epsilon-1)(Q^2+s)} \right) \\
Master_8^G &\to \bar{z}^{1-4\epsilon} \left(-\frac{2Master_1^s(4\epsilon-3)}{s(Q^2+s)} \right) \\
Master_9^G &\to \bar{z}^{2-4\epsilon} \left(-\frac{Master_1^s(4\epsilon-3)}{(2\epsilon-1)(Q^2+s)} \right)
\end{aligned}
$$

$$
\begin{aligned}
Master^G_{10} &\to \bar{z}^{3-4\epsilon} Master^s_1 \\
Master^G_{11} &\to \bar{z}^{2-4\epsilon} \frac{Master^s_1(4\epsilon-3)}{s} \\
Master^G_{12} &\to \bar{z}^{2-4\epsilon} \left(-\frac{Master^s_1(4\epsilon-3)}{(2\epsilon-1)\left(Q^2+s\right)}\right) \\
Master^G_{13} &\to \bar{z}^{1-4\epsilon} \frac{Master^s_1(2\epsilon-1)(4\epsilon-3)}{\epsilon^2\left(Q^2+s\right)^2}
\end{aligned} \tag{6.18}
$$

- t^G_2:

$$
\begin{aligned}
Master^G_{14} \to & Master^s_1 \bar{z}^{3-4\text{ep}} \\
Master^G_{15} \to & \frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{s} \\
Master^G_{16} \to & -\frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{(2\epsilon-1)\left(Q^2+s\right)} \\
Master^G_{17} \to & -Master^s_1 \bar{z}^{3-4\text{ep}} \\
Master^G_{18} \to & -\frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{s} \\
Master^G_{19} \to & Master^s_1 \bar{z}^{3-4\text{ep}} \\
Master^G_{20} \to & \frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{(2\epsilon-1)\left(Q^2+s\right)} \\
Master^G_{21} \to & \frac{2Master^s_1(4\epsilon-3)\bar{z}^{1-4\text{ep}}}{s\left(Q^2+s\right)} \\
Master^G_{22} \to & Master^s_4 \bar{z}^{1-4\text{ep}} \\
Master^G_{23} \to & \frac{(4\epsilon-1)\bar{z}^{-4\text{ep}}\left(Master^s_1\left(-8\epsilon^2+10\epsilon-3\right)+Master^s_4 s\epsilon^2\left(Q^2+s\right)\right)}{3s^2\epsilon^2\left(Q^2+s\right)} \\
Master^G_{24} \to & \frac{(4\epsilon-1)\bar{z}^{-4\text{ep}}\left(Master^s_1\left(8\epsilon^2-10\epsilon+3\right)+2Master^s_4 s\epsilon^2\left(Q^2+s\right)\right)}{3s^2\epsilon^2\left(Q^2+s\right)} \\
Master^G_{25} \to & -Master^s_4 \bar{z}^{1-4\text{ep}} \\
Master^G_{26} \to & \frac{(4\epsilon-1)\bar{z}^{-4\text{ep}}\left(2Master^s_1\left(8\epsilon^2-10\epsilon+3\right)+Master^s_4 s\epsilon^2\left(Q^2+s\right)\right)}{3s\epsilon^3\left(Q^2+s\right)^2}
\end{aligned} \tag{6.19}
$$

- t^G_3:

$$
\begin{aligned}
Master^G_{27} \to & Master^s_1 \bar{z}^{3-4\text{ep}} \\
Master^G_{28} \to & \frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{s} \\
Master^G_{29} \to & -\frac{Master^s_1(4\epsilon-3)\bar{z}^{2-4\text{ep}}}{s(2\epsilon-1)} \\
Master^G_{30} \to & -\frac{Master^s_1(2\epsilon-1)(4\epsilon-3)\bar{z}^{1-4\text{ep}}}{s^2\epsilon} \\
Master^G_{31} \to & \frac{Master^s_1(2\epsilon-1)\left(32\epsilon^2-32\epsilon+6\right)\bar{z}^{-4\text{ep}-1}}{s^2\epsilon^3\left(Q^2+s\right)}
\end{aligned} \tag{6.20}
$$

Canonical Differential Equations and Boundary Conditions for $g+W^* \to t+\bar{b}$ at 1-loop.

$$
\begin{aligned}
M_1^G =& \frac{(1-2\epsilon)(\epsilon-1)G(\{1,1,0,0,0,0,1\})}{(1-z)z} \\
M_2^G =& \frac{y(1-2\epsilon)G(\{1,2,0,1,0,0,1\})}{1-z} \\
M_3^G =& \frac{(1-2\epsilon)\left(\frac{(d-3)G(1,\{1,1,0,0,0,1,1\})}{\mathrm{mt}^2-s} - \frac{(d-2)G(1,\{1,1,0,0,0,0,1\})}{2\mathrm{mt}^2(\mathrm{mt}^2-s)}\right)}{1-z} \\
M_4^G =& 2(2\epsilon(y+z+1)(G(\{1,0,1,0,1,0,2\}) - G(\{1,0,2,0,1,0,1\})) + 3\epsilon(y+z+1)G(\{1,0,2,0,1,0,1\}) \\
& + z(y+z+1)G(\{1,0,1,0,2,0,2\})) \\
M_5^G =& \frac{1}{2(z-1)^2 z(2\epsilon+1)(y+z)^2(yz+1)} \times \\
& \times \Big[(1-z)\left(2(y+1)\left(2\epsilon^2-3\epsilon+1\right)\left(y^2(3z\epsilon+z-\epsilon)-y(z-1)z+z^2((z-3)\epsilon-1)\right)G(1,\{1,1,0,0,0,0,1\})\right. \\
& + z(2\epsilon+1)\left(2(y+1)\left((z-1)^2(y+z)\left((y+1)z(3\epsilon+1)(y+z)G(1,\{1,1,0,1,2,0,1\})\right.\right.\right. \\
& \left.\left. -2\epsilon\left(z\left(z-y^2\right)G(1,\{1,2,0,1,1,0,1\}) + \epsilon(yz+y+2z)G(1,\{1,1,0,1,1,0,1\})\right)\right)\right) \\
& \left. -2y(y+1)z(1-2\epsilon)^2 G(1,\{1,1,0,1,0,0,1\})\right) \\
& \left.\left.\left. + \left(6\epsilon^2-5\epsilon+1\right)\left(3y^3z+y^2\left(4z^2+6z-1\right)+yz^2(z+8)+z^2(2z+1)\right)G(1,\{1,0,0,1,1,0,1\})\right)\right)\Big] \\
M_6^G =& \frac{1}{(z-1)z(y+z)} \times \Big[\left(2\epsilon^2-3\epsilon+1\right)(y-z)G(1,\{1,1,0,0,0,0,1\}) \\
& + z\left((y+z)\left(2(y+1)(z-1)\epsilon(zG(1,\{1,2,0,1,1,0,1\}) + \epsilon G(1,\{1,1,0,1,1,0,1\}))\right.\right. \\
& \left. + \left(6\epsilon^2-5\epsilon+1\right)G(1,\{1,0,0,1,1,0,1\})\right) \\
& \left. -2y(1-2\epsilon)^2 G(1,\{1,1,0,1,0,0,1\})\right)\Big] \\
M_7^G =& 2(y+1)\epsilon^2 G(1,\{1,1,0,1,1,0,1\})
\end{aligned} \tag{6.21}
$$

$$
\begin{aligned}
M_8^G =& \frac{(1-2\epsilon)(\epsilon-1)G(2,\{1,0,0,0,1,1,0\})}{(1-z)z} \\
M_9^G =& \frac{(1-2\epsilon)(\epsilon-1)G(2,\{1,0,0,0,0,1,1\})}{(1-z)z} \\
M_{10}^G =& \frac{y(1-2\epsilon)G(2,\{1,0,1,0,2,1,0\})}{1-z} \\
M_{11}^G =& \frac{1}{(z-1)z} \times \\
& \times \Big[z\left((z((y+5)\epsilon-1)-(y+1)\epsilon)G(2,\{1,1,0,0,2,1,0\}) + (y+1)z\epsilon G(2,\{1,2,0,0,1,1,0\})\right. \\
& \left. + \left(6\epsilon^2-5\epsilon+1\right)G(2,\{1,1,0,0,1,1,0\})\right) \\
& + \left(-2\epsilon^2+3\epsilon-1\right)G(2,\{1,0,0,0,1,1,0\})\Big] \\
M_{12}^G =& (y+1)\epsilon G(2,\{1,2,0,0,1,1,0\}) \\
M_{13}^G =& (y+1)\epsilon G(2,\{1,1,0,0,2,1,0\})
\end{aligned}
$$

$$
\begin{aligned}
M_{14}^G =&(1-2\epsilon)\left(\frac{(1-2\epsilon)G(2,\{1,0,1,0,0,1,1\})}{(1-z)^2}-\frac{(1-\epsilon)G(2,\{1,0,0,0,1,1,0\})}{(1-z)^2}\right)\\
M_{15}^G =&2(y+1)\epsilon^2 G(2,\{1,1,1,0,1,1,0\})\\
M_{16}^G =&\frac{(y+1)(1-2\epsilon)\epsilon G(2,\{1,0,1,0,1,1,1\})}{1-z}\\
M_{17}^G =&4(y+1)z\epsilon G(2,\{1,1,0,0,1,2,1\})\\
M_{18}^G =&4(y+1)\epsilon^2 G(2,\{1,1,0,0,1,1,1\})\\
M_{19}^G =&2(y+1)^2(1-z)\epsilon^2 G(2,\{1,1,1,1,1,1,1\})
\end{aligned}
\tag{6.22}
$$

$$
\begin{aligned}
M_{20}^G =&\frac{(1-2\epsilon)(\epsilon-1)G(3,\{1,0,0,0,0,1,1\})}{(1-z)z}\\
M_{21}^G =&\frac{(1-3\epsilon)(1-2\epsilon)G(3,\{1,0,1,0,1,0,1\})}{z-1}\\
M_{22}^G =&\frac{y(1-2\epsilon)G(3,\{1,0,1,0,0,2,1\})}{1-z}\\
M_{23}^G =&\frac{1}{(z-1)z}\times\\
&\times\left[z\left((z((y+5)\epsilon-1)-(y+1)\epsilon)G(3,\{1,1,0,0,0,2,1\})+(y+1)z\epsilon G(3,\{1,2,0,0,0,1,1\})\right.\right.\\
&\left.+\left(6\epsilon^2-5\epsilon+1\right)G(3,\{1,1,0,0,0,1,1\})\right)\\
&\left.+\left(-2\epsilon^2+3\epsilon-1\right)G(3,\{1,0,0,0,0,1,1\})\right]\\
M_{24}^G =&(y+1)\epsilon G(3,\{1,1,0,0,0,2,1\})\\
M_{25}^G =&(y+1)\epsilon G(3,\{1,2,0,0,0,1,1\})\\
M_{26}^G =&\frac{1}{2(z-1)^2z(2\epsilon+1)(y+z)^2(yz+1)}\times\\
&\times\left[(1-z)\left(2(y+1)\left(2\epsilon^2-3\epsilon+1\right)\left(y^2(3z\epsilon+z-\epsilon)-y(z-1)z+z^2((z-3)\epsilon-1)\right)G(3,\{1,0,0,0,0,1,1\})\right.\right.\\
&+z(2\epsilon+1)\left(2(y+1)\left((z-1)^2(y+z)\left((y+1)z(3\epsilon+1)(y+z)G(3,\{1,1,0,1,2,0,1\})\right.\right.\right.\\
&\left.\left.-2\epsilon\left(z\left(z-y^2\right)G(3,\{1,2,0,1,1,0,1\})+\epsilon(yz+y+2z)G(3,\{1,1,0,1,1,0,1\})\right)\right)\\
&\left.-2y(y+1)z(1-2\epsilon)^2G(3,\{1,0,1,0,0,1,1\})\right)\\
&\left.\left.\left.+\left(6\epsilon^2-5\epsilon+1\right)\left(3y^3z+y^2\left(4z^2+6z-1\right)+yz^2(z+8)+z^2(2z+1)\right)G(3,\{1,0,1,0,1,0,1\})\right)\right)\right]\\
M_{27}^G =&\frac{1}{(z-1)z(y+z)}\times\\
&\times\left[\left(2\epsilon^2-3\epsilon+1\right)(y-z)G(3,\{1,0,0,0,0,1,1\})\right.\\
&+z\left((y+z)\left(2(y+1)(z-1)\epsilon(zG(3,\{1,2,0,1,1,0,1\})+\epsilon G(3,\{1,1,0,1,1,0,1\}))\right.\right.\\
&\left.+\left(6\epsilon^2-5\epsilon+1\right)G(3,\{1,0,1,0,1,0,1\})\right)\\
&\left.\left.-2y(1-2\epsilon)^2G(3,\{1,0,1,0,0,1,1\})\right)\right]\\
M_{28}^G =&2(y+1)\epsilon^2 G(3,\{1,1,0,1,1,0,1\})\\
M_{29}^G =&(y+1)\epsilon^2 G(3,\{1,1,1,0,0,1,1\})\\
M_{30}^G =&2(y+1)^2\epsilon^2 G(3,\{1,1,1,0,1,1,1\})
\end{aligned}
\tag{6.23}
$$

$$M_{31}^G = \frac{(1-2\epsilon)(\epsilon-1)G(4,\{1,1,0,0,0,0,1\})}{(1-z)z}$$
$$M_{32}^G = \frac{(y+1)(1-\epsilon)\epsilon G(4,\{1,1,1,0,0,0,1\})}{z}$$
$$M_{33}^G = \frac{(1-3\epsilon)(1-2\epsilon)G(4,\{1,0,0,1,1,0,1\})}{z-1}$$
$$M_{34}^G = \frac{(1-2\epsilon)G(4,\{1,2,0,0,0,1,1\})}{1-z}$$
$$M_{35}^G = \frac{y(1-2\epsilon)G(4,\{1,2,0,1,0,0,1\})}{1-z}$$
$$M_{36}^G = y(y+1)\epsilon G(4,\{1,2,1,1,0,0,1\})$$
$$M_{37}^G = (y+1)\epsilon G(4,\{1,2,1,0,0,1,1\})$$
$$M_{38}^G = (y+1)(1-2\epsilon)\epsilon G(4,\{1,0,1,1,1,0,1\})$$
$$\begin{aligned} M_{39}^G = & \frac{1}{2(z-1)^2 z(2\epsilon+1)(y+z)^2(yz+1)} \times \\ & \times \Big[(1-z)\Big(2(y+1)\Big(2\epsilon^2-3\epsilon+1\Big)\Big(y^2(3z\epsilon+z-\epsilon)-y(z-1)z+z^2((z-3)\epsilon-1)\Big)G(4,\{1,1,0,0,0,0,1\}) \\ & + z(2\epsilon+1)\Big(2(y+1)\Big((z-1)^2(y+z)\left((y+1)z(3\epsilon+1)(y+z)G(1,\{1,1,0,1,2,0,1\})\right. \\ & - 2\epsilon\Big(z\Big(z-y^2\Big)G(1,\{1,2,0,1,1,0,1\})+\epsilon(yz+y+2z)G(1,\{1,1,0,1,1,0,1\})\Big)\Big) \\ & -2y(y+1)z(1-2\epsilon)^2 G(4,\{1,1,1,0,0,0,1\})\Big) \\ & + \Big(6\epsilon^2-5\epsilon+1\Big)\Big(3y^3z+y^2\Big(4z^2+6z-1\Big)+yz^2(z+8)+z^2(2z+1)\Big)G(4,\{1,1,0,0,0,1,1\})\Big)\Big)\Big] \end{aligned}$$
$$\begin{aligned} M_{40}^G = & \frac{1}{(z-1)z(y+z)} \times \\ & \times \Big[\Big(2\epsilon^2-3\epsilon+1\Big)(y-z)G(4,\{1,1,0,0,0,0,1\}) \\ & + z\left((y+z)\left(2(y+1)(z-1)\epsilon(zG(1,\{1,2,0,1,1,0,1\})+\epsilon G(1,\{1,1,0,1,1,0,1\}))\right.\right. \\ & + \Big(6\epsilon^2-5\epsilon+1\Big)G(4,\{1,1,0,0,0,1,1\})\Big) \\ & -2y(1-2\epsilon)^2 G(4,\{1,1,1,0,0,0,1\})\Big)\Big] \end{aligned}$$
$$M_{41}^G = 2(y+1)\epsilon^2 G(1,\{1,1,0,1,1,0,1\})$$
$$M_{42}^G = (y+1)\epsilon^2\sqrt{4yz+z^2+2z+1}\,G(4,\{1,1,1,1,1,0,1\})$$
$$M_{43}^G = 2(y+1)^2(1-z)\epsilon^2 G(4,\{1,1,1,1,1,1,1\}) \tag{6.24}$$

$$M_{44}^G = \frac{(1-2\epsilon)(\epsilon-1)G(5,\{1,0,0,0,1,1,0\})}{(1-z)z}$$
$$M_{45}^G = \frac{(1-2\epsilon)(\epsilon-1)G(5,\{1,0,0,0,0,1,1\})}{(1-z)z}$$
$$M_{46}^G = \frac{(y+1)(1-\epsilon)\epsilon G(5,\{1,0,0,1,1,1,0\})}{z}$$
$$\begin{aligned} M_{47}^G = & \frac{1}{(z-1)z} \times \\ & \times \left[z\left((z((y+5)\epsilon-1)-(y+1)\epsilon)G(5,\{1,1,0,0,2,1,0\})+(y+1)z\epsilon G(5,\{1,2,0,0,1,1,0\})\right.\right. \end{aligned}$$

$$
\begin{aligned}
&+\left(6\epsilon^2-5\epsilon+1\right)G(5,\{1,1,0,0,1,1,0\})\Big) \\
&+\left(-2\epsilon^2+3\epsilon-1\right)G(5,\{1,0,0,0,1,1,0\})\Big] \\
M_{48}^G =&(y+1)\epsilon G(5,\{1,2,0,0,1,1,0\}) \\
M_{49}^G =&(y+1)\epsilon G(5,\{1,1,0,0,2,1,0\}) \\
M_{50}^G =&\frac{y(1-2\epsilon)G(5,\{1,0,1,0,2,1,0\})}{1-z} \\
M_{51}^G =&\frac{(y+1)(1-\epsilon)\epsilon G(5,\{1,0,0,1,0,1,1\})}{z} \\
M_{52}^G =&\frac{(1-2\epsilon)G(5,\{1,0,1,0,0,1,2\})}{1-z} \\
M_{53}^G =&y(y+1)\epsilon G(5,\{1,0,1,1,1,1,0\}) \\
M_{54}^G =&2(y+1)\epsilon^2G(5,\{1,1,1,0,1,1,0\}) \\
M_{55}^G =&(y+1)\epsilon G(5,\{1,0,1,1,0,1,2\}) \\
M_{56}^G =&4(y+1)z\epsilon G(5,\{1,1,0,0,1,2,1\}) \\
M_{57}^G =&4(y+1)\epsilon^2G(5,\{1,1,0,0,1,1,1\}) \\
M_{58}^G =&\frac{(y+1)(1-2\epsilon)\epsilon G(5,\{1,0,1,0,1,1,1\})}{1-z} \\
M_{59}^G =&2(y+1)^2\epsilon^2G(5,\{1,0,1,1,1,1,1\})
\end{aligned}
\tag{6.25}
$$

$$
\begin{aligned}
dM_1 =&M_1(-2\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(z)) \\
dM_2 =&M_1(\epsilon \mathrm{dL}(y+z)-\epsilon \mathrm{dL}(z))+M_2(-2\epsilon \mathrm{dL}(y+z)+\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(1-z)) \\
dM_3 =&M_1(\epsilon \mathrm{dL}(z)-\epsilon \mathrm{dL}(1-z))-4M_3\epsilon \mathrm{dL}(1-z) \\
dM_4 =&M_4(-\epsilon \mathrm{dL}(y+1)-3\epsilon \mathrm{dL}(1-z)) \\
dM_5 =&M_1(-2\epsilon \mathrm{dL}(y+1)+2\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(z))+M_2(-2\epsilon \mathrm{dL}(y)+4\epsilon \mathrm{dL}(y+1)-4\epsilon \mathrm{dL}(1-z)+2\epsilon \mathrm{dL}(z)) \\
&+M_4(3\epsilon \mathrm{dL}(y+1)-3\epsilon \mathrm{dL}(1-z)+2\epsilon \mathrm{dL}(z))-4M_5\epsilon \mathrm{dL}(1-z) \\
&+M_6(2\epsilon \mathrm{dL}(y+z)-2\epsilon \mathrm{dL}(y+1)-\epsilon \mathrm{dL}(z))+M_7\epsilon \mathrm{dL}(z) \\
dM_6 =&M_1(\epsilon \mathrm{dL}(y)-\epsilon \mathrm{dL}(z))+M_4(\epsilon \mathrm{dL}(z)-\epsilon \mathrm{dL}(y)) \\
&+M_5(\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(y+1)+2\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(z)) \\
&+M_6(2\epsilon \mathrm{dL}(y)-4\epsilon \mathrm{dL}(y+z))+M_7(\epsilon \mathrm{dL}(y)-\epsilon \mathrm{dL}(z)) \\
dM_7 =&M_1(\epsilon \mathrm{dL}(z)-\epsilon \mathrm{dL}(y))+M_2(2\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(z))+M_4(\epsilon \mathrm{dL}(y)-\epsilon \mathrm{dL}(z)) \\
&+M_5(-\epsilon \mathrm{dL}(y)-\epsilon \mathrm{dL}(z))+M_6(\epsilon \mathrm{dL}(z)-2\epsilon \mathrm{dL}(y))+M_7(-\epsilon \mathrm{dL}(y)+2\epsilon \mathrm{dL}(y+1)-2\epsilon \mathrm{dL}(z))
\end{aligned}
\tag{6.26}
$$

$$
\begin{aligned}
dM_8 =&M_8(-2\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(z)) \\
dM_9 =&M_9(-2\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(z)) \\
dM_{10} =&M_{10}(-2\epsilon \mathrm{dL}(y+z)+\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(1-z))+M_8(\epsilon \mathrm{dL}(y+z)-\epsilon \mathrm{dL}(z)) \\
dM_{11} =&M_{11}(\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(y+1)-2\epsilon \mathrm{dL}(1-z)+\epsilon \mathrm{dL}(z)) \\
&+M_{12}(\epsilon \mathrm{dL}(1-z)-\epsilon \mathrm{dL}(y))+M_{13}(-2\epsilon \mathrm{dL}(y)+3\epsilon \mathrm{dL}(y+1)+2\epsilon \mathrm{dL}(1-z)-3\epsilon \mathrm{dL}(z)) \\
&+M_8(\epsilon \mathrm{dL}(z)-\epsilon \mathrm{dL}(y+1)) \\
dM_{12} =&M_{11}(\epsilon \mathrm{dL}(y-z+1)-\epsilon \mathrm{dL}(y)+\epsilon \mathrm{dL}(z))+M_{12}(\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(y+1)-2\epsilon \mathrm{dL}(z)) \\
&+M_{13}(2\epsilon \mathrm{dL}(y)-2\epsilon \mathrm{dL}(y+1)-\epsilon \mathrm{dL}(z))+M_8\epsilon \mathrm{dL}(z) \\
dM_{13} =&M_{11}(-\epsilon \mathrm{dL}(y-z+1)+\epsilon \mathrm{dL}(y)+\epsilon \mathrm{dL}(z))-M_{12}\epsilon \mathrm{dL}(y)+M_{13}(-2\epsilon \mathrm{dL}(y)+2\epsilon \mathrm{dL}(y+1)-3\epsilon \mathrm{dL}(z))
\end{aligned}
$$

$$
\begin{aligned}
dM_{14} =& -4M_{14}\epsilon \mathrm{dL}(1-z) - M_8\epsilon \mathrm{dL}(1-z) \\
dM_{15} =& -2M_{10}\epsilon \mathrm{dL}(y) - 2M_{11}\epsilon \mathrm{dL}(y) + M_{12}(-2\epsilon \mathrm{dL}(y+z) + 2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(1-z)) \\
& + M_{13}(4\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(y+z)) + M_{15}(-2\epsilon \mathrm{dL}(y+z) + 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(1-z)) \\
dM_{16} =& M_{10}(\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z)) + M_{14}\epsilon \mathrm{dL}(z) + M_{16}(-2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) + M_8\epsilon \mathrm{dL}(z) \\
dM_{17} =& 4M_{11}\epsilon \mathrm{dL}(y-z+1) + M_{12}(4\epsilon \mathrm{dL}(1-z) - 4\epsilon \mathrm{dL}(y+1)) - 8M_{13}\epsilon \mathrm{dL}(y+1) \\
& + M_{17}(-2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(z)) - 2M_{18}\epsilon \mathrm{dL}(1-z) + 2M_8\epsilon \mathrm{dL}(z) \\
dM_{18} =& -2M_{12}\epsilon \mathrm{dL}(z) + M_{13}(4\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(z)) + M_{17}\epsilon \mathrm{dL}(z) - M_{18}\epsilon \mathrm{dL}(z) \\
dM_{19} =& M_{10}(4\epsilon \mathrm{dL}(y+z) - 4\epsilon \mathrm{dL}(z)) + M_{11}(4\epsilon \mathrm{dL}(y+1) - 4\epsilon \mathrm{dL}(z)) \\
& + M_{12}(-2\epsilon \mathrm{dL}(y+z) + 2\epsilon \mathrm{dL}(y+1) - 4\epsilon \mathrm{dL}(1-z) + 4\epsilon \mathrm{dL}(z)) \\
& + M_{13}(-2\epsilon \mathrm{dL}(y+z) - 2\epsilon \mathrm{dL}(y+1) + 4\epsilon \mathrm{dL}(z)) + M_{14}(4\epsilon \mathrm{dL}(z) - 4\epsilon \mathrm{dL}(1-z)) \\
& + M_{15}(2\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(y+z)) + M_{16}(4\epsilon \mathrm{dL}(1-z) - 4\epsilon \mathrm{dL}(z)) \\
& + M_{17}(\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) + M_{18}(\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) - 2M_{19}\epsilon \mathrm{dL}(1-z) \\
& + M_8(-2\epsilon \mathrm{dL}(y+z) + 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z)) \qquad (6.27)
\end{aligned}
$$

$$
\begin{aligned}
dM_{20} =& M_{20}(-2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) \\
dM_{21} =& M_{21}(-\epsilon \mathrm{dL}(y+1) - 3\epsilon \mathrm{dL}(1-z)) \\
dM_{22} =& M_{20}(\epsilon \mathrm{dL}(y+z) - \epsilon \mathrm{dL}(z)) + M_{22}(-2\epsilon \mathrm{dL}(y+z) + \epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(1-z)) \\
dM_{23} =& M_{20}(\epsilon \mathrm{dL}(z) - \epsilon \mathrm{dL}(y+1)) + M_{23}(\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(1-z) + \epsilon \mathrm{dL}(z)) \\
& + M_{24}(-2\epsilon \mathrm{dL}(y) + 3\epsilon \mathrm{dL}(y+1) + 2\epsilon \mathrm{dL}(1-z) - 3\epsilon \mathrm{dL}(z)) + M_{25}(\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(y)) \\
dM_{24} =& M_{23}(-\epsilon \mathrm{dL}(y-z+1) + \epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z)) + M_{24}(-2\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(y+1) - 3\epsilon \mathrm{dL}(z)) - M_{25}\epsilon \mathrm{dL}(y) \\
dM_{25} =& M_{20}\epsilon \mathrm{dL}(z) + M_{23}(\epsilon \mathrm{dL}(y-z+1) - \epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(z)) + M_{24}(2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(y+1) - \epsilon \mathrm{dL}(z)) \\
& + M_{25}(\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(z)) \\
dM_{26} =& M_{20}(-4\epsilon \mathrm{dL}(y+1) + 4\epsilon \mathrm{dL}(1-z) - 2\epsilon \mathrm{dL}(z)) + M_{21}(3\epsilon \mathrm{dL}(y+1) - 3\epsilon \mathrm{dL}(1-z) + 2\epsilon \mathrm{dL}(z)) \\
& + M_{22}(-4\epsilon \mathrm{dL}(y) + 8\epsilon \mathrm{dL}(y+1) - 8\epsilon \mathrm{dL}(1-z) + 4\epsilon \mathrm{dL}(z)) - 4M_{26}\epsilon \mathrm{dL}(1-z) \\
& + M_{27}(2\epsilon \mathrm{dL}(y+z) - 2\epsilon \mathrm{dL}(y+1) - \epsilon \mathrm{dL}(z)) + M_{28}\epsilon \mathrm{dL}(z) \\
dM_{27} =& M_{20}(2\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(z)) + M_{21}(\epsilon \mathrm{dL}(z) - \epsilon \mathrm{dL}(y)) \\
& + M_{26}(\epsilon \mathrm{dL}(y) - 2\epsilon \mathrm{dL}(y+1) + 2\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) \\
& + M_{27}(2\epsilon \mathrm{dL}(y) - 4\epsilon \mathrm{dL}(y+z)) + M_{28}(\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z)) \\
dM_{28} =& M_{20}(2\epsilon \mathrm{dL}(z) - 2\epsilon \mathrm{dL}(y)) + M_{21}(\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z)) \\
& + M_{22}(4\epsilon \mathrm{dL}(y) - 4\epsilon \mathrm{dL}(z)) + M_{26}(-\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(z)) \\
& + M_{27}(\epsilon \mathrm{dL}(z) - 2\epsilon \mathrm{dL}(y)) + M_{28}(-\epsilon \mathrm{dL}(y) + 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(z)) \\
dM_{29} =& -M_{22}\epsilon \mathrm{dL}(y) - M_{23}\epsilon \mathrm{dL}(y) + M_{24}(2\epsilon \mathrm{dL}(y) - \epsilon \mathrm{dL}(y+z)) + M_{25}(-\epsilon \mathrm{dL}(y+z) + \epsilon \mathrm{dL}(y) + \epsilon \mathrm{dL}(1-z)) \\
& + M_{29}(-2\epsilon \mathrm{dL}(y+z) + 2\epsilon \mathrm{dL}(y+1) - 2\epsilon \mathrm{dL}(1-z)) \\
dM_{30} =& M_{20}(2\epsilon \mathrm{dL}(y+z) - 2\epsilon \mathrm{dL}(1-z)) + M_{21}(3\epsilon \mathrm{dL}(z) - 3\epsilon \mathrm{dL}(1-z)) \\
& + M_{22}(4\epsilon \mathrm{dL}(z) - 4\epsilon \mathrm{dL}(y+z)) + M_{23}(4\epsilon \mathrm{dL}(z) - 4\epsilon \mathrm{dL}(1-z)) \\
& + M_{24}(2\epsilon \mathrm{dL}(y+z) + 4\epsilon \mathrm{dL}(1-z) - 6\epsilon \mathrm{dL}(z)) + M_{25}(2\epsilon \mathrm{dL}(y+z) + 4\epsilon \mathrm{dL}(1-z) - 6\epsilon \mathrm{dL}(z)) \\
& + M_{26}(\epsilon \mathrm{dL}(1-z) - \epsilon \mathrm{dL}(z)) + M_{27}(\epsilon \mathrm{dL}(y+z) - \epsilon \mathrm{dL}(z)) \\
& + M_{29}(4\epsilon \mathrm{dL}(y+z) - 4\epsilon \mathrm{dL}(1-z)) - 2M_{30}\epsilon \mathrm{dL}(y+1) \qquad (6.28)
\end{aligned}
$$

$$
\begin{aligned}
dM_{31} =& -M_{31}\epsilon(-3\mathrm{dL}(a-b) + 2\mathrm{dL}(2a-b) + \mathrm{dL}(a)) \\
dM_{32} =& M_{31}\epsilon(\mathrm{dL}(a-b) - \mathrm{dL}(a)) - 2M_{32}\epsilon(\mathrm{dL}(a) - \mathrm{dL}(a-b)) \\
dM_{33} =& M_{33}\epsilon(4\mathrm{dL}(a-b) - 3\mathrm{dL}(2a-b) - \mathrm{dL}(a(b-1)+b) - \mathrm{dL}(b+1)) \\
dM_{34} =& M_{31}\epsilon(\mathrm{dL}(a) - \mathrm{dL}(2a-b)) + 4M_{34}\epsilon(\mathrm{dL}(a-b) - \mathrm{dL}(2a-b)) \\
dM_{35} =& M_{31}\epsilon\left(\mathrm{dL}\left((a+1)b^2 + a\right) - \mathrm{dL}(a)\right)
\end{aligned}
$$

$$
\begin{aligned}
& + M_{35}\epsilon\left(-2\left(\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(2a-b)-\mathrm{dL}(b)\right)+3\mathrm{dL}(a-b)+\mathrm{dL}(a+1)\right)\\
dM_{36} =& M_{32}\epsilon\left(\mathrm{dL}(a)-\mathrm{dL}\left((a+1)b^2+a\right)\right)\\
& + M_{36}\epsilon\left(2\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a-b)+\mathrm{dL}(b)\right)-\mathrm{dL}(a)+\mathrm{dL}(a+1)\right)\\
& + M_{35}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(a-b))\\
dM_{37} =& M_{32}\epsilon(\mathrm{dL}(2a-b)-\mathrm{dL}(a))+M_{34}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(a-b))-M_{37}\epsilon(-3\mathrm{dL}(a-b)+2\mathrm{dL}(2a-b)+\mathrm{dL}(a))\\
dM_{38} =& M_{33}\epsilon(\mathrm{dL}(a-b)-\mathrm{dL}(a))-M_{38}\epsilon(-2\mathrm{dL}(a-b)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1))\\
dM_{39} =& -M_{40}\epsilon\left(2\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1)\right)-\mathrm{dL}(a-b)+\mathrm{dL}(a)\right)\\
& - 2M_{31}\epsilon(-\mathrm{dL}(a-b)+2(-\mathrm{dL}(2a-b)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1))+\mathrm{dL}(a))\\
& + M_{33}\epsilon(-2\mathrm{dL}(a-b)+3(-\mathrm{dL}(2a-b)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1))+2\mathrm{dL}(a))\\
& + 4M_{35}\epsilon(-2(\mathrm{dL}(2a-b)-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b)-\mathrm{dL}(b+1))+\mathrm{dL}(a)-\mathrm{dL}(a+1))\\
& + 4M_{39}\epsilon(\mathrm{dL}(a-b)-\mathrm{dL}(2a-b))+M_{41}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(a-b))\\
dM_{40} =& 2M_{40}\epsilon\left(-2\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a-b)+\mathrm{dL}(a+1)+2\mathrm{dL}(b)\right)\\
& + 2M_{31}\epsilon(-\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b))+M_{33}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(a+1)-2\mathrm{dL}(b))\\
& - M_{39}\epsilon(-2(\mathrm{dL}(2a-b)-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b)-\mathrm{dL}(b+1))+\mathrm{dL}(a)-\mathrm{dL}(a+1))\\
& + M_{41}\epsilon(-\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b))\\
dM_{41} =& 2M_{31}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(a+1)-2\mathrm{dL}(b))+M_{33}\epsilon(-\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b))\\
& + 4M_{35}\epsilon(-\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b))-M_{39}\epsilon(-2\mathrm{dL}(a-b)+\mathrm{dL}(a)+\mathrm{dL}(a+1)\\
& + 2\mathrm{dL}(b))+M_{40}\epsilon(\mathrm{dL}(a-b)+\mathrm{dL}(a)-2\mathrm{dL}(a+1)-4\mathrm{dL}(b))\\
& - M_{41}\epsilon(-\mathrm{dL}(a-b)-2\mathrm{dL}(a(b-1)+b)+2\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b)-2\mathrm{dL}(b+1))\\
dM_{42} =& -\frac{1}{2}M_{31}\epsilon(-2\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b+1)+\mathrm{dL}(4))\\
& + M_{32}\epsilon(-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1))\\
& + \frac{1}{4}M_{33}\epsilon(-2\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b+1)+\mathrm{dL}(4))\\
& + M_{35}\epsilon(-2\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b+1)\\
& + \mathrm{dL}(4))+M_{36}\epsilon(-2\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(a+1)+2\mathrm{dL}(b+1)+\mathrm{dL}(4))\\
& - M_{38}\epsilon(-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1))\\
& - \frac{1}{4}M_{39}\epsilon(2\mathrm{dL}(a(b-1)+b)-3\mathrm{dL}(a)+\mathrm{dL}(a+1)-2\mathrm{dL}(b+1)+\mathrm{dL}(4))\\
& - \frac{1}{2}M_{40}\epsilon(-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a+1)+\mathrm{dL}(b+1)+\mathrm{dL}(4))\\
& + 2M_{42}\epsilon(\mathrm{dL}(a-b)-\mathrm{dL}(a(b-1)+b)-\mathrm{dL}(a)+\mathrm{dL}(2a+1)+\mathrm{dL}(b)-\mathrm{dL}(b+1))\\
& - \frac{1}{4}M_{41}\epsilon(-\mathrm{dL}(a)+\mathrm{dL}(a+1)+\mathrm{dL}(4))\\
dM_{43} =& -2M_{31}\epsilon\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1)\right)\\
& + 2M_{32}\epsilon\left(-\mathrm{dL}\left((a+1)b^2+a\right)-\mathrm{dL}(2a-b)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1)\right)\\
& + 4M_{35}\epsilon\left(-\mathrm{dL}\left((a+1)b^2+a\right)-\mathrm{dL}(2a-b)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1)\right)\\
& + 4M_{36}\epsilon\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1)\right)\\
& - M_{40}\epsilon\left(-\mathrm{dL}\left((a+1)b^2+a\right)+\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(b+1)\right)+3M_{33}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(2a-b))\\
& + 4M_{34}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(2a-b))+4M_{37}\epsilon(\mathrm{dL}(2a-b)-\mathrm{dL}(a))\\
& + M_{39}\epsilon(\mathrm{dL}(a)-\mathrm{dL}(2a-b))-4M_{42}\epsilon(-\mathrm{dL}(a(b-1)+b)+\mathrm{dL}(a)+\mathrm{dL}(b+1))\\
& + 2M_{43}\epsilon(\mathrm{dL}(a-b)-\mathrm{dL}(2a-b)) \qquad (6.29)
\end{aligned}
$$

$$
\begin{aligned}
dM_{44} =& M_{44}(-2\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(z))\\
dM_{45} =& M_{45}(-2\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(z))\\
dM_{46} =& - M_{44}\epsilon\mathrm{dL}(z) - 2M_{46}\epsilon\mathrm{dL}(z)\\
dM_{47} =& M_{44}(\epsilon\mathrm{dL}(z) - \epsilon\mathrm{dL}(y+1)) + M_{47}(\epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(y+1) - 2\epsilon\mathrm{dL}(1-z) + \epsilon\mathrm{dL}(z))\\
&+ M_{48}(\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(y))\\
&+ M_{49}(-2\epsilon\mathrm{dL}(y) + 3\epsilon\mathrm{dL}(y+1) + 2\epsilon\mathrm{dL}(1-z) - 3\epsilon\mathrm{dL}(z))\\
dM_{48} =& M_{44}\epsilon\mathrm{dL}(z) + M_{47}(\epsilon\mathrm{dL}(y-z+1) - \epsilon\mathrm{dL}(y) + \epsilon\mathrm{dL}(z))\\
&+ M_{48}(\epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(y+1) - 2\epsilon\mathrm{dL}(z))\\
&+ M_{49}(2\epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(y+1) - \epsilon\mathrm{dL}(z))\\
dM_{49} =& M_{47}(-\epsilon\mathrm{dL}(y-z+1) + \epsilon\mathrm{dL}(y) + \epsilon\mathrm{dL}(z)) - M_{48}\epsilon\mathrm{dL}(y) + M_{49}(-2\epsilon\mathrm{dL}(y) + 2\epsilon\mathrm{dL}(y+1) - 3\epsilon\mathrm{dL}(z))\\
dM_{50} =& M_{44}(\epsilon\mathrm{dL}(y+z) - \epsilon\mathrm{dL}(z)) + M_{50}(-2\epsilon\mathrm{dL}(y+z) + \epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(1-z))\\
dM_{51} =& - M_{44}\epsilon\mathrm{dL}(z) - 2M_{51}\epsilon\mathrm{dL}(z)\\
dM_{52} =& M_{44}(\epsilon\mathrm{dL}(z) - \epsilon\mathrm{dL}(1-z)) - 4M_{52}\epsilon\mathrm{dL}(1-z)\\
dM_{53} =& M_{46}(\epsilon\mathrm{dL}(z) - \epsilon\mathrm{dL}(y+z)) + M_{50}\epsilon\mathrm{dL}(z) + M_{53}(-2\epsilon\mathrm{dL}(y+z) + \epsilon\mathrm{dL}(y) - \epsilon\mathrm{dL}(z))\\
dM_{54} =& - 2M_{47}\epsilon\mathrm{dL}(y) + M_{48}(-2\epsilon\mathrm{dL}(y+z) + 2\epsilon\mathrm{dL}(y) + 2\epsilon\mathrm{dL}(1-z))\\
&+ M_{49}(4\epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(y+z)) - 2M_{50}\epsilon\mathrm{dL}(y)\\
&+ M_{54}(-2\epsilon\mathrm{dL}(y+z) + 2\epsilon\mathrm{dL}(y+1) - 2\epsilon\mathrm{dL}(1-z))\\
dM_{55} =& M_{51}(\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(z)) + M_{52}\epsilon\mathrm{dL}(z) + M_{55}(-2\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(z))\\
dM_{56} =& 2M_{44}\epsilon\mathrm{dL}(z) + 4M_{47}\epsilon\mathrm{dL}(y-z+1) + M_{48}(4\epsilon\mathrm{dL}(1-z)\\
&- 4\epsilon\mathrm{dL}(y+1)) - 8M_{49}\epsilon\mathrm{dL}(y+1) + M_{56}(-2\epsilon\mathrm{dL}(1-z) - 2\epsilon\mathrm{dL}(z))\\
&- 2M_{57}\epsilon\mathrm{dL}(1-z)\\
dM_{57} =& - 2M_{48}\epsilon\mathrm{dL}(z) + M_{49}(4\epsilon\mathrm{dL}(y+1) - 2\epsilon\mathrm{dL}(z)) + M_{56}\epsilon\mathrm{dL}(z) - M_{57}\epsilon\mathrm{dL}(z)\\
dM_{58} =& M_{50}(\epsilon\mathrm{dL}(y) - \epsilon\mathrm{dL}(z)) - M_{52}\epsilon\mathrm{dL}(z) + M_{58}(-2\epsilon\mathrm{dL}(1-z) - \epsilon\mathrm{dL}(z))\\
dM_{59} =& - M_{46}\epsilon\mathrm{dL}(z) + M_{51}\epsilon\mathrm{dL}(z) + M_{53}(2\epsilon\mathrm{dL}(y) - 2\epsilon\mathrm{dL}(z)) - 2M_{55}\epsilon\mathrm{dL}(z) + 2M_{58}\epsilon\mathrm{dL}(z) - 2M_{59}\epsilon\mathrm{dL}(z)
\end{aligned} \tag{6.30}
$$

Appendix B: Master Integrals at NNLO

We report in the following the expressions for the entire set of Master Integrals describing CC-DIS massive Form Factors. We indicate with $\tilde{M}_i$ the dimensionless master integrals $Master_i$. In other words $Master_i = \tilde{M}_i s^{4-2\epsilon+N_n-N_d}$, with N_n being the number of inverse propagator at numerator and N_d being the number of inverse propagator at denominator.

