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A bottom-up approach to fermion masses

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Avec l'assistance de Dieu et avec son concours précieux, je dis : L'algèbre est un art scientifique. Son objet, ce sont le nombre absolu et les grandeurs mesurables, étant inconnus, mais rapportés à quelque chose de connu de manière à pouvoir être déterminés; cette chose connue est une quantité ou un rapport individuellement déterminé, ainsi qu'on le reconnaît en les examinant attentivement; ce qu'on cherche dans cet art, ce sont les relations qui joignent les données des problèmes à l'inconnue, qui de la manière susdite forme l'objet de l'algèbre. La perfection de cet art consiste dans la connaissance des méthodes mathématiques au moyen desquelles on est en état d'effectuer le susdit genre de détermination des inconnues, soit numériques, soit géométriques.

Omar Khayyám
Démonstrations de problèmes d'algèbre (1070)
Trad. F. Woepcke in "L'algèbre d'Omar Alkayyani" 1851

Contents

Foreword

Over the past decades, the gauge symmetry structure of the Standard Model of strong and electroweak interactions has been thoroughly checked and confirmed in great detail. However, a whole sector of the theory has not yet been observed. This Higgs sector has been eagerly anticipated since the start of the LEP and Tevatron experiments. The combination of the direct lower bound (coming from LEP) and indirect upper bound (from electroweak precision data) has significantly constrained the range of possible values for the Higgs mass. The Tevatron and LHC will thus be soon able to confirm or not the existence of the Higgs particle.

If the Higgs particle turns out to be exactly as expected, then the Standard Model is closed from a mathematical point of view. In that case, it is conceivable that any new physics will be far beyond the reach of future colliders. On the contrary, if the data do not reveal any scalar particle, it is likely that the LHC will spot some unexpected or unexplained events since the unitarity of the theory is broken around 1 TeV. From a mathematical point of view, the first possibility is probably more appealing but the latter is certainly the most interesting one for us, *now*. Indeed, in spite of its simplicity, the Brout-Englert-Higgs mechanism also raises some questions.

In the Standard Model, all forces are explained in terms of boson exchange. These bosons are associated with a gauge symmetry. However, the scalar sector of the theory as it stands cannot be considered as a fifth force. The Higgs boson is moreover the only particle that knows the difference between the fermion families as well as between the generations. The coupling is indeed proportional to the mass instead of some conserved charge. This particular status is quite intriguing. Furthermore, these masses seem to be completely arbitrary and display a huge hierarchy, not to mention the astonishingly small mass of the neutrinos.

The mass generation mechanism is also intimately connected with the $SU(2)_L$ gauge symmetry. This connection is at the root of some of the most interesting properties of the Standard Model, namely flavour mixing and CP violation.

The Standard Model requires thus some new flavour physics, in particular to explain the fermion mass spectrum and the number of families and generations. These numbers must be somehow connected with the mass generation mechanism. Most of the proposed extensions of the Standard Model fail to meet these criteria. Some problems of the Standard Model have indeed been solved by grand unification theories, supersymmetry, technicolour, horizontal symmetries and others. However, it has always been at the expense of a complexification of the theory, for instance a zoo of new particles or a rather large group of symmetry. The LHC will certainly help us to select the best candidate for new physics. We hope that it will also surprise us.

Thesis Aims

Fermion mixings arise naturally in the theory of electroweak interactions and result from a mismatch between mass- and weak-eigenstates. Within the assumption of a Higgs mechanism, mixings and masses have the same origin and are bound together in the Yukawa couplings. This mechanism however fails to give an explanation of the observed fermion spectrum. Therefore, the Standard Model is probably not the end of the story but only a low energy effective theory.

In this work, we do not aim at finding a new mechanism that could explain this spectrum, but we rather assume that fermion masses and mixings are calculable in a yet-to-be-found more fundamental theory. Our goal is to glean as much information as possible from the observed fermion masses and mixings in order to identify some hidden structures that could significantly lower the number of free parameters and help us to get some clues about what could be this fundamental theory.

To achieve this goal, we follow two distinct paths:

- The analysis of the various parametrizations of the flavour mixing matrix points us to a specific decomposition. We note that the parameters of this decomposition can be independently and accurately computed if we impose some simple textures to the Yukawa couplings. We propose then a straightforward combination of these interesting textures in order to recover the observed quark flavour mixing.
- We study the properties of a successful mass relation for the charged leptons. We propose some generalizations of this relation which also apply to the neutrinos and the quarks. One of them successfully combines the masses and mixings in a kind of weak-eigenstate mass. Another one describes the lepton masses through a well-defined geometric picture.

Hopefully, these two paths lead to similar conclusions and allow us to speculate about some interesting properties new flavour physics should display.

1

A Mass Definition

1.1 Three Concepts

Giving the definition of a word is usually not within the scope of physics. Yet, if one concept of physics deserves such a privilege, mass is probably that one. In day-to-day life, people mix up the notions of mass and volume. A big box *looks* always heavier than a small one. On the contrary, everybody has a very good intuition about its properties: on Earth, an object is harder to set in motion or is harder to lift than another one. However, we do not usually pay to much attention on what these properties really mean. They refer indeed to two very different concepts: inertia and gravitation, respectively.