Bottom channel

Master Integrals for $\mathbf{b}+\mathbf{W}^* \to \mathbf{t}+\mathbf{g}+\mathbf{g}$.

$$
\begin{aligned}
\tilde{M}_1 = &\left(-\frac{1}{2}(y+1)zG(\{0\},z) - \frac{1}{12}(y+1)(z((z-6)z+3)+2)\right)\\
&+\epsilon\left(-\frac{1}{12}(y+1)z((z-6)z+18)G(\{0\},z) + \frac{1}{3}(y+1)(z((z-6)z+3)+2)G(\{1\},z)\right.\\
&+\frac{1}{2}(y+1)zG(\{0,0\},z) + 2(y+1)zG(\{0,1\},z)\\
&\left.+\frac{1}{24}(y+1)\left(z\left((74-13z)z+8\pi^2-35\right)-26\right)\right)\\
&+\epsilon^2\left(\frac{1}{24}(y+1)z\left((70-13z)z+2\pi^2-76\right)G(\{0\},z)\right.\\
&+\frac{1}{6}(y+1)(z-1)(z(13z-61)-26)G(\{1\},z) + \frac{1}{12}(y+1)z((z-6)z+18)G(\{0,0\},z)\\
&+\frac{1}{3}(y+1)z((z-6)z+18)G(\{0,1\},z) + \frac{1}{6}(y+1)(z((z-6)z+3)+2)G(\{1,0\},z)\\
&-\frac{4}{3}(y+1)(z((z-6)z+3)+2)G(\{1,1\},z) - \frac{1}{2}(y+1)zG(\{0,0,0\},z)\\
&-2(y+1)zG(\{0,0,1\},z) + (y+1)zG(\{0,1,0\},z) - 8(y+1)zG(\{0,1,1\},z)\\
&\left.+\frac{1}{144}(y+1)\left(576z\zeta(3) - 15(z-1)(z(23z-103)-46) + 2\pi^2(3z(3(z-6)z+29)+10)\right)\right)\\
&+O\left(\epsilon^3\right)
\end{aligned}
\tag{6.31}
$$

$$
\tilde{M}_2 = \left(zG(\{0\},z) + \frac{1}{2}\left(1-z^2\right)\right)
$$

$$
\begin{aligned}
&+\epsilon\left(2\left(z^2-1\right)G(\{1\},z)-\frac{1}{2}(z-6)zG(\{0\},z)-zG(\{0,0\},z)-4zG(\{0,1\},z)\right.\\
&\left.+\frac{1}{12}\left(-39z^2-8\pi^2 z+39\right)\right)\\
&+\epsilon^2\left(13\left(z^2-1\right)G(\{1\},z)+\left(z^2-1\right)G(\{1,0\},z)-8\left(z^2-1\right)G(\{1,1\},z)\right.\\
&-\frac{1}{12}\left(39z+2\pi^2-72\right)zG(\{0\},z)+\frac{1}{2}(z-6)zG(\{0,0\},z)+2(z-6)zG(\{0,1\},z)\\
&+zG(\{0,0,0\},z)+4zG(\{0,0,1\},z)-2zG(\{0,1,0\},z)+16zG(\{0,1,1\},z)\\
&\left.-\frac{115}{8}\left(z^2-1\right)-8z\zeta(3)+\frac{1}{12}\pi^2(3z(3z-8)-5)\right)\\
&+O\left(\epsilon^3\right)
\end{aligned}
\tag{6.32}
$$

$$
\begin{aligned}
\tilde{M}_3=&\left(-\frac{(y+z)G(\{-1\},y)}{y}+\frac{(y+z)G(\{0\},y)}{y}-\frac{zG(\{0\},z)}{y}+\frac{(y+z)G(\{-y\},z)}{y}\right)\\
&+\epsilon\left(-\frac{5zG(\{0\},z)}{y}+\frac{5(y+z)G(\{-y\},z)}{y}+\frac{(y+z)G(\{-1\},y)(3G(\{-y\},z)-5)}{y}\right.\\
&-\frac{(y+z)G(\{0\},y)(3G(\{-y\},z)-5)}{y}+\frac{(y+z)G(\{-1,-1\},y)}{y}-\frac{(y+z)G(\{-1,0\},y)}{y}\\
&+\frac{(y+z)G(\{0,-1\},y)}{y}-\frac{(y+z)G(\{0,0\},y)}{y}+\frac{zG(\{0,0\},z)}{y}+\frac{4zG(\{0,1\},z)}{y}\\
&\left.+\frac{2(y+z)G(\{-y,0\},z)}{y}-\frac{4(y+z)G(\{-y,1\},z)}{y}-\frac{3(y+z)G(\{-y,-y\},z)}{y}+\frac{\pi^2(z-y)}{3y}\right)\\
&+\epsilon^2\left(\frac{\left(\pi^2-114\right)zG(\{0\},z)}{6y}+\frac{5(y+z)G(\{-1,-1\},y)}{y}-\frac{5(y+z)G(\{-1,0\},y)}{y}\right.\\
&+\frac{5(y+z)G(\{0,-1\},y)}{y}-\frac{5(y+z)G(\{0,0\},y)}{y}\\
&+\frac{(y+z)}{6y}G(\{-y\},z)\left(-18G(\{-1,-1\},y)+18G(\{-1,0\},y)\right.\\
&\left.-18G(\{0,-1\},y)+18G(\{0,0\},y)-7\pi^2+114\right)\\
&+\frac{5zG(\{0,0\},z)}{y}+\frac{20zG(\{0,1\},z)}{y}+\frac{10(y+z)G(\{-y,0\},z)}{y}-\frac{20(y+z)G(\{-y,1\},z)}{y}\\
&+\frac{(y+z)G(\{-1\},y)\left(90G(\{-y\},z)-54G(\{-y,-y\},z)+7\pi^2-114\right)}{6y}\\
&-\frac{(y+z)G(\{0\},y)\left(30G(\{-y\},z)-18G(\{-y,-y\},z)+\pi^2-38\right)}{2y}\\
&-\frac{15(y+z)G(\{-y,-y\},z)}{y}-\frac{(y+z)G(\{-1,-1,-1\},y)}{y}+\frac{(y+z)G(\{-1,-1,0\},y)}{y}\\
&-\frac{(y+z)G(\{-1,0,-1\},y)}{y}+\frac{(y+z)G(\{-1,0,0\},y)}{y}-\frac{(y+z)G(\{0,-1,-1\},y)}{y}\\
&+\frac{(y+z)G(\{0,-1,0\},y)}{y}-\frac{(y+z)G(\{0,0,-1\},y)}{y}+\frac{(y+z)G(\{0,0,0\},y)}{y}\\
&-\frac{zG(\{0,0,0\},z)}{y}-\frac{4zG(\{0,0,1\},z)}{y}+\frac{2zG(\{0,1,0\},z)}{y}-\frac{16zG(\{0,1,1\},z)}{y}\\
&-\frac{2(y+z)G(\{-y,0,0\},z)}{y}-\frac{8(y+z)G(\{-y,0,1\},z)}{y}-\frac{2(y+z)G(\{-y,1,0\},z)}{y}
\end{aligned}
$$

$$+\frac{16(y+z)G(\{-y,1,1\},z)}{y}-\frac{6(y+z)G(\{-y,-y,0\},z)}{y}+\frac{12(y+z)G(\{-y,-y,1\},z)}{y}$$
$$+\frac{9(y+z)G(\{-y,-y,-y\},z)}{y}-\frac{6\zeta(3)(y-3z)+5\pi^2(y-z)}{3y}\Bigg)$$
$$+O\left(\epsilon^3\right) \tag{6.33}$$

$$\tilde{M}_4=\Bigg(-\frac{\left(2G(\{0,-1\},y)-2G(\{0,0\},y)+\pi^2\right)G(\{0\},z)}{2(y+1)}$$
$$-\frac{G(\{0\},y)\left(-6G(\{0,-y\},z)+6G(\{0,1\},z)+\pi^2\right)}{6(y+1)}$$
$$+\frac{G(\{-1\},y)(G(\{0,1\},z)-G(\{0,-y\},z))}{y+1}$$
$$-\frac{G(\{0,1,-y\},z)}{y+1}+\frac{G(\{0,-y,-y\},z)}{y+1}-\frac{G(\{0,-1,-1\},y)}{y+1}+\frac{G(\{0,-1,0\},y)}{y+1}$$
$$+\frac{2G(\{0,0,-1\},y)}{y+1}-\frac{2G(\{0,0,0\},y)}{y+1}+\frac{\zeta(3)}{y+1}\Bigg)$$
$$.+O\left(\mathrm{ep}^2\right) \tag{6.34}$$

$$\tilde{M}_5=-\frac{1}{3\epsilon^3(y+1)^2}$$
$$+\frac{1}{\epsilon^2}\left(-\frac{2G(\{0\},z)}{3(y+1)^2}+\frac{4G(\{1\},z)}{3(y+1)^2}-\frac{2G(\{-y\},z)}{3(y+1)^2}+\frac{2G(\{-1\},y)}{3(y+1)^2}-\frac{2G(\{0\},y)}{3(y+1)^2}\right)$$
$$+\frac{1}{\epsilon}\Bigg(-\frac{4G(\{0\},y)G(\{0\},z)}{3(y+1)^2}-\frac{G(\{0\},y)(G(\{1\},z)-5G(\{-y\},z))}{3(y+1)^2}$$
$$+\frac{G(\{-1\},y)(-5G(\{-y\},z)+4G(\{0\},z)+G(\{1\},z))}{3(y+1)^2}+\frac{2G(\{0,0\},z)}{3(y+1)^2}$$
$$+\frac{8G(\{0,1\},z)}{3(y+1)^2}-\frac{4G(\{0,-y\},z)}{3(y+1)^2}+\frac{2G(\{1,0\},z)}{3(y+1)^2}$$
$$-\frac{16G(\{1,1\},z)}{3(y+1)^2}-\frac{G(\{1,-y\},z)}{3(y+1)^2}-\frac{4G(\{-y,0\},z)}{3(y+1)^2}$$
$$+\frac{8G(\{-y,1\},z)}{3(y+1)^2}+\frac{5G(\{-y,-y\},z)}{3(y+1)^2}-\frac{4G(\{-1,-1\},y)}{3(y+1)^2}$$
$$+\frac{4G(\{-1,0\},y)}{3(y+1)^2}-\frac{5G(\{0,-1\},y)}{3(y+1)^2}+\frac{5G(\{0,0\},y)}{3(y+1)^2}+\frac{\pi^2}{(y+1)^2}\Bigg)$$
$$+\Bigg(-\frac{G(\{1\},z)\left(-G(\{-1,-1\},y)+G(\{-1,0\},y)+G(\{0,-1\},y)-G(\{0,0\},y)+2\pi^2\right)}{3(y+1)^2}$$
$$+\frac{2G(\{0\},z)(-4G(\{-1,-1\},y)+4G(\{-1,0\},y)+G(\{0,-1\},y)-G(\{0,0\},y))}{3(y+1)^2}$$
$$+\frac{G(\{-y\},z)\left(7G(\{-1,-1\},y)-7G(\{-1,0\},y)-G(\{0,-1\},y)+G(\{0,0\},y)+2\pi^2\right)}{3(y+1)^2}$$
$$+\frac{G(\{0\},y)}{3(y+1)^2}\left(4G(\{0,0\},z)-2G(\{0,1\},z)+10G(\{0,-y\},z)-2G(\{1,0\},z)\right.$$
$$\left.+4G(\{1,1\},z)+G(\{1,-y\},z)-2G(\{-y,0\},z)-2G(\{-y,1\},z)-11G(\{-y,-y\},z)+2\pi^2\right)$$
$$-\frac{G(\{-1\},y)}{3(y+1)^2}\left(4G(\{0,0\},z)-2G(\{0,1\},z)+10G(\{0,-y\},z)-2G(\{1,0\},z)\right.$$

$$+4G(\{1,1\},z)+G(\{1,-y\},z)-2G(\{-y,0\},z)-2G(\{-y,1\},z)-11G(\{-y,-y\},z)+6\pi^2\Big)$$
$$+\frac{8G(\{-1,-1,-1\},y)}{3(y+1)^2}-\frac{8G(\{-1,-1,0\},y)}{3(y+1)^2}+\frac{10G(\{-1,0,-1\},y)}{3(y+1)^2}$$
$$-\frac{10G(\{-1,0,0\},y)}{3(y+1)^2}+\frac{7G(\{0,-1,-1\},y)}{3(y+1)^2}-\frac{7G(\{0,-1,0\},y)}{3(y+1)^2}$$
$$-\frac{G(\{0,0,-1\},y)}{3(y+1)^2}+\frac{G(\{0,0,0\},y)}{3(y+1)^2}-\frac{2G(\{0,0,0\},z)}{3(y+1)^2}$$
$$-\frac{8G(\{0,0,1\},z)}{3(y+1)^2}+\frac{4G(\{0,0,-y\},z)}{3(y+1)^2}+\frac{4G(\{0,1,0\},z)}{3(y+1)^2}$$
$$-\frac{32G(\{0,1,1\},z)}{3(y+1)^2}-\frac{2G(\{0,1,-y\},z)}{3(y+1)^2}-\frac{8G(\{0,-y,0\},z)}{3(y+1)^2}$$
$$+\frac{16G(\{0,-y,1\},z)}{3(y+1)^2}+\frac{10G(\{0,-y,-y\},z)}{3(y+1)^2}-\frac{2G(\{1,0,0\},z)}{3(y+1)^2}$$
$$-\frac{8G(\{1,0,1\},z)}{3(y+1)^2}-\frac{2G(\{1,0,-y\},z)}{3(y+1)^2}-\frac{8G(\{1,1,0\},z)}{3(y+1)^2}$$
$$+\frac{64G(\{1,1,1\},z)}{3(y+1)^2}+\frac{4G(\{1,1,-y\},z)}{3(y+1)^2}-\frac{2G(\{1,-y,0\},z)}{3(y+1)^2}$$
$$+\frac{4G(\{1,-y,1\},z)}{3(y+1)^2}+\frac{G(\{1,-y,-y\},z)}{3(y+1)^2}+\frac{4G(\{-y,0,0\},z)}{3(y+1)^2}$$
$$+\frac{16G(\{-y,0,1\},z)}{3(y+1)^2}-\frac{2G(\{-y,0,-y\},z)}{3(y+1)^2}+\frac{4G(\{-y,1,0\},z)}{3(y+1)^2}$$
$$-\frac{32G(\{-y,1,1\},z)}{3(y+1)^2}-\frac{2G(\{-y,1,-y\},z)}{3(y+1)^2}+\frac{10G(\{-y,-y,0\},z)}{3(y+1)^2}$$
$$-\frac{20G(\{-y,-y,1\},z)}{3(y+1)^2}-\frac{11G(\{-y,-y,-y\},z)}{3(y+1)^2}+\frac{59\zeta(3)}{9(y+1)^2}\Big)$$
$$+O\left(\mathrm{ep}^3\right) \tag{6.35}$$

$$\tilde{M}_6 =\frac{1}{6y(y+1)}\times$$
$$\times\Big(-6\left(y^2(z+5)+3yz+y-2z\right)G(\{0,-1\},y)+6\left(y^2(z+5)+3yz+y-2z\right)G(\{0,0\},y)$$
$$+6G(\{-1\},y)\,(-(y-3)y(z-1)G(\{1\},z)+y(y-2z-1)G(\{0,1\},z)$$
$$+y(-y+2z+1)G(\{0,-y\},z)+(y+1)(y(z+2)-z)G(\{0\},z)$$
$$-(3y-1)(y+z)G(\{-y\},z)+2y^2+2yz+2y+2z\Big)$$
$$+6(y-3)y(z-1)G(\{1,-y\},z)-6y(y-2z-1)G(\{0,-1,-1\},y)$$
$$+6y(y-2z-1)G(\{0,-1,0\},y)+12y(y-2z-1)G(\{0,0,-1\},y)$$
$$-12y(y-2z-1)G(\{0,0,0\},y)-6y(y-2z-1)G(\{0,1,-y\},z)$$
$$+6y(y-2z-1)G(\{0,-y,-y\},z)-12(y+1)(y+z)G(\{-y\},z)$$
$$+6(y+1)(y(z+2)-z)G(\{-1,-1\},y)-6(y+1)(y(z+2)-z)G(\{-1,0\},y)$$
$$+3G(\{0\},z)\,(2y(y-2z-1)(G(\{0,0\},y)-G(\{0,-1\},y))$$
$$+2\left(\left(2+\pi^2\right)y+2\right)z+y\left(-\left(4+\pi^2\right)y+\pi^2-4\right)\Big)$$
$$-6(y+1)(y(z+2)-z)G(\{0,-y\},z)$$
$$+G(\{0\},y)\,(-6(y+1)(y(z+2)-z)G(\{0\},z)+6(y-3)y(z-1)G(\{1\},z)$$
$$+6((3y-1)(y+z)G(\{-y\},z)+y(y-2z-1)(G(\{0,-y\},z)-G(\{0,1\},z)))$$
$$+y\left(\pi^2(-y+2z+1)-12(y+z+1)\right)-12z\Big)$$

$$+ 6(3y-1)(y+z)G(\{-y,-y\},z) + 4\pi^2 y^2 z + 12y^2 z + 6y^2\zeta(3) - \pi^2 y^2 - 12y^2$$
$$-12yz\zeta(3) - 9\pi^2 yz + 12yz - 6y\zeta(3) + 11\pi^2 y - 12y - \pi^2 z\Big)$$
$$+ O(\epsilon) \tag{6.36}$$

$$\tilde{M}_7 = \frac{1}{6y(y+1)} \times$$
$$\times \left(-6(y+1)y(z+1)G(\{-1,-1\},y) + 6(y+1)y(z+1)G(\{-1,0\},y)\right.$$
$$+ 6y(y(z+3)+3z+1)G(\{0,-1\},y) - 6y(y(z+3)+3z+1)G(\{0,0\},y)$$
$$+ 6(y+1)y(z+1)G(\{0,-y\},z) + 6y(y(-z)+y+z-1)G(\{1,-y\},z)$$
$$- 12y(y+z)G(\{-y,-y\},z) + 6y(y-z)G(\{0,-1,-1\},y) + 6y(z-y)G(\{0,-1,0\},y)$$
$$+ 12y(z-y)G(\{0,0,-1\},y) + 12y(y-z)G(\{0,0,0\},y) + 6y(y-z)G(\{0,1,-y\},z)$$
$$+ 6y(z-y)G(\{0,-y,-y\},z) + 6(y+1)(y+z)G(\{-y\},z)$$
$$+ 3(y-z)\left(2yG(\{0,-1\},y) - 2yG(\{0,0\},y) + \left(2+\pi^2\right)y + 2\right)G(\{0\},z)$$
$$+ 6G(\{-1\},y)(-y(y+1)(z+1)G(\{0\},z) + (y-1)y(z-1)G(\{1\},z)$$
$$+ y(2(y+z)G(\{-y\},z) - (y-z)(G(\{0,1\},z) - G(\{0,-y\},z))) - (y+1)(y+z))$$
$$+ G(\{0\},y)\,(6y(y+1)(z+1)G(\{0\},z) - 6(y-1)y(z-1)G(\{1\},z)$$
$$+ 6y((y-z)(G(\{0,1\},z) - G(\{0,-y\},z)) - 2(y+z)G(\{-y\},z)) + \left(6 - \left(\pi^2-6\right)y\right)z$$
$$+y\left(\left(6+\pi^2\right)y + 6\right)\Big)$$
$$-4\pi^2 y^2 z - 6y^2 z - 6y^2\zeta(3) + 2\pi^2 y^2 + 6y^2 + 6yz\zeta(3) + 2\pi^2 yz - 6yz - 4\pi^2 y + 6y\Big)$$
$$+ O(\epsilon) \tag{6.37}$$

$$\tilde{M}_{10} = \frac{1}{6}\left(6G(\{0,0\},z) - 6G(\{1,0\},z) + \pi^2\right)$$
$$\frac{1}{6}\epsilon\left(-5\pi^2 G(\{0\},z) + 2\pi^2 G(\{1\},z) + 12G(\{0,0\},z) - 12G(\{1,0\},z)\right.$$
$$- 12G(\{0,0,0\},z) - 24G(\{0,0,1\},z) + 6G(\{0,1,0\},z) - 6G(\{1,0,0\},z)$$
$$\left.+24G(\{1,0,1\},z) + 12G(\{1,1,0\},z) - 6\zeta(3) + 2\pi^2\right)$$
$$+ O(\epsilon^2) \tag{6.38}$$

$$\tilde{M}_{11} = \frac{1}{\epsilon^3(y+1)^2(z-1)}$$
$$+ \frac{1}{\epsilon^2}\left(\frac{2G(\{0\},z) - 4G(\{1\},z)}{(y+1)^2(z-1)}\right)$$
$$+ \frac{1}{\epsilon}\,\frac{-48G(\{0,1\},z) - 24G(\{1,0\},z) + 96G(\{1,1\},z) - 11\pi^2}{6(y+1)^2(z-1)}$$
$$+ \frac{1}{3(y+1)^2(z-1)}\left(-6\pi^2 G(\{0\},z) + 17\pi^2 G(\{1\},z) - 6G(\{0,0,0\},z) - 6G(\{0,1,0\},z)\right.$$
$$+ 96G(\{0,1,1\},z) + 6G(\{1,0,0\},z) + 48G(\{1,0,1\},z)$$
$$\left.+30G(\{1,1,0\},z) - 192G(\{1,1,1\},z) - 74\zeta(3)\right) \tag{6.39}$$

$$
\begin{aligned}
\tilde{M}_4 =& \frac{1}{3y(y+1)} \Big(-\pi^2 yz - 3(y+1)G(\{0\},z)z + 3yG(\{-1,-1\},y)z - 3yG(\{-1,0\},y)z \\
&+ 3yG(\{0,-1\},y)z - 3yG(\{0,0\},y)z + 3yG(\{0,0\},z)z + 3yG(\{0,-y\},z)z \\
&- 3yG(\{-y-1,0\},z)z - 3yG(\{-y-1,-y\},z)z \\
&- 3G(\{-1\},y)((y+1)(y+z) + yz(G(\{0\},z) - G(\{-y-1\},z))) \\
&+3G(\{0\},y)((y+1)(y+z) + yz(G(\{0\},z) - G(\{-y-1\},z))) + 3(y+1)(y+z)G(\{-y\},z)) \\
&+ \frac{\epsilon}{3y(y+1)} \times \\
&\times \Big(-\pi^2 y^2 + 3(y+z+1)G(\{0,-1\},y)y - 3(y+z+1)G(\{0,0\},y)y + 2zG(\{-y-1\},z) \\
&\Big(-3G(\{-1,-1\},y) + 3G(\{-1,0\},y) - 3G(\{0,-1\},y) + 3G(\{0,0\},y) + 2\pi^2 \Big) y \\
&+ 3(y-z+1)G(\{-y-1,0\},z)y + 3(y-z+1)G(\{-y-1,-y\},z)y \\
&- 6zG(\{0,-1,-1\},y)y + 6zG(\{0,-1,0\},y)y - 9zG(\{0,0,0\},z)y \\
&- 12zG(\{0,0,1\},z)y - 6zG(\{0,0,-y\},z)y + 9zG(\{0,-y-1,0\},z)y \\
&+ 9zG(\{0,-y-1,-y\},z)y + 3zG(\{0,-y,0\},z)y - 12zG(\{0,-y,1\},z)y \\
&- 6zG(\{0,-y,-y\},z)y + 12zG(\{-y-1,0,1\},z)y - 3zG(\{-y-1,0,-y\},z)y \\
&- 3zG(\{-y-1,-y,0\},z)y + 12zG(\{-y-1,-y,1\},z)y + 6zG(\{-y-1,-y,-y\},z)y \\
&- 6z\zeta(3)y - \pi^2 y + \pi^2 z + 15(y+1)(y+z)G(\{-y\},z) \\
&+ 3\left(y^2 + 3zy + y + 2z\right) G(\{-1,-1\},y) - 3\left(y^2 + 3zy + y + 2z\right) G(\{-1,0\},y) \\
&- zG(\{0\},z)\,(3(G(\{-1,-1\},y) - G(\{-1,0\},y) + G(\{0,-1\},y) - G(\{0,0\},y))y \\
&+ \left(15 + \pi^2\right) y + 15\Big) \\
&+ 3(3y+2)zG(\{0,0\},z) + 12(y+1)zG(\{0,1\},z) - 3zG(\{0,-y\},z) \\
&+ G(\{0\},y)\left(15y(y+1) + \left(15 - \left(-15 + \pi^2\right) y\right) z \right. \\
&- 3zG(\{0\},z) + 3y(y-z+1)G(\{-y-1\},z) - 3(2(y+1)(y+z)G(\{-y\},z) + yz(2G(\{0,0\},z) \\
&-3G(\{0,-y-1\},z) + 2G(\{0,-y\},z) + G(\{-y-1,0\},z) - 2G(\{-y-1,-y\},z)))) \\
&+ G(\{-1\},y)\Big(-15y^2 + 2\pi^2 zy - 15zy - 3(y-z+1)G(\{-y-1\},z)y \\
&+ 6zG(\{0,0\},z)y - 9zG(\{0,-y-1\},z)y + 6zG(\{0,-y\},z)y + 3zG(\{-y-1,0\},z)y \\
&-6zG(\{-y-1,-y\},z)y - 15y - 15z + 3zG(\{0\},z) + 6(y+1)(y+z)G(\{-y\},z)) \\
&+ 3(y+1)(y+z)G(\{-y,0\},z) - 12(y+1)(y+z)G(\{-y,1\},z) \\
&-6(y+1)(y+z)G(\{-y,-y\},z)) \\
&+ O(\epsilon^2)
\end{aligned} \tag{6.40}
$$

$$
\begin{aligned}
\tilde{M}_5 =& \frac{1}{12y^2(y+1)(y+z+1)} \times \\
&\times \Big(3zy^4 - 3y^4 - 2\pi^2 z^2 y^3 + 6z^2 y^3 - 4\pi^2 zy^3 - 6y^3 - 2\pi^2 z^3 y^2 + 3z^3 y^2 - 6\pi^2 z^2 y^2 \\
&+ 6z^2 y^2 - 4\pi^2 zy^2 - 6zy^2 + 6z(z+2)(y+z+1)G(\{-1,-1\},y)y^2 \\
&- 6z(z+2)(y+z+1)G(\{-1,0\},y)y^2 + 6z(z+2)(y+z+1)G(\{0,-1\},y)y^2 \\
&- 6z(z+2)(y+z+1)G(\{0,0\},y)y^2 + 6z(z+2)(y+z+1)G(\{0,0\},z)y^2 \\
&+ 6z(z+2)(y+z+1)G(\{0,-y\},z)y^2 - 6z(z+2)(y+z+1)G(\{-y-1,0\},z)y^2
\end{aligned}
$$

$$
\begin{aligned}
& - 6z(z+2)(y+z+1)G(\{-y-1,-y\},z)y^2 - 3y^2 + 3z^3y - 3zy \\
& - 3(y+1)z\left(3zy^2 + (3z(z+2)+2)y + z^2 + z\right)G(\{0\},z) \\
& + 3G(\{0\},y)\left(2y^2z(z+2)(y+z+1)(G(\{0\},z) - G(\{-y-1\},z))\right. \\
& \left. -(y+1)(y+z)\left(y^3 - (z(3z+5)+1)y - z(z+1)\right)\right) \\
& + 3G(\{-1\},y)\left(2z(z+2)(y+z+1)(G(\{-y-1\},z) - G(\{0\},z))y^2\right. \\
& \left. +(y+1)(y+z)\left(y^3 - (z(3z+5)+1)y - z(z+1)\right)\right) \\
& \left. -3(y+1)(y+z)\left(y^3 - (z(3z+5)+1)y - z(z+1)\right)G(\{-y\},z)\right) \\
& + \frac{\epsilon}{24y^2(y+1)(y+z+1)} \times \\
& \times \left(2\pi^2y^5 - 2\pi^2zy^4 + 39zy^4 + 2\pi^2y^4 - 39y^4 + 78z^2y^3 - 20\pi^2zy^3 - 24z^2\zeta(3)y^3\right. \\
& - 48z\zeta(3)y^3 - 2\pi^2y^3 - 78y^3 + 4\pi^2z^3y^2 + 39z^3y^2 + 12\pi^2z^2y^2 + 78z^2y^2 - 14\pi^2zy^2 - 78zy^2 \\
& + 6\left(z^3 + 5(y+1)z^2 + 3(y+1)(y+3)z - (y-1)(y+1)^2\right)G(\{0,-1\},y)y^2 \\
& + 6\left(-z^3 - 5(y+1)z^2 - 3(y+1)(y+3)z + (y-1)(y+1)^2\right)G(\{0,0\},y)y^2 \\
& + 8z(z+2)(y+z+1)G(\{-y-1\},z)\left(-3G(\{-1,-1\},y) + 3G(\{-1,0\},y)\right. \\
& \left. -3G(\{0,-1\},y) + 3G(\{0,0\},y) + 2\pi^2\right)y^2 \\
& - 6(y+z+1)\left(y^2 + z^2 - 1\right)G(\{-y-1,0\},z)y^2 \\
& - 6(y+z+1)\left(y^2 + z^2 - 1\right)G(\{-y-1,-y\},z)y^2 \\
& - 24z(z+2)(y+z+1)G(\{0,-1,-1\},y)y^2 + 24z(z+2)(y+z+1)G(\{0,-1,0\},y)y^2 \\
& - 36z(z+2)(y+z+1)G(\{0,0,0\},z)y^2 - 48z(z+2)(y+z+1)G(\{0,0,1\},z)y^2 \\
& - 24z(z+2)(y+z+1)G(\{0,0,-y\},z)y^2 + 36z(z+2)(y+z+1)G(\{0,-y-1,0\},z)y^2 \\
& + 36z(z+2)(y+z+1)G(\{0,-y-1,-y\},z)y^2 + 12z(z+2)(y+z+1)G(\{0,-y,0\},z)y^2 \\
& - 48z(z+2)(y+z+1)G(\{0,-y,1\},z)y^2 - 24z(z+2)(y+z+1)G(\{0,-y,-y\},z)y^2 \\
& + 48z(z+2)(y+z+1)G(\{-y-1,0,1\},z)y^2 - 12z(z+2)(y+z+1)G(\{-y-1,0,-y\},z)y^2 \\
& - 12z(z+2)(y+z+1)G(\{-y-1,-y,0\},z)y^2 + 48z(z+2)(y+z+1)G(\{-y-1,-y,1\},z)y^2 \\
& + 24z(z+2)(y+z+1)G(\{-y-1,-y,-y\},z)y^2 - 24z^3\zeta(3)y^2 - 72z^2\zeta(3)y^2 \\
& - 48z\zeta(3)y^2 - 2\pi^2y^2 - 39y^2 + 8\pi^2z^3y + 39z^3y + 14\pi^2z^2y + 4\pi^2zy - 39zy \\
& - 24(y+1)(z-1)(y+z)(y+z+1)G(\{1\},z)y + 2\pi^2z^3 + 2\pi^2z^2 \\
& - 3(y+1)(y+z)\left(-9(3y+1)z^2 - (y(6y+47)+9)z + y(y+1)(9y-11)\right)G(\{-y\},z) \\
& + \left(6(y(7y+8)+2)z^3 + 6(y+1)(y(7y+12)+2)z^2 - 6y(y+1)((y-9)y-4)z\right. \\
& \left. -6(y-1)y^2(y+1)^2\right)G(\{-1,-1\},y) \\
& + \left(-6(y(7y+8)+2)z^3 - 6(y+1)(y(7y+12)+2)z^2 + 6y(y+1)((y-9)y-4)z\right. \\
& \left. +6(y-1)y^2(y+1)^2\right)G(\{-1,0\},y) \\
& + zG(\{0\},z)\left(12(z+2)(y+z+1)(-G(\{-1,-1\},y) + G(\{-1,0\},y)\right. \\
& - G(\{0,-1\},y) + G(\{0,0\},y))y^2 \\
& + 2(y+1)\left(y\left(6y - 4\pi^2 - 3\right) - 27\right)y
\end{aligned}
$$

$$
\begin{aligned}
&- \left(y \left(\left(75 + 4\pi^2\right) y + 102\right) + 27\right) z^2 \\
&- \left(y \left(y \left(63y + 4\pi^2(y+3) + 219\right) + 183\right) + 27\right) z\Big) \\
&+ 6z(y+z+1)(4y(y+1) + (y(7y+8)+2)z)G(\{0,0\},z) \\
&+ 24(y+1)z \left(3zy^2 + (3z(z+2)+2)y + z^2 + z\right) G(\{0,1\},z) \\
&- 6z \left(-4y^4 - 2(z+4)y^3 + 2(z(z+2)-1)y^2 + (z(4z+7)+2)y + z^2 + z\right) G(\{0,-y\},z) \\
&+ G(\{0\},y) \Big(\left(y \left(\left(81 - 4\pi^2\right) y + 108\right) + 27\right) z^3 \\
&+ \left(y \left(y \left(99y - 4\pi^2(y+3) + 267\right) + 195\right) + 27\right) z^2 \\
&- y(y+1) \left(y \left(9y + 8\pi^2 - 147\right) - 60\right) z \\
&- 6 \left(-4y^4 - 2(z+4)y^3 + 2(z(z+2)-1)y^2 + (z(4z+7)+2)y + z^2 + z\right) G(\{0\},z)z \\
&- 3y^2(y+1)^2(9y-11) - 6y^2(y+z+1) \left(y^2 + z^2 - 1\right) G(\{-y-1\},z) \\
&+ 12 \Big((y+1)(y+z) \left(y^3 - (z(3z+5)+1)y - z(z+1)\right) G(\{-y\},z) \\
&- y^2 z(z+2)(y+z+1)(2G(\{0,0\},z) - 3G(\{0,-y-1\},z) + 2G(\{0,-y\},z) \\
&+G(\{-y-1,0\},z) - 2G(\{-y-1,-y\},z)))) \\
&+ G(\{-1\},y) \Big(\left(y \left(\left(-81 + 8\pi^2\right) y - 108\right) - 27\right) z^3 \\
&+ \left(y \left(y \left(8\pi^2(y+3) - 3(33y+89)\right) - 195\right) - 27\right) z^2 \\
&+ y(y+1) \left(y \left(9y + 16\pi^2 - 147\right) - 60\right) z \\
&+ 6 \left(-4y^4 - 2(z+4)y^3 + 2(z(z+2)-1)y^2 + (z(4z+7)+2)y + z^2 + z\right) G(\{0\},z)z \\
&+ 3y^2(y+1)^2(9y-11) + 6y^2(y+z+1) \left(y^2 + z^2 - 1\right) G(\{-y-1\},z) \\
&+ 12 \Big(y^2 z(z+2)(y+z+1)(2G(\{0,0\},z) - 3G(\{0,-y-1\},z) + 2G(\{0,-y\},z) \\
&+ G(\{-y-1,0\},z) - 2G(\{-y-1,-y\},z)) \\
&-(y+1)(y+z) \left(y^3 - (z(3z+5)+1)y - z(z+1)\right) G(\{-y\},z)\Big)\Big) \\
&- 6(y+1)(y+z) \left(y^3 - (z(3z+5)+1)y - z(z+1)\right) G(\{-y,0\},z) \\
&+ 24(y+1)(y+z) \left(y^3 - (z(3z+5)+1)y - z(z+1)\right) G(\{-y,1\},z) \\
&+12(y+1)(y+z) \left(y^3 - (z(3z+5)+1)y - z(z+1)\right) G(\{-y,-y\},z)\Big) \\
&+ O(\epsilon^2)
\end{aligned}
\tag{6.41}
$$

$$
\begin{aligned}
\tilde{M}_6 =& \frac{1}{12y(y+1)(y+z+1)} \times \\
&\times \Big(-9z^2y^3 + 4\pi^2 zy^3 + 12zy^3 - 3y^3 - 9z^3y^2 - 6z^2y^2 + 4\pi^2 zy^2 + 21zy^2 - 6y^2 \\
&- 4\pi^2 z^3 y - 9z^3 y - 4\pi^2 z^2 y + 3z^2 y + 9zy - 12(y-z)z(y+z+1)G(\{-1,-1\},y)y \\
&+ 12(y-z)z(y+z+1)G(\{-1,0\},y)y - 12(y-z)z(y+z+1)G(\{0,-1\},y)y \\
&+ 12(y-z)z(y+z+1)G(\{0,0\},y)y \\
&- 12(y-z)z(y+z+1)G(\{0,0\},z)y - 12(y-z)z(y+z+1)G(\{0,-y\},z)y \\
&+ 12(y-z)z(y+z+1)G(\{-y-1,0\},z)y \\
&+ 12(y-z)z(y+z+1)G(\{-y-1,-y\},z)y
\end{aligned}
$$