The classical, or naive, idea that the mass is intimately connected with the amount of matter in an object finds its roots in the etymology of the word. In Latin, "massa" meant lump or dough [1]. This notion of "aggregation of matter" has been conserved through the centuries [2]. In 1828, the Webster's dictionnary did not make any connection with physics but gave only the following definition:

> *A lump; a body of matter concreted, collected or formed into a lump; applied to any solid body; as a mass of iron or lead; a mass of flesh; as mass of ice; a mass of dough.*

Later in the century, in 1863, in the first edition of the Hachette dictionary Émile Littré gave a similar definition:

> *Amas de parties qui font un corps ensemble.*

Together with

> *Terme de physique. Somme des points matériels que chaque corps renferme, par opposition à volume qui exprime l'espace occupé. La gravitation s'exerce en raison directe des masses et en raison inverse du carré des distances.*

Mass is thus defined as **the amount of matter**. This mere fact implies however the existence of fundamental, and countable, pieces of matter. In this view, mass is similar to volume: more matter implies a bigger volume and thus a bigger mass. This connection is also found in the original definition of the kilogram: the mass of one litre of water at zero degree [3].

This definition is however disturbing in regard to modern physics. At the atomic level, one mole of helium or one mole of an heavy gas like the radon both correspond to the same number of fundamental particles. Consequently, they will share almost the same volume under equivalent conditions of temperature and pressure. One has to go deeper into the atoms and the nuclei to explain in terms of nucleons the mass difference, or rather the density difference. In elementary particle physics, the fundamental building blocks of matter have all a zero spatial size but present a large variety of mass scales, from the light electron to the heavy top quark. As a result, the concept of amount of matter has again lost its meaning. Unless, of course, these particles are found to be themselves composite.

In addition to this intrinsic definition, a mass m may also be defined with respect to its various properties:

The inertial mass is a measure of an object's resistance to changing its state of motion when a force is applied. This concept is behind the second Newton's law: $F = ma$.

The gravitational mass is a measure of the strength of an object's interaction with a gravitational field [1]. This concept is at the heart of Newton's theory of gravitation: $F = G\frac{mM}{r^2}$.

[1] In principle, there is an active and a passive gravitational mass. The former is defined as the mass that induces gravitation, whereas the latter is defined as the mass susceptible to gravitation. They are here represented by M and m, respectively . Newton's third law (action equals reaction) ensures the equality of these two concepts.

Here, the mass is defined in terms of its reaction with respect to an inertial force and a gravitational force. The priority of force over mass is therefore implicitly assumed. A more modern way of thinking is to consider energy as more fundamental than force. In that case, the inertial mass is easily defined thanks to the kinetic energy: $E = \frac{mv^2}{2}$. It requires more work, or energy, to impart a certain velocity to a hammer than a feather.

The inertial and gravitational mass definitions are conceptually completely different. However, at the experimental level, no difference has ever been found between them. Galileo illustrated this equality in a famous thought experiment [2]: the feather and the hammer both fall at the same speed (at least in the vacuum); in opposition to what Aristotle taught. This empirical fact has been raised to the status of fundamental law in general relativity, under the name of equivalence principle.

Once again, elementary particle physics cannot accommodate itself to these definitions. For instance, the definition in terms of inertia assumes the existence of a frame in which the particle is at rest. However, special relativity has proved that such a frame does not exist for massless particles. Moreover, this definition is only valid if, at some point, the particle is free from any interaction. Yet, the confinement of the quarks prevents them to move freely and their inertial mass does not make sense.

1.2 Modern View

In special relativity the mass is identified as the norm of the four-momentum

$$p^\mu = (E/c, \vec{p}) \quad . \tag{1.1}$$

[2]The experiments did not really take place at the leaning tower of Pisa but were actually conducted by rolling balls down inclined planes. It is worth mentioning that similar results were found some years before by Simon Stevin while throwing lead balls from the Delft churchtower. Isaac Newton also confirmed the equivalence by measuring the period of pendulums of different mass but identical length. Nowadays, the best measure is achieved via a torsion balance. The observed difference of acceleration between two different masses is smaller than 10^{-15} m/s^2 [4]. These tests are of crucial importance since many attempts to quantify general relativity allow for a violation of the equivalence principle.

This value is scalar and therefore invariant under any Lorentz transformation.

$$m^2c^2 = p^\mu p_\mu = \left(\frac{E}{c}\right)^2 - (\vec{p})^2 \quad . \tag{1.2}$$

At the classical level, a change of mass was interpreted as a flux of matter. The indestructibility of matter was indeed implied, together with the identification of matter and mass. Within special relativity, the mass is no longer an additive property. For instance, the mass of an atom of hydrogen is smaller than the mass of its constituents, the difference being the binding energy between the proton and the electron. At rest, there is a full equivalence between the mass of a free particle and its energy, i.e. mass is just a form of energy: $E_0 = mc^2$.

In quantum field theory, the situation is much the same. The particles are assumed to be point-like and not made up of other particles. Each of these particles has a defined mass, spin and charge.