$$
\begin{aligned}
&-3y+6(y+1)z\left((y-1)z^2+y(y+3)z+2y(y+2)+1\right)G(\{0\},z)\\
&+6G(\{-1\},y)\left((y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)\right.\\
&\left.+2y(y-z)z(y+z+1)(G(\{0\},z)-G(\{-y-1\},z))\right)\\
&+6G(\{0\},y)\left(2y(y-z)z(y+z+1)(G(\{-y-1\},z)-G(\{0\},z))\right.\\
&\left.-(y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)\right)\\
&\left.-6(y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y\},z)\right)\\
&+\frac{\epsilon}{24y(y+1)(y+z+1)}\times\\
&\times\left(8\pi^2y^4-8\pi^2z^2y^3-105z^2y^3+4\pi^2zy^3+144zy^3+48z\zeta(3)y^3+20\pi^2y^3-39y^3\right.\\
&-8\pi^2z^3y^2-105z^3y^2-32\pi^2z^2y^2-66z^2y^2-16\pi^2zy^2+249zy^2+48z\zeta(3)y^2\\
&+16\pi^2y^2-78y^2-4\pi^2z^3y-105z^3y-24\pi^2z^2y+39z^2y-24\pi^2zy+105zy\\
&+24(y+1)(z-1)(y+z+1)(3z-1)G(\{1\},z)y\\
&-12(y+1)\left(2y^2-(z-3)(z+1)y-(z+1)(z(z+2)-1)\right)G(\{0,-1\},y)y\\
&+12(y+1)\left(2y^2-(z-3)(z+1)y-(z+1)(z(z+2)-1)\right)G(\{0,0\},y)y\\
&-16(y-z)z(y+z+1)G(\{-y-1\},z)\left(-3G(\{-1,-1\},y)+3G(\{-1,0\},y)\right.\\
&\left.-3G(\{0,-1\},y)+3G(\{0,0\},y)+2\pi^2\right)y\\
&-12(y+1)\left((z-1)^2+2y\right)(y+z+1)G(\{-y-1,0\},z)y\\
&-12(y+1)\left((z-1)^2+2y\right)(y+z+1)G(\{-y-1,-y\},z)y\\
&+48(y-z)z(y+z+1)G(\{0,-1,-1\},y)y-48(y-z)z(y+z+1)G(\{0,-1,0\},y)y\\
&+72(y-z)z(y+z+1)G(\{0,0,0\},z)y+96(y-z)z(y+z+1)G(\{0,0,1\},z)y\\
&+48(y-z)z(y+z+1)G(\{0,0,-y\},z)y-72(y-z)z(y+z+1)G(\{0,-y-1,0\},z)y\\
&-72(y-z)z(y+z+1)G(\{0,-y-1,-y\},z)y-24(y-z)z(y+z+1)G(\{0,-y,0\},z)y\\
&+96(y-z)z(y+z+1)G(\{0,-y,1\},z)y+48(y-z)z(y+z+1)G(\{0,-y,-y\},z)y\\
&-96(y-z)z(y+z+1)G(\{-y-1,0,1\},z)y+24(y-z)z(y+z+1)G(\{-y-1,0,-y\},z)y\\
&+24(y-z)z(y+z+1)G(\{-y-1,-y,0\},z)y-96(y-z)z(y+z+1)G(\{-y-1,-y,1\},z)y\\
&-48(y-z)z(y+z+1)G(\{-y-1,-y,-y\},z)y-48z^3\zeta(3)y-48z^2\zeta(3)y+4\pi^2y-39y\\
&+4\pi^2z^3-4\pi^2z-12(y+1)(y+z)\left((3y-5)z^2+8yz+5y(2y+3)+5\right)G(\{-y\},z)\\
&-12(y+1)\left((y-2)z^3+y(y+3)z^2+(y(8y+7)+2)z+y(y+1)(2y+1)\right)G(\{-1,-1\},y)\\
&+12(y+1)\left((y-2)z^3+y(y+3)z^2+(y(8y+7)+2)z+y(y+1)(2y+1)\right)G(\{-1,0\},y)\\
&+2zG(\{0\},z)\left(\left(y\left(9y-4\pi^2-21\right)-30\right)z^2\right.\\
&+y\left(3(y+1)(3y+19)-4\pi^2\right)z+2(y+1)\left(2y\left(\left(15+\pi^2\right)y+27\right)+15\right)\\
&\left.+12y(y-z)(y+z+1)(G(\{-1,-1\},y)-G(\{-1,0\},y)+G(\{0,-1\},y)-G(\{0,0\},y))\right)\\
&-12(y+1)z(y+z+1)(-2z+y(z+6)+2)G(\{0,0\},z)\\
&-48(y+1)z\left((y-1)z^2+y(y+3)z+2y(y+2)+1\right)G(\{0,1\},z)\\
&+12(y+1)z\left((2y-1)z^2+2y(y+2)z+2y(y+2)+1\right)G(\{0,-y\},z)\\
&+4G(\{-1\},y)\left(\left(y\left(9y+4\pi^2-6\right)-15\right)z^3+y\left(9(y+1)^2+4\pi^2\right)z^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&- (y+1)\left(y\left(\left(-54+4\pi^2\right)y-45\right)-15\right)z \\
&- 3(y+1)\left((2y-1)z^2+2y(y+2)z+2y(y+2)+1\right)G(\{0\},z)z \\
&+ 15y(y+1)^2(2y+1)+3y(y+1)\left((z-1)^2+2y\right)(y+z+1)G(\{-y-1\},z) \\
&+ 6\left(-(y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y\},z)\right. \\
&- y(y-z)z(y+z+1)(2G(\{0,0\},z)-3G(\{0,-y-1\},z)+2G(\{0,-y\},z) \\
&+G(\{-y-1,0\},z)-2G(\{-y-1,-y\},z)))) \\
&+ 4G(\{0\},y)\left(\left(\left(\left(-9y-2\pi^2+6\right)y+15\right)z^3\right.\right. \\
&- y\left(9(y+1)^2+2\pi^2\right)z^2+(y+1)\left(y\left(2\left(-27+\pi^2\right)y-45\right)-15\right)z \\
&+ 3(y+1)\left((2y-1)z^2+2y(y+2)z+2y(y+2)+1\right)G(\{0\},z)z \\
&- 15y(y+1)^2(2y+1)-3y(y+1)\left((z-1)^2+2y\right)(y+z+1)G(\{-y-1\},z) \\
&+ 6\left((y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y\},z)\right. \\
&+ y(y-z)z(y+z+1)(2G(\{0,0\},z)-3G(\{0,-y-1\},z) \\
&+2G(\{0,-y\},z)+G(\{-y-1,0\},z)-2G(\{-y-1,-y\},z)))) \\
&- 12(y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y,0\},z) \\
&+ 48(y+1)(y+z)\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y,1\},z) \\
&+24(y+1)(y+z)\left.\left((y-1)z^2+2yz+y(2y+3)+1\right)G(\{-y,-y\},z)\right) \\
&+ O(\epsilon^2)
\end{aligned}
\tag{6.42}
$$

$$
\begin{aligned}
\tilde{M}_7 =&\frac{1}{3y(y+1)}\times\left(-\pi^2y-3zG(\{0\},y)G(\{0\},z)+G(\{-1\},y)(3zG(\{0\},z)+3(z-1)G(\{1\},z)\right. \\
&+ 6(2y+z+1)G(\{-2y-1\},z)-3(y+z+1)G(\{-y-1\},z)-9(y+z)G(\{-y\},z)) \\
&+ G(\{0\},y)(-3(z-1)G(\{1\},z)-6(2y+z+1)G(\{-2y-1\},z) \\
&+ 3(y+z+1)G(\{-y-1\},z)+9(y+z)G(\{-y\},z))-3(2y+z+2)G(\{-1,-1\},y) \\
&+ 3(2y+z+2)G(\{-1,0\},y)+6(2y+z+1)G\left(\left\{-\frac{1}{2},-1\right\},y\right) \\
&- 6(2y+z+1)G\left(\left\{-\frac{1}{2},0\right\},y\right)+(-6y-9z)G(\{0,-1\},y) \\
&+ (6y+9z)G(\{0,0\},y)-3zG(\{0,0\},z)-3zG(\{0,-y\},z)+3(z-1)G(\{1,0\},z) \\
&+ (3-3z)G(\{1,-y\},z)-6(2y+z+1)G(\{-2y-1,-y\},z)+3(y+z+1)G(\{-y-1,0\},z) \\
&+3(y+z+1)G(\{-y-1,-y\},z)-3(y+z)G(\{-y,0\},z)+9(y+z)G(\{-y,-y\},z)) \\
&+ \frac{\epsilon}{6y(y+1)}\times \\
&\times\left(-60\zeta(3)y+6\pi^2\log(4)y-12\pi^2y\right. \\
&+ 6\pi^2(2y+z+1)G\left(\left\{-\frac{1}{2}\right\},y\right)-36(2y+z+2)G(\{-1,-1\},y) \\
&+ 36(2y+z+2)G(\{-1,0\},y)+72(2y+z+1)G\left(\left\{-\frac{1}{2},-1\right\},y\right) \\
&- 72(2y+z+1)G\left(\left\{-\frac{1}{2},0\right\},y\right)
\end{aligned}
$$

$$
\begin{aligned}
&+6(2y+z+1)G(\{-2y-1\},z)\left(-4G\left(\left\{-\frac{1}{2},-1\right\},y\right)+4G\left(\left\{-\frac{1}{2},0\right\},y\right)+\pi^2\right)\\
&-36(2y+3z)G(\{0,-1\},y)+2(y+z)G(\{-y\},z)\,(18G(\{-1,-1\},y)-18G(\{-1,0\},y)\\
&-24G\left(\left\{-\frac{1}{2},-1\right\},y\right)+24G\left(\left\{-\frac{1}{2},0\right\},y\right)\\
&+36G(\{0,-1\},y)-36G(\{0,0\},y)+\pi^2\Big)\\
&+2zG(\{0\},z)\left(3G(\{-1,-1\},y)-3G(\{-1,0\},y)+3G(\{0,-1\},y)-3G(\{0,0\},y)+\pi^2\right)\\
&+36(2y+3z)G(\{0,0\},y)-4(y+z+1)G(\{-y-1\},z)\,(-3G(\{-1,-1\},y)+3G(\{-1,0\},y)\\
&-3G(\{0,-1\},y)+3G(\{0,0\},y)+2\pi^2\Big)\\
&-2(z-1)G(\{1\},z)\left(9G(\{-1,-1\},y)-9G(\{-1,0\},y)-9G(\{0,-1\},y)+9G(\{0,0\},y)+\pi^2\right)\\
&-36zG(\{0,0\},z)-36zG(\{0,-y\},z)+36(z-1)G(\{1,0\},z)-36(z-1)G(\{1,-y\},z)\\
&-72(2y+z+1)G(\{-2y-1,-y\},z)+36(y+z+1)G(\{-y-1,0\},z)\\
&+36(y+z+1)G(\{-y-1,-y\},z)-36(y+z)G(\{-y,0\},z)+108(y+z)G(\{-y,-y\},z)\\
&+2G(\{0\},y)\,(-18G(\{0\},z)z+6G(\{0,0\},z)z\\
&-9G(\{0,-y-1\},z)z+6G(\{0,-y\},z)z+\pi^2z-18(z-1)G(\{1\},z)\\
&-36(2y+z+1)G(\{-2y-1\},z)+18(y+z+1)G(\{-y-1\},z)+54(y+z)G(\{-y\},z)\\
&+6(z-1)G(\{1,0\},z)+6(z-1)G(\{1,1\},z)-3(z-1)G(\{1,-y\},z)\\
&+12(2y+z+1)G(\{-2y-1,-2y-1\},z)-6(2y+z+1)G(\{-2y-1,-y-1\},z)\\
&+6(2y+z+1)G(\{-2y-1,-y\},z)+3(y+z+1)G(\{-y-1,0\},z)\\
&-6(y+z+1)G(\{-y-1,-y\},z)+3(y+z)G(\{-y,0\},z)+12(y+z)G(\{-y,1\},z)\\
&+24(y+z)G(\{-y,-2y-1\},z)-3(y+z)G(\{-y,-y-1\},z)\\
&-57(y+z)G(\{-y,-y\},z))\\
&+2G(\{-1\},y)\Big(-4\pi^2y-4\pi^2z+18zG(\{0\},z)\\
&+18(z-1)G(\{1\},z)+36(2y+z+1)G(\{-2y-1\},z)-18(y+z+1)G(\{-y-1\},z)\\
&-54(y+z)G(\{-y\},z)-6zG(\{0,0\},z)+9zG(\{0,-y-1\},z)\\
&-6zG(\{0,-y\},z)-6(z-1)G(\{1,0\},z)-6(z-1)G(\{1,1\},z)\\
&+3(z-1)G(\{1,-y\},z)-12(2y+z+1)G(\{-2y-1,-2y-1\},z)\\
&+6(2y+z+1)G(\{-2y-1,-y-1\},z)-6(2y+z+1)G(\{-2y-1,-y\},z)\\
&-3(y+z+1)G(\{-y-1,0\},z)+6(y+z+1)G(\{-y-1,-y\},z)-3(y+z)G(\{-y,0\},z)\\
&-12(y+z)G(\{-y,1\},z)-24(y+z)G(\{-y,-2y-1\},z)\\
&+3(y+z)G(\{-y,-y-1\},z)+57(y+z)G(\{-y,-y\},z)-2\pi^2\Big)\\
&+36(y+1)G(\{-1,-1,-1\},y)-36(y+1)G(\{-1,-1,0\},y)\\
&+24(z-1)G\left(\left\{-1,-\frac{1}{2},-1\right\},y\right)-24(z-1)G\left(\left\{-1,-\frac{1}{2},0\right\},y\right)\\
&-12(y+2z-1)G(\{-1,0,-1\},y)+12(y+2z-1)G(\{-1,0,0\},y)\\
&-24(2y+z+1)G\left(\left\{-\frac{1}{2},-\frac{1}{2},-1\right\},y\right)+24(2y+z+1)G\left(\left\{-\frac{1}{2},-\frac{1}{2},0\right\},y\right)\\
&-12(y-2z+2)G(\{0,-1,-1\},y)+12(y-2z+2)G(\{0,-1,0\},y)\\
&-24(z-1)G\left(\left\{0,-\frac{1}{2},-1\right\},y\right)+24(z-1)G\left(\left\{0,-\frac{1}{2},0\right\},y\right)\\
&+36(y+z)G(\{0,0,-1\},y)-36(y+z)G(\{0,0,0\},y)+18zG(\{0,0,0\},z)\\
&+24zG(\{0,0,1\},z)+12zG(\{0,0,-y\},z)-18zG(\{0,-y-1,0\},z)\\
&-18zG(\{0,-y-1,-y\},z)-6zG(\{0,-y,0\},z)+24zG(\{0,-y,1\},z)
\end{aligned}
$$

$$
\begin{aligned}
&+12zG(\{0,-y,-y\},z)+(6-6z)G(\{1,0,0\},z)-24(z-1)G(\{1,0,1\},z)\\
&+12(z-1)G(\{1,0,-y\},z)-12(z-1)G(\{1,1,0\},z)+12(z-1)G(\{1,1,-y\},z)\\
&+24(z-1)G(\{1,-y,1\},z)+(6-6z)G(\{1,-y,-y\},z)\\
&+24(2y+z+1)G(\{-2y-1,-2y-1,-y\},z)-12(2y+z+1)G(\{-2y-1,-y-1,0\},z)\\
&-12(2y+z+1)G(\{-2y-1,-y-1,-y\},z)-12(2y+z+1)G(\{-2y-1,-y,0\},z)\\
&+48(2y+z+1)G(\{-2y-1,-y,1\},z)+12(2y+z+1)G(\{-2y-1,-y,-y\},z)\\
&-24(y+z+1)G(\{-y-1,0,1\},z)+6(y+z+1)G(\{-y-1,0,-y\},z)\\
&+6(y+z+1)G(\{-y-1,-y,0\},z)-24(y+z+1)G(\{-y-1,-y,1\},z)\\
&-12(y+z+1)G(\{-y-1,-y,-y\},z)+24(y+z)G(\{-y,0,0\},z)+24(y+z)G(\{-y,0,1\},z)\\
&+6(y+z)G(\{-y,0,-y\},z)-24(y+z)G(\{-y,1,0\},z)+24(y+z)G(\{-y,1,-y\},z)\\
&+48(y+z)G(\{-y,-2y-1,-y\},z)-6(y+z)G(\{-y,-y-1,0\},z)\\
&-6(y+z)G(\{-y,-y-1,-y\},z)+48(y+z)G(\{-y,-y,0\},z)\\
&-72(y+z)G(\{-y,-y,1\},z)-114(y+z)G(\{-y,-y,-y\},z)\\
&-48z\zeta(3)+3\pi^2 z\log(4)+3\pi^2\log(4)\Big)\\
&+O(\epsilon^2) \qquad (6.43)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_8 =&\frac{1}{18y^2(y+1)}\times\\
&\times\Big(-2\pi^2y^3-3\pi^2zy^2-6zy^2-3\pi^2y^2+6y^2-6zy+6y\\
&+6(y+1)zG(\{0\},z)+6\left(y^2-1\right)(y+z)G(\{-y\},z)\\
&+G(\{-1\},y)\Big(-6y^3-6zy^2+6y+6z+3z(6y-(z-3)z)G(\{0\},z)\\
&-3(z-1)^3G(\{1\},z)+6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1\},z)\\
&-3\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1\},z)\\
&-9(2y-z+3)(y+z)^2G(\{-y\},z)\Big)\\
&+3G(\{0\},y)\Big(G(\{1\},z)(z-1)^3+2\left(y^2-1\right)(y+z)+z((z-3)z-6y)G(\{0\},z)\\
&-2(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1\},z)\\
&+\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1\},z)\\
&+3(2y-z+3)(y+z)^2G(\{-y\},z)\Big)\\
&-3\left(4y^3+6(z+1)y^2+18zy+z(6-(z-3)z)-2\right)G(\{-1,-1\},y)\\
&+3\left(4y^3+6(z+1)y^2+18zy+z(6-(z-3)z)-2\right)G(\{-1,0\},y)\\
&+6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G\left(\left\{-\frac{1}{2},-1\right\},y\right)\\
&-6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G\left(\left\{-\frac{1}{2},0\right\},y\right)\\
&-3\left(4y^3+6(z+1)y^2+18zy-3(z-3)z^2\right)G(\{0,-1\},y)\\
&+3\left(4y^3+6(z+1)y^2+18zy-3(z-3)z^2\right)G(\{0,0\},y)\\
&+3z((z-3)z-6y)G(\{0,0\},z)+3z((z-3)z-6y)G(\{0,-y\},z)\\
&-3(z-1)^3G(\{1,0\},z)+3(z-1)^3G(\{1,-y\},z)
\end{aligned}
$$

$$
\begin{aligned}
&- 6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y\},z)\\
&+\left(6y^3+9(z+1)y^2+36zy-3(z+1)((z-4)z+1)\right)G(\{-y-1,0\},z)\\
&+\left(6y^3+9(z+1)y^2+36zy-3(z+1)((z-4)z+1)\right)G(\{-y-1,-y\},z)\\
&-3(2y-z+3)(y+z)^2G(\{-y,0\},z)+9(2y-z+3)(y+z)^2G(\{-y,-y\},z)\Big)\\
&+\frac{\epsilon}{36y^2(y+1)}\times\\
&\times\Big(-120\zeta(3)y^3+12\pi^2\log(4)y^3-28\pi^2y^3-24\pi^2zy^2-84zy^2\\
&-180z\zeta(3)y^2-180\zeta(3)y^2+18\pi^2z\log(4)y^2+18\pi^2\log(4)y^2-28\pi^2y^2\\
&+84y^2+6\pi^2z^2y-34\pi^2zy-84zy-288z\zeta(3)y+36\pi^2z\log(4)y+8\pi^2y\\
&+84y-4\pi^2z+6\pi^2(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G\left(\left\{-\frac{1}{2}\right\},y\right)\\
&-6\left(22y^3+(34z+44)y^2+2(2z(z+20)+7)y+z((13-4z)z+37)-8\right)G(\{-1,-1\},y)\\
&+6\left(22y^3+(34z+44)y^2+2(2z(z+20)+7)y+z((13-4z)z+37)-8\right)G(\{-1,0\},y)\\
&+48(2y+z+1)\left(3y^2+3(z+1)y-(z-5)z-1\right)G\left(\left\{-\frac{1}{2},-1\right\},y\right)\\
&-48(2y+z+1)\left(3y^2+3(z+1)y-(z-5)z-1\right)G\left(\left\{-\frac{1}{2},0\right\},y\right)\\
&+6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1\},z)\\
&\times\left(-4G\left(\left\{-\frac{1}{2},-1\right\},y\right)\right.\\
&\left.+4G\left(\left\{-\frac{1}{2},0\right\},y\right)+\pi^2\right)\\
&-6\left(22y^3+(58z+28)y^2+\left(22z^2+86z-2\right)y+z((55-12z)z-5)\right)G(\{0,-1\},y)\\
&+2(y+z)G(\{-y\},z)\left(30\left(y^2-1\right)\right.\\
&+\pi^2(2y-z+3)(y+z)+6(2y-z+3)(y+z)\left(3G(\{-1,-1\},y)-3G(\{-1,0\},y)\right.\\
&-4G\left(\left\{-\frac{1}{2},-1\right\},y\right)+4G\left(\left\{-\frac{1}{2},0\right\},y\right)\\
&\left.\left.+6G(\{0,-1\},y)-6G(\{0,0\},y)\right)\right)\\
&+2zG(\{0\},z)\left(-12y^2+6\left(3+\pi^2\right)y-\pi^2(z-3)z\right.\\
&\left.+3(6y-(z-3)z)(G(\{-1,-1\},y)-G(\{-1,0\},y)+G(\{0,-1\},y)-G(\{0,0\},y))+30\right)\\
&+6\left(22y^3+(58z+28)y^2+\left(22z^2+86z-2\right)y+z((55-12z)z-5)\right)G(\{0,0\},y)\\
&-4\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1\},z)\left(-3G(\{-1,-1\},y)\right.\\
&\left.+3G(\{-1,0\},y)-3G(\{0,-1\},y)+3G(\{0,0\},y)+2\pi^2\right)\\
&+2(z-1)G(\{1\},z)\left(9G(\{-1,-1\},y)(z-1)^2-9G(\{-1,0\},y)(z-1)^2-9G(\{0,-1\},y)(z-1)^2\right.\\
&\left.+\pi^2(z-1)^2+24y(y+1)+9(z-2)zG(\{0,0\},y)+9G(\{0,0\},y)\right)\\
&-6z\left(8y^2+2(4z+9)y+(17-4z)z+1\right)G(\{0,0\},z)-48(y+1)zG(\{0,1\},z)\\
&-6z\left(16y^2+2(5z+11)y+(19-4z)z-3\right)G(\{0,-y\},z)
\end{aligned}
$$

$$
\begin{aligned}
&+24(z-1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{1,0\},z)\\
&+24(z-1)\left(-2y^2-2(z+1)y+(z-4)z+1\right)G(\{1,-y\},z)\\
&-48(2y+z+1)\left(3y^2+3(z+1)y-(z-5)z-1\right)G(\{-2y-1,-y\},z)\\
&+6\left(16y^3+(27z+25)y^2+(z(7z+52)+5)y-4(z+1)((z-5)z+1)\right)G(\{-y-1,0\},z)\\
&+6\left(16y^3+(27z+25)y^2+(z(7z+52)+5)y-4(z+1)((z-5)z+1)\right)G(\{-y-1,-y\},z)\\
&-6(y+z)\left(12y^2+(11z+17)y+(19-4z)z+1\right)G(\{-y,0\},z)-48\left(y^2-1\right)(y+z)G(\{-y,1\},z)\\
&+6(y+z)\left(34y^2+(31z+45)y+(55-12z)z-1\right)G(\{-y,-y\},z)\\
&+2G(\{0\},y)\left(30y^3+30zy^2+6\pi^2zy-30y-\pi^2z^3+3\pi^2z^2-30z\right.\\
&+3z\left(-16y^2-2(5z+11)y+z(4z-19)+3\right)G(\{0\},z)\\
&+12(z-1)\left(-2y^2-2(z+1)y+(z-4)z+1\right)G(\{1\},z)\\
&-24(2y+z+1)\left(3y^2+3(z+1)y-(z-5)z-1\right)G(\{-2y-1\},z)\\
&+3\left(16y^3+(27z+25)y^2+(z(7z+52)+5)y-4(z+1)((z-5)z+1)\right)G(\{-y-1\},z)\\
&+3(y+z)\left(34y^2+(31z+45)y+(55-12z)z-1\right)G(\{-y\},z)\\
&+6z(6y-(z-3)z)G(\{0,0\},z)\\
&+9z((z-3)z-6y)G(\{0,-y-1\},z)+6z(6y-(z-3)z)G(\{0,-y\},z)\\
&-6(z-1)^3G(\{1,0\},z)-6(z-1)^3G(\{1,1\},z)\\
&+3(z-1)^3G(\{1,-y\},z)\\
&+12(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-2y-1\},z)\\
&-6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y-1\},z)\\
&+6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y\},z)\\
&+3\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,0\},z)\\
&-6\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,-y\},z)\\
&+3(2y-z+3)(y+z)^2G(\{-y,0\},z)\\
&+12(2y-z+3)(y+z)^2G(\{-y,1\},z)+24(2y-z+3)(y+z)^2G(\{-y,-2y-1\},z)\\
&-3(2y-z+3)(y+z)^2G(\{-y,-y-1\},z)\\
&\left.-57(2y-z+3)(y+z)^2G(\{-y,-y\},z)\right)+2G(\{-1\},y)\left(-8\pi^2y^3-30y^3-12\pi^2zy^2-30zy^2\right.\\
&-12\pi^2y^2-36\pi^2zy+30y+4\pi^2z^3-12\pi^2z^2-6\pi^2z+30z\\
&+3z\left(16y^2+2(5z+11)y+(19-4z)z-3\right)G(\{0\},z)\\
&+12(z-1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{1\},z)\\
&+24(2y+z+1)\left(3y^2+3(z+1)y-(z-5)z-1\right)G(\{-2y-1\},z)\\
&-3\left(16y^3+(27z+25)y^2+(z(7z+52)+5)y-4(z+1)((z-5)z+1)\right)G(\{-y-1\},z)\\
&-3(y+z)\left(34y^2+(31z+45)y+(55-12z)z-1\right)G(\{-y\},z)\\
&+6z((z-3)z-6y)G(\{0,0\},z)
\end{aligned}
$$

$$+ 9z(6y-(z-3)z)G(\{0,-y-1\},z)+6z((z-3)z-6y)G(\{0,-y\},z)$$
$$+ 6(z-1)^3G(\{1,0\},z)+6(z-1)^3G(\{1,1\},z)$$
$$- 3(z-1)^3G(\{1,-y\},z)-12(2y+z+1)\Big(2y^2$$
$$+2(z+1)y-(z-4)z-1\Big)G(\{-2y-1,-2y-1\},z)$$
$$+ 6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y-1\},z)$$
$$- 6(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y\},z)$$
$$- 3\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,0\},z)$$
$$+ 6\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,-y\},z)$$
$$- 3(2y-z+3)(y+z)^2G(\{-y,0\},z)$$
$$- 12(2y-z+3)(y+z)^2G(\{-y,1\},z)-24(2y-z+3)(y+z)^2G(\{-y,-2y-1\},z)$$
$$+ 3(2y-z+3)(y+z)^2G(\{-y,-y-1\},z)$$
$$+57(2y-z+3)(y+z)^2G(\{-y,-y\},z)+2\pi^2\Big)$$
$$+ 36(y+1)^2(2y+3z-1)G(\{-1,-1,-1\},y)$$
$$- 36(y+1)^2(2y+3z-1)G(\{-1,-1,0\},y)-24(z-1)^3G\left(\left\{-1,-\frac{1}{2},-1\right\},y\right)$$
$$+ 24(z-1)^3G\left(\left\{-1,-\frac{1}{2},0\right\},y\right)$$
$$- 12\left(2y^3+3(z+1)y^2+6zy+z(-2(z-3)z-3)+1\right)G(\{-1,0,-1\},y)$$
$$+ 12\left(2y^3+3(z+1)y^2+6zy+z(-2(z-3)z-3)+1\right)G(\{-1,0,0\},y)$$
$$- 24(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G\left(\left\{-\frac{1}{2},-\frac{1}{2},-1\right\},y\right)$$
$$+ 24(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G\left(\left\{-\frac{1}{2},-\frac{1}{2},0\right\},y\right)$$
$$- 12\left(2y^3+3(z+1)y^2+2(z-1)^3\right)G(\{0,-1,-1\},y)$$
$$+ 12\left(2y^3+3(z+1)y^2+2(z-1)^3\right)G(\{0,-1,0\},y)$$
$$+ 24(z-1)^3G\left(\left\{0,-\frac{1}{2},-1\right\},y\right)-24(z-1)^3G\left(\left\{0,-\frac{1}{2},0\right\},y\right)$$
$$+ 36(2y-z+3)(y+z)^2G(\{0,0,-1\},y)-36(2y-z+3)(y+z)^2G(\{0,0,0\},y)$$
$$+ 18z(6y-(z-3)z)G(\{0,0,0\},z)$$
$$+ 24z(6y-(z-3)z)G(\{0,0,1\},z)-12z((z-3)z-6y)G(\{0,0,-y\},z)$$
$$+ 18z((z-3)z-6y)G(\{0,-y-1,0\},z)$$
$$+ 18z((z-3)z-6y)G(\{0,-y-1,-y\},z)+6z((z-3)z-6y)G(\{0,-y,0\},z)$$
$$+ 24z(6y-(z-3)z)G(\{0,-y,1\},z)$$
$$- 12z((z-3)z-6y)G(\{0,-y,-y\},z)+6(z-1)^3G(\{1,0,0\},z)$$
$$+ 24(z-1)^3G(\{1,0,1\},z)$$
$$- 12(z-1)^3G(\{1,0,-y\},z)+12(z-1)^3G(\{1,1,0\},z)-12(z-1)^3G(\{1,1,-y\},z)$$
$$- 24(z-1)^3G(\{1,-y,1\},z)+6(z-1)^3G(\{1,-y,-y\},z)$$
$$+ 24(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-2y-1,-y\},z)$$
$$- 12(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y-1,0\},z)$$

$$
\begin{aligned}
&- 12(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y-1,-y\},z)\\
&- 12(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y,0\},z)\\
&+ 48(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y,1\},z)\\
&+ 12(2y+z+1)\left(2y^2+2(z+1)y-(z-4)z-1\right)G(\{-2y-1,-y,-y\},z)\\
&- 24\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,0,1\},z)\\
&+ 6\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,0,-y\},z)\\
&+ 6\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,-y,0\},z)\\
&- 24\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,-y,1\},z)\\
&- 12\left(2y^3+3(z+1)y^2+12zy-(z+1)((z-4)z+1)\right)G(\{-y-1,-y,-y\},z)\\
&+ 24(2y-z+3)(y+z)^2G(\{-y,0,0\},z)+24(2y-z+3)(y+z)^2G(\{-y,0,1\},z)\\
&+ 6(2y-z+3)(y+z)^2G(\{-y,0,-y\},z)\\
&- 24(2y-z+3)(y+z)^2G(\{-y,1,0\},z)+24(2y-z+3)(y+z)^2G(\{-y,1,-y\},z)\\
&+ 48(2y-z+3)(y+z)^2G(\{-y,-2y-1,-y\},z)\\
&- 6(2y-z+3)(y+z)^2G(\{-y,-y-1,0\},z)-6(2y-z+3)(y+z)^2G(\{-y,-y-1,-y\},z)\\
&+ 48(2y-z+3)(y+z)^2G(\{-y,-y,0\},z)\\
&- 72(2y-z+3)(y+z)^2G(\{-y,-y,1\},z)-114(2y-z+3)(y+z)^2G(\{-y,-y,-y\},z)\\
&+ 48z^3\zeta(3)-144z^2\zeta(3)-3\pi^2z^3\log(4)\\
&+9\pi^2z^2\log(4)+9\pi^2z\log(4)-3\pi^2\log(4)\Big)\\
&+ O(\epsilon^2) \qquad (6.44)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_9 =& \frac{1}{\epsilon}\frac{3G(\{-y,0\},z)-3G(\{0,-1\},y)+3G(\{0,0\},y)-3G(\{1,0\},z)+\pi^2}{3(y+1)}\\
&+ \frac{1}{3(y+1)}\times\\
&\times\Big(\left(-3G(\{-1,-1\},y)+3G(\{-1,0\},y)+9G(\{0,-1\},y)-9G(\{0,0\},y)-4\pi^2\right)\\
&G(\{-y\},z)+G(\{-1\},y)\left(-3G(\{-y,0\},z)-3G(\{-y,-y-1\},z)\right.\\
&\left.+6G(\{1,0\},z)+2\pi^2\right)\\
&+ 3G(\{0\},y)\left(G(\{-y,0\},z)+G(\{-y,-y-1\},z)-2G(\{1,0\},z)-\pi^2\right)\\
&- 6G(\{1,0,-y\},z)-6G(\{1,-y,0\},z)-6G(\{-y,0,0\},z)\\
&- 12G(\{-y,0,1\},z)+3G(\{-y,0,-y\},z)+6G(\{-y,1,0\},z)\\
&+ 3G(\{-y,-y-1,0\},z)+3G(\{-y,-y-1,-y\},z)-6G(\{-y,-y,0\},z)\\
&- 6G(\{-1,0,-1\},y)+6G(\{-1,0,0\},y)+3G(\{0,-1,-1\},y)\\
&- 3G(\{0,-1,0\},y)+9G(\{0,0,-1\},y)-9G(\{0,0,0\},y)\\
&+ \pi^2G(\{1\},z)+3G(\{1,0,0\},z)+12G(\{1,0,1\},z)\\
&+6G(\{1,1,0\},z)+6\zeta(3)\Big)\\
&+ O(\epsilon) \qquad (6.45)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{10} = &- \frac{1}{2\mathrm{ep}^3(y+1)^2} \\
&\frac{1}{\epsilon^2}\frac{-G(\{-y\},z)+G(\{-1\},y)-G(\{0\},y)-G(\{0\},z)+2G(\{1\},z)}{(y+1)^2} \\
&+\frac{1}{\epsilon}\frac{1}{12(y+1)^2}\left(-24G(\{0\},y)G(\{0\},z)+24G(\{-1\},y)(G(\{0\},z)\right. \\
&-G(\{-y\},z))+24G(\{0\},y)G(\{-y\},z)-24G(\{0,-y\},z)-24G(\{-y,0\},z) \\
&+48G(\{-y,1\},z)+24G(\{-y,-y\},z)-24G(\{-1,-1\},y) \\
&+24G(\{-1,0\},y)-24G(\{0,-1\},y)+24G(\{0,0\},y) \\
&\left.+24G(\{0,0\},z)+48G(\{0,1\},z)-96G(\{1,1\},z)+23\pi^2\right) \\
&+\frac{1}{6(y+1)^2}\left(\left(-24G(\{-1,-1\},y)+24G(\{-1,0\},y)\right.\right. \\
&\left.+12G(\{0,-1\},y)-12G(\{0,0\},y)-\pi^2\right)G(\{0\},z) \\
&+\left(24G(\{-1,-1\},y)-24G(\{-1,0\},y)-12G(\{0,-1\},y)\right. \\
&\left.+12G(\{0,0\},y)+13\pi^2\right)G(\{-y\},z) \\
&+G(\{0\},y)\left(24G(\{0,-y\},z)-12G(\{-y,0\},z)-24G(\{-y,-y\},z)\right. \\
&\left.+24G(\{0,0\},z)-12G(\{1,0\},z)+13\pi^2\right) \\
&+G(\{-1\},y)\left(-24G(\{0,-y\},z)+12G(\{-y,0\},z)+24G(\{-y,-y\},z)\right. \\
&\left.-24G(\{0,0\},z)+12G(\{1,0\},z)-23\pi^2\right) \\
&+24G(\{0,0,-y\},z)-12G(\{0,-y,0\},z)+48G(\{0,-y,1\},z) \\
&+24G(\{0,-y,-y\},z)-12G(\{1,0,-y\},z)-12G(\{1,-y,0\},z) \\
&+24G(\{-y,0,0\},z)+48G(\{-y,0,1\},z)-12G(\{-y,0,-y\},z) \\
&-96G(\{-y,1,1\},z)+24G(\{-y,-y,0\},z)-48G(\{-y,-y,1\},z) \\
&-24G(\{-y,-y,-y\},z)+24G(\{-1,-1,-1\},y)-24G(\{-1,-1,0\},y) \\
&+24G(\{-1,0,-1\},y)-24G(\{-1,0,0\},y)+24G(\{0,-1,-1\},y) \\
&-24G(\{0,-1,0\},y)-12G(\{0,0,-1\},y)+12G(\{0,0,0\},y) \\
&-12\pi^2G(\{1\},z)-24G(\{0,0,0\},z)-48G(\{0,0,1\},z) \\
&+12G(\{0,1,0\},z)-96G(\{0,1,1\},z)-12G(\{1,1,0\},z) \\
&\left.+192G(\{1,1,1\},z)+146\zeta(3)\right) \\
&+O(\epsilon)
\end{aligned}
\tag{6.46}
$$

Master Integrals for $[\mathbf{b}+\mathbf{W}^*\to\mathbf{t}+\mathbf{g}]_{\mathbf{1loop}}$.

$$
\begin{aligned}
\tilde{M}_1 = &-\frac{1}{\epsilon}\frac{G(\{0\},z)}{8\pi(y+1)} \\
&+\frac{1}{16\pi(y+1)}\times\left(2G(\{-1\},y)(G(\{y+1\},z)+G(\{0\},z))-2G(\{y+1,0\},z)\right. \\
&+2G(\{-1,-1\},y)-2G(\{0,-1\},y)+4G(\{0,0\},z) \\
&\left.+4G(\{0,1\},z)-2\log(4\pi)G(\{0\},z)+\pi^2\right) \\
&+\frac{\epsilon}{48\pi(y+1)}\times \\
&\times\left(\left(-12G(\{-1,-1\},y)+6G(\{0,-1\},y)+\pi^2\right)G(\{y+1\},z)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 6G(\{0, y + 1, 0\}, z) + 12G(\{y + 1, 0, 1\}, z) + 6G(\{y + 1, 1, 0\}, z)\\
&+ G(\{0\}, z) \left(-3 \left(6G(\{-1, -1\}, y) - 4G(\{0, -1\}, y) + \pi^2 + \log^2(4)\right)\right.\\
&\left.- \log(\pi) \log \left(4096\pi^3\right)\right)\\
&+ G(\{-1\}, y) \left(6 \log(4\pi)(G(\{y + 1\}, z) + G(\{0\}, z))\right.\\
&- 3 \left(2G(\{0, y + 1\}, z) - 2G(\{y + 1, 0\}, z) + 4G(\{y + 1, 1\}, z)\right.\\
&\left.\left.+4G(\{0, 0\}, z) + 4G(\{0, 1\}, z) + \pi^2\right)\right)\\
&- 6 \log(4\pi)G(\{y + 1, 0\}, z) - 12G(\{-1, -1, -1\}, y) + 6G(\{-1, 0, -1\}, y)\\
&+ 6G(\{0, -1, -1\}, y) + 6 \log(4\pi)G(\{-1, -1\}, y) - 6 \log(4\pi)G(\{0, -1\}, y)\\
&- 24G(\{0, 0, 0\}, z) - 24G(\{0, 0, 1\}, z) - 6G(\{0, 1, 0\}, z)\\
&- 24G(\{0, 1, 1\}, z) + 12 \log(4\pi)G(\{0, 0\}, z) + 12 \log(4\pi)G(\{0, 1\}, z)\\
&\left.+18\zeta(3) + 3\pi^2 \log(4\pi)\right)\\
&+ O(\epsilon^2) \qquad (6.47)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_2 =& + \frac{1}{48\pi(y + 1)} \left(6G(\{-1\}, y)(G(\{0\}, z) - G(\{y + 1\}, z)) + 6G(\{y + 1, 0\}, z)\right.\\
&\left.-6G(\{-1, -1\}, y) + 6G(\{0, -1\}, y) - 6G(\{0, 0\}, z) - \pi^2\right)\\
&\frac{\epsilon}{48\pi(y + 1)} \left(\left(6G(\{-1, -1\}, y) + 4 \left(\pi^2 - 3G(\{0, -1\}, y)\right)\right) G(\{0\}, z)\right.\\
&+ \left(12G(\{-1, -1\}, y) - 6G(\{0, -1\}, y) - \pi^2\right) G(\{y + 1\}, z) - 18G(\{0, y + 1, 0\}, z)\\
&- 12G(\{y + 1, 0, 1\}, z) - 6G(\{y + 1, 1, 0\}, z)\\
&+ G(\{-1\}, y) \left(6 \log(4\pi)(G(\{0\}, z) - G(\{y + 1\}, z))\right.\\
&- 3 \left(-6G(\{0, y + 1\}, z) + 2G(\{y + 1, 0\}, z) - 4G(\{y + 1, 1\}, z)\right.\\
&\left.\left.+4G(\{0, 0\}, z) + 4G(\{0, 1\}, z) + \pi^2\right)\right)\\
&+ 6 \log(4\pi)G(\{y + 1, 0\}, z) + 6G(\{-1, 0, -1\}, y) - 6G(\{0, -1, -1\}, y)\\
&- 6 \log(4\pi)G(\{-1, -1\}, y) + 6 \log(4\pi)G(\{0, -1\}, y) + 18G(\{0, 0, 0\}, z)\\
&\left.+12G(\{0, 0, 1\}, z) + 6G(\{0, 1, 0\}, z) - 6 \log(4\pi)G(\{0, 0\}, z) + 18\zeta(3) - \pi^2 \log(4\pi)\right)\\
&+ O(\epsilon^2) \qquad (6.48)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_3 =& \frac{1}{\epsilon}\frac{1 - z}{8\pi}\\
&+ \frac{1}{48\pi(y + 1)} \left(6(y + 1)zG(\{0\}, z) + 12(y + 1)(z - 1)G(\{1\}, z)\right.\\
&+ 6G(\{-1\}, y)(-zG(\{y + 1\}, z) + zG(\{0\}, z) + (y + 1)(z - 1))\\
&- 6zG(\{-1, -1\}, y) + 6zG(\{0, -1\}, y) + 6zG(\{y + 1, 0\}, z) - 6zG(\{0, 0\}, z)\\
&\left.-30yz - 6yz \log(4\pi) + 30y + 6y \log(4\pi) - \pi^2 z - 30z - 6z \log(4\pi) + 30 + 6 \log(4\pi)\right)\\
&+ \frac{\epsilon}{48\pi(y + 1)}\times\\
&\times \left(-z \left(-12G(\{-1, -1\}, y) + 6G(\{0, -1\}, y) + \pi^2\right) G(\{y + 1\}, z)\right.\\
&- 12(y + 1)zG(\{0, 1\}, z) - 6(y + 1)(z - 1)G(\{1, 0\}, z) - 24(y + 1)(z - 1)G(\{1, 1\}, z)\\
&+ 6zG(\{-1, 0, -1\}, y) - 6zG(\{0, -1, -1\}, y) - 18zG(\{0, y + 1, 0\}, z)
\end{aligned}
$$

$$
\begin{aligned}
&- 12zG(\{y+1,0,1\},z) - 6zG(\{y+1,1,0\},z)\\
&+ 6z(y+2+\log(\pi)+\log(4))G(\{0,-1\},y) - 6z(2y+3+\log(\pi)+\log(4))G(\{0,0\},z)\\
&+ 6(-2yz+y-z(3+\log(4)+\log(\pi))+1)G(\{-1,-1\},y)\\
&+ 2zG(\{0\},z)\left(3G(\{-1,-1\},y) - 6G(\{0,-1\},y) + 3(y+1)(5+\log(4)+\log(\pi)) + 2\pi^2\right)\\
&+ 6(y+z+z\log(4\pi)+1)G(\{y+1,0\},z) + 3G(\{-1\},y)\,(-4(y+1)(z-1)G(\{1\},z)\\
&- 2((y+z+1)G(\{y+1\},z) + z(-3G(\{0,y+1\},z) + G(\{y+1,0\},z)\\
&- 2G(\{y+1,1\},z) + 2G(\{0,0\},z) + 2G(\{0,1\},z)))\\
&+ 2z(\log(4\pi)-y)G(\{0\},z) + 2\log(4\pi)(-zG(\{y+1\},z) + y(z-1) + z)\\
&+10y(z-1) - \left(\pi^2-10\right)z - 2(5+\log(4)+\log(\pi))\Big)\\
&+ 12(y+1)(z-1)(5+\log(4)+\log(\pi))G(\{1\},z) + 18zG(\{0,0,0\},z) + 12zG(\{0,0,1\},z)\\
&+ 6zG(\{0,1,0\},z) - 114yz - 3yz\log^2(4) - yz\log(\pi)\log\left(4096\pi^3\right)\\
&- 30yz\log(4\pi) - 3\pi^2 y + 114y + 3y\log^2(4) + y\log(\pi)\log\left(4096\pi^3\right)\\
&+ 30y\log(4\pi) + 18z\zeta(3) - \pi^2 z - 114z - 3z\log^2(4)\\
&- z\log(\pi)\log\left(4096\pi^3\right) - \pi^2 z\log(4\pi)\\
&- 30z\log(4\pi) - 3\pi^2 + 114 + 3\log^2(4)\\
&+ \log(\pi)\log\left(4096\pi^3\right) + 30\log(4\pi)\Big)\\
&+ O(\epsilon^2)
\end{aligned}
\tag{6.49}
$$

$$
\begin{aligned}
\tilde{M}_4 =& -\frac{1}{\epsilon}\frac{(z-1)z}{8\pi}\\
&+ \frac{1}{8\pi}\Big((z-1)zG(\{0\},z) + 2(z-1)zG(\{1\},z) - 3z^2\\
&+z^2(-\log(4\pi)) + 3z + z\log(4\pi)\Big)\\
&+ \frac{\epsilon}{48\pi}\,(-6(z-1)zG(\{0,0\},z) - 12(z-1)zG(\{0,1\},z)\\
&- 12(z-1)zG(\{1,0\},z) - 24(z-1)zG(\{1,1\},z) + 6(z-1)z(3+\log(4)+\log(\pi))G(\{0\},z)\\
&+ 12(z-1)z(3+\log(4)+\log(\pi))G(\{1\},z) + \pi^2 z^2 - 42z^2 - 3z^2\log^2(4)\\
&+ z^2(-\log(\pi))\log\left(4096\pi^3\right) - 18z^2\log(4\pi) - \pi^2 z + 42z\\
&+3z\log^2(4) + z\log(\pi)\log\left(4096\pi^3\right) + 18z\log(4\pi)\Big)\\
&+ \frac{\epsilon^2}{48\pi}\,(6(z-1)zG(\{0,0,0\},z) + 12(z-1)zG(\{0,0,1\},z)\\
&+ 12(z-1)zG(\{0,1,0\},z) + 24(z-1)zG(\{0,1,1\},z) + 12(z-1)zG(\{1,0,0\},z)\\
&+ 24(z-1)zG(\{1,0,1\},z) + 24(z-1)zG(\{1,1,0\},z) + 48(z-1)zG(\{1,1,1\},z)\\
&- (z-1)z\left(\pi^2 - 3\left(14+\log^2(4)+6\log(4\pi)\right)\right.\\
&\left.-\log(\pi)\log\left(4096\pi^3\right)\right)G(\{0\},z)\\
&- 2(z-1)z\left(\pi^2 - 3\left(14+\log^2(4)+6\log(4\pi)\right)\right.\\
&\left.-\log(\pi)\log\left(4096\pi^3\right)\right)G(\{1\},z)\\
&- 6(z-1)z(3+\log(4)+\log(\pi))G(\{0,0\},z)\\
&- 12(z-1)z(3+\log(4)+\log(\pi))G(\{0,1\},z)
\end{aligned}
$$

$$
\begin{aligned}
&- 12(z-1)z(3+\log(4)+\log(\pi))G(\{1,0\},z)\\
&- 24(z-1)z(3+\log(4)+\log(\pi))G(\{1,1\},z)\\
&+ 16z^2\zeta(3)+3\pi^2z^2-90z^2-z^2\log^2(\pi)\log(64\pi)\\
&- 12z^2\log^2(2)\log(\pi)-z^2\log^2(2)\log(256)-9z^2\log^2(4)\\
&- 3z^2\log(\pi)\log\left(4096\pi^3\right)+\pi^2z^2\log(4\pi)\\
&- 42z^2\log(4\pi)-16z\zeta(3)-3\pi^2z+90z+z\log^2(\pi)\log(64\pi)\\
&+ 12z\log^2(2)\log(\pi)+z\log^2(2)\log(256)+9z\log^2(4)\\
&+3z\log(\pi)\log\left(4096\pi^3\right)-\pi^2z\log(4\pi)+42z\log(4\pi)\Big)\\
&+ O(\epsilon^3)
\end{aligned}
\tag{6.50}
$$

$$
\begin{aligned}
\tilde{M}_5 =&\frac{1}{\epsilon}\left(\frac{3G(\{-y,0\},z)-3G(\{0,-1\},y)+3G(\{0,0\},y)-3G(\{1,0\},z)+\pi^2}{24\pi(y+1)}\right)\\
&+\frac{1}{48\pi(y+1)}\Big(\left(6G(\{-1,-1\},y)+6G(\{0,-1\},y)-12G(\{0,0\},y)-\pi^2\right)G(\{1\},z)\\
&-6\left(-2G(\{0,-1\},y)+2G(\{0,0\},y)+\pi^2\right)G(\{-y\},z)\\
&+G(\{-1\},y)\left(6G(\{1,y+1\},z)+4\left(\pi^2-3G(\{-y,0\},z)\right)+6G(\{1,0\},z)\right)\\
&-12G(\{1,-y,0\},z)-6G(\{1,y+1,0\},z)-6G(\{-y,0,0\},z)-12G(\{-y,0,1\},z)\\
&+12G(\{-y,1,0\},z)-12G(\{-y,-y,0\},z)+6\log(4\pi)G(\{-y,0\},z)-5\pi^2G(\{0\},y)\\
&-12G(\{-1,0,-1\},y)+12G(\{-1,0,0\},y)+6G(\{0,-1,-1\},y)+12G(\{0,0,-1\},y)\\
&-18G(\{0,0,0\},y)-6\log(4\pi)G(\{0,-1\},y)+6\log(4\pi)G(\{0,0\},y)\\
&+12G(\{1,0,0\},z)+12G(\{1,0,1\},z)+12G(\{1,1,0\},z)\\
&-6\log(4\pi)G(\{1,0\},z)-6\zeta(3)+2\pi^2\log(4\pi)\Big)\\
&+O(\epsilon)
\end{aligned}
\tag{6.51}
$$

$$
\begin{aligned}
\tilde{M}_6 =&\frac{1}{\epsilon}\frac{1-z}{8\pi}\\
&+\frac{1}{8\pi y}\left(-4zy+2(z-1)G(\{1\},z)y-z\log(4\pi)y+\log(4\pi)y\right.\\
&\left.+4y+(z-1)(y+z)G(\{0\},y)-(z-1)zG(\{0\},z)+(z-1)(y+z)G(\{-y\},z)\right)\\
&+\frac{\epsilon}{48\pi y}\Big(\pi^2z^2+2\pi^2yz-72yz+6(z-1)G(\{0,0\},z)z\\
&+12(z-1)G(\{0,1\},z)z+12(z-1)G(\{1,0\},z)z-y\log(\pi)\log\left(4096\pi^3\right)z\\
&-24y\log(4\pi)z-6(z-1)G(\{0\},z)(4+\log(4)+\log(\pi))z-3y\log^2(4)z\\
&-\pi^2z-2\pi^2y+72y-6(z-1)(y+z)G(\{0,0\},y)-24y(z-1)G(\{1,1\},z)\\
&-12(z-1)(y+z)G(\{1,-y\},z)+6(z-1)(y+z)G(\{-y,0\},z)-12(z-1)(y+z)G(\{-y,1\},z)\\
&-12(z-1)(y+z)G(\{-y,-y\},z)-6(z-1)(y+z)G(\{0\},y)(2(G(\{1\},z)+G(\{-y\},z)-2)\\
&-\log(4\pi))+y\log(\pi)\log\left(4096\pi^3\right)+24y\log(4\pi)\\
&+12y(z-1)G(\{1\},z)(4+\log(4)+\log(\pi))\\
&+6(z-1)(y+z)G(\{-y\},z)(4+\log(4)+\log(\pi))+3y\log^2(4)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\epsilon^2}{48\pi y}\Big(12\zeta(3)z^2+\pi^2\log(4\pi)z^2+4\pi^2z^2 \\
&+8\pi^2yz-192yz-6(z-1)G(\{0,0,0\},z)z-12(z-1)G(\{0,0,1\},z)z \\
&-12(z-1)G(\{0,1,0\},z)z-24(z-1)G(\{0,1,1\},z)z-12(z-1)G(\{1,0,0\},z)z \\
&-24(z-1)G(\{1,0,1\},z)z-24(z-1)G(\{1,1,0\},z)z+28y\zeta(3)z-12\zeta(3)z \\
&-4y\log(\pi)\log\Big(4096\pi^3\Big)z-y\log^2(\pi)\log(64\pi)z \\
&+2\pi^2y\log(4\pi)z-72y\log(4\pi)z-\pi^2\log(4\pi)z \\
&+(z-1)G(\{0\},z)\Big(-72+\pi^2-12\log(2)(4+\log(2))-3\log(\pi)(8+\log(16)+\log(\pi))\Big)z \\
&+6(z-1)G(\{0,0\},z)(4+\log(4)+\log(\pi))z+12(z-1)G(\{0,1\},z)(4+\log(4)+\log(\pi))z \\
&+12(z-1)G(\{1,0\},z)(4+\log(4)+\log(\pi))z-12y\log^2(2)\log(\pi)z \\
&-y\log^2(2)\log(256)z-12y\log^2(4)z-4\pi^2z-8\pi^2y+192y \\
&+6(z-1)(y+z)G(\{0,0,0\},y)+48y(z-1)G(\{1,1,1\},z)+24(z-1)(y+z)G(\{1,1,-y\},z) \\
&-12(z-1)(y+z)G(\{1,-y,0\},z)+24(z-1)(y+z)G(\{1,-y,1\},z) \\
&+24(z-1)(y+z)G(\{1,-y,-y\},z)-6(z-1)(y+z)G(\{-y,0,0\},z) \\
&-12(z-1)(y+z)G(\{-y,0,1\},z)-12(z-1)(y+z)G(\{-y,1,0\},z) \\
&+24(z-1)(y+z)G(\{-y,1,1\},z)+24(z-1)(y+z)G(\{-y,1,-y\},z) \\
&-12(z-1)(y+z)G(\{-y,-y,0\},z)+24(z-1)(y+z)G(\{-y,-y,1\},z) \\
&+24(z-1)(y+z)G(\{-y,-y,-y\},z) \\
&-3(z-1)(y+z)G(\{-y\},z)\,(-4G(\{0,0\},y)-\log(\pi)(8+\log(16)+\log(\pi)) \\
&-4\log(2)(4+\log(2))+\pi^2-24\Big) \\
&-2(z-1)G(\{1\},z)\Big(\Big(-72+2\pi^2-12\log(2)(4+\log(2))-3\log(\pi)(8+\log(16)+\log(\pi))\Big)y \\
&+\pi^2z-6(y+z)G(\{0,0\},y)\Big)-(z-1)(y+z)G(\{0\},y)\,(12(4 \\
&+\log(4)+\log(\pi))G(\{1\},z)-24G(\{1,1\},z)-24G(\{1,-y\},z)-24G(\{-y,1\},z) \\
&-24G(\{-y,-y\},z)-\log(\pi)\log\Big(4096\pi^3\Big)-3\Big(24+\log^2(4)+8\log(4\pi)\Big) \\
&+12G(\{-y\},z)(4+\log(4)+\log(\pi))+2\pi^2\Big) \\
&-28y\zeta(3)+4y\log(\pi)\log\Big(4096\pi^3\Big) \\
&+y\log^2(\pi)\log(64\pi)-2\pi^2y\log(4\pi)+72y\log(4\pi) \\
&-6(z-1)(y+z)G(\{0,0\},y)(4+\log(4)+\log(\pi)) \\
&-24y(z-1)G(\{1,1\},z)(4+\log(4)+\log(\pi)) \\
&-12(z-1)(y+z)G(\{1,-y\},z)(4+\log(4)+\log(\pi)) \\
&+6(z-1)(y+z)G(\{-y,0\},z)(4+\log(4)+\log(\pi)) \\
&-12(z-1)(y+z)G(\{-y,1\},z)(4+\log(4)+\log(\pi)) \\
&-12(z-1)(y+z)G(\{-y,-y\},z)(4+\log(4)+\log(\pi))+12y\log^2(2)\log(\pi) \\
&+y\log^2(2)\log(256)+12y\log^2(4)\Big) \\
&+O(\epsilon^3)
\end{aligned}
\tag{6.52}
$$

$$
\begin{aligned}
\tilde{M}_7 =&\frac{1}{\epsilon}\frac{1-z}{8\pi} \\
&+\frac{1}{8\pi}\left((z-1)G(\{-1\},y)+3(z-1)G(\{1\},z)-5z+z(-\log(4\pi))+5+\log(4\pi)\right) \\
&+\frac{\epsilon}{16\pi}\left((2-2z)G(\{-1,-1\},y)-2(z-1)G(\{-1\},y)(3G(\{1\},z)-5-\log(4\pi))\right.
\end{aligned}
$$

$$
\begin{aligned}
&- 18(z-1)G(\{1,1\},z) + 6(z-1)(5+\log(4)+\log(\pi))G(\{1\},z) + \pi^2 z - 38z - 4z\log^2(2) \\
&+ z(-\log(\pi))\log(16\pi) - 10z\log(4\pi) - \pi^2 + 38 + 4\log^2(2) \\
&+ \log(\pi)\log(16\pi) + 10\log(4\pi)) \\
&+ \frac{\epsilon^2}{48\pi}\Big(6(z-1)G(\{-1,-1,-1\},y) - 9(z-1)G(\{1\},z)\Big(-2G(\{-1,-1\},y) + \pi^2 \\
&-38 - 4\log^2(2) - \log(\pi)\log(16\pi) - 10\log(4\pi)\Big) \\
&- 3(z-1)G(\{-1\},y)\Big(-18G(\{1,1\},z) + 6(5+\log(4)+\log(\pi))G(\{1\},z) + \pi^2 - 38 - 4\log^2(2) \\
&- \log(\pi)\log(16\pi) - 10\log(4\pi)) - 6(z-1)(5+\log(4)+\log(\pi))G(\{-1,-1\},y) \\
&+ 162(z-1)G(\{1,1,1\},z) - 54(z-1)(5+\log(4)+\log(\pi))G(\{1,1\},z) + 52z\zeta(3) + 15\pi^2 z - 390z \\
&- 4z\log^2(2)\log\Big(4\pi^3\Big) - z\log^2(\pi)\log(64\pi) \\
&- 60z\log^2(2) - 15z\log(\pi)\log(16\pi) + 3\pi^2 z\log(4\pi) - 114z\log(4\pi) \\
&- 52\zeta(3) - 15\pi^2 + 390 + 4\log^2(2)\log\Big(4\pi^3\Big) + \log^2(\pi)\log(64\pi) \\
&+60\log^2(2) + 15\log(\pi)\log(16\pi) - 3\pi^2\log(4\pi) + 114\log(4\pi)\Big) \qquad (6.53)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_8 =& \frac{1}{\epsilon}\frac{zy^2 + y^2 + 2z^2y + 3zy - y + 4z^2 - 2z}{24\pi(y+1)^2(z-1)z(y+z)} \\
&+ \frac{1}{24\pi(y+1)^2(z-1)z(y+z)}\Big(z\log(4\pi)y^2 + \log(4\pi)y^2 - 6y^2 - 3z^2y - 9zy \\
&+ 2z^2\log(4\pi)y + 3z\log(4\pi)y - \log(4\pi)y - 9z^2 + 3z \\
&+ \Big(-2(z+1)y^2 - (z(z+9)-2)y - 5z^2 + z\Big)G(\{-1\},y) \\
&+ 2\Big((z+1)y^2 - ((z-6)z+1)y + z^2 + z\Big)G(\{0\},y) \\
&+ \Big(-(z+1)y^2 + ((z-6)z+1)y - z(z+1)\Big)G(\{0\},z) \\
&+ \Big(-4(z+1)y^2 + (4-5z(z+3))y + (5-13z)z\Big)G(\{1\},z) \\
&+2\Big((z+1)y^2 - ((z-6)z+1)y + z^2 + z\Big)G(\{-y\},z) + 4z^2\log(4\pi) - 2z\log(4\pi)\Big) \\
&+ \frac{\epsilon}{288\pi(y+1)^2(z-1)z(y+z)}\times \\
&\times\Big(2\pi^2 zy^2 - 72zy^2 - z\log(\pi)\log\Big(4096\pi^3\Big)y^2 \\
&- 4\log(\pi)\log\Big(4096\pi^3\Big)y^2 + 9z\log(\pi)\log(16\pi)y^2 + 18\log(\pi)\log(16\pi)y^2 \\
&- 72\log(4\pi)y^2 - 6z\log^2(4)y^2 - 12\log^2(4)y^2 + 48z\log^2(2)y^2 + 72\log^2(2)y^2 + 2\pi^2y^2 \\
&+ 288y^2 + 4\pi^2z^2y + 36z^2y - 18\pi^2zy + 324zy + z^2\log(\pi)\log\Big(4096\pi^3\Big)y \\
&- 12z\log(\pi)\log\Big(4096\pi^3\Big)y + \log(\pi)\log\Big(4096\pi^3\Big)y \\
&+ 9z^2\log(\pi)\log(16\pi)y + 54z\log(\pi)\log(16\pi)y - 9\log(\pi)\log(16\pi)y \\
&- 36z^2\log(4\pi)y - 108z\log(4\pi)y - 36z\log^2(4)y + 48z^2\log^2(2)y + 216z\log^2(2)y \\
&- 24\log^2(2)y + 22\pi^2y + 72y - 16\pi^2z^2 + 252z^2 + 20\pi^2z - 36z - 12\Big(2(z-2)y^2 + ((z-9)z+4)y \\
&+(5-7z)z\Big)G(\{-1,-1\},y) + 48\Big((y-1)z^2 + 2\Big(y^2+1\Big)z - (y-1)y\Big)G(\{-1,0\},y) \\
&+ 24\Big((z-2)y^2 - \Big(z^2+1\Big)y - 2z^2 + z\Big)G(\{0,-1\},y)
\end{aligned}
$$

$$+ 24 \left((2-4z)y^2 + ((z-6)z+1)y + 2(z-2)z\right) G(\{0,0\},y)$$
$$+ 12 \left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{0,0\},z)$$
$$- 24(y(z-2)-z)(y+2z-1)G(\{0,1\},z)$$
$$+ 24 \left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{0,-y\},z)$$
$$+ 24(2y+z+1)(y+(y+2)z)G(\{1,0\},z)$$
$$+ 12 \left(16(z+1)y^2 + (z(17z+57)-10)y + z(43z-11)\right) G(\{1,1\},z)$$
$$- 48(2y+z+1)(y+(y+2)z)G(\{1,-y\},z)$$
$$+ 24(y-z+2)(z+y(2z-1))G(\{-y,0\},z) - 24 \left((z+4)y^2\right.$$
$$\left.-((z-12)z+1)y + 4z^2 + z\right) G(\{-y,1\},z)$$
$$- 48(y-z+2)(z+y(2z-1))G(\{-y,-y\},z)$$
$$+ 24G(\{0\},y) \left(-3(z+1)y^2 + 3((z-6)z+1)y - 3z(z+1)\right.$$
$$+ \left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{0\},z) - 2(2y+z+1)(y+(y+2)z)G(\{1\},z)$$
$$\left.-2(y-z+2)(z+y(2z-1))G(\{-y\},z) + \left(-(y-1)z^2 + y(y+6)z + z + (y-1)y\right)\log(4\pi)\right)$$
$$+ G(\{0\},z) \left(72\left(y^2 - (z-3)zy + z\right) - 12\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right)\log(4\pi)\right)$$
$$+ G(\{1\},z) \left(36\left(2(z+3)y^2 + (3z(z+5)-2)y + z(11z-3)\right)\right.$$
$$\left.-12\left(4(z+1)y^2 + (5z(z+3)-4)y + z(13z-5)\right)\log(4\pi)\right)$$
$$+ 12G(\{-1\},y) \left(3(y+5)z^2 + 3(y(2y+9)-1)z + 6(y-1)y - 2\left((y-1)z^2\right.\right.$$
$$\left.+2\left(y^2+1\right)z - (y-1)y\right) G(\{0\},z) + \left(8(z+1)y^2 + (z(7z+27)-2)y + z(17z-1)\right) G(\{1\},z)$$
$$+ 2\left((z-2)y^2 - \left(z^2+1\right)y - 2z^2 + z\right) G(\{-y\},z)$$
$$\left.- \left(2(z+1)y^2 + (z(z+9)-2)y + z(5z-1)\right)\log(4\pi)\right)$$
$$- 4z^2\log(\pi)\log\left(4096\pi^3\right) - z\log(\pi)\log\left(4096\pi^3\right)$$
$$+ 36z^2\log(\pi)\log(16\pi) - 9z\log(\pi)\log(16\pi) - 108z^2\log(4\pi)$$
$$+ 36z\log(4\pi) + 24\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{-y\},z)(-3+\log(4)+\log(\pi))$$
$$\left.-12z^2\log^2(4) - 6z\log^2(4) + 144z^2\log^2(2) - 24z\log^2(2)\right)$$
$$+ \frac{\epsilon^2}{288\pi(y+1)^2(z-1)z(y+z)} \times$$
$$\left(-12\pi^2 zy^2 + 360zy^2 - 152z\zeta(3)y^2 - 152\zeta(3)y^2\right.$$
$$+ 9z\log(\pi)\log\left(4096\pi^3\right)y^2 + 6\log(\pi)\log\left(4096\pi^3\right)y^2$$
$$+ 12z\log^2(2)\log\left(4\pi^3\right)y^2 + 24\log^2(2)\log\left(4\pi^3\right)y^2$$
$$+ 2z\log^2(\pi)\log(64\pi)y^2 + 2\log^2(\pi)\log(64\pi)y^2 - 27z\log(\pi)\log(16\pi)y^2$$
$$- 54\log(\pi)\log(16\pi)y^2 + 2\pi^2 z\log(4\pi)y^2 - 72z\log(4\pi)y^2 + 2\pi^2\log(4\pi)y^2$$
$$+ 288\log(4\pi)y^2 - 12z\log^2(2)\log(\pi)y^2 - 48\log^2(2)\log(\pi)y^2 - 6\log^2(2)\log(256)y^2$$
$$+ 36z\log^2(4)y^2 + 18\log^2(4)y^2 - 8z\log^3(2)y^2 + 16\log^3(2)y^2 - 144z\log^2(2)y^2 - 216\log^2(2)y^2$$
$$- 1008y^2 - 18\pi^2 z^2 y + 36z^2 y + 54\pi^2 zy - 972zy - 376z^2\zeta(3)y - 456z\zeta(3)y + 224\zeta(3)y$$

$$+ 3z^2 \log(\pi) \log\left(4096\pi^3\right) y + 36z \log(\pi) \log\left(4096\pi^3\right) y$$
$$- 9 \log(\pi) \log\left(4096\pi^3\right) y + 12z^2 \log^2(2) \log\left(4\pi^3\right) y$$
$$+ 72z \log^2(2) \log\left(4\pi^3\right) y - 12 \log^2(2) \log\left(4\pi^3\right) y$$
$$+ 4z^2 \log^2(\pi) \log(64\pi) y + 6z \log^2(\pi) \log(64\pi) y - 2 \log^2(\pi) \log(64\pi) y$$
$$- 27z^2 \log(\pi) \log(16\pi) y - 162z \log(\pi) \log(16\pi) y + 27 \log(\pi) \log(16\pi) y$$
$$+ 4\pi^2 z^2 \log(4\pi) y + 36z^2 \log(4\pi) y - 18\pi^2 z \log(4\pi) y + 324z \log(4\pi) y$$
$$+ 22\pi^2 \log(4\pi) y + 72 \log(4\pi) y + 12z^2 \log^2(2) \log(\pi) y$$
$$- 144z \log^2(2) \log(\pi) y + 12 \log^2(2) \log(\pi) y - 12z \log^2(2) \log(256) y$$
$$+ 18z^2 \log^2(4) y + 108z \log^2(4) y - 18 \log^2(4) y + 8z^2 \log^3(2) y + 8 \log^3(2) y$$
$$- 144z^2 \log^2(2) y - 648z \log^2(2) y + 72 \log^2(2) y - 60\pi^2 y - 360y + 42\pi^2 z^2 - 612z^2$$
$$- 54\pi^2 z - 36z + 12\left(2(5z-4)y^2 + (z(5z-9)+8)y + (13-11z)z\right) G(\{-1,-1,-1\},y)$$
$$- 96\left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{-1,-1,0\},y)$$
$$- 48\left((z+1)y^2 + 2\left(z^2+1\right)y + z^2 + z\right) G(\{-1,0,-1\},y)$$
$$+ 48\left((z+1)y^2 + 2\left(z^2+1\right)y + z^2 + z\right) G(\{-1,0,0\},y)$$
$$- 24\left((5z-4)y^2 + \left(4z^2-2\right)y + z(2z-1)\right) G(\{0,-1,-1\},y)$$
$$+ 48\left(4zy^2 - (2y+1)y + (5y+4)z^2 - 2z\right) G(\{0,-1,0\},y)$$
$$+ 24\left(2(z+1)y^2 + 7z^2 y + y + 4z(2z-1)\right) G(\{0,0,-1\},y)$$
$$- 24\left(2(z+1)y^2 + (z(13z-6)+1)y + 2z(7z-5)\right) G(\{0,0,0\},y)$$
$$- 12\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{0,0,0\},z)$$
$$+ 24(y(z-2)-z)(y+2z-1) G(\{0,0,1\},z)$$
$$- 24\left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{0,0,-y\},z)$$
$$+ 24\left((z-2)y^2 + (z(5z-6)-1)y + z(4z-5)\right) G(\{0,1,0\},z)$$
$$+ 24\left((11y-2)z^2 + 2y(5y-6)z + 4z + (5-8y)y\right) G(\{0,1,1\},z)$$
$$- 48\left(4zy^2 - (2y+1)y + (5y+4)z^2 - 2z\right) G(\{0,1,-y\},z)$$
$$+ 24\left(-(4y+5)z^2 - 2\left(y^2-2\right)z + y(y+2)\right) G(\{0,-y,0\},z)$$
$$- 24\left((8z-4)y^2 + 7z^2 y + y + 2z(z+1)\right) G(\{0,-y,1\},z)$$
$$+ 48\left((4y+5)z^2 + 2\left(y^2-2\right)z - y(y+2)\right) G(\{0,-y,-y\},z)$$
$$- 24(2y+z+1)(y+(y+2)z) G(\{1,0,0\},z)$$
$$- 48\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{1,0,1\},z)$$
$$- 48\left((z+1)y^2 + 2\left(z^2+1\right)y + z^2 + z\right) G(\{1,0,-y\},z)$$
$$- 48\left(4(z+1)y^2 + (z(5z+6)+5)y + 4z(z+1)\right) G(\{1,1,0\},z)$$
$$- 12\left(64(z+1)y^2 + (z(71z+195)-10)y + z(145z-17)\right) G(\{1,1,1\},z)$$
$$+ 96\left(4(z+1)y^2 + (z(5z+6)+5)y + 4z(z+1)\right) G(\{1,1,-y\},z)$$

$$
\begin{aligned}
&- 48\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{1,-y,0\},z)\\
&+ 48\left(5(z+1)y^2+4(z(z+3)+1)y+5z(z+1)\right)G(\{1,-y,1\},z)\\
&+ 96\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{1,-y,-y\},z)\\
&- 24(y-z+2)(z+y(2z-1))G(\{-y,0,0\},z)\\
&- 48\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{-y,0,1\},z)\\
&- 48\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-y,0,-y\},z)\\
&- 48\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{-y,1,0\},z)\\
&+ 24\left((z+10)y^2-((z-24)z+1)y+10z^2+z\right)G(\{-y,1,1\},z)\\
&+ 96\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{-y,1,-y\},z)\\
&+ 48\left((4y+5)z^2-2(y(2y+3)+2)z+y(5y+4)\right)G(\{-y,-y,0\},z)\\
&+ 96\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)G(\{-y,-y,1\},z)\\
&+ 96\left((4z-5)y^2+\left(-4z^2+6z-4\right)y+(4-5z)z\right)G(\{-y,-y,-y\},z)\\
&+ G(\{0,1\},z)\left(72\left(-3y^2+2(y-3)zy+y+(3y-1)z^2\right)-24(y(z-2)-z)(y+2z-1)\log(4\pi)\right)\\
&+ G(\{1,0\},z)\left(24(2y+z+1)(y+(y+2)z)\log(4\pi)-72\left((z+3)y^2+(6z+2)y+z(z+3)\right)\right)\\
&+ G(\{0,0\},z)\left(12\left((z+1)y^2-((z-6)z+1)y+z^2+z\right)\log(4\pi)-72\left(y^2-(z-3)zy+z\right)\right)\\
&+ 12G(\{1,1\},z)\left(-39(y+3)z^2-9y(4y+19)z+21z+6(3-10y)y\right.\\
&\left.+\left(16(z+1)y^2+(z(17z+57)-10)y+z(43z-11)\right)\log(4\pi)\right)\\
&+ 4G(\{-y\},z)\left(\left(\pi^2(z-11)+54(z+1)\right)y^2-\left(54((z-6)z+1)+\pi^2(z(z+18)+1)\right)y\right.\\
&+ z\left(\pi^2(1-11z)+54(z+1)\right)+6\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-1,-1\},y)\\
&- 24\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-1,0\},y)\\
&+ 3\left(\left(\log^2(4)-6\log(4\pi)\right)\left(-(y-1)z^2+y(y+6)z+z+(y-1)y\right)\right.\\
&+ 8\left((y+2)z^2-\left(y^2+1\right)z+y(2y+1)\right)G(\{0,-1\},y)\\
&+4\left(-(5y+7)z^2+(y(5y+6)+5)z-y(7y+5)\right)G(\{0,0\},y)\Big)\\
&+\left.\left(-(y-1)z^2+y(y+6)z+z+(y-1)y\right)\log(\pi)\log\left(4096\pi^3\right)\right)\\
&+ G(\{0\},z)\left(10\pi^2zy^2+72zy^2+36z\log(\pi)\log(16\pi)y^2\right.\\
&- 18\log(\pi)\log(16\pi)y^2+72\log(4\pi)y^2-24z\log^2(2)y^2-24\log^2(2)y^2\\
&- 2\pi^2y^2-288y^2+26\pi^2z^2y+288z^2y+12\pi^2zy-648zy+27z^2\log(\pi)\log(16\pi)y\\
&+ 9\log(\pi)\log(16\pi)y-72z^2\log(4\pi)y+216z\log(4\pi)y+24z^2\log^2(2)y-144z\log^2(2)y\\
&+ 24\log^2(2)y-22\pi^2y-72y+46\pi^2z^2+72z^2-38\pi^2z-288z\\
&+ 48\left((y-1)z^2+2\left(y^2+1\right)z-(y-1)y\right)G(\{-1,-1\},y)\\
&- 48\left((y-1)z^2+2\left(y^2+1\right)z-(y-1)y\right)G(\{-1,0\},y)\\
&- 24\left(4zy^2-(2y+1)y+(5y+4)z^2-2z\right)G(\{0,-1\},y)
\end{aligned}
$$

$$+ 24 \left(4zy^2 - (2y+1)y + (5y+4)z^2 - 2z\right) G(\{0,0\},y)$$
$$- \left(-4y^2 + y + (7y+2)z^2 + 2(y(7y+6)+4)z\right) \log(\pi) \log\left(4096\pi^3\right)$$
$$+18z\log(\pi)\log(16\pi) + 72z\log(4\pi) - 24z^2\log^2(2) - 24z\log^2(2)\Big)$$
$$+ G(\{0\},y) \left(4\pi^2 zy^2 + 216zy^2 - 18z\log(\pi)\log(16\pi)y^2\right.$$
$$+ 18\log(\pi)\log(16\pi)y^2 - 72z\log(4\pi)y^2 - 72\log(4\pi)y^2 + 48z\log^2(2)y^2$$
$$+ 48\log^2(2)y^2 - 44\pi^2y^2 + 216y^2 - 16\pi^2z^2y - 216z^2y - 48\pi^2zy + 1296zy$$
$$- 9z^2\log(\pi)\log(16\pi)y + 9\log(\pi)\log(16\pi)y + 72z^2\log(4\pi)y$$
$$- 432z\log(4\pi)y + 72\log(4\pi)y - 48z^2\log^2(2)y + 288z\log^2(2)y - 48\log^2(2)y$$
$$- 16\pi^2y - 216y - 44\pi^2z^2 + 216z^2 + 4\pi^2z + 216z$$
$$- 24\left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{0,0\},z)$$
$$- 48\left(4zy^2 - (2y+1)y + (5y+4)z^2 - 2z\right) G(\{0,1\},z)$$
$$+ 48\left((4y+5)z^2 + 2\left(y^2-2\right)z - y(y+2)\right) G(\{0,-y\},z)$$
$$- 48\left((z+1)y^2 + 2\left(z^2+1\right)y + z^2 + z\right) G(\{1,0\},z)$$
$$+ 96\left(4(z+1)y^2 + (z(5z+6)+5)y + 4z(z+1)\right) G(\{1,1\},z)$$
$$+ 96\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{1,-y\},z)$$
$$- 48\left((z-2)y^2 - \left(z^2+1\right)y - 2z^2 + z\right) G(\{-y,0\},z)$$
$$+ 96\left((z+1)y^2 - ((z-6)z+1)y + z^2 + z\right) G(\{-y,1\},z)$$
$$+ 96\left((4z-5)y^2 + \left(-4z^2 + 6z - 4\right)y + (4-5z)z\right) G(\{-y,-y\},z)$$
$$+ \left(2(5z-1)y^2 - ((z-24)z+7)y + 4z(z+1)\right) \log(\pi) \log\left(4096\pi^3\right)$$
$$- 72z^2\log(4\pi) - 72z\log(4\pi) + 24\Big((y-1)z^2$$
$$+2\left(y^2+1\right)z - (y-1)y\Big) G(\{0\},z)(-3 + \log(4) + \log(\pi))$$
$$- 48(2y+z+1)(y+(y+2)z)G(\{1\},z)(-3+\log(4)+\log(\pi))$$
$$-48(y-z+2)(z+y(2z-1))G(\{-y\},z)(-3+\log(4)+\log(\pi)) + 48z^2\log^2(2) + 48z\log^2(2)\Big)$$
$$+ 2G(\{-1\},y)\left(-2\pi^2zy^2 - 108zy^2 + 9z\log(\pi)\log(16\pi)y^2\right.$$
$$- 9\log(\pi)\log(16\pi)y^2 + 36z\log(4\pi)y^2 + 36\log(4\pi)y^2 - 24z\log^2(2)y^2 - 24\log^2(2)y^2$$
$$+ 22\pi^2y^2 - 108y^2 + 11\pi^2z^2y - 54z^2y + 27\pi^2zy - 486zy + 9z^2\log(\pi)\log(16\pi)y$$
$$- 27z\log(\pi)\log(16\pi)y + 18\log(\pi)\log(16\pi)y + 18z^2\log(4\pi)y + 162z\log(4\pi)y$$
$$- 36\log(4\pi)y - 12z^2\log^2(2)y - 108z\log^2(2)y + 24\log^2(2)y + 2\pi^2y + 108y + 31\pi^2z^2 - 270z^2$$
$$- 11\pi^2z + 54z + 12\left((y-1)z^2 + 2\left(y^2+1\right)z - (y-1)y\right) G(\{0,0\},z)$$
$$+ 12\left((8z-4)y^2 + 7z^2y + y + 2z(z+1)\right) G(\{0,1\},z)$$
$$- 12\left(4zy^2 - (2y+1)y + (5y+4)z^2 - 2z\right) G(\{0,-y\},z)$$
$$+ 24\left((z+1)y^2 + 2\left(z^2+1\right)y + z^2 + z\right) G(\{1,0\},z)$$
$$- 6\left(32(z+1)y^2 + (z(37z+81)+10)y + z(59z+5)\right) G(\{1,1\},z)$$