1.2.1 Self-Energy Contribution

It is known that the environment contributes to the effective mass of a moving object. Let's imagine two identical dense objects, one in water, the other one in the air. Their mass, measured at rest with a balance, is exactly the same in both cases[3]. However, if a given force is applied on these objects, their acceleration will be different. When moving, the object in water will behave as if it were heavier than the one in the air. The same phenomenon appears with charged particles moving in an external electromagnetic field. For instance, an electron moving in a crystal behaves as a free particle in the vacuum but with a different mass, called effective mass.

Up to now, we did not make any distinction between charged and neutral particles. However, the mass-energy of a moving charged particle in its *own* electromagnetic field should, in principle, also contribute to the observed mass.

In classical electrodynamics, this reasoning leads to inconsistencies because the electric field ($\vec{E}_q$) is divergent at small distance. Indeed,

[3]The Archimedes' principle can be safely neglected here.

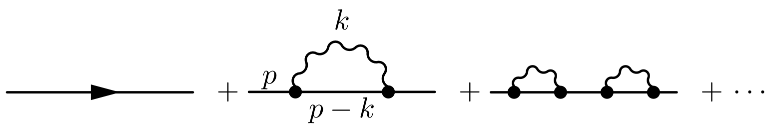


Figure 1.1: *Self-energy contributions to the electron propagator.*

if we assume that the electron is a charged spherical shell of radius r_e, the energy in the field is given by

$$\delta m\ c^2 = \int \mathrm{d}V \frac{\epsilon_0 E_q^2}{2} = \int_{r_e}^{\infty} \mathrm{d}r\ 4\pi r^2 \frac{\epsilon_0}{2} \left(\frac{e}{4\pi\epsilon_0 r^2}\right)^2 = \frac{e^2}{8\pi\epsilon_0 r_e} \quad . \tag{1.3}$$

This correction is infinite if the electron is point like. The value of r_e that makes δm equal to the measured electron mass is called the classical electron radius:

$$r_e^c \equiv \frac{e^2}{8\pi\epsilon_0 m_e c^2} = \frac{\alpha}{2} \frac{\hbar}{m_e c} \sim 10^{-15}\ \mathrm{m} \tag{1.4}$$

where $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$ is the fine-structure constant. This classical radius is however much smaller than the Compton wavelength of the electron:

$$\lambda_e = 2\pi \frac{\hbar}{m_e c} \sim 10^{-12}\ \mathrm{m} \tag{1.5}$$

below which quantum corrections cannot be neglected.

In quantum field theory, the behaviour at small distance is only slightly better [5]. What we want to compute is the correction to the mass due to the self-interaction of the electron in his own field. We need then to take into account the photon loop contributions to the electron propagator. The thorough computation can be found in many books, for instance [6]. Here, we shall only sketch the various steps.

The propagator of the electron $S(p)$ gets new contributions proportional to the one loop contribution $\Sigma(p)$ (Fig. 1.1):

$$\begin{aligned} iS(p) \rightsquigarrow & iS(p) + iS(p)\Sigma(p)S(p) + iS(p)\Sigma(p)S(p)\Sigma(p)S(p) + \cdots \\ &= iS(p)\left[1 + \Sigma(p)S(p) + (\Sigma(p)S(p))^2 + \cdots\right] \\ &= \frac{iS(p)}{1 - \Sigma(p)S(p)} = \frac{i}{S^{-1}(p) - \Sigma(p)} \end{aligned} \tag{1.6}$$

with $S(p) = \frac{1}{\not{p}-m+i\epsilon}$. Then the corrected propagator $S'(p)$ is

$$iS'(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon} \quad . \tag{1.7}$$

The pole of the propagator is no more the mass m but a loop corrected mass $m' = m + \Sigma(p)$. The self-energy contribution to the mass of the electron is then the one loop contribution $\Sigma(p)$.

Its computation requires an integration over all possible values of the momentum k running in the loop. In the Feynman gauge, one finds:

$$ie_0^2\Sigma(p) = -\frac{e_0^2}{(2\pi)^4}\int \mathrm{d}^4k \frac{1}{k^2 + i\epsilon}\gamma^\alpha \frac{m + \not{p} - \not{k}}{(p-k)^2 - m^2 + i\epsilon}\gamma_\alpha \quad . \tag{1.8}$$

The term linear in $\not{k}$ disappears through the angular integration and the leading term at short distance, i.e. large momentum, scales like

$$\sim \int \frac{\mathrm{d}^4k}{k^4} \tag{1.9}$$

which is "only" logarithmically divergent compared to the linear divergence at the classical level (1.3). This divergence is at first sight disturbing. Fortunately, mathematical tools have been developed to deal with this infinity.

Regularization

To make the integral (1.8) mathematically meaningful, we need to write it as a suitable limit of a convergent integral. This process, called regularization, is not unique.

The simplest one is probably the cut-off method in which the high momentum region — the source of the divergence — is cut off in the integral. A quite similar method is the lattice regularization where the space-time is discretized. As a consequence, the short distance contribution to the space-time integration is under control. This is quite similar to considering that the electron has a finite size in the classical electrodynamics. However, both methods have some drawbacks, the breaking of the gauge or Lorentz invariance, respectively.