$$+ 24\left((z+1)y^2+2\left(z^2+1\right)y+z^2+z\right)G(\{1,-y\},z)$$
$$+ 24\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-y,0\},z)$$
$$- 12\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-y,1\},z)$$
$$- 48\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-y,-y\},z)$$
$$+ \left((1-5z)y^2-4\left(z^2+1\right)y+(z-5)z\right)\log(\pi)\log\left(4096\pi^3\right)$$
$$- 18z^2\log(\pi)\log(16\pi)+18z\log(\pi)\log(16\pi)+90z^2\log(4\pi)-18z\log(4\pi)$$
$$- 12\left((y-1)z^2+2\left(y^2+1\right)z-(y-1)y\right)G(\{0\},z)(-3+\log(4)+\log(\pi))$$
$$+ 6\left(8(z+1)y^2+(z(7z+27)-2)y+z(17z-1)\right)G(\{1\},z)(-3+\log(4)+\log(\pi))$$
$$+ 12\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{-y\},z)(-3+\log(4)+\log(\pi))$$
$$-60z^2\log^2(2)+12z\log^2(2)\Big)$$
$$+ 2G(\{1\},z)\Big(-4\pi^2zy^2-36zy^2-27z\log(\pi)\log(16\pi)y^2$$
$$- 27\log(\pi)\log(16\pi)y^2+36z\log(4\pi)y^2+108\log(4\pi)y^2-48z\log^2(2)y^2$$
$$- 48\log^2(2)y^2-4\pi^2y^2-396y^2-17\pi^2z^2y-90z^2y+45\pi^2zy-810zy$$
$$- 27z^2\log(\pi)\log(16\pi)y-81z\log(\pi)\log(16\pi)y+54z^2\log(4\pi)y$$
$$+ 270z\log(4\pi)y-36\log(4\pi)y-60z^2\log^2(2)y-180z\log^2(2)y+48\log^2(2)y$$
$$- 44\pi^2y+36y+23\pi^2z^2-522z^2-31\pi^2z+90z$$
$$- 6\left(4(z+1)y^2-((z-27)z+10)y+z(13z-5)\right)G(\{-1,-1\},y)$$
$$- 48\left((z+1)y^2+2\left(z^2+1\right)y+z^2+z\right)G(\{-1,0\},y)$$
$$+ 24\left((z+1)y^2+2\left(z^2+1\right)y+z^2+z\right)G(\{0,-1\},y)$$
$$+ 24(2y+z+1)(y+(y+2)z)G(\{0,0\},y)$$
$$+ \left(5(z+1)y^2+4(z(z+3)+1)y+5z(z+1)\right)\log(\pi)\log\left(4096\pi^3\right)$$
$$-54z^2\log(\pi)\log(16\pi)+198z^2\log(4\pi)-54z\log(4\pi)-156z^2\log^2(2)+60z\log^2(2)\Big)$$
$$- 752z^2\zeta(3)+448z\zeta(3)+18z^2\log(\pi)\log\left(4096\pi^3\right)$$
$$- 3z\log(\pi)\log\left(4096\pi^3\right)+48z^2\log^2(2)\log\left(4\pi^3\right)$$
$$- 12z\log^2(2)\log\left(4\pi^3\right)+8z^2\log^2(\pi)\log(64\pi)$$
$$- 4z\log^2(\pi)\log(64\pi)-108z^2\log(\pi)\log(16\pi)+27z\log(\pi)\log(16\pi)$$
$$- 16\pi^2z^2\log(4\pi)+252z^2\log(4\pi)+20\pi^2z\log(4\pi)-36z\log(4\pi)$$
$$- 12\left(2(z-2)y^2+((z-9)z+4)y+(5-7z)z\right)G(\{-1,-1\},y)(-3+\log(4)+\log(\pi))$$
$$+ 48\left((y-1)z^2+2\left(y^2+1\right)z-(y-1)y\right)G(\{-1,0\},y)(-3+\log(4)+\log(\pi))$$
$$+ 24\left((z-2)y^2-\left(z^2+1\right)y-2z^2+z\right)G(\{0,-1\},y)(-3+\log(4)+\log(\pi))$$
$$+ 24\left((2-4z)y^2+((z-6)z+1)y+2(z-2)z\right)G(\{0,0\},y)(-3+\log(4)+\log(\pi))$$
$$+ 24\left((y-1)z^2+2\left(y^2+1\right)z-(y-1)y\right)G(\{0,-y\},z)(-3+\log(4)+\log(\pi))$$
$$- 48(2y+z+1)(y+(y+2)z)G(\{1,-y\},z)(-3+\log(4)+\log(\pi))$$

$$+ 24(y-z+2)(z+y(2z-1))G(\{-y,0\},z)(-3+\log(4)+\log(\pi))$$
$$- 24\left((z+4)y^2-((z-12)z+1)y+4z^2+z\right)G(\{-y,1\},z)(-3+\log(4)+\log(\pi))$$
$$- 48(y-z+2)(z+y(2z-1))G(\{-y,-y\},z)(-3+\log(4)+\log(\pi)) - 48z^2\log^2(2)\log(\pi)$$
$$- 12z\log^2(2)\log(\pi) - 6z^2\log^2(2)\log(256) + 54z^2\log^2(4) + 16z^2\log^3(2)$$
$$-8z\log^3(2) - 432z^2\log^2(2) + 72z\log^2(2)\Big) \qquad (6.54)$$

$$\begin{aligned}
\tilde{M}_9 =& \frac{1}{\epsilon^2}\frac{1}{48\pi(y+1)z} \\
&+ \frac{1}{\epsilon}\frac{-2G(\{-1\},y)+2G(\{0\},y)-G(\{0\},z)-4G(\{1\},z)+2G(\{-y\},z)+\log(4\pi)}{48\pi(y+1)z} \\
&+ \frac{1}{288\pi(y+1)z}\times(-24\log(4\pi)G(\{1\},z)+24G(\{-1,-1\},y)-24G(\{-1,0\},y) \\
&- 24G(\{0,-1\},y)+24G(\{0,0\},y)+6G(\{0,0\},z)+24G(\{0,1\},z)-12G(\{0,-y\},z) \\
&+ 24G(\{1,0\},z)+96G(\{1,1\},z)-48G(\{1,-y\},z)-12G(\{-y,0\},z)-48G(\{-y,1\},z) \\
&+ 24G(\{-y,-y\},z)+12G(\{-1\},y)(G(\{0\},z)+4G(\{1\},z)-2G(\{-y\},z)-\log(4\pi)) \\
&- 6G(\{0\},z)(2G(\{0\},y)+\log(4\pi))+12G(\{0\},y)(-4G(\{1\},z)+2G(\{-y\},z) \\
&+ \log(4\pi))-2\log(\pi)\log\left(4096\pi^3\right)+9\log(\pi)\log(16\pi) \\
&+12G(\{-y\},z)\log(4\pi)-6\log^2(4)+36\log^2(2)+\pi^2\Big) \\
&+ \frac{\epsilon}{288\pi(y+1)z}(24\log(4\pi)G(\{-1,-1\},y)-48G(\{-1,-1,-1\},y) \\
&+ 48G(\{-1,-1,0\},y)-24G(\{-1,0,-1\},y)+24G(\{-1,0,0\},y)+48G(\{0,-1,-1\},y) \\
&- 48G(\{0,-1,0\},y)+24G(\{0,0,-1\},y)-24G(\{0,0,0\},y)-6G(\{0,0,0\},z) \\
&- 24G(\{0,0,1\},z)+12G(\{0,0,-y\},z)-24G(\{0,1,0\},z)-96G(\{0,1,1\},z) \\
&+ 48G(\{0,1,-y\},z)+12G(\{0,-y,0\},z)+48G(\{0,-y,1\},z)-24G(\{0,-y,-y\},z) \\
&- 24G(\{1,0,0\},z)-24G(\{1,0,1\},z)-24G(\{1,0,-y\},z)-96G(\{1,1,0\},z) \\
&- 384G(\{1,1,1\},z)+192G(\{1,1,-y\},z)-24G(\{1,-y,0\},z)+120G(\{1,-y,1\},z) \\
&+ 48G(\{1,-y,-y\},z)+12G(\{-y,0,0\},z)-24G(\{-y,0,1\},z)+48G(\{-y,0,-y\},z) \\
&- 24G(\{-y,1,0\},z)+120G(\{-y,1,1\},z)+48G(\{-y,1,-y\},z)+120G(\{-y,-y,0\},z) \\
&+ 48G(\{-y,-y,1\},z)-240G(\{-y,-y,-y\},z)+G(\{0\},z)\left(-24G(\{-1,-1\},y)\right. \\
&+ 24G(\{-1,0\},y)+24G(\{0,-1\},y)-24G(\{0,0\},y)-\log(\pi)\log\left(4096\pi^3\right) \\
&-12\log^2(2)-\pi^2\Big)-4G(\{1\},z)\left(6G(\{-1,-1\},y)+12G(\{-1,0\},y)\right. \\
&-6G(\{0,-1\},y)-12G(\{0,0\},y)+\log(\pi)\log\left(4096\pi^3\right)+12\log^2(2)+\pi^2\Big) \\
&+ G(\{-y\},z)\left(-24G(\{-1,-1\},y)+96G(\{-1,0\},y)+96G(\{0,-1\},y)-168G(\{0,0\},y)\right. \\
&+2\log(\pi)\log\left(4096\pi^3\right)+6\log^2(4)-22\pi^2\Big) \\
&+ G(\{-1\},y)\left(12\log(4\pi)(G(\{0\},z)+4G(\{1\},z)-2G(\{-y\},z))\right. \\
&- 12G(\{0,0\},z)-48G(\{0,1\},z)+24G(\{0,-y\},z)+24G(\{1,0\},z)-192G(\{1,1\},z) \\
&+ 24G(\{1,-y\},z)-48G(\{-y,0\},z)+24G(\{-y,1\},z)+96G(\{-y,-y\},z) \\
&- \log(\pi)\log\left(16777216\pi^6\right)-24\log^2(2)+22\pi^2\Big) \\
&+ G(\{0\},y)\left(-12\log(4\pi)(G(\{0\},z)+4G(\{1\},z)-2G(\{-y\},z))\right. \\
&+ 12G(\{0,0\},z)+48G(\{0,1\},z)-24G(\{0,-y\},z)-24G(\{1,0\},z)+192G(\{1,1\},z) \\
&+ 48G(\{1,-y\},z)+48G(\{-y,0\},z)+48G(\{-y,1\},z)-240G(\{-y,-y\},z)
\end{aligned}$$

$$
\begin{aligned}
&+\log(\pi)\log\left(16777216\pi^6\right)+24\log^2(2)-22\pi^2\Big)\\
&-76\zeta(3)+12\log^2(2)\log\left(4\pi^3\right)+\log^2(\pi)\log(64\pi)\\
&-24G(\{-1,0\},y)\log(4\pi)-24G(\{0,-1\},y)\log(4\pi)+24G(\{0,0\},y)\log(4\pi)\\
&+6G(\{0,0\},z)\log(4\pi)+24G(\{0,1\},z)\log(4\pi)-12G(\{0,-y\},z)\log(4\pi)\\
&+24G(\{1,0\},z)\log(4\pi)+96G(\{1,1\},z)\log(4\pi)-48G(\{1,-y\},z)\log(4\pi)\\
&-12G(\{-y,0\},z)\log(4\pi)-48G(\{-y,1\},z)\log(4\pi)\\
&+24G(\{-y,-y\},z)\log(4\pi)+\pi^2\log(4\pi)-24\log^2(2)\log(\pi)\\
&-3\log^2(2)\log(256)+8\log^3(2)\Big)\\
+O(\epsilon^2) \qquad (6.55)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{10}=&\frac{1}{192\pi(y+1)}\times\left(8\left(-3G(\{0,-1\},y)+3G(\{0,0\},y)+\pi^2\right)G(\{0\},z)\right.\\
&-48G(\{0\},y)G(\{0,1\},z)+24G(\{-1\},y)(G(\{0,1\},z)-G(\{0,-y\},z))\\
&+48G(\{0\},y)G(\{0,-y\},z)-48G(\{0,1,-y\},z)-24G(\{0,-y,0\},z)\\
&-24G(\{0,-y,1\},z)+48G(\{0,-y,-y\},z)-24G(\{0,-1,-1\},y)\\
&+48G(\{0,-1,0\},y)+24G(\{0,0,-1\},y)-48G(\{0,0,0\},y)+24G(\{0,1,0\},z)\\
&+24G(\{0,1,1\},z)-48\zeta(3)) \qquad (6.56)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{11}=&-\frac{1}{\epsilon^3}\frac{5}{48\pi(y+1)^2}\\
&+\frac{1}{\epsilon^2}\frac{10G(\{-1\},y)-4G(\{0\},y)-10G(\{0\},z)+14G(\{1\},z)-4G(\{-y\},z)-5\log(4\pi)}{48\pi(y+1)^2}\\
&+\frac{1}{\epsilon}\frac{1}{288\pi(y+1)^2}\left(84\log(4\pi)G(\{1\},z)-120G(\{-1,-1\},y)\right.\\
&+48G(\{-1,0\},y)+12G(\{0,-1\},y)+24G(\{0,0\},y)+96G(\{0,0\},z)\\
&+168G(\{0,1\},z)-48G(\{0,-y\},z)-48G(\{1,0\},z)-228G(\{1,1\},z)\\
&+24G(\{1,-y\},z)-48G(\{-y,0\},z)+60G(\{-y,1\},z)+24G(\{-y,-y\},z)\\
&+24G(\{0\},y)(G(\{1\},z)+G(\{-y\},z)-\log(4\pi))-12G(\{0\},z)(4G(\{0\},y)+5\log(4\pi))\\
&+12G(\{-1\},y)(10G(\{0\},z)-11G(\{1\},z)+G(\{-y\},z)+5\log(4\pi))\\
&\left.-15\log^2(4\pi)-24G(\{-y\},z)\log(4\pi)+61\pi^2\right)\\
&+\frac{1}{288\pi(y+1)^2}\left(-120\log(4\pi)G(\{-1,-1\},y)+240G(\{-1,-1,-1\},y)\right.\\
&-96G(\{-1,-1,0\},y)-24G(\{-1,0,-1\},y)-48G(\{-1,0,0\},y)+12G(\{0,-1,-1\},y)\\
&-48G(\{0,-1,0\},y)-120G(\{0,0,-1\},y)+120G(\{0,0,0\},y)-168G(\{0,0,0\},z)\\
&-240G(\{0,0,1\},z)+48G(\{0,0,-y\},z)-24G(\{0,1,0\},z)-456G(\{0,1,1\},z)\\
&+48G(\{0,1,-y\},z)-24G(\{0,-y,0\},z)+120G(\{0,-y,1\},z)+48G(\{0,-y,-y\},z)\\
&+120G(\{1,0,0\},z)+120G(\{1,0,1\},z)-24G(\{1,0,-y\},z)+48G(\{1,1,0\},z)\\
&+588G(\{1,1,1\},z)+48G(\{1,1,-y\},z)-24G(\{1,-y,0\},z)-24G(\{1,-y,1\},z)\\
&-96G(\{1,-y,-y\},z)+48G(\{-y,0,0\},z)+120G(\{-y,0,1\},z)-24G(\{-y,0,-y\},z)\\
&-24G(\{-y,1,0\},z)-132G(\{-y,1,1\},z)-96G(\{-y,1,-y\},z)+48G(\{-y,-y,0\},z)\\
&-96G(\{-y,-y,1\},z)+48G(\{-y,-y,-y\},z)+2G(\{0\},z)\left(-120G(\{-1,-1\},y)\right.\\
&\left.+48G(\{-1,0\},y)+48G(\{0,-1\},y)-12G(\{0,0\},y)-15\log^2(4\pi)+\pi^2\right)
\end{aligned}
$$

$$+ 2G(\{-1\},y)\left(-96G(\{0,0\},z) - 132G(\{0,1\},z) + 12G(\{0,-y\},z)\right.$$
$$+ 48G(\{1,0\},z) + 138G(\{1,1\},z) + 12G(\{1,-y\},z) + 48G(\{-y,0\},z)$$
$$- 6G(\{-y,1\},z) - 24G(\{-y,-y\},z) + 3\log(4\pi)(20G(\{0\},z) - 22G(\{1\},z)$$
$$\left. +2G(\{-y\},z) + 5\log(4\pi)) - 61\pi^2\right)$$
$$+ 4G(\{-y\},z)\left(3G(\{-1,-1\},y) - 12G(\{-1,0\},y) - 30G(\{0,-1\},y)\right.$$
$$\left. +30G(\{0,0\},y) - \log(\pi)\log\left(4096\pi^3\right) - 3\log^2(4) + 14\pi^2\right)$$
$$+ 4G(\{0\},y)\left(6\log(4\pi)(-2G(\{0\},z) + G(\{1\},z) + G(\{-y\},z))\right.$$
$$+ 12G(\{0,0\},z) + 12G(\{0,1\},z) + 12G(\{0,-y\},z) - 6G(\{1,0\},z)$$
$$+ 12G(\{1,1\},z) - 24G(\{1,-y\},z) - 6G(\{-y,0\},z) - 24G(\{-y,1\},z)$$
$$\left. +12G(\{-y,-y\},z) - \log(\pi)\log\left(4096\pi^3\right) - 3\log^2(4) + 14\pi^2\right)$$
$$+ G(\{1\},z)\left(228G(\{-1,-1\},y) - 48G(\{-1,0\},y) + 24G(\{0,-1\},y)\right.$$
$$\left. -96G(\{0,0\},y) + 14\log(\pi)\log\left(4096\pi^3\right) + 168\log^2(2) - 58\pi^2\right)$$
$$+ 848\zeta(3) - 5\log^3(4\pi) + 48G(\{-1,0\},y)\log(4\pi) + 12G(\{0,-1\},y)\log(4\pi)$$
$$+ 24G(\{0,0\},y)\log(4\pi) + 96G(\{0,0\},z)\log(4\pi) + 168G(\{0,1\},z)\log(4\pi)$$
$$- 48G(\{0,-y\},z)\log(4\pi) - 48G(\{1,0\},z)\log(4\pi) - 228G(\{1,1\},z)\log(4\pi)$$
$$+ 24G(\{1,-y\},z)\log(4\pi) - 48G(\{-y,0\},z)\log(4\pi) + 60G(\{-y,1\},z)\log(4\pi)$$
$$\left. +24G(\{-y,-y\},z)\log(4\pi) + 61\pi^2\log(4\pi)\right)$$
$$+ O(\epsilon) \tag{6.57}$$

$$\tilde{M}_{16} = \frac{1}{48\pi(y+1)} \times \left(\left(-6G(\{-1,-1\},y) + 6G(\{0,-1\},y) - \pi^2\right)G(\{1\},z)\right.$$
$$+ 2\left(-3G(\{0,-1\},y) + 3G(\{0,0\},y) + \pi^2\right)G(\{0\},z)$$
$$+ 6G(\{-1\},y)(G(\{1,0\},z) - G(\{1,y+1\},z)) + 6G(\{0,-y,0\},z) + 6G(\{1,y+1,0\},z)$$
$$- 2\pi^2G(\{0\},y) + 6G(\{0,-1,-1\},y) + 6G(\{0,0,-1\},y) - 12G(\{0,0,0\},y)$$
$$\left. -6G(\{0,1,0\},z) - 6G(\{1,0,0\},z)\right)$$
$$+ O(\epsilon) \tag{6.58}$$

$$\tilde{M}_{17} = \frac{1}{\epsilon}\frac{G(\{0\},z)}{8\pi(y+1)(y+z)}$$
$$+ \frac{1}{8\pi(y+1)(y+z)} \times \left(-2G(\{-1\},y)G(\{0\},z) - 2G(\{-y,0\},z)\right.$$
$$+ 2G(\{0,-1\},y) - 2G(\{0,0\},y) - G(\{0,0\},z) - 2G(\{0,1\},z)$$
$$\left. +2G(\{1,0\},z) + \log(4\pi)G(\{0\},z) - \pi^2\right)$$
$$\frac{\epsilon}{48\pi(y+1)(y+z)} \times$$
$$\times \left(12\left(-2G(\{0,-1\},y) + 2G(\{0,0\},y) + \pi^2\right)G(\{-y\},z)\right.$$
$$+ 2\left(-6G(\{-1,-1\},y) - 6G(\{0,-1\},y) + 12G(\{0,0\},y) + \pi^2\right)G(\{1\},z)$$
$$+ 12G(\{0,y+1,0\},z) + 24G(\{1,-y,0\},z) + 12G(\{1,y+1,0\},z) + 12G(\{-y,0,0\},z)$$
$$+ 24G(\{-y,0,1\},z) - 24G(\{-y,1,0\},z) + 24G(\{-y,-y,0\},z) + G(\{0\},z)\,(12G(\{-1,-1\},y)$$

$$-\pi^2 + 3\log^2(4) + \log(\pi)\log\left(4096\pi^3\right)\Big)$$
$$- 2G(\{-1\}, y)\,(6G(\{0, y+1\}, z) + 6G(\{1, y+1\}, z) - 12G(\{-y, 0\}, z) - 12G(\{0, 0\}, z)$$
$$-12G(\{0, 1\}, z) + 6G(\{1, 0\}, z) + 6\log(4\pi)G(\{0\}, z) + \pi^2\Big)$$
$$- 12\log(4\pi)G(\{-y, 0\}, z) + 10\pi^2 G(\{0\}, y) + 12G(\{-1, -1, -1\}, y)$$
$$+ 12G(\{-1, 0, -1\}, y) - 24G(\{-1, 0, 0\}, y) - 12G(\{0, -1, -1\}, y) - 24G(\{0, 0, -1\}, y)$$
$$+ 36G(\{0, 0, 0\}, y) + 12\log(4\pi)G(\{0, -1\}, y) - 12\log(4\pi)G(\{0, 0\}, y)$$
$$+ 6G(\{0, 0, 0\}, z) + 12G(\{0, 0, 1\}, z) + 24G(\{0, 1, 1\}, z) - 24G(\{1, 0, 0\}, z)$$
$$- 24G(\{1, 0, 1\}, z) - 24G(\{1, 1, 0\}, z) - 6\log(4\pi)G(\{0, 0\}, z) - 12\log(4\pi)G(\{0, 1\}, z)$$
$$+12\log(4\pi)G(\{1, 0\}, z) - 24\zeta(3) - 6\pi^2\log(4\pi)\Big)$$
$$+ O(\epsilon^2) \tag{6.59}$$

$$\tilde{M}_{18} = \frac{1}{48\pi y^2(y+1)(z-1)^2 z} \times \Big(-3zy^3 + 3y^3 + \pi^2 z^2 y^2 + 3z^2 y^2 - 6zy^2$$
$$+ 6z^2 G(\{-1, -1\}, y)y^2 - 6z^2 G(\{0, -1\}, y)y^2 + 6z^2 G(\{0, 0\}, z)y^2$$
$$- 6z^2 G(\{y+1, 0\}, z)y^2 + 3y^2 + 3z^2 y - 3zy + 3(y+1)(y+z)\left(y^2 - 3zy + y + z\right)G(\{0\}, y)$$
$$+ 3(y+1)z(y(3z-2) - z)G(\{0\}, z) + 3(y+1)(y+z)\left(y^2 - 3zy + y + z\right)G(\{-y\}, z)$$
$$-3G(\{-1\}, y)\left(2y^2(G(\{0\}, z) - G(\{y+1\}, z))z^2 + (y+1)\left(y^3 + (1-2z)y^2 - z^2 y + z^2\right)\right)\Big)$$
$$+ \frac{\epsilon}{96\pi y^2(y+1)(z-1)^2 z}\times$$
$$\times \Big(\pi^2 y^4 - 2\pi^2 zy^3 - 6z\log(4\pi)y^3 + 6\log(4\pi)y^3 + 2\pi^2 y^3$$
$$- 9\pi^2 z^2 y^2 + 12\pi^2 zy^2 + 2z^2 G(\{y+1\}, z)\left(-12G(\{-1, -1\}, y) + 6G(\{0, -1\}, y) + \pi^2\right)y^2$$
$$+ 12(y - 3z + 2)(y+z)G(\{-y, 0\}, z)y^2 - 12z^2 G(\{-1, 0, -1\}, y)y^2$$
$$+ 36z^2 G(\{0, -1, -1\}, y)y^2 + 24z^2 G(\{0, 0, -1\}, y)y^2 - 48z^2 G(\{0, 0, 0\}, y)y^2$$
$$- 36z^2 G(\{0, 0, 0\}, z)y^2 - 24z^2 G(\{0, 0, 1\}, z)y^2 - 36z^2 G(\{0, 1, 0\}, z)y^2$$
$$+ 24z^2 G(\{0, -y, 0\}, z)y^2 + 36z^2 G(\{0, y+1, 0\}, z)y^2 - 24z^2 G(\{1, 0, 0\}, z)y^2$$
$$+ 24z^2 G(\{1, y+1, 0\}, z)y^2 + 24z^2 G(\{y+1, 0, 1\}, z)y^2 + 12z^2 G(\{y+1, 1, 0\}, z)y^2$$
$$- 36z^2\zeta(3)y^2 + 2\pi^2 z^2\log(4\pi)y^2 + 6z^2\log(4\pi)y^2$$
$$- 12z\log(4\pi)y^2 + 6\log(4\pi)y^2 - 3\pi^2 y^2 + 2\pi^2 z^2 y - 2\pi^2 zy$$
$$- 4G(\{1\}, z)\Big(-3z(y+1)^2 + 3y(y+1) + \left(\left(3 + \pi^2\right)y + 3\right)z^2$$
$$+6yz^2(G(\{-1, -1\}, y) - G(\{0, -1\}, y))\Big)y$$
$$+ 12G(\{y+1, 0\}, z)\left((y+1)(y-z)(y-z+1) - yz^2\log(4\pi)\right)y + 6z^2\log(4\pi)y$$
$$- 6z\log(4\pi)y - \pi^2 z^2 + 6\Big((y(y+2) - 1)y^2 - 2\left(y^2 + 3\right)zy$$
$$+3(y(y+2) - 1)z^2\Big)G(\{0, 0\}, y) - 12(y+1)z(y(3z-2) - z)G(\{0, 1\}, z)$$
$$- 6(y+1)\left((y+1)y^2 - 2(y+1)zy + (3y-1)z^2\right)G(\{0, -y\}, z)$$
$$- 12\left(\left(z^2 + z - 1\right)y^2 + 3(z-1)zy - z^2\right)G(\{1, 0\}, z)$$
$$- 12(y+1)(y+z)\left(y^2 - 3zy + y + z\right)G(\{1, -y\}, z)$$

$$
\begin{aligned}
&- 12(y+1)(y+z)\left(y^2 - 3zy + y + z\right) G(\{-y,1\},z) \\
&- 12(y+1)(y+z)\left(y^2 - 3zy + y + z\right) G(\{-y,-y\},z) \\
&+ 2G(\{0\},y)\left(3y^4 - \left(\left(3+4\pi^2\right)z^2 + 3\right)y^2 + 3z^2\right. \\
&- 3(y+1)\left((y+1)y^2 - 2(y+1)zy + (3y-1)z^2\right) G(\{0\},z) \\
&+3(y+1)(y+z)\left(y^2 - 3zy + y + z\right)\left(-2(G(\{1\},z) + G(\{-y\},z)) + \log(\pi) + \log(4)\right)\Big) \\
&+ 12zG(\{-1,-1\},y)\left(((-2+\log(4)+\log(\pi))z+1)y^2 - zy + y + z\right) \\
&- 6zG(\{0,0\},z)\left((y+1)(y(3z-2)-z) - 2y^2 z\log(4\pi)\right) \\
&+ 6(y+1)(y+z)G(\{-y\},z)\left((y-1)(y-z) + \left(y^2 - 3zy + y + z\right)\log(4\pi)\right) \\
&+ 6G(\{-1\},y)\left(-y^4 + y^2 - 2\left((y+1)(y-z)(y-z+1)G(\{y+1\},z) - yz^2(2G(\{0,0\},z)\right.\right. \\
&+ 2G(\{0,1\},z) - 3G(\{0,y+1\},z) + 2G(\{1,0\},z) - 2G(\{1,y+1\},z) \\
&+G(\{y+1,0\},z) - 2G(\{y+1,1\},z)))\,y + \left(\left(1+\pi^2\right)y^2 - 1\right)z^2 \\
&+ 2(y+1)\left(y^3 + (1-2z)y^2 - z^2 y + z^2\right) G(\{1\},z) \\
&+ G(\{0\},z)\left((y+1)^2(y-z)^2 - 2y^2z^2\log(4\pi)\right) \\
&- \left((y+1)\left(y^3 - 2zy^2 + y^2 - z^2 y + z^2\right) - 2y^2z^2 G(\{y+1\},z)\right)\log(4\pi)\Big) \\
&- 6G(\{0\},z)\left(2y^2 G(\{-1,-1\},y)z^2 - 4y^2 G(\{0,0\},y)z^2\right. \\
&+(y+1)\left((y-z)^2 + z(-3zy+2y+z)\log(4\pi)\right)\Big) \\
&-6G(\{0,-1\},y)\left(y^4 - 2(z-1)y^3 + (z((-3+2\log(4\pi))z+4)-1)y^2 + 2(z-1)zy - z^2\right)\Big) \\
&+ O(\epsilon^2) \qquad (6.60)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{19} =& + \frac{6i\pi G(\{0\},z) + 6G(\{0,0\},z) - 6G(\{0,1\},z) - \pi^2}{48\pi} \\
& \frac{\epsilon}{48\pi}\left(2\pi^2 G(\{1\},z) - 12i\pi G(\{1,0\},z) - 12G(\{0,0,0\},z) - 6G(\{0,0,1\},z)\right. \\
& - 18G(\{0,1,0\},z) + 36G(\{0,1,1\},z) - 12G(\{1,0,0\},z) + 12G(\{1,0,1\},z) \\
& - 3\pi(\pi - 2i(2+\log(4)+\log(\pi)))G(\{0\},z) + 6(-2-4i\pi-\log(4\pi))G(\{0,1\},z) \\
& \left.+6(2 - i\pi + \log(4) + \log(\pi))G(\{0,0\},z) + 6\zeta(3) - 2i\pi^3 - 2\pi^2 - \pi^2\log(4\pi)\right) \\
& + O(\epsilon^2) \qquad (6.61)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{24} =& \frac{1}{\epsilon^2}\frac{1}{12\pi(y+1)(y+z)} \\
& \frac{1}{\epsilon}\frac{-G(\{-1\},y) - 2G(\{0\},y) + G(\{0\},z) - 5G(\{1\},z) - 2G(\{-y\},z) + 2\log(\pi) + \log(16)}{24\pi(y+1)(y+z)} \\
& + \frac{1}{72\pi(y+1)(y+z)}\left(-15\log(4\pi)G(\{1\},z) - 3G(\{-1,-1\},y) + 12G(\{-1,0\},y)\right. \\
& + 12G(\{0,-1\},y) - 12G(\{0,0\},y) - 3G(\{0,0\},z) - 12G(\{0,1\},z) + 6G(\{0,-y\},z) \\
& - 12G(\{1,0\},z) + 33G(\{1,1\},z) + 24G(\{1,-y\},z) + 6G(\{-y,0\},z) + 24G(\{-y,1\},z)
\end{aligned}
$$

$$
\begin{aligned}
&- 12G(\{-y,-y\},z) + 6G(\{0\},y)(4G(\{1\},z) - 2G(\{-y\},z) - \log(4\pi)) \\
&+ 3G(\{-1\},y)(-2G(\{0\},z) + G(\{1\},z) + 4G(\{-y\},z) - \log(4\pi)) \\
&+3G(\{0\},z)(2G(\{0\},y) + \log(4\pi)) + 3\log^2(4\pi) - 6G(\{-y\},z)\log(4\pi) - 5\pi^2\Big) \\
&+ \frac{\epsilon}{144\pi(y+1)(y+z)}\times \\
&\times (-6\log(4\pi)G(\{-1,-1\},y) + 30G(\{-1,-1,-1\},y) - 48G(\{-1,-1,0\},y) \\
&+ 24G(\{-1,0,-1\},y) - 24G(\{-1,0,0\},y) - 12G(\{0,-1,-1\},y) - 24G(\{0,-1,0\},y) \\
&- 60G(\{0,0,-1\},y) + 96G(\{0,0,0\},y) + 6G(\{0,0,0\},z) + 24G(\{0,0,1\},z) \\
&- 12G(\{0,0,-y\},z) - 12G(\{0,1,0\},z) + 60G(\{0,1,1\},z) + 24G(\{0,1,-y\},z) \\
&+ 24G(\{0,-y,0\},z) - 12G(\{0,-y,1\},z) - 48G(\{0,-y,-y\},z) + 24G(\{1,0,0\},z) \\
&+ 24G(\{1,0,1\},z) + 24G(\{1,0,-y\},z) + 96G(\{1,1,0\},z) - 102G(\{1,1,1\},z) \\
&- 192G(\{1,1,-y\},z) + 24G(\{1,-y,0\},z) - 120G(\{1,-y,1\},z) - 48G(\{1,-y,-y\},z) \\
&- 12G(\{-y,0,0\},z) + 24G(\{-y,0,1\},z) - 48G(\{-y,0,-y\},z) + 24G(\{-y,1,0\},z) \\
&- 120G(\{-y,1,1\},z) - 48G(\{-y,1,-y\},z) - 120G(\{-y,-y,0\},z) - 48G(\{-y,-y,1\},z) \\
&+ 240G(\{-y,-y,-y\},z) + G(\{-y\},z)\,(24G(\{-1,-1\},y) - 96G(\{-1,0\},y) \\
&-96G(\{0,-1\},y) + 168G(\{0,0\},y) - 6\log^2(4\pi) + 22\pi^2\Big) \\
&+ G(\{0\},z)\,(24G(\{-1,-1\},y) - 24G(\{-1,0\},y) + 12G(\{0,-1\},y) - 12G(\{0,0\},y) \\
&+18\log(\pi)\log(16\pi) - 15\log^2(4\pi) + 72\log^2(2) - 11\pi^2\Big) \\
&+ G(\{-1\},y)\,(12G(\{0,0\},z) + 12G(\{0,1\},z) + 12G(\{0,-y\},z) - 24G(\{1,0\},z) \\
&+ 30G(\{1,1\},z) - 24G(\{1,-y\},z) + 48G(\{-y,0\},z) - 24G(\{-y,1\},z) - 96G(\{-y,-y\},z) \\
&+ 27\log(\pi)\log(16\pi) + 6(-2G(\{0\},z) + G(\{1\},z) + 4G(\{-y\},z) - 5\log(4\pi))\log(4\pi) \\
&+108\log^2(2) - 13\pi^2\Big) + G(\{1\},z)\,(-30G(\{-1,-1\},y) + 48G(\{-1,0\},y) \\
&- 24G(\{0,-1\},y) - 48G(\{0,0\},y) - 15\log(\pi)\log\left(4096\pi^3\right) + 30\log^2(4\pi) \\
&-180\log^2(2) + 31\pi^2\Big) + G(\{0\},y)\,(-12G(\{0,0\},z) + 24G(\{0,1\},z) \\
&- 48G(\{0,-y\},z) + 24G(\{1,0\},z) - 192G(\{1,1\},z) - 48G(\{1,-y\},z) - 48G(\{-y,0\},z) \\
&- 48G(\{-y,1\},z) + 240G(\{-y,-y\},z) + 3\log(4\pi)(4G(\{0\},z) + 16G(\{1\},z) \\
&-8G(\{-y\},z) + \log(4\pi)) - 3\log(\pi)\log\left(4096\pi^3\right) - 36\log^2(2) + 22\pi^2\Big) \\
&- 8\zeta(3) + 2\log^2(\pi)\log(64\pi) + 24G(\{-1,0\},y)\log(4\pi) \\
&+ 24G(\{0,-1\},y)\log(4\pi) - 24G(\{0,0\},y)\log(4\pi) - 6G(\{0,0\},z)\log(4\pi) \\
&- 24G(\{0,1\},z)\log(4\pi) + 12G(\{0,-y\},z)\log(4\pi) - 24G(\{1,0\},z)\log(4\pi) \\
&+ 66G(\{1,1\},z)\log(4\pi) + 48G(\{1,-y\},z)\log(4\pi) + 12G(\{-y,0\},z)\log(4\pi) \\
&+ 48G(\{-y,1\},z)\log(4\pi) - 24G(\{-y,-y\},z)\log(4\pi) - 10\pi^2\log(4\pi) \\
&+24\log^2(2)\log(\pi) + 16\log^3(2)\Big) \\
&+ O(\epsilon^2) \qquad (6.62)
\end{aligned}
$$

Gluon channel

Master Integrals for $\mathbf{g} + \mathbf{W}^* \to \mathbf{t} + \bar{\mathbf{b}} + \mathbf{g}$.

The variables e, f in terms of which some of the following masters are expressed are defined as $e = 1/c, d = 1/f$, being c, d defined in Eq.(5.165).

$$
\begin{aligned}
\tilde{M}_9 =& \frac{(f-1)(ef-e+f)}{6(f-e)}\Bigg(-6G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)+6G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)\\
&-6G(\{0\},e)G(\{0\},f)-6G(\{0\},f)G(\{f\},e)+12G(\{0\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)\\
&+6G(\{-1\},f)G(\{f\},e)-6G(\{-1\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)+6G(\{0,f\},e)+6G(\{f,0\},e)\\
&-6G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)-6G\left(\left\{-\frac{f}{f-2},0\right\},e\right)-6G\left(\left\{-\frac{f}{f-2},f\right\},e\right)\\
&+6G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)-6G(\{-1,0\},f)-6G(\{0,-1\},f)\\
&+18G(\{0,0\},f)-6G(\{1,0\},f)+\pi^2\Bigg)\\
&+\epsilon\frac{(f-1)(fe-e+f)}{6(f-e)}\Bigg(12G(\{f\},e)G(\{-1\},f)-12G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)\\
&-6G(\{0,f\},e)G(\{-1\},f)+18G\left(\left\{0,-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)\\
&-12G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)G(\{-1\},f)-6G(\{f,0\},e)G(\{-1\},f)-6G(\{f,f\},e)G(\{-1\},f)\\
&+6G\left(\left\{f,-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)+6G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)G(\{-1\},f)\\
&+6G\left(\left\{-\frac{f}{f-2},0\right\},e\right)G(\{-1\},f)-12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)\\
&-6G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)G(\{-1\},f)+12G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)G(\{-1\},f)\\
&+12G\left(\left\{-\frac{f}{f-1},f\right\},e\right)G(\{-1\},f)-12G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)\\
&+\pi^2G(\{-1\},f)+\pi^2G(\{0\},e)-12G(\{0\},e)G(\{0\},f)+\pi^2G(\{0\},f)+3\pi^2G(\{1\},f)\\
&-12G(\{0\},f)G(\{f\},e)-5\pi^2G(\{f\},e)+24G(\{0\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)\\
&+2\pi^2G\left(\left\{-\frac{f}{f-2}\right\},e\right)+2\pi^2G\left(\left\{-\frac{f}{f-1}\right\},e\right)-6G(\{f\},e)G(\{-1,-1\},f)\\
&+6G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{-1,-1\},f)+6G(\{0\},e)G(\{-1,0\},f)+12G(\{f\},e)G(\{-1,0\},f)\\
&-6G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{-1,0\},f)-12G\left(\left\{-\frac{f}{f-1}\right\},e\right)G(\{-1,0\},f)\\
&-12G(\{-1,0\},f)+6G(\{0\},e)G(\{0,-1\},f)+12G(\{f\},e)G(\{0,-1\},f)\\
&-6G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{0,-1\},f)\\
&-12G\left(\left\{-\frac{f}{f-1}\right\},e\right)G(\{0,-1\},f)-12G(\{0,-1\},f)+6G(\{0\},f)G(\{0,0\},e)\\
&-12G(\{0\},e)G(\{0,0\},f)-24G(\{f\},e)G(\{0,0\},f)+36G\left(\left\{-\frac{f}{f-1}\right\},e\right)G(\{0,0\},f)\\
&+36G(\{0,0\},f)+12G(\{0\},f)G(\{0,f\},e)+12G(\{0,f\},e)-36G(\{0\},f)G\left(\left\{0,-\frac{f}{f-2}\right\},e\right)
\end{aligned}
$$

$$- 12G(\{0\},f)G\left(\left\{0,-\frac{f}{f-1}\right\},e\right)+30G(\{0\},f)G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)$$
$$- 12G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)-6G(\{0\},e)G(\{1,0\},f)+6G(\{f\},e)G(\{1,0\},f)$$
$$+ 12G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{1,0\},f)-12G\left(\left\{-\frac{f}{f-1}\right\},e\right)G(\{1,0\},f)$$
$$- 12G(\{1,0\},f)+12G(\{0\},f)G(\{f,0\},e)+12G(\{f,0\},e)+6G(\{0\},f)G(\{f,f\},e)$$
$$- 12G(\{0\},f)G\left(\left\{f,-\frac{f}{f-2}\right\},e\right)-6G(\{0\},f)G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)$$
$$- 12G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)-6G(\{0\},f)G\left(\left\{-\frac{f}{f-2},0\right\},e\right)$$
$$- 12G\left(\left\{-\frac{f}{f-2},0\right\},e\right)-6G(\{0\},f)G\left(\left\{-\frac{f}{f-2},f\right\},e\right)$$
$$- 12G\left(\left\{-\frac{f}{f-2},f\right\},e\right)+24G(\{0\},f)G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right)$$
$$+ 18G(\{0\},f)G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)+12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)$$
$$- 30G(\{0\},f)G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+12G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)$$
$$- 12G(\{0\},f)G\left(\left\{-\frac{f}{f-1},0\right\},e\right)-12G(\{0\},f)G\left(\left\{-\frac{f}{f-1},f\right\},e\right)$$
$$+ 24G(\{0\},f)G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2}\right\},e\right)+6G(\{-1,-1,0\},f)+6G(\{-1,0,-1\},f)$$
$$- 12G(\{-1,0,0\},f)-6G(\{-1,1,0\},f)+6G(\{0,-1,-1\},f)-12G(\{0,-1,0\},f)$$
$$- 12G(\{0,0,-1\},f)+12G(\{0,0,0\},f)$$
$$- 6G(\{0,0,f\},e)+6G\left(\left\{0,0,\frac{f^2}{2f-1}\right\},e\right)+18G(\{0,1,0\},f)-6G(\{0,f,0\},e)$$
$$- 24G\left(\left\{0,f,\frac{f}{2}\right\},e\right)-6G(\{0,f,f\},e)+30G\left(\left\{0,f,-\frac{f}{f-1}\right\},e\right)$$
$$+ 6G\left(\left\{0,f,\frac{f^2}{2f-1}\right\},e\right)+18G\left(\left\{0,-\frac{f}{f-2},0\right\},e\right)$$
$$+ 18G\left(\left\{0,-\frac{f}{f-2},f\right\},e\right)-18G\left(\left\{0,-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)$$
$$- 18G\left(\left\{0,-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+12G\left(\left\{0,-\frac{f}{f-1},f\right\},e\right)$$
$$- 12G\left(\left\{0,-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)-12G\left(\left\{0,\frac{f^2}{2f-1},0\right\},e\right)$$
$$+ 24G\left(\left\{0,\frac{f^2}{2f-1},\frac{f}{2}\right\},e\right)-18G\left(\left\{0,\frac{f^2}{2f-1},f\right\},e\right)$$
$$- 12G\left(\left\{0,\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)+18G\left(\left\{0,\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right)$$
$$- 6G(\{1,-1,0\},f)-6G(\{1,0,-1\},f)+24G(\{1,0,0\},f)-18G(\{1,1,0\},f)-6G(\{f,0,0\},e)$$
$$- 24G\left(\left\{f,0,\frac{f}{2}\right\},e\right)-6G(\{f,0,f\},e)+30G\left(\left\{f,0,-\frac{f}{f-1}\right\},e\right)$$

$$
\begin{aligned}
&+6G\left(\left\{f,0,\frac{f^2}{2f-1}\right\},e\right)-6G(\{f,f,0\},e)+6G\left(\left\{f,f,-\frac{f}{f-1}\right\},e\right)\\
&+6G\left(\left\{f,-\frac{f}{f-2},0\right\},e\right)+6G\left(\left\{f,-\frac{f}{f-2},f\right\},e\right)\\
&-6G\left(\left\{f,-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)-6G\left(\left\{f,-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)\\
&+6G\left(\left\{f,-\frac{f}{f-1},0\right\},e\right)+24G\left(\left\{f,-\frac{f}{f-1},\frac{f}{2}\right\},e\right)\\
&-30G\left(\left\{f,-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)+6G\left(\left\{-\frac{f}{f-2},0,0\right\},e\right)\\
&+24G\left(\left\{-\frac{f}{f-2},0,\frac{f}{2}\right\},e\right)-30G\left(\left\{-\frac{f}{f-2},0,-\frac{f}{f-1}\right\},e\right)\\
&+24G\left(\left\{-\frac{f}{f-2},f,\frac{f}{2}\right\},e\right)+6G\left(\left\{-\frac{f}{f-2},f,f\right\},e\right)\\
&-24G\left(\left\{-\frac{f}{f-2},f,-\frac{f}{f-1}\right\},e\right)-6G\left(\left\{-\frac{f}{f-2},f,\frac{f^2}{2f-1}\right\},e\right)\\
&-12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},0\right\},e\right)-12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},f\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)+12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)\\
&-6G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},0\right\},e\right)-24G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},\frac{f}{2}\right\},e\right)\\
&-12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},f\right\},e\right)+30G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)+12G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},0\right\},e\right)\\
&-24G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},\frac{f}{2}\right\},e\right)+18G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},f\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)-18G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-1},0,f\right\},e\right)-12G\left(\left\{-\frac{f}{f-1},0,\frac{f^2}{2f-1}\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-1},f,0\right\},e\right)-12G\left(\left\{-\frac{f}{f-1},f,-\frac{f}{f-1}\right\},e\right)\\
&-12G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2},0\right\},e\right)-12G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2},f\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)\\
&+12G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+6\zeta(3)+2\pi^2\Bigg)\\
&+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{6.63}
$$

$$
\tilde{M}_{10} = \frac{1}{4\left(1-\frac{f(e(f-2)+f)}{(f-1)(e(f-1)+f)}\right)}\times
$$

$$
\begin{aligned}
&\times \Bigg(4G\left(\left\{0,\frac{1}{f}\right\},e\right)G(\{-1\},f)-4G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)G(\{-1\},f)\\
&+4G(\{f,0\},e)G(\{-1\},f)-4G\left(\left\{f,\frac{1}{f}\right\},e\right)G(\{-1\},f)\\
&+4G\left(\left\{f,-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)-4G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)G(\{-1\},f)\\
&-4G\left(\left\{-\frac{f}{f-2},0\right\},e\right)G(\{-1\},f)-4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right)G(\{-1\},f)\\
&+4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)G(\{-1\},f)+4G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)G(\{-1\},f)\\
&+\frac{2}{3}\pi^2G(\{-1\},f)+\frac{2}{3}\pi^2G(\{0\},e)+\frac{2}{3}\pi^2G(\{1\},f)-\frac{4}{3}\pi^2G(\{f\},e)\\
&+\frac{2}{3}\pi^2G\left(\left\{-\frac{f}{f-2}\right\},e\right)+4G(\{f\},e)G(\{-1,-1\},f)\\
&-4G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{-1,-1\},f)-4G(\{0\},e)G(\{-1,0\},f)\\
&+4G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{-1,0\},f)-4G(\{0\},e)G(\{0,-1\},f)\\
&+4G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{0,-1\},f)-4G(\{0\},f)G(\{0,0\},e)+8G(\{0\},e)G(\{0,0\},f)\\
&-8G(\{f\},e)G(\{0,0\},f)+4G(\{0\},f)G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)\\
&-4G(\{0\},e)G(\{1,0\},f)+8G(\{f\},e)G(\{1,0\},f)-4G\left(\left\{-\frac{f}{f-2}\right\},e\right)G(\{1,0\},f)\\
&-8G(\{0\},f)G\left(\left\{f,-\frac{f}{f-2}\right\},e\right)+8G(\{0\},f)G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)\\
&+4G(\{0\},f)G\left(\left\{-\frac{f}{f-2},0\right\},e\right)+8G(\{0\},f)G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right)\\
&-8G(\{0\},f)G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)\\
&-4G(\{0\},f)G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)-4G(\{-1,-1,0\},f)-4G(\{-1,0,-1\},f)\\
&+8G(\{-1,0,0\},f)-4G(\{-1,1,0\},f)-4G(\{0,-1,-1\},f)+8G(\{0,-1,0\},f)\\
&+8G(\{0,0,-1\},f)-8G(\{0,0,0\},f)+4G(\{0,0,f\},e)-4G\left(\left\{0,0,\frac{f^2}{2f-1}\right\},e\right)\\
&+4G\left(\left\{0,\frac{1}{f},0\right\},e\right)-4G\left(\left\{0,\frac{1}{f},f\right\},e\right)\\
&-4G\left(\left\{0,\frac{1}{f},-\frac{f}{f-1}\right\},e\right)+4G\left(\left\{0,\frac{1}{f},\frac{f^2}{2f-1}\right\},e\right)\\
&-4G\left(\left\{0,\frac{f^2}{2f-1},0\right\},e\right)+4G\left(\left\{0,\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)\\
&-4G(\{1,-1,0\},f)-4G(\{1,0,-1\},f)+8G(\{1,0,0\},f)-4G(\{1,1,0\},f)\\
&+4G(\{f,0,0\},e)-4G(\{f,0,f\},e)-4G\left(\left\{f,0,-\frac{f}{f-1}\right\},e\right)\\
&+4G\left(\left\{f,0,\frac{f^2}{2f-1}\right\},e\right)-4G\left(\left\{f,\frac{1}{f},0\right\},e\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4G\left(\left\{f,\frac{1}{f},f\right\},e\right)+4G\left(\left\{f,\frac{1}{f},-\frac{f}{f-1}\right\},e\right)\\
&-4G\left(\left\{f,\frac{1}{f},\frac{f^2}{2f-1}\right\},e\right)+4G\left(\left\{f,-\frac{f}{f-2},0\right\},e\right)\\
&+4G\left(\left\{f,-\frac{f}{f-2},f\right\},e\right)-4G\left(\left\{f,-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)\\
&-4G\left(\left\{f,-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)-4G\left(\left\{f,-\frac{f}{f-1},0\right\},e\right)\\
&-4G\left(\left\{f,-\frac{f}{f-1},f\right\},e\right)+4G\left(\left\{f,-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)\\
&+4G\left(\left\{f,-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)-4G\left(\left\{-\frac{f}{f-2},0,0\right\},e\right)\\
&+4G\left(\left\{-\frac{f}{f-2},0,-\frac{f}{f-1}\right\},e\right)-4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},0\right\},e\right)\\
&-4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},f\right\},e\right)+4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)\\
&+4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},0\right\},e\right)\\
&+4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},f\right\},e\right)-4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)\\
&-4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)+4G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},0\right\},e\right)\\
&-4G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)+12\zeta(3)\Bigg)\\
&+\mathcal{O}(\epsilon)
\end{aligned}
\tag{6.64}
$$

$$
\begin{aligned}
\tilde{M}_{11}=&\frac{(f-1)^2(e(f-1)+f)^2}{e(f+1)(e-f)^2}\times\\
&\Bigg(e(-f)G(\{0\},f)+eG(\{0\},f)-efG\left(\left\{-\frac{f}{f-1}\right\},e\right)\\
&+eG\left(\left\{-\frac{f}{f-1}\right\},e\right)+(e(f-1)+f)G(\{-1\},f)+(e(f-1)+f)G(\{0\},e)\\
&-fG\left(\left\{-\frac{f}{f-1}\right\},e\right)-fG(\{0\},f)-2e+f\Bigg)\\
&+\frac{\epsilon(f-1)(e(f-1)+f)^2}{3e(e-f)^2(f+1)}\times\\
&\times\Bigg(9eG(\{0\},f)f^2+3G(\{0\},f)f^2-12G\left(\left\{\frac{f}{2}\right\},e\right)f^2\\
&+12eG(\{0\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)f^2+12G(\{0\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)f^2\\
&+9eG\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2-15eG(\{0\},f)G\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2\\
&-15G(\{0\},f)G\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2+15G\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2\\
&-6G(\{0\},f)G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)f^2-9eG(\{-1,-1\},f)f^2-9G(\{-1,-1\},f)f^2
\end{aligned}
$$

$$
\begin{aligned}
&+3eG(\{-1,0\},f)f^2+9G(\{-1,0\},f)f^2+3eG(\{0,-1\},f)f^2\\
&+9G(\{0,-1\},f)f^2-9eG(\{0,0\},e)f^2\\
&-9G(\{0,0\},e)f^2+9eG(\{0,0\},f)f^2-3G(\{0,0\},f)f^2-12eG\left(\left\{0,\frac{f}{2}\right\},e\right)f^2\\
&-12G\left(\left\{0,\frac{f}{2}\right\},e\right)f^2+6eG(\{0,f\},e)f^2+21eG\left(\left\{0,-\frac{f}{f-1}\right\},e\right)f^2\\
&+21G\left(\left\{0,-\frac{f}{f-1}\right\},e\right)f^2-6eG\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)f^2\\
&-12eG(\{1,0\},f)f^2-6G(\{1,0\},f)f^2+6eG\left(\left\{\frac{1}{f},0\right\},e\right)f^2\\
&-6eG\left(\left\{\frac{1}{f},f\right\},e\right)f^2-6eG\left(\left\{\frac{1}{f},-\frac{f}{f-1}\right\},e\right)f^2\\
&+6eG\left(\left\{\frac{1}{f},\frac{f^2}{2f-1}\right\},e\right)f^2-6eG\left(\left\{-\frac{f}{f-2},0\right\},e\right)f^2\\
&-6G\left(\left\{-\frac{f}{f-2},0\right\},e\right)f^2-6eG\left(\left\{-\frac{f}{f-2},f\right\},e\right)f^2\\
&-6G\left(\left\{-\frac{f}{f-2},f\right\},e\right)f^2+6eG\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)f^2\\
&+6G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)f^2+6eG\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)f^2\\
&+6G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)f^2+9eG\left(\left\{-\frac{f}{f-1},0\right\},e\right)f^2\\
&+9G\left(\left\{-\frac{f}{f-1},0\right\},e\right)f^2+12eG\left(\left\{-\frac{f}{f-1},\frac{f}{2}\right\},e\right)f^2\\
&+12G\left(\left\{-\frac{f}{f-1},\frac{f}{2}\right\},e\right)f^2+6eG\left(\left\{-\frac{f}{f-1},f\right\},e\right)f^2\\
&+6G\left(\left\{-\frac{f}{f-1},f\right\},e\right)f^2-21eG\left(\left\{-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)f^2\\
&-21G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)f^2-6eG\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)f^2\\
&-6G\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)f^2+6G\left(\left\{\frac{f^2}{2f-1},0\right\},e\right)f^2\\
&-6G\left(\left\{\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)f^2-\pi^2f^2+3f^2-6ef-6eG(\{0\},f)f\\
&-3G(\{0\},f)f+24eG\left(\left\{\frac{f}{2}\right\},e\right)f+12G\left(\left\{\frac{f}{2}\right\},e\right)f\\
&-6G(\{0\},f)G(\{f\},e)f-24eG(\{0\},f)G\left(\left\{-\frac{f}{f-2}\right\},e\right)f\\
&-30eG\left(\left\{-\frac{f}{f-1}\right\},e\right)f+30eG(\{0\},f)G\left(\left\{-\frac{f}{f-1}\right\},e\right)f\\
&+15G(\{0\},f)G\left(\left\{-\frac{f}{f-1}\right\},e\right)f-15G\left(\left\{-\frac{f}{f-1}\right\},e\right)f\\
&+12eG(\{0\},f)G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)f+18eG(\{-1,-1\},f)f+9G(\{-1,-1\},f)f\\
&-18eG(\{-1,0\},f)f-9G(\{-1,0\},f)f-18eG(\{0,-1\},f)f-9G(\{0,-1\},f)f
\end{aligned}
$$

$$
\begin{aligned}
&+ 18eG(\{0,0\},e)f + 9G(\{0,0\},e)f + 6eG(\{0,0\},f)f + 9G(\{0,0\},f)f \\
&+ 24eG\left(\left\{0,\frac{f}{2}\right\},e\right)f + 12G\left(\left\{0,\frac{f}{2}\right\},e\right)f \\
&- 42eG\left(\left\{0,-\frac{f}{f-1}\right\},e\right)f - 21G\left(\left\{0,-\frac{f}{f-1}\right\},e\right)f \\
&+ 12eG(\{1,0\},f)f + 6G(\{1,0\},f)f - 6G\left(\left\{\frac{1}{f},0\right\},e\right)f \\
&+ 6G\left(\left\{\frac{1}{f},f\right\},e\right)f + 6G\left(\left\{\frac{1}{f},-\frac{f}{f-1}\right\},e\right)f \\
&- 6G\left(\left\{\frac{1}{f},\frac{f^2}{2f-1}\right\},e\right)f + 6G(\{f,0\},e)f - 6G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)f \\
&+ 12eG\left(\left\{-\frac{f}{f-2},0\right\},e\right)f + 12eG\left(\left\{-\frac{f}{f-2},f\right\},e\right)f \\
&- 12eG\left(\left\{-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)f - 12eG\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)f \\
&- 18eG\left(\left\{-\frac{f}{f-1},0\right\},e\right)f - 9G\left(\left\{-\frac{f}{f-1},0\right\},e\right)f \\
&- 24eG\left(\left\{-\frac{f}{f-1},\frac{f}{2}\right\},e\right)f - 12G\left(\left\{-\frac{f}{f-1},\frac{f}{2}\right\},e\right)f \\
&- 12eG\left(\left\{-\frac{f}{f-1},f\right\},e\right)f - 6G\left(\left\{-\frac{f}{f-1},f\right\},e\right)f \\
&+ 42eG\left(\left\{-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)f + 21G\left(\left\{-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right)f \\
&+ 12eG\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)f + 6G\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)f \\
&- 12eG\left(\left\{\frac{f^2}{2f-1},0\right\},e\right)f + 12eG\left(\left\{\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)f \\
&+ 2e\pi^2 f + \pi^2 f - 3f + 6e - 3eG(\{0\},f) \\
&+ 3(f-1)G(\{0\},e)(-f - e(3f+1) + (e(f-5)+3f)G(\{0\},f)) \\
&- 24eG\left(\left\{\frac{f}{2}\right\},e\right) + 6eG(\{0\},f)G(\{f\},e) + 21eG\left(\left\{-\frac{f}{f-1}\right\},e\right) \\
&- 15eG(\{0\},f)G\left(\left\{-\frac{f}{f-1}\right\},e\right) - 6eG(\{0\},f)G\left(\left\{\frac{f^2}{2f-1}\right\},e\right) \\
&- 3G(\{-1\},f)\left(3ef^2 + 2eG\left(\left\{-\frac{f}{f-2}\right\},e\right)f^2 + 2G\left(\left\{-\frac{f}{f-2}\right\},e\right)f^2\right. \\
&- 3eG\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2 - 3G\left(\left\{-\frac{f}{f-1}\right\},e\right)f^2 \\
&- 2G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)f^2 + f^2 - 2ef - 2(ef-1)G\left(\left\{\frac{1}{f}\right\},e\right)f \\
&- 2G(\{f\},e)f - 4eG\left(\left\{-\frac{f}{f-2}\right\},e\right)f + 6eG\left(\left\{-\frac{f}{f-1}\right\},e\right)f \\
&+ 3G\left(\left\{-\frac{f}{f-1}\right\},e\right)f + 4eG\left(\left\{\frac{f^2}{2f-1}\right\},e\right)f \\
&- f - e + 3(f-1)(e(f-1)+f)G(\{0\},e) + 2eG(\{f\},e) - 3eG\left(\left\{-\frac{f}{f-1}\right\},e\right)
\end{aligned}
$$

$$
\begin{aligned}
&-2eG\left(\left\{\frac{f^2}{2f-1}\right\},e\right)\Bigg)\Bigg) - 9eG(\{-1,-1\},f) + 15eG(\{-1,0\},f) \\
&+ 15eG(\{0,-1\},f) - 9eG(\{0,0\},e) - 21eG(\{0,0\},f) - 12eG\left(\left\{0,\frac{f}{2}\right\},e\right) \\
&- 6eG(\{0,f\},e) + 21eG\left(\left\{0,-\frac{f}{f-1}\right\},e\right) + 6eG\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right) \\
&- 6eG(\{f,0\},e) + 6eG\left(\left\{f,-\frac{f}{f-1}\right\},e\right) + 9eG\left(\left\{-\frac{f}{f-1},0\right\},e\right) \\
&+ 12eG\left(\left\{-\frac{f}{f-1},\frac{f}{2}\right\},e\right) + 6eG\left(\left\{-\frac{f}{f-1},f\right\},e\right) \\
&- 21eG\left(\left\{-\frac{f}{f-1},-\frac{f}{f-1}\right\},e\right) - 6eG\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right) \\
&+6eG\left(\left\{\frac{f^2}{2f-1},0\right\},e\right) - 6eG\left(\left\{\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right) - 2e\pi^2\Bigg) \\
&+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{6.65}
$$

$$
\begin{aligned}
\tilde{M}_{12} =& -\frac{(f-1)(e(f-1)+f)}{2(f-e)\sqrt{\frac{(f^2+f-2e)^2}{(f-1)^2(e(f-1)+f)^2}}}\times \\
&\left(-\frac{4}{3}\pi^2 G(\{1\},f) + \frac{1}{3}G\left(\left\{-\frac{f}{f-2}\right\},e\right)(-12G(\{-1,-1\},f)\right. \\
&\left.+6G(\{-1,0\},f) + 6G(\{0,-1\},f) + 24G(\{0,0\},f) - 18G(\{1,0\},f) + \pi^2\right) \\
&- \frac{2}{3}G(\{0\},e)\left(6G(\{-1,0\},f) - 6G(\{0,-1\},f) + 9G(\{0,0\},f) - 6G(\{1,0\},f) + 2\pi^2\right) \\
&+ G(\{f\},e)\,(4G(\{-1,-1\},f) + 2G(\{-1,0\},f) - 6G(\{0,-1\},f) \\
&\left.-2G(\{0,0\},f) + 2G(\{1,0\},f) + \pi^2\right) \\
&+ \frac{2}{3}G(\{-1\},f)\left(-6G(\{0,f\},e) + 6G\left(\left\{0,-\frac{f}{f-2}\right\},e\right) + 6G(\{f,0\},e)\right. \\
&- 6G\left(\left\{f,\frac{f}{2}\right\},e\right) + 3G(\{f,f\},e) - 3G\left(\left\{f,-\frac{f}{f-2}\right\},e\right) \\
&- 6G\left(\left\{-\frac{f}{f-2},0\right\},e\right) + 6G\left(\left\{-\frac{f}{f-2},\frac{f}{2}\right\},e\right) \\
&\left.+3G\left(\left\{-\frac{f}{f-2},f\right\},e\right) - 3G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right) + 2\pi^2\right) \\
&+ G(\{0\},f)\left(4G(\{0,0\},e) + 6G(\{0,f\},e) - 8G\left(\left\{0,-\frac{f}{f-2}\right\},e\right)\right. \\
&- 8G\left(\left\{0,\frac{f+1}{2}\right\},e\right) + 6G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right) - 6G(\{f,0\},e) \\
&+ 8G\left(\left\{f,\frac{f}{2}\right\},e\right) - 2G(\{f,f\},e) + 4G\left(\left\{f,-\frac{f}{f-2}\right\},e\right) \\
&+ 8G\left(\left\{f,\frac{f+1}{2}\right\},e\right) - 12G\left(\left\{f,\frac{f^2}{2f-1}\right\},e\right) \\
&+ 2G\left(\left\{-\frac{f}{f-2},0\right\},e\right) - 8G\left(\left\{-\frac{f}{f-2},\frac{f}{2}\right\},e\right) \\
&- 4G\left(\left\{-\frac{f}{f-2},f\right\},e\right) + 4G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2}\right\},e\right)
\end{aligned}
$$

$$+6G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+\pi^2\Bigg)$$

$$+4G(\{-1,-1,0\},f)-4G(\{-1,0,-1\},f)+6G(\{-1,0,0\},f)-4G(\{-1,1,0\},f)$$

$$+4G(\{0,-1,-1\},f)+2G(\{0,-1,0\},f)-6G(\{0,0,-1\},f)-2G(\{0,0,0\},f)$$

$$-4G(\{0,0,f\},e)+4G\left(\left\{0,0,\frac{f^2}{2f-1}\right\},e\right)+2G(\{0,1,0\},f)$$

$$-4G(\{0,f,0\},e)-2G(\{0,f,f\},e)+4G\left(\left\{0,f,-\frac{f}{f-1}\right\},e\right)$$

$$+2G\left(\left\{0,f,\frac{f^2}{2f-1}\right\},e\right)+4G\left(\left\{0,-\frac{f}{f-2},0\right\},e\right)$$

$$+4G\left(\left\{0,-\frac{f}{f-2},f\right\},e\right)-4G\left(\left\{0,-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)$$

$$-4G\left(\left\{0,-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)+8G\left(\left\{0,\frac{f+1}{2},f\right\},e\right)$$

$$-8G\left(\left\{0,\frac{f+1}{2},\frac{f^2}{2f-1}\right\},e\right)-6G\left(\left\{0,\frac{f^2}{2f-1},f\right\},e\right)$$

$$+6G\left(\left\{0,\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right)-4G(\{1,-1,0\},f)+4G(\{1,0,-1\},f)$$

$$-6G(\{1,0,0\},f)+4G(\{1,1,0\},f)+4G(\{f,0,0\},e)+2G(\{f,0,f\},e)$$

$$-4G\left(\left\{f,0,-\frac{f}{f-1}\right\},e\right)$$

$$-2G\left(\left\{f,0,\frac{f^2}{2f-1}\right\},e\right)-4G\left(\left\{f,\frac{f}{2},0\right\},e\right)$$

$$-4G\left(\left\{f,\frac{f}{2},f\right\},e\right)+4G\left(\left\{f,\frac{f}{2},-\frac{f}{f-1}\right\},e\right)$$

$$+4G\left(\left\{f,\frac{f}{2},\frac{f^2}{2f-1}\right\},e\right)+2G(\{f,f,0\},e)$$

$$-2G\left(\left\{f,f,-\frac{f}{f-1}\right\},e\right)-2G\left(\left\{f,-\frac{f}{f-2},0\right\},e\right)$$

$$-2G\left(\left\{f,-\frac{f}{f-2},f\right\},e\right)+2G\left(\left\{f,-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)$$

$$+2G\left(\left\{f,-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right)-8G\left(\left\{f,\frac{f+1}{2},f\right\},e\right)$$

$$+8G\left(\left\{f,\frac{f+1}{2},\frac{f^2}{2f-1}\right\},e\right)+12G\left(\left\{f,\frac{f^2}{2f-1},f\right\},e\right)$$

$$-12G\left(\left\{f,\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right)-4G\left(\left\{-\frac{f}{f-2},0,0\right\},e\right)$$

$$+2G\left(\left\{-\frac{f}{f-2},0,f\right\},e\right)+4G\left(\left\{-\frac{f}{f-2},0,-\frac{f}{f-1}\right\},e\right)$$

$$-2G\left(\left\{-\frac{f}{f-2},0,\frac{f^2}{2f-1}\right\},e\right)+4G\left(\left\{-\frac{f}{f-2},\frac{f}{2},0\right\},e\right)$$

$$+4G\left(\left\{-\frac{f}{f-2},\frac{f}{2},f\right\},e\right)-4G\left(\left\{-\frac{f}{f-2},\frac{f}{2},-\frac{f}{f-1}\right\},e\right)$$

$$
\begin{aligned}
&- 4G\left(\left\{-\frac{f}{f-2},\frac{f}{2},\frac{f^2}{2f-1}\right\},e\right) + 2G\left(\left\{-\frac{f}{f-2},f,0\right\},e\right)\\
&+ 2G\left(\left\{-\frac{f}{f-2},f,f\right\},e\right) - 2G\left(\left\{-\frac{f}{f-2},f,-\frac{f}{f-1}\right\},e\right)\\
&- 2G\left(\left\{-\frac{f}{f-2},f,\frac{f^2}{2f-1}\right\},e\right) - 2G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},0\right\},e\right)\\
&- 2G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},f\right\},e\right) + 2G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},-\frac{f}{f-1}\right\},e\right)\\
&+ 2G\left(\left\{-\frac{f}{f-2},-\frac{f}{f-2},\frac{f^2}{2f-1}\right\},e\right) - 6G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},f\right\},e\right)\\
&+6G\left(\left\{-\frac{f}{f-2},\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right) - 18\zeta(3)\Bigg)\\
&+ \mathcal{O}(\epsilon) \qquad (6.66)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{13} =& -\frac{1}{\epsilon^2}\frac{(f-1)^3(e(f-1)+f)^3\left(G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)-G(\{f\},e)+G(\{0\},f)\right)}{f(e-f)^2(2e-f-1)}\\
&\frac{1}{\epsilon}\frac{f(2e-f-1)}{(e-f)^2}\\
&\times\Bigg(-(f-1)^3(e(f-1)+f)^3G(\{0\},f)\left(-3G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)+G(\{f\},e)\right.\\
&\left.+2G\left(\left\{-\frac{f}{f-1}\right\},e\right)-2G\left(\left\{\frac{f+1}{2}\right\},e\right)\right)\\
&+2(f-1)^3(e(f-1)+f)^3G(\{-1\},f)\left(G(\{f\},e)-G\left(\left\{\frac{f^2}{2f-1}\right\},e\right)\right)\\
&-2(f-1)^3(e(f-1)+f)^3G\left(\left\{0,\frac{f^2}{2f-1}\right\},e\right)\\
&+(f-1)^3(e(f-1)+f)^3G\left(\left\{f,\frac{f^2}{2f-1}\right\},e\right)\\
&-2(f-1)^3(e(f-1)+f)^3G\left(\left\{-\frac{f}{f-1},\frac{f^2}{2f-1}\right\},e\right)\\
&+2(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f+1}{2},\frac{f^2}{2f-1}\right\},e\right)\\
&-2(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f^2}{2f-1},0\right\},e\right)\\
&+4(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f^2}{2f-1},\frac{f}{2}\right\},e\right)\\
&-(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f^2}{2f-1},f\right\},e\right)\\
&-2(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f^2}{2f-1},-\frac{f}{f-1}\right\},e\right)\\
&+(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f^2}{2f-1},\frac{f^2}{2f-1}\right\},e\right)
\end{aligned}
$$

$$
\begin{aligned}
&-2(f-1)^3(e(f-1)+f)^3G(\{0\},e)G(\{0\},f)\\
&-2(f-1)^3(e(f-1)+f)^3G(\{0,-1\},f)+5(f-1)^3(e(f-1)+f)^3G(\{0,0\},f)\\
&+2(f-1)^3(e(f-1)+f)^3G(\{0,f\},e)-4(f-1)^3(e(f-1)+f)^3G(\{1,0\},f)\\
&+2(f-1)^3(e(f-1)+f)^3G(\{f,0\},e)-4(f-1)^3(e(f-1)+f)^3G\left(\left\{f,\frac{f}{2}\right\},e\right)\\
&-(f-1)^3(e(f-1)+f)^3G(\{f,f\},e)+2(f-1)^3(e(f-1)+f)^3G\left(\left\{f,-\frac{f}{f-1}\right\},e\right)\\
&+2(f-1)^3(e(f-1)+f)^3G\left(\left\{-\frac{f}{f-1},f\right\},e\right)\\
&-2(f-1)^3(e(f-1)+f)^3G\left(\left\{\frac{f+1}{2},f\right\},e\right)+\frac{5}{6}\pi^2(f-1)^3(e(f-1)+f)^3\Big)
\end{aligned}
\tag{6.67}
$$

$$
\begin{aligned}
\tilde{M}_{25}=&\frac{1}{6\sqrt{\frac{(2a+1)^2b^2}{(a-b)^2}}(b+1)(ba-a+b)}\Big(-3G(\{0\},a)\left(-2G(\{-1,-1\},b)+2G(\{0,-1\},b)+3\pi^2\right)(a-b)\\
&+2G\left(\left\{-\frac{b}{b-1}\right\},a\right)\Big(3G(\{-1,-1\},b)+4\Big(-3G\left(\left\{-\frac{1}{2},-1\right\},b\right)\\
&+3G\left(\left\{-\frac{1}{2},0\right\},b\right)+6G(\{0,-1\},b)-6G(\{0,0\},b)+\pi^2\Big)\Big)(a-b)\\
&+G(\{-1\},a)\Big(-12G(\{-1,-1\},b)+24G\left(\left\{-\frac{1}{2},-1\right\},b\right)\\
&-24G\left(\left\{-\frac{1}{2},0\right\},b\right)-42G(\{0,-1\},b)+48G(\{0,0\},b)+\pi^2\Big)(a-b)\\
&+2G(\{0\},b)\Big(6G(\{-1,-1\},a)-3G(\{-1,0\},a)-12G\left(\left\{-1,-1-\frac{1}{2b}\right\},a\right)\\
&-6G(\{-1,b\},a)+15G\left(\left\{-1,\frac{1}{b^2+1}-1\right\},a\right)-12G(\{0,-1\},a)+3G(\{0,0\},a)\\
&-18G\left(\left\{0,\frac{b}{2}\right\},a\right)+12G(\{0,b\},a)+15G\left(\left\{0,\frac{1}{b^2+1}-1\right\},a\right)\\
&+6G\left(\left\{-\frac{b}{b-1},-1\right\},a\right)+12G\left(\left\{-\frac{b}{b-1},-1-\frac{1}{2b}\right\},a\right)\\
&+18G\left(\left\{-\frac{b}{b-1},\frac{b}{2}\right\},a\right)-6G\left(\left\{-\frac{b}{b-1},b\right\},a\right)\\
&-30G\left(\left\{-\frac{b}{b-1},\frac{1}{b^2+1}-1\right\},a\right)+4\pi^2\Big)(a-b)\\
&-2G(\{-1\},b)\Big(3G(\{-1,-1\},a)-6G(\{-1,0\},a)-12G\left(\left\{-1,-1-\frac{1}{2b}\right\},a\right)\\
&-3G(\{-1,b\},a)+6G\left(\left\{-1,-\frac{b+1}{b}\right\},a\right)+12G\left(\left\{-1,\frac{1}{b^2+1}-1\right\},a\right)\\
&-6G(\{0,-1\},a)+3G(\{0,0\},a)-12G\left(\left\{0,\frac{b}{2}\right\},a\right)+6G(\{0,b\},a)\\
&-3G\left(\left\{0,-\frac{b+1}{b}\right\},a\right)+12G\left(\left\{0,\frac{1}{b^2+1}-1\right\},a\right)\\
&+3G\left(\left\{-\frac{b}{b-1},-1\right\},a\right)+3G\left(\left\{-\frac{b}{b-1},0\right\},a\right)\\
&+12G\left(\left\{-\frac{b}{b-1},-1-\frac{1}{2b}\right\},a\right)+12G\left(\left\{-\frac{b}{b-1},\frac{b}{2}\right\},a\right)\\
&-3G\left(\left\{-\frac{b}{b-1},b\right\},a\right)-3G\left(\left\{-\frac{b}{b-1},-\frac{b+1}{b}\right\},a\right)
\end{aligned}
$$

$$-24G\left(\left\{-\frac{b}{b-1},\frac{1}{b^2+1}-1\right\},a\right)+4\pi^2\Big)(a-b)$$
$$-6G(\{-1,-1,-1\},b)(a-b)-6G(\{-1,-1,0\},a)(a-b)+6G(\{-1,-1,b\},a)(a-b)$$
$$-6G\left(\left\{-1,-1,-\frac{b}{b-1}\right\},a\right)(a-b)+6G\left(\left\{-1,-1,\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$+24G\left(\left\{-1,-\frac{1}{2},-1\right\},b\right)(a-b)-24G\left(\left\{-1,-\frac{1}{2},0\right\},b\right)(a-b)$$
$$-48G(\{-1,0,-1\},b)(a-b)-6G(\{-1,0,0\},a)(a-b)+48G(\{-1,0,0\},b)(a-b)$$
$$+6G(\{-1,0,b\},a)(a-b)+12G\left(\left\{-1,0,-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$-12G\left(\left\{-1,0,\frac{1}{b^2+1}-1\right\},a\right)(a-b)+24G\left(\left\{-1,-1-\frac{1}{2b},-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$-24G\left(\left\{-1,-1-\frac{1}{2b},\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$+6G(\{-1,b,0\},a)(a-b)-6G(\{-1,b,b\},a)(a-b)$$
$$+6G\left(\left\{-1,b,-\frac{b}{b-1}\right\},a\right)(a-b)-6G\left(\left\{-1,b,\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$+12G\left(\left\{-1,-\frac{b+1}{b},0\right\},a\right)(a-b)-12G\left(\left\{-1,-\frac{b+1}{b},b\right\},a\right)(a-b)$$
$$-12G\left(\left\{-1,-\frac{b+1}{b},-\frac{b}{b-1}\right\},a\right)(a-b)+12G\left(\left\{-1,-\frac{b+1}{b},\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$-6G\left(\left\{-1,\frac{1}{b^2+1}-1,0\right\},a\right)(a-b)+6G\left(\left\{-1,\frac{1}{b^2+1}-1,b\right\},a\right)(a-b)$$
$$-24G\left(\left\{-1,\frac{1}{b^2+1}-1,-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$+24G\left(\left\{-1,\frac{1}{b^2+1}-1,\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$+6G(\{0,-1,-1\},b)(a-b)+12G(\{0,-1,0\},a)(a-b)-12G(\{0,-1,b\},a)(a-b)$$
$$+12G\left(\left\{0,-1,-\frac{b}{b-1}\right\},a\right)(a-b)-12G\left(\left\{0,-1,\frac{1}{b^2+1}-1\right\},a\right)(a-b)$$
$$-24G\left(\left\{0,-\frac{1}{2},-1\right\},b\right)(a-b)+24G\left(\left\{0,-\frac{1}{2},0\right\},b\right)(a-b)$$
$$+48G(\{0,0,-1\},b)(a-b)-48G(\{0,0,0\},b)(a-b)-6G\left(\left\{0,0,-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$+6G\left(\left\{0,0,\frac{1}{b^2+1}-1\right\},a\right)(a-b)+12G\left(\left\{0,\frac{b}{2},0\right\},a\right)(a-b)$$
$$-12G\left(\left\{0,\frac{b}{2},b\right\},a\right)(a-b)+24G\left(\left\{0,\frac{b}{2},-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$-24G\left(\left\{0,\frac{b}{2},\frac{1}{b^2+1}-1\right\},a\right)(a-b)-12G(\{0,b,0\},a)(a-b)$$
$$+12G(\{0,b,b\},a)(a-b)-12G\left(\left\{0,b,-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$+12G\left(\left\{0,b,\frac{1}{b^2+1}-1\right\},a\right)(a-b)-6G\left(\left\{0,-\frac{b+1}{b},0\right\},a\right)(a-b)$$
$$+6G\left(\left\{0,-\frac{b+1}{b},b\right\},a\right)(a-b)+6G\left(\left\{0,-\frac{b+1}{b},-\frac{b}{b-1}\right\},a\right)(a-b)$$
$$-6G\left(\left\{0,-\frac{b+1}{b},\frac{1}{b^2+1}-1\right\},a\right)(a-b)-6G\left(\left\{0,\frac{1}{b^2+1}-1,0\right\},a\right)(a-b)$$

$$
\begin{aligned}
&+ 6G\left(\left\{0, \frac{1}{b^2+1} - 1, b\right\}, a\right)(a-b) - 24G\left(\left\{0, \frac{1}{b^2+1} - 1, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&+ 24G\left(\left\{0, \frac{1}{b^2+1} - 1, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) - 6G\left(\left\{-\frac{b}{b-1}, -1, 0\right\}, a\right)(a-b)\\
&+ 6G\left(\left\{-\frac{b}{b-1}, -1, b\right\}, a\right)(a-b) - 6G\left(\left\{-\frac{b}{b-1}, -1, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&+ 6G\left(\left\{-\frac{b}{b-1}, -1, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) + 6G\left(\left\{-\frac{b}{b-1}, 0, 0\right\}, a\right)(a-b)\\
&- 6G\left(\left\{-\frac{b}{b-1}, 0, b\right\}, a\right)(a-b) - 6G\left(\left\{-\frac{b}{b-1}, 0, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&+ 6G\left(\left\{-\frac{b}{b-1}, 0, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) - 24G\left(\left\{-\frac{b}{b-1}, -1-\frac{1}{2b}, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&+ 24G\left(\left\{-\frac{b}{b-1}, -1-\frac{1}{2b}, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) - 12G\left(\left\{-\frac{b}{b-1}, \frac{b}{2}, 0\right\}, a\right)(a-b)\\
&+ 12G\left(\left\{-\frac{b}{b-1}, \frac{b}{2}, b\right\}, a\right)(a-b) - 24G\left(\left\{-\frac{b}{b-1}, \frac{b}{2}, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&+ 24G\left(\left\{-\frac{b}{b-1}, \frac{b}{2}, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) + 6G\left(\left\{-\frac{b}{b-1}, b, 0\right\}, a\right)(a-b)\\
&- 6G\left(\left\{-\frac{b}{b-1}, b, b\right\}, a\right)(a-b) + 6G\left(\left\{-\frac{b}{b-1}, b, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&- 6G\left(\left\{-\frac{b}{b-1}, b, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b) - 6G\left(\left\{-\frac{b}{b-1}, -\frac{b+1}{b}, 0\right\}, a\right)(a-b)\\
&+ 6G\left(\left\{-\frac{b}{b-1}, -\frac{b+1}{b}, b\right\}, a\right)(a-b) + 6G\left(\left\{-\frac{b}{b-1}, -\frac{b+1}{b}, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&- 6G\left(\left\{-\frac{b}{b-1}, -\frac{b+1}{b}, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b)\\
&+ 12G\left(\left\{-\frac{b}{b-1}, \frac{1}{b^2+1} - 1, 0\right\}, a\right)(a-b)\\
&- 12G\left(\left\{-\frac{b}{b-1}, \frac{1}{b^2+1} - 1, b\right\}, a\right)(a-b)\\
&+ 48G\left(\left\{-\frac{b}{b-1}, \frac{1}{b^2+1} - 1, -\frac{b}{b-1}\right\}, a\right)(a-b)\\
&- 48G\left(\left\{-\frac{b}{b-1}, \frac{1}{b^2+1} - 1, \frac{1}{b^2+1} - 1\right\}, a\right)(a-b)\\
&- \left(-4\pi^2 + 18\zeta(3)\right)(a-b)\Big)\\
&+ \mathcal{O}(\epsilon)
\end{aligned}
\tag{6.68}
$$

$$
\begin{aligned}
\tilde{M}_{29} =& \frac{\pi^2 G(\{0\}, z) + 6G(\{0,0,0\}, z) - 6G(\{0,1,0\}, z) + 12\zeta(3)}{6(y+1)}\\
&+ \mathcal{O}(\epsilon)
\end{aligned}
\tag{6.69}
$$

$$
\begin{aligned}
\tilde{M}_{30} =& \frac{1}{\epsilon}\frac{G(\{0\}, z)}{2(y+1)z}\\
&+ \frac{-3(y+1)G(\{0,0\}, z) - 12(y+1)G(\{0,1\}, z) + 6G(\{0\}, z) - 2\pi^2 y - 6z - 2\pi^2 + 6}{6(y+1)^2 z}\\
&+ \frac{\epsilon}{12(y+1)^2 z}\Big(-3\left(\pi^2(y+1) + 8z + 4\right)G(\{0\}, z) - 6(y+1)G(\{0,0,0\}, z)\\
&+ 24(y+1)G(\{0,0,1\}, z) + 96(y+1)G(\{0,1,1\}, z) + 48(z-1)G(\{1\}, z) + (24z-36)G(\{0,0\}, z)
\end{aligned}
$$

$$-48G(\{0,1\},z)-24(z-1)G(\{1,0\},z)-72y\zeta(3)+4\pi^2z-12z-72\zeta(3)-12\pi^2+12\Big) \tag{6.70}$$

$$\begin{aligned}\tilde{M}_{31}=&+\frac{1}{\epsilon^3(y+1)^2(z-1)}\\ &\frac{2G(\{0\},z)-4G(\{1\},z)}{\epsilon^2(y+1)^2(z-1)}\\ &+\frac{-48G(\{0,1\},z)-24G(\{1,0\},z)+96G(\{1,1\},z)-11\pi^2}{6\text{ep}(y+1)^2(z-1)}\\ &+\frac{1}{3(y+1)^2(z-1)}\Big(-6\pi^2G(\{0\},z)+17\pi^2G(\{1\},z)-6G(\{0,0,0\},z)-6G(\{0,1,0\},z)\\ &+96G(\{0,1,1\},z)+6G(\{1,0,0\},z)+48G(\{1,0,1\},z)+30G(\{1,1,0\},z)\\ &-192G(\{1,1,1\},z)-74\zeta(3))+\mathcal{O}(\epsilon)\end{aligned} \tag{6.71}$$