Before presenting another regularization method, let's first go back to the classical electrodynamics. The mass of a spherical charged particle includes the mass of the spherical shell. If the shell's mass is allowed to be negative, it might be possible to have a regular point-like limit. This is called renormalization since the bare mass, i.e. the mass of a fictitious non-interacting particle, is tuned to compensate the infinities coming from its self-energy. The observed mass is the renormalized mass. The bare mass and the corrections coming from self-energy cannot be disentangled from one another.

The same trick is used in quantum field theory within a dimensional regularization. The dimensional regularization assumes that space-time dimension is not four but rather $d = 4 - \epsilon$. When ϵ is positive, the integral (1.8) is convergent. The renormalization of the electron mass and charge is not unique. We have to choose how the divergent piece will be subtracted out in the renormalization process. This prescription is called renormalization scheme.

The on-shell scheme assumes that the renormalized mass is the pole of the propagator once self-energy are taken into account (see (1.7)). This pole mass is really the physical mass since it is energy and gauge invariant. Unfortunately, the propagator of the quarks is ill-defined. Indeed, quarks are always confined in colourless bound states, such as mesons ($q\overline{q}$) or baryons (qqq), and can never be observed directly. As the strong coupling constant is increasing with decreasing energy, the pole mass for quarks is therefore infrared divergent. This is particularly true for the light quarks u, d and s.

As a consequence, another prescription is commonly used for the quarks, the modified minimal subtraction scheme ($\overline{MS}$). Its main advantage is that the renormalization constants in this scheme are independent of mass parameters of the theory. However, the renormalized parameters are now energy dependent and are called running parameters. In contrast to the on-shell scheme, the running masses for the quarks at a given energy, $m(\mu)$, is well-defined. The running mass is not really a physical mass but reflects the contribution of the mass to a given process. It is worth mentioning that the renormalization point is not itself a physical quantity. The prediction of the theory, calculated to all orders in perturbation, should in principle be independent of the renormalization point. However, in practice, the infinite summation is truncated at a given order. When computing a physical process, one chooses thus a renormalization point close to the typical energies and momenta involved in the interaction.

1.2.2 Mass Issues

Defining a word or a concept does not means it is totally under control. At this stage, we still have no clue about its origin. Moreover, in contrast with the electric charge which takes only quantized values (± 1, $\pm\frac{2}{3}$, $\pm\frac{1}{3}$ and 0), the mass spectrum of the elementary particles does not show any obvious pattern. One should also be puzzled by the huge mass scale between the lightest fermions (the neutrinos, lighter [4] than 1 eV) and the heaviest one (the top quark, around 200 GeV). The same mass ratio ($\sim 10^{11}$) is found when comparing a mosquito and a blue whale. One can also wonder how the top quark is so heavy while it is as point-like as the electron in the Standard Model for electroweak and strong interactions. It has indeed the same mass as one molecule of vitamin C ($C_6H_8O_6$), an antioxidant found in abundance in potato [8]. If a mechanism for generating masses has to be found, it should explained this amazing hierarchy.

Another open question concerns the so-called missing mass problem. In spiral galaxies, if one assumes that the mass distribution is proportional to the luminosity, most of the mass is concentrated in the central bulge. In the Newtonian approximation, the orbital velocity of the stars in the spiral arms is then given by

$$\frac{GM}{r^2} = \frac{v^2}{r} \tag{1.10}$$

where r is the distance from the centre of the galaxy. This velocity should therefore vary in $r^{-1/2}$. However, one observes an almost constant velocity over a long range of distances. So, the galactic rotation curve cannot be explained by the visible matter alone. This result suggests that most of the galaxy mass is contained in a relatively dark spherical halo. In other words, a big part of the mass seems to be invisible.

A missing mass problem is also present in the cosmic microwave background radiation (CMB) [7]. When the universe was much younger and hotter, the electrons, protons and photons were highly interacting through Thomson scattering. As the universe expanded, this plasma cooled down until it allowed the formation of neutral hydrogen. At this moment, the universe was roughly 400 000 year old. The photons

[4] The stringent constraint comes from a model dependent analysis of the Cosmic Microwave Background [7]: $\sum m_\nu < 0.61$ eV at 95%CL

scattered then off the newly formed neutral atoms and began to travel freely. Because of the fluctuation of matter at the recombination, the last scattering surface was not completely homogeneous. The CMB is really a picture of the early universe since the fluctuations depend on its energetic content. Within the context of general relativity, the size and distribution of the tiny observed variations allow us to conclude that the baryonic matter, i.e. protons and neutrons, represents less than 5% of the total energy content of the universe in striking agreement with our understanding of the Big Bang nucleosynthesis. The 95% remaining are totally unknown. The only thing we know about them is that 23% consist of massive but non-baryonic matter, i.e. matter which interacts only gravitationally with the "normal" matter. This dark matter may explain the missing mass in galaxies. The remainder, 72%, is simply called dark energy since it has no mass and behaves as a diffuse substance.