Master Integrals for $\left[\mathbf{g}+\mathbf{W}^*\to\mathbf{t}+\bar{\mathbf{b}}\right]_{\mathbf{1loop}}$.

$$\begin{aligned}\tilde{M}_{15}=&\frac{1}{\epsilon}\frac{3G(\{-y,0\},z)-3G(\{0,-1\},y)+3G(\{0,0\},y)-3G(\{1,0\},z)+\pi^2}{24\pi(y+1)}\\ &+\frac{1}{48\pi(y+1)}\left(6G(\{-1\},y)G(\{1,0\},z)+6G(\{-1\},y)G(\{1,y+1\},z)\right.\\ &-12G(\{-1\},y)G(\{-y,0\},z)-6\pi^2G(\{-y\},z)+6G(\{-1,-1\},y)G(\{1\},z)\\ &+6G(\{0,-1\},y)G(\{1\},z)+12G(\{0,-1\},y)G(\{-y\},z)-12G(\{0,0\},y)G(\{1\},z)\\ &-12G(\{0,0\},y)G(\{-y\},z)-12G(\{1,-y,0\},z)-6G(\{1,y+1,0\},z)-6G(\{-y,0,0\},z)\\ &-12G(\{-y,0,1\},z)+12G(\{-y,1,0\},z)-12G(\{-y,-y,0\},z)+6\log(\pi)G(\{-y,0\},z)\\ &+6\log(4)G(\{-y,0\},z)+4\pi^2G(\{-1\},y)-5\pi^2G(\{0\},y)-12G(\{-1,0,-1\},y)\\ &+12G(\{-1,0,0\},y)+6G(\{0,-1,-1\},y)+12G(\{0,0,-1\},y)-18G(\{0,0,0\},y)\\ &-6\log(\pi)G(\{0,-1\},y)+6\log(\pi)G(\{0,0\},y)-6\log(4)G(\{0,-1\},y)\\ &+6\log(4)G(\{0,0\},y)-\pi^2G(\{1\},z)+12G(\{1,0,0\},z)+12G(\{1,0,1\},z)\\ &\left.+12G(\{1,1,0\},z)-6\log(\pi)G(\{1,0\},z)-6\log(4)G(\{1,0\},z)-6\zeta(3)+2\pi^2\log(4\pi)\right)\\ &+\mathcal{O}(\epsilon)\end{aligned} \tag{6.72}$$

$$\begin{aligned}\tilde{M}_{16}=&\frac{(z-1)\left(-6G(\{0\},y)G(\{0\},z)-6G(\{0,-y\},z)+6G(\{0,0\},y)-6i\pi G(\{0\},z)+6G(\{0,1\},z)+\pi^2\right)}{48\pi(y+1)}\\ &+\epsilon\frac{(z-1)}{48\pi(y+1)}\left(-12G(\{0\},y)G(\{0\},z)+6G(\{0\},y)G(\{0,0\},z)\right.\\ &+12G(\{0\},y)G(\{0,1\},z)+12G(\{0\},y)G(\{0,-y\},z)+12G(\{0\},y)G(\{1,0\},z)\\ &-12G(\{0,0\},y)G(\{1\},z)-12G(\{0,-y\},z)+6G(\{0,0,-y\},z)+12G(\{0,1,-y\},z)\\ &-6G(\{0,-y,0\},z)+12G(\{0,-y,1\},z)+12G(\{0,-y,-y\},z)+12G(\{1,0,-y\},z)\\ &-6\log(\pi)G(\{0\},y)G(\{0\},z)-6\log(4)G(\{0\},y)G(\{0\},z)-6\log(\pi)G(\{0,-y\},z)\\ &-6\log(4)G(\{0,-y\},z)+\pi^2G(\{0\},y)+12G(\{0,0\},y)-6G(\{0,0,0\},y)\\ &+6\log(\pi)G(\{0,0\},y)+6\log(4)G(\{0,0\},y)+2\pi^2G(\{0\},z)-12i\pi G(\{0\},z)\\ &-2\pi^2G(\{1\},z)+6i\pi G(\{0,0\},z)+24i\pi G(\{0,1\},z)+12G(\{0,1\},z)\\ &+12i\pi G(\{1,0\},z)-6G(\{0,0,1\},z)+6G(\{0,1,0\},z)-36G(\{0,1,1\},z)\\ &-12G(\{1,0,1\},z)-6i\pi\log(4\pi)G(\{0\},z)+6\log(\pi)G(\{0,1\},z)\end{aligned}$$

$$+6\log(4)G(\{0,1\},z) - 36\zeta(3) + 2i\pi^3 + 2\pi^2 + \pi^2\log(4\pi)\Big)$$
$$+ \epsilon^2 \frac{(z-1)}{24\pi(y+1)} \left(-12G(\{0\},y)G(\{0\},z) + 6G(\{0\},y)G(\{0,0\},z)\right.$$
$$+ 12G(\{0\},y)G(\{0,1\},z) + 12G(\{0\},y)G(\{0,-y\},z) + 12G(\{0\},y)G(\{1,0\},z)$$
$$- 12G(\{0,0\},y)G(\{1\},z) - 12G(\{0,-y\},z) + 6G(\{0,0,-y\},z) + 12G(\{0,1,-y\},z)$$
$$- 6G(\{0,-y,0\},z) + 12G(\{0,-y,1\},z) + 12G(\{0,-y,-y\},z) + 12G(\{1,0,-y\},z)$$
$$- 6\log(\pi)G(\{0\},y)G(\{0\},z) - 6\log(4)G(\{0\},y)G(\{0\},z) - 6\log(\pi)G(\{0,-y\},z)$$
$$- 6\log(4)G(\{0,-y\},z) + \pi^2 G(\{0\},y) + 12G(\{0,0\},y) - 6G(\{0,0,0\},y)$$
$$+ 6\log(\pi)G(\{0,0\},y) + 6\log(4)G(\{0,0\},y) + 2\pi^2 G(\{0\},z) - 12i\pi G(\{0\},z)$$
$$- 2\pi^2 G(\{1\},z) + 6i\pi G(\{0,0\},z) + 24i\pi G(\{0,1\},z) + 12G(\{0,1\},z)$$
$$+ 12i\pi G(\{1,0\},z) - 6G(\{0,0,1\},z) + 6G(\{0,1,0\},z) - 36G(\{0,1,1\},z)$$
$$- 12G(\{1,0,1\},z) - 6i\pi\log(4\pi)G(\{0\},z) + 6\log(\pi)G(\{0,1\},z)$$
$$+6\log(4)G(\{0,1\},z) - 36\zeta(3) + 2i\pi^3 + 2\pi^2 + \pi^2\log(4\pi)\Big)$$
$$+ \mathcal{O}(\epsilon) \tag{6.73}$$

$$\tilde{M}_{17} = \frac{1}{48\pi(y+1)} \left(6(G(\{-1,-1\},y)G(\{0\},z) - G(\{0,y+1,0\},z) - G(\{-1,-1,-1\},y)\right.$$
$$+ G(\{-1,0,-1\},y) + G(\{0,0,0\},z) + 2\zeta(3))$$
$$-G(\{-1\},y)\left(-6G(\{0,y+1\},z) + 6G(\{0,0\},z) + \pi^2\right)\Big)$$
$$+ \mathcal{O}(\epsilon) \tag{6.74}$$

$$\tilde{M}_{18} = \frac{G(\{0\},z)}{\epsilon(16\pi yz + 16\pi z)}$$
$$- \frac{G(\{0\},z)(6G(\{-1\},y) - 3(2+\log(4\pi))) + 3G(\{0,0\},z) + 6G(\{0,1\},z) + \pi^2}{48(\pi(y+1)z)}$$
$$- \frac{\epsilon}{96\left(\pi(y+1)^2 z\right)} \times$$
$$\times \left(2\left(18yG(\{0,0\},z) + 12yG(\{0,1\},z) - 12yG(\{y+1,0\},z) + 3yG(\{0,0,0\},z)\right.\right.$$
$$- 6yG(\{0,0,1\},z) - 12yG(\{0,1,1\},z) - 12yG(\{0,y+1,0\},z) + 12(y-z+1)G(\{-1,-1\},y)$$
$$+ 12zG(\{y+1,0\},z) - 12G(\{y+1,0\},z) - 12G(\{0,y+1,0\},z) + 3y\log(\pi)G(\{0,0\},z)$$
$$+ 6y\log(\pi)G(\{0,1\},z) + y\log(64)G(\{0,0\},z) + 6y\log(4)G(\{0,1\},z)$$
$$- 12yG(\{0,-1\},y) - 12yG(\{-1,-1,-1\},y) + 12yG(\{-1,0,-1\},y) - 12G(\{-1,-1,-1\},y)$$
$$+ 12G(\{-1,0,-1\},y) - 12zG(\{0,0\},z) + 18G(\{0,0\},z) + 12G(\{0,1\},z)$$
$$+ 3G(\{0,0,0\},z) - 6G(\{0,0,1\},z) - 12G(\{0,1,1\},z) + 3\log(\pi)G(\{0,0\},z)$$
$$+ 6\log(\pi)G(\{0,1\},z) + \log(64)G(\{0,0\},z) + 6\log(4)G(\{0,1\},z) + 30y\zeta(3)$$
$$+4\pi^2 y + \pi^2 y\log(4\pi) + 30\zeta(3) + 2\pi^2 + \pi^2\log(4\pi)\Big)$$
$$+ (y+1)\left(\pi^2 - 3\log^2(4) - 12\log(4) - \log(\pi)(12 + \log(4096) + 3\log(\pi))\right)G(\{0\},z)$$
$$- 4G(\{-1\},y)\left(-6(y-z+1)G(\{y+1\},z) + (y+1)\left(-6G(\{0,y+1\},z)\right.\right.$$
$$\left.+9G(\{0,0\},z) + 6G(\{0,1\},z) + 2\pi^2\right) - 3((y+1)\log(4\pi) + 2z)G(\{0\},z)\Big)\Big)$$
$$+ \frac{\epsilon^2}{48\pi(y+1)^2 z} \left(2G(\{-1\},y)\left(12G(\{0,1\},z)z + 6G(\{1,0\},z)z\right.\right.$$
$$- 6G(\{1,y+1\},z)z + 6G(\{y+1,0\},z)z - 12G(\{y+1,1\},z)z - 6G(\{0\},z)\log(\pi)z$$

$$
\begin{aligned}
&+ 6G(\{y+1\},z)\log(\pi)z - 6G(\{0\},z)\log(4)z + 6G(\{y+1\},z)\log(4)z + \pi^2 z \\
&+ 6(2y-z+2)G(\{0,y+1\},z) - 6G(\{1,0\},z) + 6G(\{1,y+1\},z) - 6yG(\{y+1,0\},z) \\
&- 6G(\{y+1,0\},z) + 12yG(\{y+1,1\},z) + 12G(\{y+1,1\},z) - 6yG(\{y+1\},z)\log(\pi) \\
&-6G(\{y+1\},z)\log(\pi) - 6yG(\{y+1\},z)\log(4) - 6G(\{y+1\},z)\log(4)) \\
&+ G(\{0\},z)\left(-24G(\{0,-1\},y)y + \log(\pi)\log\left(4096\pi^3\right)y\right. \\
&+ 3\log^2(4)y + 7\pi^2 y - 4\pi^2 z + 12(2y+z+2)G(\{-1,-1\},y) + \log(\pi)\log\left(4096\pi^3\right) \\
&\left.+3\log^2(4) + 3\pi^2\right) + 2\,(6G(\{-1,-1,-1\},y)y - 6G(\{0,-1,-1\},y)y \\
&+ 21G(\{0,0,0\},z)y + 18G(\{0,0,1\},z)y + 6G(\{0,1,0\},z)y + 12G(\{0,1,1\},z)y \\
&- 12G(\{0,y+1,0\},z)y - 12G(\{y+1,0,1\},z)y - 6G(\{y+1,1,0\},z)y \\
&- 2\pi^2\log(4\pi)y - 6G(\{-1,-1\},y)\log(\pi)y + 6G(\{0,-1\},y)\log(\pi)y \\
&- 9G(\{0,0\},z)\log(\pi)y - 6G(\{0,1\},z)\log(\pi)y + 6G(\{y+1,0\},z)\log(\pi)y \\
&- 6G(\{-1,-1\},y)\log(4)y + 6G(\{0,-1\},y)\log(4)y - 9G(\{0,0\},z)\log(4)y \\
&- 6G(\{0,1\},z)\log(4)y + 6G(\{y+1,0\},z)\log(4)y \\
&+ (z-1)G(\{1\},z)\left(-6G(\{-1,-1\},y) - 6G(\{0,-1\},y) + \pi^2\right) \\
&- (y-z+1)G(\{y+1\},z)\left(-12G(\{-1,-1\},y) + 6G(\{0,-1\},y) + \pi^2\right) \\
&- 12zG(\{-1,-1,-1\},y) + 6G(\{-1,-1,-1\},y) + 6zG(\{-1,0,-1\},y) - 6zG(\{0,0,0\},z) \\
&+ 21G(\{0,0,0\},z) - 12zG(\{0,0,1\},z) + 18G(\{0,0,1\},z) - 6zG(\{0,1,0\},z) \\
&+ 6G(\{0,1,0\},z) + 12G(\{0,1,1\},z) + 6zG(\{0,y+1,0\},z) - 12G(\{0,y+1,0\},z) \\
&- 6zG(\{1,0,0\},z) + 6G(\{1,0,0\},z) + 6zG(\{1,y+1,0\},z) - 6G(\{1,y+1,0\},z) \\
&+ 12zG(\{y+1,0,1\},z) - 12G(\{y+1,0,1\},z) + 6zG(\{y+1,1,0\},z) - 6G(\{y+1,1,0\},z) \\
&- 24z\zeta(3) + 30\zeta(3) - \pi^2\log(4\pi) + 6zG(\{-1,-1\},y)\log(\pi) \\
&- 6G(\{-1,-1\},y)\log(\pi) + 6zG(\{0,0\},z)\log(\pi) - 9G(\{0,0\},z)\log(\pi) \\
&- 6G(\{0,1\},z)\log(\pi) - 6zG(\{y+1,0\},z)\log(\pi) + 6G(\{y+1,0\},z)\log(\pi) \\
&+ 6zG(\{-1,-1\},y)\log(4) - 6G(\{-1,-1\},y)\log(4) + 6zG(\{0,0\},z)\log(4) \\
&- 9G(\{0,0\},z)\log(4) - 6G(\{0,1\},z)\log(4) - 6zG(\{y+1,0\},z)\log(4) \\
&+6G(\{y+1,0\},z)\log(4))) \qquad (6.75)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{19} = &- \frac{1}{16\epsilon^3\,(\pi(y+1)^2(z-1))} \\
&- \frac{2G(\{-y\},z) - 2G(\{-1\},y) + 2G(\{0\},y) + 2G(\{0\},z) - 4G(\{1\},z) + 2i\pi + \log(4\pi)}{16\epsilon^2\,(\pi(y+1)^2(z-1))} \\
&+ \frac{1}{32\pi\epsilon(y+1)^2(z-1)}\,(8G(\{0\},y)G(\{-y\},z) - 8G(\{0,-y\},z) \\
&+ 8G(\{1,-y\},z) - 8G(\{-y,0\},z) + 8G(\{-y,1\},z) + 8G(\{-y,-y\},z) \\
&+ G(\{0\},z)(8G(\{-1\},y) - 8G(\{0\},y) - 8i\pi - 4\log(\pi) - 4\log(4)) \\
&+ 8G(\{1\},z)(-G(\{-1\},y) + G(\{0\},y) + 2i\pi + \log(4\pi)) - 4\log(\pi)G(\{-y\},z) \\
&- 4\log(4)G(\{-y\},z) - 8G(\{-1,-1\},y) + 8G(\{0,0\},y) + 4\log(\pi)G(\{-1\},y) \\
&+ 4\log(4)G(\{-1\},y) - 4\log(\pi)G(\{0\},y) - 4\log(4)G(\{0\},y) \\
&+ 16G(\{0,1\},z) + 8G(\{1,0\},z) - 32G(\{1,1\},z) + 5\pi^2 \\
&\left. - \log^2(4) - \log(\pi)\log(16\pi) - 4i\pi\log(4\pi)\right) \\
&+ \frac{1}{96\pi(y+1)^2(z-1)}\,(-24G(\{-1,-1\},y)G(\{0\},z) + 24G(\{0,-1\},y)G(\{0\},z) \\
&- 2\log(\pi)\log\left(4096\pi^3\right)G(\{0\},z) - 24i\pi\log(4\pi)G(\{0\},z)
\end{aligned}
$$

$$
\begin{aligned}
&- 6\log^2(4)G(\{0\},z) + 10\pi^2 G(\{0\},z) - 44\pi^2 G(\{1\},z) + 18\pi^2 G(\{-y\},z)\\
&+ 48G(\{1\},z)G(\{-1,-1\},y) - 24G(\{-y\},z)G(\{0,-1\},y) - 48G(\{1\},z)G(\{0,0\},y)\\
&+ 24i\pi G(\{0,0\},z) + 96i\pi G(\{0,1\},z) + 24i\pi G(\{1,0\},z)\\
&- 192i\pi G(\{1,1\},z) + 24G(\{-1,-1,-1\},y) - 24G(\{0,0,-1\},y) - 24G(\{0,0,1\},z)\\
&+ 24G(\{0,0,-y\},z) + 24G(\{0,1,0\},z) - 192G(\{0,1,1\},z) + 48G(\{0,1,-y\},z)\\
&- 24G(\{0,-y,0\},z) + 48G(\{0,-y,1\},z) + 48G(\{0,-y,-y\},z) - 24G(\{0,y+1,0\},z)\\
&- 72G(\{1,0,1\},z) + 24G(\{1,0,-y\},z) - 72G(\{1,1,0\},z) + 384G(\{1,1,1\},z)\\
&- 48G(\{1,1,-y\},z) + 24G(\{1,-y,0\},z) - 48G(\{1,-y,1\},z) - 48G(\{1,-y,-y\},z)\\
&+ 24G(\{-y,0,0\},z) + 48G(\{-y,0,1\},z) - 48G(\{-y,1,1\},z) - 48G(\{-y,1,-y\},z)\\
&+ 48G(\{-y,-y,0\},z) - 48G(\{-y,-y,1\},z) - 48G(\{-y,-y,-y\},z)\\
&+ 2G(\{0\},y)\left(-12\log(\pi)G(\{0\},z) - 12\log(4)G(\{0\},z)\right.\\
&+ 12G(\{0,0\},z) + 24G(\{0,1\},z) + 24G(\{0,-y\},z) + 12G(\{1,0\},z) - 24G(\{1,1\},z)\\
&- 24G(\{1,-y\},z) - 24G(\{-y,1\},z) - 24G(\{-y,-y\},z)\\
&- \log(\pi)\log\left(4096\pi^3\right) + 12G(\{1\},z)\log(\pi)\\
&+ 12G(\{-y\},z)\log(\pi) - 3\log^2(4)\\
&\left.+12G(\{1\},z)\log(4) + 12G(\{-y\},z)\log(4) + 9\pi^2\right)\\
&+ G(\{-1\},y)\left(24\log(\pi)G(\{0\},z) + 24\log(4)G(\{0\},z) - 24G(\{0,0\},z)\right.\\
&- 48G(\{0,1\},z) + 24G(\{0,y+1\},z) - 24G(\{1,0\},z) + 48G(\{1,1\},z)\\
&+ 24G(\{-y,0\},z) + 2\log(\pi)\log\left(4096\pi^3\right) - 24G(\{1\},z)\log(\pi)\\
&\left.+6\log^2(4) - 24G(\{1\},z)\log(4) - 14\pi^2\right) + 76\zeta(3)\\
&+ 2G(\{1\},z)\log(\pi)\log\left(4096\pi^3\right)\\
&- 2G(\{-y\},z)\log(\pi)\log\left(4096\pi^3\right) - \log^2(\pi)\log(64\pi)\\
&+ 6G(\{1\},z)\log(\pi)\log(16\pi) - 6i\pi\log^2(4\pi)\\
&+ 48i\pi G(\{1\},z)\log(4\pi) + 15\pi^2\log(4\pi)\\
&- 24G(\{-1,-1\},y)\log(\pi) + 24G(\{0,0\},y)\log(\pi) + 48G(\{0,1\},z)\log(\pi)\\
&- 24G(\{0,-y\},z)\log(\pi) + 24G(\{1,0\},z)\log(\pi) - 96G(\{1,1\},z)\log(\pi)\\
&+ 24G(\{1,-y\},z)\log(\pi) - 24G(\{-y,0\},z)\log(\pi) + 24G(\{-y,1\},z)\log(\pi)\\
&+ 24G(\{-y,-y\},z)\log(\pi) - 12\log^2(2)\log(\pi) + 12G(\{1\},z)\log^2(4)\\
&- 6G(\{-y\},z)\log^2(4) - 24G(\{-1,-1\},y)\log(4) + 24G(\{0,0\},y)\log(4)\\
&+ 48G(\{0,1\},z)\log(4) - 24G(\{0,-y\},z)\log(4) + 24G(\{1,0\},z)\log(4)\\
&- 96G(\{1,1\},z)\log(4) + 24G(\{1,-y\},z)\log(4) - 24G(\{-y,0\},z)\log(4)\\
&+ 24G(\{-y,1\},z)\log(4) + 24G(\{-y,-y\},z)\log(4)\\
&\left.-8\log^3(2) + 18i\pi^3\right)\\
&+ \mathcal{O}(\epsilon)
\end{aligned}
\tag{6.76}
$$

$$
\begin{aligned}
\tilde{M}_{32} =& \frac{1}{\epsilon}\frac{a(-G(\{b\},a) + G(\{0\},a) - G(\{0\},b))}{8\pi(b+1)(a(b-1)+b)}\\
&- \frac{1}{24(\pi(b+1)(a(b-1)+b))}\left(a\left(3G(\{b\},a)\right.\right.\\
&+ 6G\left(\left\{0,\frac{b}{2}\right\},a\right) - 15G(\{0,b\},a) - 9G(\{b,0\},a)
\end{aligned}
$$

$$
\begin{aligned}
&- 6G\left(\left\{b, \frac{b}{2}\right\}, a\right) + 15G(\{b,b\},a) + 3\log(\pi)G(\{b\},a) \\
&+ \log(64)G(\{b\},a) - 3G(\{0\},a)(3G(\{0\},b) + 1 + \log(4\pi)) \\
&+ 3G(\{0\},b)(3G(\{b\},a) + 1 + \log(4\pi)) + 9G(\{0,0\},a) \\
&+9G(\{0,0\},b) + \pi^2\Big)\Big) \\
&- \frac{\epsilon}{48(\pi(b+1)(a(b-1)+b))}\,(a\,(42G(\{0,0\},b)G(\{b\},a) \\
&+ 3\pi^2 G(\{b\},a) + 6G(\{b\},a) + 12G\left(\left\{0, \frac{b}{2}\right\}, a\right) \\
&- 30G(\{0,b\},a) - 18G(\{b,0\},a) - 12G\left(\left\{b, \frac{b}{2}\right\}, a\right) \\
&+ 30G(\{b,b\},a) - 36G\left(\left\{0, 0, \frac{b}{2}\right\}, a\right) + 78G(\{0,0,b\},a) \\
&- 12G\left(\left\{0, \frac{b}{2}, 0\right\}, a\right) - 24G\left(\left\{0, \frac{b}{2}, \frac{b}{2}\right\}, a\right) \\
&+ 36G\left(\left\{0, \frac{b}{2}, b\right\}, a\right) + 54G(\{0,b,0\},a) \\
&+ 60G\left(\left\{0, b, \frac{b}{2}\right\}, a\right) - 114G(\{0,b,b\},a) + 42G(\{b,0,0\},a) \\
&+ 36G\left(\left\{b, 0, \frac{b}{2}\right\}, a\right) - 78G(\{b,0,b\},a) \\
&+ 12G\left(\left\{b, \frac{b}{2}, 0\right\}, a\right) + 24G\left(\left\{b, \frac{b}{2}, \frac{b}{2}\right\}, a\right) \\
&- 36G\left(\left\{b, \frac{b}{2}, b\right\}, a\right) - 54G(\{b,b,0\},a) \\
&- 60G\left(\left\{b, b, \frac{b}{2}\right\}, a\right) + 114G(\{b,b,b\},a) \\
&+ 3\log^2(4)G(\{b\},a) - G(\{0\},a)\,(42G(\{0,0\},b) \\
&+ 18(1+\log(4\pi))G(\{0\},b) + 3\pi^2 + 6 + 3\log^2(4) \\
&+ \log(\pi)\log\left(4096\pi^3\right) + 6\log(\pi) + 6\log(4)\Big) \\
&+ G(\{0\},b)\left(12G\left(\left\{0, \frac{b}{2}\right\}, a\right) - 54G(\{0,b\},a)\right. \\
&- 42G(\{b,0\},a) - 12G\left(\left\{b, \frac{b}{2}\right\}, a\right) + 54G(\{b,b\},a) \\
&+ 18(1+\log(4\pi))G(\{b\},a) + 42G(\{0,0\},a) + 3\pi^2 + 6 + 3\log^2(4) \\
&+ \log(\pi)\log\left(4096\pi^3\right) + 6\log(\pi) + 6\log(4)\Big) \\
&+ \log(\pi)\log\left(4096\pi^3\right)G(\{b\},a) + 6\log(\pi)G(\{b\},a) \\
&+ 6\log(4)G(\{b\},a) + 12\log(\pi)G\left(\left\{0, \frac{b}{2}\right\}, a\right) \\
&- 30\log(\pi)G(\{0,b\},a) - 18\log(\pi)G(\{b,0\},a) \\
&- 12\log(\pi)G\left(\left\{b, \frac{b}{2}\right\}, a\right) + 30\log(\pi)G(\{b,b\},a) \\
&+ 12\log(4)G\left(\left\{0, \frac{b}{2}\right\}, a\right) - 30\log(4)G(\{0,b\},a) \\
&- 18\log(4)G(\{b,0\},a) - 12\log(4)G\left(\left\{b, \frac{b}{2}\right\}, a\right)
\end{aligned}
$$

$$+ 30\log(4)G(\{b,b\},a) + 18G(\{0,0\},a) - 42G(\{0,0,0\},a)$$
$$+ 18\log(\pi)G(\{0,0\},a) + 18\log(4)G(\{0,0\},a) + 18G(\{0,0\},b)$$
$$+ 42G(\{0,0,0\},b) + 18\log(\pi)G(\{0,0\},b) + 18\log(4)G(\{0,0\},b)$$
$$+12\zeta(3) + 2\pi^2 + 2\pi^2\log(4\pi)\Big)\Big) \tag{6.77}$$

$$\tilde{M}_{36} = -\frac{1}{\epsilon}\frac{G(\{b\},a) - G(\{0\},a) + G(\{0\},b)}{8\pi\left(1 - \frac{(a+1)b^2}{a-b}\right)}$$
$$\frac{(a-b)}{24\pi(b+1)(a(b-1)+b)}\times$$
$$\times\left(\frac{3\left(ab^2+a+b^2\right)}{(a+1)b^2}\left(-G(\{0\},b)\left(G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)\right.\right.\right.$$
$$+2G(\{b\},a)) + G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right) - G\left(\left\{b,-\frac{b^2}{b^2+1}\right\},a\right)$$
$$+ G\left(\left\{-\frac{b^2}{b^2+1},0\right\},a\right) - G\left(\left\{-\frac{b^2}{b^2+1},b\right\},a\right)$$
$$+3G(\{0\},a)G(\{0\},b) + G(\{0,b\},a) + G(\{b,0\},a) - 2G(\{0,0\},a) - 4G(\{0,0\},b))$$
$$- 9G(\{0\},a)G(\{0\},b) + 9G(\{0\},b)G(\{b\},a) + 6G(\{b\},a)$$
$$+ 6G\left(\left\{0,\frac{b}{2}\right\},a\right) - 15G(\{0,b\},a) - 9G(\{b,0\},a)$$
$$- 6G\left(\left\{b,\frac{b}{2}\right\},a\right) + 15G(\{b,b\},a) + 3\log(\pi)G(\{b\},a)$$
$$+ \log(64)G(\{b\},a) - 6G(\{0\},a) + 9G(\{0,0\},a) - 3\log(\pi)G(\{0\},a)$$
$$- 3\log(4)G(\{0\},a) + 6G(\{0\},b) + 9G(\{0,0\},b) + 3\log(\pi)G(\{0\},b)$$
$$\left.+ \log(64)G(\{0\},b) + \pi^2\right)$$
$$\frac{(a-b)\epsilon}{48(b+1)(a(b-1)+b)\pi}(-36G(\{0\},b)G(\{0\},a) - 42G(\{0,0\},b)G(\{0\},a)$$
$$- \log(\pi)\log\left(4096\pi^3\right)G(\{0\},a) - 18G(\{0\},b)\log(\pi)G(\{0\},a)$$
$$- 12\log(\pi)G(\{0\},a) - 3\log^2(4)G(\{0\},a) - 18G(\{0\},b)\log(4)G(\{0\},a)$$
$$- 12\log(4)G(\{0\},a) - 3\pi^2G(\{0\},a) - 24G(\{0\},a) + 3\pi^2G(\{0\},b)$$
$$+ 24G(\{0\},b) + 36G(\{0\},b)G(\{b\},a) + 3\pi^2G(\{b\},a) + 24G(\{b\},a)$$
$$+ 42G(\{0\},b)G(\{0,0\},a) + 36G(\{0,0\},a) + 42G(\{b\},a)G(\{0,0\},b) + 36G(\{0,0\},b)$$
$$+ 12G(\{0\},b)G\left(\left\{0,\frac{b}{2}\right\},a\right) + 24G\left(\left\{0,\frac{b}{2}\right\},a\right)$$
$$- 54G(\{0\},b)G(\{0,b\},a) - 60G(\{0,b\},a) - 42G(\{0\},b)G(\{b,0\},a)$$
$$- 36G(\{b,0\},a) - 12G(\{0\},b)G\left(\left\{b,\frac{b}{2}\right\},a\right)$$
$$- 24G\left(\left\{b,\frac{b}{2}\right\},a\right) + 54G(\{0\},b)G(\{b,b\},a) + 60G(\{b,b\},a)$$
$$- 42G(\{0,0,0\},a) + 42G(\{0,0,0\},b) - 36G\left(\left\{0,0,\frac{b}{2}\right\},a\right)$$
$$+ 78G(\{0,0,b\},a) - 12G\left(\left\{0,\frac{b}{2},0\right\},a\right)$$
$$- 24G\left(\left\{0,\frac{b}{2},\frac{b}{2}\right\},a\right) + 36G\left(\left\{0,\frac{b}{2},b\right\},a\right)$$

$$+ 54G(\{0,b,0\},a) + 60G\left(\left\{0,b,\frac{b}{2}\right\},a\right) - 114G(\{0,b,b\},a)$$
$$+ 42G(\{b,0,0\},a) + 36G\left(\left\{b,0,\frac{b}{2}\right\},a\right) - 78G(\{b,0,b\},a)$$
$$+ 12G\left(\left\{b,\frac{b}{2},0\right\},a\right) + 24G\left(\left\{b,\frac{b}{2},\frac{b}{2}\right\},a\right)$$
$$- 36G\left(\left\{b,\frac{b}{2},b\right\},a\right) - 54G(\{b,b,0\},a) - 60G\left(\left\{b,b,\frac{b}{2}\right\},a\right)$$
$$+ 114G(\{b,b,b\},a) + \frac{(ab^2+b^2+a)}{(a+1)b^2}\,(-48G(\{0,0\},b)G(\{b\},a)$$
$$- \pi^2 G(\{b\},a) - 2\pi^2 G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right) - 24G(\{0,0\},a)$$
$$- 24G(\{-1\},a)G(\{0,0\},b) + 30G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)G(\{0,0\},b)$$
$$- 48G(\{0,0\},b) + 12G(\{0,b\},a) + 12G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right) + 12G(\{b,0\},a)$$
$$- 12G\left(\left\{b,-\frac{b^2}{b^2+1}\right\},a\right) + 12G\left(\left\{-\frac{b^2}{b^2+1},0\right\},a\right)$$
$$- 12G\left(\left\{-\frac{b^2}{b^2+1},b\right\},a\right) - 12G(\{-1,0,0\},a) + 6G(\{-1,0,b\},a)$$
$$+ 6G\left(\left\{-1,0,-\frac{b^2}{b^2+1}\right\},a\right) + 6G(\{-1,b,0\},a)$$
$$- 6G\left(\left\{-1,b,-\frac{b^2}{b^2+1}\right\},a\right) + 6G\left(\left\{-1,-\frac{b^2}{b^2+1},0\right\},a\right)$$
$$- 6G\left(\left\{-1,-\frac{b^2}{b^2+1},b\right\},a\right) - 6G(\{0,-1,0\},a)$$
$$+ 6G\left(\left\{0,-1,-\frac{b^2}{b^2+1}\right\},a\right) + 36G(\{0,0,0\},a) - 36G(\{0,0,0\},b)$$
$$+ 24G\left(\left\{0,0,\frac{b}{2}\right\},a\right) - 54G(\{0,0,b\},a)$$
$$- 6G\left(\left\{0,0,-\frac{b^2}{b^2+1}\right\},a\right) + 12G\left(\left\{0,\frac{b}{2},0\right\},a\right)$$
$$- 12G\left(\left\{0,\frac{b}{2},-\frac{b^2}{b^2+1}\right\},a\right) - 42G(\{0,b,0\},a)$$
$$- 12G\left(\left\{0,b,\frac{b}{2}\right\},a\right) + 30G(\{0,b,b\},a)$$
$$+ 24G\left(\left\{0,b,-\frac{b^2}{b^2+1}\right\},a\right) - 12G\left(\left\{0,-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right)$$
$$+ 24G\left(\left\{0,-\frac{b^2}{b^2+1},b\right\},a\right) - 12G\left(\left\{0,-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right)$$
$$+ 6G(\{b,-1,0\},a) - 6G\left(\left\{b,-1,-\frac{b^2}{b^2+1}\right\},a\right) - 30G(\{b,0,0\},a)$$

$$
\begin{aligned}
& -12G\left(\left\{b,0,\frac{b}{2}\right\},a\right)+30G(\{b,0,b\},a)+12G\left(\left\{b,0,-\frac{b^2}{b^2+1}\right\},a\right) \\
& -12G\left(\left\{b,\frac{b}{2},0\right\},a\right)+12G\left(\left\{b,\frac{b}{2},-\frac{b^2}{b^2+1}\right\},a\right) \\
& +30G(\{b,b,0\},a)-30G\left(\left\{b,b,-\frac{b^2}{b^2+1}\right\},a\right)+6G\left(\left\{b,-\frac{b^2}{b^2+1},0\right\},a\right) \\
& +12G\left(\left\{b,-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right)-30G\left(\left\{b,-\frac{b^2}{b^2+1},b\right\},a\right) \\
& +12G\left(\left\{b,-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right)+6G\left(\left\{-\frac{b^2}{b^2+1},0,0\right\},a\right) \\
& -12G\left(\left\{-\frac{b^2}{b^2+1},0,\frac{b}{2}\right\},a\right)+18G\left(\left\{-\frac{b^2}{b^2+1},0,b\right\},a\right) \\
& -12G\left(\left\{-\frac{b^2}{b^2+1},0,-\frac{b^2}{b^2+1}\right\},a\right)+6G\left(\left\{-\frac{b^2}{b^2+1},b,0\right\},a\right) \\
& +12G\left(\left\{-\frac{b^2}{b^2+1},b,\frac{b}{2}\right\},a\right)-30G\left(\left\{-\frac{b^2}{b^2+1},b,b\right\},a\right) \\
& +12G\left(\left\{-\frac{b^2}{b^2+1},b,-\frac{b^2}{b^2+1}\right\},a\right)-12G\left(\left\{-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1},0\right\},a\right) \\
& +12G\left(\left\{-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1},b\right\},a\right)+3G(\{0\},a)\,(6(2+\log(4\pi))G(\{0\},b) \\
& \left.+14G(\{0,0\},b)+\pi^2\right)-G(\{0\},b)\,(12(2+\log(4\pi))G(\{b\},a) \\
& -18G(\{-1,0\},a)+12G(\{-1,b\},a)+6G\left(\left\{-1,-\frac{b^2}{b^2+1}\right\},a\right) \\
& -12G(\{0,-1\},a)+42G(\{0,0\},a)+24G\left(\left\{0,\frac{b}{2}\right\},a\right)-66G(\{0,b\},a) \\
& +12G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right)+12G(\{b,-1\},a)-42G(\{b,0\},a) \\
& -24G\left(\left\{b,\frac{b}{2}\right\},a\right)+60G(\{b,b\},a)-6G\left(\left\{b,-\frac{b^2}{b^2+1}\right\},a\right) \\
& +18G\left(\left\{-\frac{b^2}{b^2+1},0\right\},a\right)-6G\left(\left\{-\frac{b^2}{b^2+1},b\right\},a\right) \\
& -12G\left(\left\{-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right) \\
& \left.+6G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)(2+\log(4\pi))+5\pi^2\right) \\
& -12G(\{0,0\},a)\log(\pi)-24G(\{0,0\},b)\log(\pi)+6G(\{0,b\},a)\log(\pi) \\
& +6G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right)\log(\pi)+6G(\{b,0\},a)\log(\pi) \\
& -6G\left(\left\{b,-\frac{b^2}{b^2+1}\right\},a\right)\log(\pi)+6G\left(\left\{-\frac{b^2}{b^2+1},0\right\},a\right)\log(\pi)
\end{aligned}
$$

$$
\begin{aligned}
&- 6G\left(\left\{-\frac{b^2}{b^2+1}, b\right\}, a\right)\log(\pi) - 12G(\{0,0\},a)\log(4) \\
&- 24G(\{0,0\},b)\log(4) + 6G(\{0,b\},a)\log(4) + 6G\left(\left\{0,-\frac{b^2}{b^2+1}\right\}, a\right)\log(4) \\
&+ 6G(\{b,0\},a)\log(4) - 6G\left(\left\{b,-\frac{b^2}{b^2+1}\right\}, a\right)\log(4) \\
&+6G\left(\left\{-\frac{b^2}{b^2+1}, 0\right\}, a\right)\log(4) - 6G\left(\left\{-\frac{b^2}{b^2+1}, b\right\}, a\right)\log(4)\Bigg) \\
&+ 12\zeta(3) + G(\{0\},b)\log(\pi)\log\left(4096\pi^3\right) + G(\{b\},a)\log(\pi)\log\left(4096\pi^3\right) \\
&+ 2\pi^2\log(4\pi) + 12G(\{0\},b)\log(\pi) + 18G(\{0\},b)G(\{b\},a)\log(\pi) \\
&+ 12G(\{b\},a)\log(\pi) + 18G(\{0,0\},a)\log(\pi) + 18G(\{0,0\},b)\log(\pi) \\
&+ 12G\left(\left\{0,\frac{b}{2}\right\}, a\right)\log(\pi) - 30G(\{0,b\},a)\log(\pi) \\
&- 18G(\{b,0\},a)\log(\pi) - 12G\left(\left\{b,\frac{b}{2}\right\}, a\right)\log(\pi) \\
&+ 30G(\{b,b\},a)\log(\pi) + 3G(\{0\},b)\log^2(4) + 3G(\{b\},a)\log^2(4) + 12G(\{0\},b)\log(4) \\
&+ 18G(\{0\},b)G(\{b\},a)\log(4) + 12G(\{b\},a)\log(4) + 18G(\{0,0\},a)\log(4) \\
&+ 18G(\{0,0\},b)\log(4) + 12G\left(\left\{0,\frac{b}{2}\right\}, a\right)\log(4) \\
&- 30G(\{0,b\},a)\log(4) - 18G(\{b,0\},a)\log(4) - 12G\left(\left\{b,\frac{b}{2}\right\}, a\right)\log(4) \\
&+30G(\{b,b\},a)\log(4) + 4\pi^2\Big) \\
&+ \mathcal{O}(\epsilon^2) \qquad (6.78)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{37} =& -\frac{1}{\epsilon}\frac{G(\{b\},a) - G(\{0\},a) + G(\{0\},b)}{8\pi\left(1-\frac{(a+1)b^2}{a-b}\right)} \\
&+ \frac{(a-b)}{24\pi(b+1)(a(b-1)+b)}\left(\left(6G(\{b\},a) + 6G\left(\left\{0,\frac{b}{2}\right\}, a\right)\right.\right. \\
&- 15G(\{0,b\},a) - 9G(\{b,0\},a) - 6G\left(\left\{b,\frac{b}{2}\right\}, a\right) \\
&+ 15G(\{b,b\},a) + 3\log(\pi)G(\{b\},a) + \log(64)G(\{b\},a) - 3G(\{0\},a)(3G(\{0\},b) + 2 + \log(4\pi)) \\
&+3G(\{0\},b)(3G(\{b\},a) + 2 + \log(4\pi)) + 9G(\{0,0\},a) + 9G(\{0,0\},b) + \pi^2\Big) \\
&+ 3(2a-b)\left(G(\{0\},a)(2G(\{0\},b) - i\pi) + G(\{0\},b)\left(-G\left(\left\{\frac{b}{2}\right\}, a\right)\right.\right. \\
&-G(\{b\},a) + i\pi\Big) + i\pi G(\{b\},a) + G\left(\left\{0,\frac{b}{2}\right\}, a\right) \\
&+ G(\{0,b\},a) + G\left(\left\{\frac{b}{2},0\right\}, a\right) - G\left(\left\{\frac{b}{2},b\right\}, a\right) \\
&+G(\{b,0\},a) - G\left(\left\{b,\frac{b}{2}\right\}, a\right) - 2G(\{0,0\},a) - 2G(\{0,0\},b)\Big)\Big) \\
&+ \frac{\epsilon}{48(b+1)(a(b-1)+b)\pi}\left(2(2a-b)\left(3G(\{0,0\},b)G\left(\left\{\frac{b}{2}\right\}, a\right)\right.\right. \\
&- \pi^2 G\left(\left\{\frac{b}{2}\right\}, a\right) - 2\pi^2 G(\{b\},a) + 6i\pi G(\{b\},a)
\end{aligned}
$$

$$+ 3i\pi G(\{0,0\},a) - 12G(\{0,0\},a) - 21G(\{b\},a)G(\{0,0\},b) + 3i\pi G(\{0,0\},b)$$
$$- 12G(\{0,0\},b) + 12i\pi G\left(\left\{0,\frac{b}{2}\right\},a\right) + 6G\left(\left\{0,\frac{b}{2}\right\},a\right)$$
$$- 15i\pi G(\{0,b\},a) + 6G(\{0,b\},a) + 6i\pi G\left(\left\{\frac{b}{2},0\right\},a\right)$$
$$+ 6G\left(\left\{\frac{b}{2},0\right\},a\right) - 6i\pi G\left(\left\{\frac{b}{2},b\right\},a\right)$$
$$- 6G\left(\left\{\frac{b}{2},b\right\},a\right) - 9i\pi G(\{b,0\},a) + 6G(\{b,0\},a)$$
$$- 12i\pi G\left(\left\{b,\frac{b}{2}\right\},a\right) - 6G\left(\left\{b,\frac{b}{2}\right\},a\right)$$
$$+ 21i\pi G(\{b,b\},a) + 18G(\{0,0,0\},a) - 18G(\{0,0,0\},b)$$
$$+ 9G\left(\left\{0,0,\frac{b}{2}\right\},a\right) - 27G(\{0,0,b\},a) + 6G\left(\left\{0,\frac{b}{2},0\right\},a\right)$$
$$- 18G\left(\left\{0,\frac{b}{2},\frac{b}{2}\right\},a\right) + 12G\left(\left\{0,\frac{b}{2},b\right\},a\right)$$
$$- 24G(\{0,b,0\},a) + 9G\left(\left\{0,b,\frac{b}{2}\right\},a\right) + 15G(\{0,b,b\},a)$$
$$+ 3G\left(\left\{\frac{b}{2},0,0\right\},a\right) - 12G\left(\left\{\frac{b}{2},0,\frac{b}{2}\right\},a\right)$$
$$+ 9G\left(\left\{\frac{b}{2},0,b\right\},a\right) - 6G\left(\left\{\frac{b}{2},\frac{b}{2},0\right\},a\right)$$
$$+ 6G\left(\left\{\frac{b}{2},\frac{b}{2},b\right\},a\right) + 3G\left(\left\{\frac{b}{2},b,0\right\},a\right)$$
$$+ 12G\left(\left\{\frac{b}{2},b,\frac{b}{2}\right\},a\right) - 15G\left(\left\{\frac{b}{2},b,b\right\},a\right)$$
$$- 21G(\{b,0,0\},a) + 3G\left(\left\{b,0,\frac{b}{2}\right\},a\right) + 18G(\{b,0,b\},a)$$
$$+ 18G\left(\left\{b,\frac{b}{2},\frac{b}{2}\right\},a\right) - 18G\left(\left\{b,\frac{b}{2},b\right\},a\right)$$
$$+ 21G(\{b,b,0\},a) - 21G\left(\left\{b,b,\frac{b}{2}\right\},a\right)$$
$$- 3G(\{0\},b)\left((2+2i\pi+\log(4\pi))G\left(\left\{\frac{b}{2}\right\},a\right) + 6G(\{0,0\},a)\right.$$
$$+ 2G\left(\left\{0,\frac{b}{2}\right\},a\right) - 8G(\{0,b\},a) + G\left(\left\{\frac{b}{2},0\right\},a\right)$$
$$- 2G\left(\left\{\frac{b}{2},\frac{b}{2}\right\},a\right) + G\left(\left\{\frac{b}{2},b\right\},a\right)$$
$$\left.-7G(\{b,0\},a) + 7G(\{b,b\},a) + G(\{b\},a)(2-3i\pi+\log(4\pi)) - i\pi\log(4\pi) + \pi^2 - 2i\pi\right)$$
$$+ 3G(\{0\},a)((4-i\pi+\log(16)+2\log(\pi))G(\{0\},b) + 6G(\{0,0\},b) + \pi(\pi - i(2+\log(4\pi))))$$
$$+ 3i\pi G(\{b\},a)\log(4\pi) - 6G(\{0,0\},a)\log(\pi) - 6G(\{0,0\},b)\log(\pi)$$
$$+ 3G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(\pi) + 3G(\{0,b\},a)\log(\pi)$$
$$+ 3G\left(\left\{\frac{b}{2},0\right\},a\right)\log(\pi) - 3G\left(\left\{\frac{b}{2},b\right\},a\right)\log(\pi)$$
$$+ 3G(\{b,0\},a)\log(\pi) - 3G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(\pi)$$

$$
\begin{aligned}
&+ G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(64) + G(\{0,b\},a)\log(64) + G\left(\left\{\frac{b}{2},0\right\},a\right)\log(64) \\
&+ G(\{b,0\},a)\log(64) - 6G(\{0,0\},a)\log(4) - 6G(\{0,0\},b)\log(4) \\
&-3G\left(\left\{\frac{b}{2},b\right\},a\right)\log(4) - 3G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(4) + i\pi^3\Big) \\
&- (a-b)\Big(-42G(\{0,0\},b)G(\{b\},a) - \log(\pi)\log\left(4096\pi^3\right)G(\{b\},a) \\
&- 12\log(\pi)G(\{b\},a) - 3\log^2(4)G(\{b\},a) - 12\log(4)G(\{b\},a) - 3\pi^2G(\{b\},a) \\
&- 24G(\{b\},a) - 36G(\{0,0\},a) - 36G(\{0,0\},b) - 24G\left(\left\{0,\frac{b}{2}\right\},a\right) \\
&+ 60G(\{0,b\},a) + 36G(\{b,0\},a) + 24G\left(\left\{b,\frac{b}{2}\right\},a\right) - 60G(\{b,b\},a) \\
&+ 42G(\{0,0,0\},a) - 42G(\{0,0,0\},b) + 36G\left(\left\{0,0,\frac{b}{2}\right\},a\right) \\
&- 78G(\{0,0,b\},a) + 12G\left(\left\{0,\frac{b}{2},0\right\},a\right) + 24G\left(\left\{0,\frac{b}{2},\frac{b}{2}\right\},a\right) \\
&- 36G\left(\left\{0,\frac{b}{2},b\right\},a\right) - 54G(\{0,b,0\},a) - 60G\left(\left\{0,b,\frac{b}{2}\right\},a\right) \\
&+ 114G(\{0,b,b\},a) - 42G(\{b,0,0\},a) - 36G\left(\left\{b,0,\frac{b}{2}\right\},a\right) + 78G(\{b,0,b\},a) \\
&- 12G\left(\left\{b,\frac{b}{2},0\right\},a\right) - 24G\left(\left\{b,\frac{b}{2},\frac{b}{2}\right\},a\right) \\
&+ 36G\left(\left\{b,\frac{b}{2},b\right\},a\right) + 54G(\{b,b,0\},a) + 60G\left(\left\{b,b,\frac{b}{2}\right\},a\right) \\
&- 114G(\{b,b,b\},a) + G(\{0\},a)\,(18(2+\log(4\pi))G(\{0\},b) + 42G(\{0,0\},b) \\
&+ \log(\pi)\log\left(4096\pi^3\right) + 12\log(\pi) + 3\log^2(4) + 12\log(4) + 3\pi^2 + 24\Big) \\
&- G(\{0\},b)\Big(18(2+\log(4\pi))G(\{b\},a) + 42G(\{0,0\},a) + 12G\left(\left\{0,\frac{b}{2}\right\},a\right) \\
&- 54G(\{0,b\},a) - 42G(\{b,0\},a) - 12G\left(\left\{b,\frac{b}{2}\right\},a\right) + 54G(\{b,b\},a) \\
&+ \log(\pi)\log\left(4096\pi^3\right) + 12\log(\pi) + 3\log^2(4) + 12\log(4) + 3\pi^2 + 24\Big) \\
&- 12\zeta(3) - 2\pi^2\log(4\pi) - 18G(\{0,0\},a)\log(\pi) - 18G(\{0,0\},b)\log(\pi) \\
&- 12G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(\pi) + 30G(\{0,b\},a)\log(\pi) \\
&+ 18G(\{b,0\},a)\log(\pi) + 12G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(\pi) \\
&- 30G(\{b,b\},a)\log(\pi) - 18G(\{0,0\},a)\log(4) - 18G(\{0,0\},b)\log(4) \\
&- 12G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(4) + 30G(\{0,b\},a)\log(4) + 18G(\{b,0\},a)\log(4) \\
&+12G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(4) - 30G(\{b,b\},a)\log(4) - 4\pi^2\Big)\Big) \\
&+ \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{6.79}
$$

$$
\begin{aligned}
\tilde{M}_{38} = &-\frac{1}{\epsilon}\frac{G(\{b\},a) - G(\{0\},a) + G(\{0\},b)}{8\pi\left(1-\frac{(a+1)b^2}{a-b}\right)} \\
&\frac{(a-b)}{16\pi(b+1)(a(b-1)+b)}\,(-2G(\{-1\},b)G(\{b\},a) + 4G(\{b\},a)
\end{aligned}
$$

$$
\begin{aligned}
&+6G\left(\left\{0,\frac{b}{2}\right\},a\right)-10G(\{0,b\},a)+2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)\\
&-4G(\{b,0\},a)-6G\left(\left\{b,\frac{b}{2}\right\},a\right)+12G(\{b,b\},a)\\
&-2G\left(\left\{b,-\frac{b}{b-1}\right\},a\right)+2G\left(\left\{-\frac{b}{b-1},0\right\},a\right)\\
&-2G\left(\left\{-\frac{b}{b-1},b\right\},a\right)+2\log(\pi)G(\{b\},a)+\log(16)G(\{b\},a)\\
&+2G(\{0\},a)(G(\{-1\},b)-G(\{0\},b)-2-\log(4\pi))+2G(\{0\},b)\left(2G(\{b\},a)\right.\\
&-G\left(\left\{-\frac{b}{b-1}\right\},a\right)+2+\log(4\pi)\Big)\\
&\left.+2G(\{0,0\},a)-2G(\{-1,0\},b)-2G(\{0,-1\},b)+2G(\{0,0\},b)+\pi^2\right)\\
&-\frac{(a-b)\epsilon}{16(b+1)(a(b-1)+b)\pi}\left(-8G(\{b\},a)G(\{0\},b)\right.\\
&+4G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{0\},b)-2G(\{0,0\},a)G(\{0\},b)\\
&+4G(\{0,b\},a)G(\{0\},b)-2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)G(\{0\},b)\\
&+4G(\{b,0\},a)G(\{0\},b)-8G(\{b,b\},a)G(\{0\},b)+4G\left(\left\{b,-\frac{b}{b-1}\right\},a\right)G(\{0\},b)\\
&-2G\left(\left\{-\frac{b}{b-1},0\right\},a\right)G(\{0\},b)+4G\left(\left\{-\frac{b}{b-1},b\right\},a\right)G(\{0\},b)\\
&-2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right)G(\{0\},b)-\log(\pi)\log(16\pi)G(\{0\},b)\\
&-4G(\{b\},a)\log(\pi)G(\{0\},b)+2G\left(\left\{-\frac{b}{b-1}\right\},a\right)\log(\pi)G(\{0\},b)\\
&-4\log(\pi)G(\{0\},b)+G\left(\left\{-\frac{b}{b-1}\right\},a\right)\log(16)G(\{0\},b)\\
&-\log^2(4)G(\{0\},b)-4G(\{b\},a)\log(4)G(\{0\},b)-4\log(4)G(\{0\},b)-8G(\{0\},b)\\
&-\pi^2G(\{b\},a)-8G(\{b\},a)+\pi^2G\left(\left\{-\frac{b}{b-1}\right\},a\right)\\
&-2G(\{b\},a)G(\{-1,-1\},b)+4G(\{b\},a)G(\{-1,0\},b)-2G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{-1,0\},b)\\
&+4G(\{-1,0\},b)+4G(\{b\},a)G(\{0,-1\},b)-2G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{0,-1\},b)\\
&+4G(\{0,-1\},b)-4G(\{0,0\},a)-4G(\{b\},a)G(\{0,0\},b)+2G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{0,0\},b)\\
&-4G(\{0,0\},b)-12G\left(\left\{0,\frac{b}{2}\right\},a\right)+20G(\{0,b\},a)\\
&-4G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)+8G(\{b,0\},a)+12G\left(\left\{b,\frac{b}{2}\right\},a\right)\\
&-24G(\{b,b\},a)+4G\left(\left\{b,-\frac{b}{b-1}\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},0\right\},a\right)\\
&+4G\left(\left\{-\frac{b}{b-1},b\right\},a\right)-2G(\{-1,-1,0\},b)-2G(\{-1,0,-1\},b)\\
&+2G(\{-1,0,0\},b)-2G(\{0,-1,-1\},b)+2G(\{0,-1,0\},b)+2G(\{0,0,-1\},b)\\
&+2G(\{0,0,0\},a)-2G(\{0,0,0\},b)+6G\left(\left\{0,0,\frac{b}{2}\right\},a\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 10G(\{0,0,b\},a) + 2G\left(\left\{0,0,-\frac{b}{b-1}\right\},a\right) + 18G\left(\left\{0,\frac{b}{2},\frac{b}{2}\right\},a\right) \\
&- 24G\left(\left\{0,\frac{b}{2},b\right\},a\right) + 6G\left(\left\{0,\frac{b}{2},-\frac{b}{b-1}\right\},a\right) \\
&- 4G(\{0,b,0\},a) - 30G\left(\left\{0,b,\frac{b}{2}\right\},a\right) + 44G(\{0,b,b\},a) \\
&- 10G\left(\left\{0,b,-\frac{b}{b-1}\right\},a\right) + 2G\left(\left\{0,-\frac{b}{b-1},0\right\},a\right) \\
&+ 6G\left(\left\{0,-\frac{b}{b-1},\frac{b}{2}\right\},a\right) - 10G\left(\left\{0,-\frac{b}{b-1},b\right\},a\right) \\
&+ 2G\left(\left\{0,-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right) - 4G(\{b,0,0\},a) \\
&- 12G\left(\left\{b,0,\frac{b}{2}\right\},a\right) + 20G(\{b,0,b\},a) - 4G\left(\left\{b,0,-\frac{b}{b-1}\right\},a\right) \\
&- 18G\left(\left\{b,\frac{b}{2},\frac{b}{2}\right\},a\right) + 24G\left(\left\{b,\frac{b}{2},b\right\},a\right) \\
&- 6G\left(\left\{b,\frac{b}{2},-\frac{b}{b-1}\right\},a\right) + 8G(\{b,b,0\},a) + 36G\left(\left\{b,b,\frac{b}{2}\right\},a\right) \\
&- 56G(\{b,b,b\},a) + 12G\left(\left\{b,b,-\frac{b}{b-1}\right\},a\right) - 4G\left(\left\{b,-\frac{b}{b-1},0\right\},a\right) \\
&- 6G\left(\left\{b,-\frac{b}{b-1},\frac{b}{2}\right\},a\right) + 12G\left(\left\{b,-\frac{b}{b-1},b\right\},a\right) \\
&- 2G\left(\left\{b,-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right) + 2G\left(\left\{-\frac{b}{b-1},0,0\right\},a\right) \\
&+ 6G\left(\left\{-\frac{b}{b-1},0,\frac{b}{2}\right\},a\right) - 10G\left(\left\{-\frac{b}{b-1},0,b\right\},a\right) \\
&+ 2G\left(\left\{-\frac{b}{b-1},0,-\frac{b}{b-1}\right\},a\right) - 4G\left(\left\{-\frac{b}{b-1},b,0\right\},a\right) \\
&- 6G\left(\left\{-\frac{b}{b-1},b,\frac{b}{2}\right\},a\right) + 12G\left(\left\{-\frac{b}{b-1},b,b\right\},a\right) \\
&- 2G\left(\left\{-\frac{b}{b-1},b,-\frac{b}{b-1}\right\},a\right) + 2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1},0\right\},a\right) \\
&- 2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1},b\right\},a\right) + G(\{-1\},b)\,(-2(2+\log(4\pi))G(\{0\},a) \\
&+ 2G(\{0,0\},a) + 6G\left(\left\{0,\frac{b}{2}\right\},a\right) - 10G(\{0,b\},a) \\
&+ 2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right) - 4G(\{b,0\},a) - 6G\left(\left\{b,\frac{b}{2}\right\},a\right) \\
&+ 12G(\{b,b\},a) - 2G\left(\left\{b,-\frac{b}{b-1}\right\},a\right) + 2G\left(\left\{-\frac{b}{b-1},0\right\},a\right) \\
&-2G\left(\left\{-\frac{b}{b-1},b\right\},a\right) + G(\{b\},a)(4+\log(16)+2\log(\pi)) + \pi^2\Big) \\
&+ G(\{0\},a)\,((4+\log(16)+2\log(\pi))G(\{0\},b) + 2G(\{-1,-1\},b) - 2G(\{-1,0\},b) \\
&-2G(\{0,-1\},b) + 2G(\{0,0\},b) + \log(\pi)\log(16\pi) + 4\log(\pi) + \log(256) + \log^2(4) + 8\Big) \\
&- 12\zeta(3) - G(\{b\},a)\log(\pi)\log(16\pi) - \pi^2\log(4\pi) - 4G(\{b\},a)\log(\pi) \\
&+ 2G(\{-1,0\},b)\log(\pi) + 2G(\{0,-1\},b)\log(\pi) - 2G(\{0,0\},a)\log(\pi) - 2G(\{0,0\},b)\log(\pi) \\
&- 6G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(\pi) + 10G(\{0,b\},a)\log(\pi)
\end{aligned}
$$

$$
\begin{aligned}
&- 2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)\log(\pi) + 4G(\{b,0\},a)\log(\pi)\\
&+ 6G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(\pi) - 12G(\{b,b\},a)\log(\pi)\\
&+ 2G\left(\left\{b,-\frac{b}{b-1}\right\},a\right)\log(\pi) - 2G\left(\left\{-\frac{b}{b-1},0\right\},a\right)\log(\pi)\\
&+ 2G\left(\left\{-\frac{b}{b-1},b\right\},a\right)\log(\pi) + G(\{-1,0\},b)\log(16)\\
&+ G(\{0,-1\},b)\log(16) + G\left(\left\{b,-\frac{b}{b-1}\right\},a\right)\log(16)\\
&+ G\left(\left\{-\frac{b}{b-1},b\right\},a\right)\log(16) - G(\{b\},a)\log^2(4) - 4G(\{b\},a)\log(4)\\
&- 2G(\{0,0\},a)\log(4) - 2G(\{0,0\},b)\log(4) - 6G\left(\left\{0,\frac{b}{2}\right\},a\right)\log(4)\\
&+ 10G(\{0,b\},a)\log(4) - 2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)\log(4)\\
&+ 4G(\{b,0\},a)\log(4) + 6G\left(\left\{b,\frac{b}{2}\right\},a\right)\log(4)\\
&-12G(\{b,b\},a)\log(4) - 2G\left(\left\{-\frac{b}{b-1},0\right\},a\right)\log(4) - 2\pi^2\Bigg)
\end{aligned}
\tag{6.80}
$$

$$
\begin{aligned}
\tilde{M}_{42} =& \frac{(a-b)}{16(b+1)(a(b-1)+b)\sqrt{\frac{(2ab+b)^2}{(a-b)^2}}\pi}\times\\
&\times\Bigg(4G(\{-1,0\},a)G(\{0\},b) - 2G(\{-1,b\},a)G(\{0\},b) - 6G\left(\left\{-1,-\frac{b}{b-1}\right\},a\right)G(\{0\},b)\\
&+ 4G\left(\left\{-1,-\frac{b^2}{b^2+1}\right\},a\right)G(\{0\},b) - 8G(\{0,-1\},a)G(\{0\},b)\\
&+ 4G(\{0,0\},a)G(\{0\},b) - 12G\left(\left\{0,\frac{b}{2}\right\},a\right)G(\{0\},b) + 4G(\{0,b\},a)G(\{0\},b)\\
&+ 8G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)G(\{0\},b) + 4G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right)G(\{0\},b)\\
&+ 8G\left(\left\{-\frac{b}{b-1},-1\right\},a\right)G(\{0\},b) - 8G\left(\left\{-\frac{b}{b-1},0\right\},a\right)G(\{0\},b)\\
&+ 12G\left(\left\{-\frac{b}{b-1},\frac{b}{2}\right\},a\right)G(\{0\},b) - 2G\left(\left\{-\frac{b}{b-1},b\right\},a\right)G(\{0\},b)\\
&- 2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right)G(\{0\},b)\\
&- 8G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1}\right\},a\right)G(\{0\},b) - \pi^2G(\{0\},b)\\
&- \pi^2G\left(\left\{-\frac{b}{b-1}\right\},a\right) + 2G(\{-1\},a)G(\{-1,-1\},b) - 6G(\{-1\},a)G(\{-1,0\},b)\\
&- 2G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{-1,0\},b) - 4G(\{-1\},a)G(\{0,-1\},b)\\
&+ 2G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{0,-1\},b) + G(\{0\},a)\left(-2G(\{-1,-1\},b)\right.\\
&\left.+8G(\{-1,0\},b) + 2G(\{0,-1\},b) - 8G(\{0,0\},b) + \pi^2\right) + 8G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{0,0\},b)
\end{aligned}
$$

$$
\begin{aligned}
&+ G(\{-1\},b)\left(2G(\{-1,0\},a) - 2G(\{-1,b\},a) + 2G\left(\left\{-1,-\frac{b}{b-1}\right\},a\right)\right.\\
&- 2G\left(\left\{-1,-\frac{b^2}{b^2+1}\right\},a\right) - 4G(\{0,0\},a) + 4G\left(\left\{0,\frac{b}{2}\right\},a\right)\\
&+ 4G(\{0,b\},a) - 2G\left(\left\{0,-\frac{b}{b-1}\right\},a\right) - 2G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right)\\
&+ 2G\left(\left\{-\frac{b}{b-1},0\right\},a\right) - 4G\left(\left\{-\frac{b}{b-1},\frac{b}{2}\right\},a\right)\\
&- 2G\left(\left\{-\frac{b}{b-1},b\right\},a\right) + 4G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1}\right\},a\right)\\
&\left.+\pi^2\right) + 2G(\{-1,-1,0\},b) - 2G(\{-1,0,-1\},b) - 4G(\{-1,0,0\},a)\\
&- 8G(\{-1,0,0\},b) + 2G\left(\left\{-1,0,\frac{b}{2}\right\},a\right) + 2G\left(\left\{-1,0,-\frac{b}{b-1}\right\},a\right)\\
&+ 2G(\{-1,b,0\},a) - 2G\left(\left\{-1,b,\frac{b}{2}\right\},a\right) + 2G(\{-1,b,b\},a)\\
&- 2G\left(\left\{-1,b,-\frac{b}{b-1}\right\},a\right) + 2G\left(\left\{-1,-\frac{b}{b-1},0\right\},a\right)\\
&+ 2G\left(\left\{-1,-\frac{b}{b-1},\frac{b}{2}\right\},a\right) - 2G\left(\left\{-1,-\frac{b}{b-1},b\right\},a\right)\\
&+ 2G\left(\left\{-1,-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right) - 4G\left(\left\{-1,-\frac{b}{b-1},-\frac{b^2}{b^2+1}\right\},a\right)\\
&- 2G\left(\left\{-1,-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right) - 2G\left(\left\{-1,-\frac{b^2}{b^2+1},-\frac{b}{b-1}\right\},a\right)\\
&+ 4G\left(\left\{-1,-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right) + 4G(\{0,-1,0\},a) - 2G(\{0,-1,0\},b)\\
&- 4G\left(\left\{0,-1,-\frac{b^2}{b^2+1}\right\},a\right) + 2G(\{0,0,-1\},b) + 8G(\{0,0,0\},b)\\
&- 4G\left(\left\{0,0,\frac{b}{2}\right\},a\right) + 4G(\{0,0,b\},a) - 4G\left(\left\{0,0,-\frac{b}{b-1}\right\},a\right)\\
&+ 4G\left(\left\{0,0,-\frac{b^2}{b^2+1}\right\},a\right) + 4G\left(\left\{0,\frac{b}{2},0\right\},a\right)\\
&+ 4G\left(\left\{0,\frac{b}{2},\frac{b}{2}\right\},a\right) - 4G\left(\left\{0,\frac{b}{2},b\right\},a\right)\\
&+ 4G\left(\left\{0,\frac{b}{2},-\frac{b}{b-1}\right\},a\right) - 8G\left(\left\{0,\frac{b}{2},-\frac{b^2}{b^2+1}\right\},a\right)\\
&- 4G(\{0,b,0\},a) + 4G\left(\left\{0,b,\frac{b}{2}\right\},a\right) - 4G(\{0,b,b\},a)\\
&+ 4G\left(\left\{0,b,-\frac{b}{b-1}\right\},a\right) - 4G\left(\left\{0,-\frac{b}{b-1},0\right\},a\right)\\
&- 2G\left(\left\{0,-\frac{b}{b-1},\frac{b}{2}\right\},a\right) + 4G\left(\left\{0,-\frac{b}{b-1},b\right\},a\right)\\
&- 2G\left(\left\{0,-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right) + 4G\left(\left\{0,-\frac{b}{b-1},-\frac{b^2}{b^2+1}\right\},a\right)\\
&- 2G\left(\left\{0,-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right) - 2G\left(\left\{0,-\frac{b^2}{b^2+1},-\frac{b}{b-1}\right\},a\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4G\left(\left\{0,-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},-1,0\right\},a\right)\\
&+4G\left(\left\{-\frac{b}{b-1},-1,-\frac{b^2}{b^2+1}\right\},a\right)+4G\left(\left\{-\frac{b}{b-1},0,0\right\},a\right)\\
&+2G\left(\left\{-\frac{b}{b-1},0,\frac{b}{2}\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},0,b\right\},a\right)\\
&+2G\left(\left\{-\frac{b}{b-1},0,-\frac{b}{b-1}\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},0,-\frac{b^2}{b^2+1}\right\},a\right)\\
&-4G\left(\left\{-\frac{b}{b-1},\frac{b}{2},0\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},\frac{b}{2},\frac{b}{2}\right\},a\right)\\
&+4G\left(\left\{-\frac{b}{b-1},\frac{b}{2},b\right\},a\right)-4G\left(\left\{-\frac{b}{b-1},\frac{b}{2},-\frac{b}{b-1}\right\},a\right)\\
&+8G\left(\left\{-\frac{b}{b-1},\frac{b}{2},-\frac{b^2}{b^2+1}\right\},a\right)+2G\left(\left\{-\frac{b}{b-1},b,0\right\},a\right)\\
&-2G\left(\left\{-\frac{b}{b-1},b,\frac{b}{2}\right\},a\right)+2G\left(\left\{-\frac{b}{b-1},b,b\right\},a\right)\\
&-2G\left(\left\{-\frac{b}{b-1},b,-\frac{b}{b-1}\right\},a\right)+2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1},0\right\},a\right)\\
&-2G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1},b\right\},a\right)+4G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right)\\
&+4G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1},-\frac{b}{b-1}\right\},a\right)\\
&-8G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right)-6\zeta(3)\Bigg) \qquad (6.81)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}_{43}=&-\frac{1}{\epsilon^3}\frac{5(a-b)^3}{48\pi(b+1)^2(2a-b)(a(b-1)+b)^2}\\
&\frac{1}{\epsilon^2}\frac{(a-b)^3}{48\pi(b+1)^2(2a-b)(a(b-1)+b)^2}\left(-4G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)\right.\\
&+20G\left(\left\{\frac{b}{2}\right\},a\right)-10G(\{b\},a)+4G\left(\left\{-\frac{b}{b-1}\right\},a\right)\\
&\left.-10G(\{0\},a)+4G(\{-1\},b)+6G(\{0\},b)-6i\pi-5\log(\pi)-\log(1024)\right)\\
&+\frac{1}{\epsilon}\frac{(a-b)^3}{192(2a-b)(b+1)^2(a(b-1)+b)^2\pi}\left(-64G\left(\left\{\frac{b}{2}\right\},a\right)G(\{-1\},b)\right.\\
&+32G(\{b\},a)G(\{-1\},b)-8G\left(\left\{-\frac{b}{b-1}\right\},a\right)G(\{-1\},b)\\
&+8G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)G(\{-1\},b)+16\log(\pi)G(\{-1\},b)+16\log(4)G(\{-1\},b)\\
&+96i\pi G\left(\left\{\frac{b}{2}\right\},a\right)-48i\pi G(\{b\},a)-8G(\{-1,-1\},b)\\
&-48G(\{-1,0\},b)-24G(\{0,-1\},b)+16G(\{0,0\},a)+96G(\{0,0\},b)+160G\left(\left\{0,\frac{b}{2}\right\},a\right)\\
&-176G(\{0,b\},a)+32G\left(\left\{0,-\frac{b}{b-1}\right\},a\right)-32G\left(\left\{0,-\frac{b^2}{b^2+1}\right\},a\right)
\end{aligned}
$$

$$+ 64G\left(\left\{\frac{b}{2},0\right\},a\right) - 320G\left(\left\{\frac{b}{2},\frac{b}{2}\right\},a\right)$$
$$+ 256G\left(\left\{\frac{b}{2},b\right\},a\right) - 64G\left(\left\{\frac{b}{2},-\frac{b}{b-1}\right\},a\right)$$
$$+ 64G\left(\left\{\frac{b}{2},-\frac{b^2}{b^2+1}\right\},a\right) - 80G(\{b,0\},a) + 160G\left(\left\{b,\frac{b}{2}\right\},a\right)$$
$$- 80G(\{b,b\},a) + 32G\left(\left\{b,-\frac{b}{b-1}\right\},a\right) - 32G\left(\left\{b,-\frac{b^2}{b^2+1}\right\},a\right)$$
$$+ 32G\left(\left\{-\frac{b}{b-1},0\right\},a\right) - 40G\left(\left\{-\frac{b}{b-1},\frac{b}{2}\right\},a\right)$$
$$+ 32G\left(\left\{-\frac{b}{b-1},b\right\},a\right) - 8G\left(\left\{-\frac{b}{b-1},-\frac{b}{b-1}\right\},a\right)$$
$$- 16G\left(\left\{-\frac{b}{b-1},-\frac{b^2}{b^2+1}\right\},a\right) - 32G\left(\left\{-\frac{b^2}{b^2+1},0\right\},a\right)$$
$$+ 40G\left(\left\{-\frac{b^2}{b^2+1},\frac{b}{2}\right\},a\right) - 32G\left(\left\{-\frac{b^2}{b^2+1},b\right\},a\right)$$
$$+ 8G\left(\left\{-\frac{b^2}{b^2+1},-\frac{b}{b-1}\right\},a\right) + 16G\left(\left\{-\frac{b^2}{b^2+1},-\frac{b^2}{b^2+1}\right\},a\right)$$
$$+ 8G(\{0\},a)(4G(\{-1\},b) - 6G(\{0\},b) - 5\log(\pi) - 5\log(4) - 6i\pi)$$
$$+ 24G(\{0\},b)\left(2G(\{b\},a) - 2G\left(\left\{-\frac{b}{b-1}\right\},a\right)\right.$$
$$\left.+2G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right) + \log(4\pi) + 2i\pi\right)$$
$$- 10\log^2(4\pi) - 24i\pi\log(4\pi) + 80G\left(\left\{\frac{b}{2}\right\},a\right)\log(\pi)$$
$$- 40G(\{b\},a)\log(\pi) + 16G\left(\left\{-\frac{b}{b-1}\right\},a\right)\log(\pi)$$
$$- 16G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)\log(\pi) + 8G\left(\left\{\frac{b}{2}\right\},a\right)\log(1024)$$
$$- 8G(\{b\},a)\log(1024) + 40G\left(\left\{\frac{b}{2}\right\},a\right)\log(4)$$
$$+ 16G\left(\left\{-\frac{b}{b-1}\right\},a\right)\log(4) - 16G\left(\left\{-\frac{b^2}{b^2+1}\right\},a\right)\log(4)$$
$$\left.+39\pi^2 + 2\gamma^2 + 8\gamma\right) \tag{6.82}$$

$$\begin{aligned}\tilde{M}_{59} =& \frac{1}{48\pi(y+1)^2}\left(-\left(6G(\{0,0\},y) + \pi^2\right)G(\{0\},z)\right.\\ &+ 6(2(G(\{0\},y) + i\pi)G(\{0,0\},z) + 2G(\{0,0,-y\},z)\\ &\left.+G(\{0,-y,0\},z) - 2G(\{0,0,1\},z) - G(\{0,1,0\},z))\right)\\ &+ \mathcal{O}(\epsilon)\end{aligned} \tag{6.83}$$

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