1.3 An Artifact

The International System of units (SI) has defined seven base units [5]. Formally, four of them are related to metrology. The mole, ampere and candela are indeed connected to a counting of "elementary entities", electrons or photons, respectively; while the kelvin is an average on a set of particles. On the contrary, the three remaining units, length, time and mass, are the cornerstones of physics with space, time and matter.

The official unit of mass, the kilogram, has a particular feature which deserves some interest: the kilogram is the only unit which is still based on an artifact. Its official definition dated from 1889 [9].

> *The kilogram is equal to the mass of the international prototype of the kilogram kept at the Bureau International*

[5] The seven base units are:

metre	m	length
kilogram	kg	mass
second	s	time
ampere	A	electric current
kelvin	K	thermodynamic temperature
mole	mol	amount of substance
candela	cd	luminous intensity

> *des Poids et Mesures (BIPM). It follows that the mass of the international prototype of the kilogram is always 1 kilogram exactly [. . .]*

However

> *[. . .] due to the inevitable accumulation of contaminants on surfaces, the international prototype is subject to reversible surface contamination that approaches 1 μg per year in mass. For this reason, the BIPM declared that, pending further research, the reference mass of the international prototype is that immediately after cleaning and washing by a specified method.*

The first official definition of the gram can be found in a French decree of 1795 stating that the gram is equal to *the absolute weight of a volume of water equal to the cube of the hundredth part of the meter, at the temperature of melting ice* [3]. Four years later, an all-platinum kilogram prototype was manufactured with the objective that it would equal, as close as was scientifically feasible for the day, the mass of a cubic decimetre of water. Ninety years later, this prototype was replaced by the current artifact. Six sister copies were made at the same time and held in the same place. Official copies are also made available around the world and are periodically checked with the international standard.

The current definition of the kilogram does not allow it to be reproducible. By definition, the error in the measured value of the original prototype's mass is exactly zero. However, any changes in the prototype's mass over time can be deduced by comparing its mass to that of its official copies stored throughout the world. These verifications have demonstrated the stability of the artifact although a slow but inexorable divergence between the copies has also been observed.

By contrast, the other units are directly or indirectly connected to fundamental constants of nature. Their definitions allow the reproducibility of their realizations, provided you have the suitable technology. For instance, the unit of length, the meter, has been defined in 1983 as

> *the length of the path travelled by light in vacuum during a time interval of* 1/299 792 458 *of a second.*

This definition fixes the value for the speed of light but also relies on the definition of the second, this latter being

> *the duration of* 9 192 631 770 *periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium* 133 *atom.*

The meter is therefore directly or indirectly related to two important constants of Nature: the speed of light c and the Planck constant h. While the theoretical uncertainty associated with these definitions is zero, the practical accuracy of their realization will be always limited by the current technology.

Efforts are being done in order to free oneself of the kilogram prototype. The most promising techniques are based either on atom-counting approach or on electronic approach. In the former case, the Avogadro number would be fixed and the definition would refer to the amount of matter. Whereas in the latter case, depending on the technique, the gravitational constant or the electron charge would be fixed and the definition would refer to the gravitational or inertial definition of the mass, respectively. None of these techniques have been able to reach the level of precision achieved by the prototype which is of the order [6] of 20 μg with respect to 1 kg, i.e. ~ 2 parts per 10^8 [9].

Nowadays, the most precise measured mass of a fundamental particle is the electron. Its mass is actually measured with respect to the mass of a nucleon [10]. The most precise value is then given in units of atomic mass u. A conversion factor is used to recover kilograms [7]

$$m_e = 9.10938215(45)\ 10^{-31}\ \text{kg} \tag{1.11}$$

or

$$\frac{\Delta m_e}{m_e} \approx 5\ 10^{-8} \quad . \tag{1.12}$$

[6] This is actually the mean drift observed between the official copies and the original prototype since their last calibration. The relative error achieved during the calibration itself is of the order of 10^{-9}.

[7] The two digits between the parentheses denote the uncertainty at 1σ standard deviation in the two least significant digits.

The current measure has already reached the precision of the kilogram realization! The next most precise mass is the muon. Its mass is actually measured in the muonium, a μ^+e^- bound state, via a function of the masses' ratio

$$R_{e\mu} \equiv \frac{m_e}{m_\mu} = 4.83633171(12)\ 10^{-3} \tag{1.13}$$

or

$$\frac{\Delta R_{e\mu}}{R_{e\mu}} \approx 2.5\ 10^{-8} \quad . \tag{1.14}$$

which has a better relative uncertainty than the electron mass itself. Thus the error in the muon mass comes mainly from the electron mass:

$$m_\mu = 1.88353130(11)\ 10^{-28}\ \mathrm{kg} \quad . \tag{1.15}$$

Obviously, the absolute mass of these particles should not become more precise as long as the realization of the kilogram is not first improved. This is of course not true for mass ratios.

1.4 Fermion Masses

The quantum field theories [8] for fundamental interactions are based in the Lagrangian formalism. We need thus to explain what a fermion mass means in this formalism [11] since we are mainly interested in quarks and leptons throughout the next chapters.

For a free field, the only constraint for a mass term is the Lorentz invariance. The Lorentz group, $SO(3,1)$, is locally isomorphic to $SU(2) \otimes SU(2)$ whose representations are labelled by their spin. The simplest one, $(0,0)$, corresponds to a scalar field. The simplest non-trivial representation [9], $(1/2,0)$, corresponds to the so-called Weyl spinor χ of spin 1/2.

It is already possible to form a Lorentz invariant mass term from only a single Weyl spinor. This is called a Majorana mass term:

$$\mathcal{L} = \tfrac{1}{2}m(\chi^T \epsilon \chi + \mathrm{H.c.}) \tag{1.16}$$

[8] From now on, we shall use the natural units: $c = \hbar = 1$.

[9] $(1/2,0)$ and $(0,1/2)$ have both a spin 1/2 but have opposite chirality (cf. (1.30)).

where $\epsilon = i\tau_2$ is the 2×2 antisymmetric matrix and the τ_i's are the standard Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.17}$$

As it is constructed from a single Weyl spinor, this is the simplest realization of a fermion mass term. However, (1.16) is not invariant under a $U(1)$ symmetry. As a result, a Majorana mass is not allowed for fermions carrying a conserved charge, i.e. electric or colour charge as well as lepton or baryon number. More generally, if χ transforms under a complex representation of a symmetry, a Majorana mass is forbidden. Indeed, let us consider

$$\chi \rightsquigarrow U\chi \tag{1.18}$$

where U is a unitary transformation acting on a set of Weyl spinors. The mass term (1.16) transforms then as

$$\chi^T \epsilon \chi \rightsquigarrow \chi^T U^T \epsilon U \chi = \chi^T \epsilon U^T U \chi \quad . \tag{1.19}$$

In the last step we use the fact that U and ϵ act on different spaces. The term is invariant only if $U^T U = 1$, which is true only if the unitary transformation is real.

If a Weyl fermion transforms under a complex representation of an unbroken symmetry, then we need to introduce a second Weyl fermion that transforms under the complex-conjugate representation in order to construct a Lorentz invariant mass term. This is called a Dirac mass. Let χ and ξ transform under the (1/2,0) representation of the Lorentz group, and transform under some unitary symmetry as

$$\chi \rightsquigarrow U\chi \tag{1.20}$$

$$\xi \rightsquigarrow U^* \xi \quad . \tag{1.21}$$

Then, a Lorentz-invariant mass term which respects the symmetry may be formed

$$\mathcal{L} = m(\xi^T \epsilon \chi + \text{H.c.}) \tag{1.22}$$

since

$$\xi^T \epsilon \chi \rightsquigarrow \xi^T U^\dagger \epsilon U \chi = \xi^T \epsilon \chi \quad . \tag{1.23}$$

Accordingly, two Weyl spinors are required to construct a Dirac mass. We can then introduce a new object, a Dirac spinor, which is a four-component object constructed from a pair of $(1/2, 0)$ Weyl spinors χ, ξ:

$$\psi = \begin{pmatrix} \chi \\ \epsilon\xi^* \end{pmatrix} \quad . \tag{1.24}$$

In terms of a Dirac spinor, a Dirac mass is written in the familiar form

$$\mathcal{L} = -m\overline{\psi}\psi = -m \begin{pmatrix} \chi^\dagger & -\xi^T\epsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \epsilon\xi^* \end{pmatrix} \tag{1.25}$$

$$= m \left(\xi^T \epsilon\chi - \chi^\dagger \epsilon\xi^* \right) \tag{1.26}$$

where $\overline{\psi} = \psi^\dagger\gamma_0$ and $\epsilon^\dagger = -\epsilon$. The final expression is identical with (1.22). Note also that we are using a specific basis for the gamma matrices, the so-called Weyl or chiral basis:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix} \tag{1.27}$$

where each entry in the above matrices is itself a 2×2 matrix. We can also define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{1.28}$$

In this basis, the chiral projection operators $(1 \pm \gamma_5)/2$ project out the Weyl spinors,

$$\psi = \tfrac{1-\gamma_5}{2}\psi + \tfrac{1+\gamma_5}{2}\psi = \psi_L + \psi_R \tag{1.29}$$

where

$$\psi_L = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \epsilon\xi^* \end{pmatrix} \quad . \tag{1.30}$$

Thus a Dirac spinor transforms as the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group.

While a Dirac spinor is composed of two Weyl spinors, a Majorana spinor is a four-component object composed of a single Weyl spinor

$$\psi_M = \begin{pmatrix} \chi \\ \epsilon\chi^* \end{pmatrix} \tag{1.31}$$

and

$$\mathcal{L} = -\tfrac{1}{2} m \overline{\psi}_M \psi_M \tag{1.32}$$

is a Majorana mass term.

The charge-conjugation matrix C is suited to write fermion masses. In the Weyl or chiral representation of the Dirac matrices it reads

$$C \equiv i\gamma^2\gamma^0 = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \quad . \tag{1.33}$$

Given a Dirac spinor ψ, we can form the conjugate spinor via

$$\psi^c = C\gamma^0\psi^* = \begin{pmatrix} \xi \\ \epsilon\chi^* \end{pmatrix} \quad . \tag{1.34}$$

This definition implies

$$\psi_M^c = \psi_M \tag{1.35}$$

i.e. a fermion with a Majorana mass is its own antiparticle. It is easy to check that

$$\mathcal{L} = -\tfrac{1}{2} m \left(\overline{\psi_R^c}\ \psi_L + \text{H.c.} \right) \tag{1.36}$$

$$\mathcal{L} = -\tfrac{1}{2} m \left(\psi_L^T\, C\ \psi_L + \text{H.c.} \right) \tag{1.37}$$

are also Majorana masses. These expressions for the Majorana masses in terms of Dirac spinors are commonly used when working with neutrinos.

1.5 Standard Model and Symmetries

The cornerstone of the Standard Model is the gauge invariance principle. Three of the four known interactions are accurately described by a $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ local symmetry. Yet, a number of global symmetries appear, they are not imposed by construction but arise naturally in the theory. We have chosen here to build the theory step by step in order to highlight these accidental symmetries which play an important role for the fermion masses and mixings.

1.5.1 Free Field

The first step consists in a massless free field. The renormalizable Lagrangian which fulfils the Dirac equation of motion is then:

$$\mathcal{L} = i\overline{\psi}\,\partial\!\!\!/\,\psi = i\overline{\psi_L}\,\partial\!\!\!/\,\psi_L + i\overline{\psi_R}\,\partial\!\!\!/\,\psi_R \quad . \tag{1.38}$$

At this stage, we did not impose any symmetry but this Lagrangian is accidentally invariant under

$$\begin{aligned} U(1)_V &: \quad \psi \rightsquigarrow e^{i\theta}\psi \\ U(1)_A &: \quad \psi \rightsquigarrow e^{i\gamma_5\theta}\psi \end{aligned}$$

which is isomorphic to

$$\begin{aligned} U(1)_{V-A} \equiv U(1)_L &: \quad \psi_L \rightsquigarrow e^{i\theta}\psi_L \\ U(1)_{V+A} \equiv U(1)_R &: \quad \psi_R \rightsquigarrow e^{i\theta}\psi_R \quad . \end{aligned}$$

This simple description should be extended to the case of n massless particles. The new Lagrangian is simply the addition of n times (1.38). It can be equivalently written in a much concise form where the new accidental symmetries will be more apparent. The Lagrangian (1.38) keeps the same but the operator $\partial\!\!\!/$ should now be read as a diagonal operator and ψ as a vector field with n components.

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \quad . \tag{1.39}$$

Clearly, the Lagrangian is now invariant under $U(n)_L$ and $U(n)_R$. These symmetries are isomorphic to $SU(n)_L \otimes U(1)_L$ and $SU(n)_R \otimes U(1)_R$, respectively. A mixing between the n fields is then allowed but would have no physical consequence since they all have a zero mass and no charge in order to tell them apart.

In the Standard Model, the number of generations is not fixed by any symmetry. However, there is strong evidence in favour of only three generations; notably (see [12] and references therein) the three observed charged leptons, the observation of six quarks (albeit not free), the invisible width of the Z associated to three light neutrinos ($m_\nu < \frac{m_Z}{2}$), ... From now on, otherwise stated, we will only consider the case of four families of fermions with three generations (see eq. (1.42)).

1.5.2 Interacting Fields

The second step is to switch on the gauge interactions. For simplicity, we first focus on QED.

QED

The free Lagrangian (1.38) is manifestly not invariant under the $U(1)_Q$ local gauge symmetry. To achieve this goal, it requires two modifications. First, the normal derivative should be replaced by a covariant one:

$$\partial_\mu \rightsquigarrow D_\mu = \partial_\mu + ieQA_\mu \tag{1.40}$$

where e is the coupling constant of the gauge symmetry and Q is the charge of the field. The second modification consists in a kinetic term for the gauge boson A_μ: $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{1.41}$$

We need now to describe the fermion content of the theory. Let ψ be a vector containing the fields with the same electric charge:

$$\psi_\nu = \begin{pmatrix} \psi_{\nu_1} \\ \psi_{\nu_2} \\ \psi_{\nu_3} \end{pmatrix},\ \psi_l = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix},\ \psi_U = \begin{pmatrix} \psi_u \\ \psi_c \\ \psi_t \end{pmatrix},\ \psi_D = \begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix} \quad . \tag{1.42}$$

The QED Lagrangian then reads

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{D}\psi = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi_L}\not{D}\psi_l + i\overline{\psi_R}\not{D}\psi_R \tag{1.43}$$

where a sum over the various families is understood.

The particle content consisting of four families with different charges, the global symmetries are now

$$[U(3)_L]^4 = [SU(3)_L \otimes U(1)_L]^4 \tag{1.44}$$

$$[U(3)_R]^4 = [SU(3)_R \otimes U(1)_R]^4 \tag{1.45}$$

The particles being massless, they are again indistinguishable and are allow to mix, at least between particles of the same charge, without physical consequence.

QCD

When we switch on the $SU(3)_c$ gauge symmetry (QCD), the fields representing the quarks should be extended to include the three colours,

$$\psi_u = \begin{pmatrix} \psi_u^R \\ \psi_u^G \\ \psi_u^B \end{pmatrix}, \ldots \tag{1.46}$$

The same ingredients as in QED are used to make the Lagrangian invariant under this new gauge symmetry. The covariant derivative becomes

$$\partial_\mu \rightsquigarrow D_\mu = \partial_\mu + ieQA_\mu + ig_S G_\mu^a T^a \tag{1.47}$$

where g_S is the coupling constant associated with strong interactions and T^a are the eight generators of $SU(3)$. This modification requires the introduction of eight new gauge bosons, the gluons. Their kinetic term is $-\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu}$, where

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_S f^{abc} G_\mu^b G_\nu^c \quad . \tag{1.48}$$

The structure constants f^{abc} are completely determined by the symmetry group:

$$\left[T^a, T^b\right] = i f^{abc} T^c \quad . \tag{1.49}$$

As a result, in a non-abelian symmetry, the gauge bosons are self-interacting and present trilinear and quadrilinear couplings. In other words, they have a non-vanishing charge associated to the gauge symmetry.

Obviously these modifications will keep intact the global symmetries that were present with QED only. However, the complex vacuum structure of QCD calls for some precautions. Indeed, as explained in App. B, one of the $U(1)_A$ symmetries is explicitly broken by non-perturbative effects of QCD.

This approximation of massless fields has been extensively used in QCD for the light quarks u, d and s which, at first order, are massless. The natural scale for the strong interactions is indeed of $\mathcal{O}(1 \text{ GeV})$. In this context, the accidental global symmetry is called chiral symmetry and corresponds to $SU(3)_L \otimes SU(3)_R \otimes U(1)_V$. This approximate symmetry is spontaneously broken by quark condensates

$$\langle s\bar{s}\rangle = \langle d\bar{d}\rangle = \langle u\bar{u}\rangle \neq 0 \tag{1.50}$$

leaving only the vectorial sub-group $SU(3)_V \otimes U(1)_V$ unbroken. Noether theorem implies that a charge should be conserved for each unbroken symmetry. The conservation of the baryon number is associated to $U(1)_V$ while $SU(3)_V$ is the Gell-Mann $SU(3)$ of flavour.

Because of the spontaneous breaking of the chiral symmetry, eight [10] pseudoscalar Goldstone bosons should appear, in this case: the three pions $\pi^{\pm}$, π^0, the four kaons $K^{\pm}, K^0, \overline{K^0}$ and the [11] η. However, since the u, d and s quarks have different masses, the chiral symmetry is only approximate. The mass of the Goldstone bosons is therefore not exactly zero but nevertheless is much lower than $\mathcal{O}(1\ \text{GeV})$.

1.5.3 Massive Particles

So far, all the particles were assumed massless. Yet, the $SU(3)_c \otimes U(1)_Q$ gauge symmetry does not forbid a mass term for the fermions. The vanishing mass of the gauge bosons is however protected by the gauge symmetry.

The special case with the $SU(2)_L$ weak interactions switched on will be later detailed. For now, nothing prevents us from adding to the Lagrangian an explicit mass term for the fermions.

$$\mathcal{L} = i\overline{\psi}\, \not{D}\, \psi - \overline{\psi}\, M\, \psi \quad . \tag{1.51}$$

Global Symmetries

This mass term breaks most of the accidental symmetries that were present in the Lagrangian. In particular, if a mass is different from zero, the corresponding axial symmetry is broken.

In the most general case where all the masses are different, the fields are unambiguously defined and (1.51) is no more invariant under $SU(3)_L$ and $SU(3)_R$. For each family, the remaining unbroken symmetries are $[U(1)_V]^3$ corresponding to the individual vectorial symmetries. This result will be slightly modified once the weak interactions will be switched on.

[10] One for each broken generator of the chiral symmetry.

[11] To be precise, the η and the η' share the same quantum numbers and therefore mix.

In the particular case of massless neutrinos, the $[U(1)_V]^3$ symmetries coming from the charged leptons sector can be associated to the conservation of the individual lepton numbers: L_e, L_μ and L_τ. It is readily seen that the Lagrangian is also invariant under a global rephasing of the field ψ. However, this latter symmetry is only a particular case of the former ones and corresponds to the conservation of the global lepton number: $L = L_e + L_\mu + L_\tau$. With massive Dirac neutrinos[12] or in the quark sector, this picture will be modified by the weak interactions, keeping only the global lepton and baryon numbers unbroken.

Mass Matrix

The mass M in (1.51) is not necessarily diagonal but could be any three-by-three complex matrix. In the most general case, the Hermitian conjugate of the mass term should then be added to the Lagrangian in order to keep it self-adjoint.

The mass matrix can be easily diagonalized thanks to the accidental symmetries. The interaction term in the Lagrangian is indeed invariant under $[U(3)_L \otimes U(3)_R]^4$ which allows the bi-diagonalization of the four mass matrices (one for each family).

However, one $U(1)_A$ of the quark symmetries is broken by the vacuum structure of QCD (Cf. App. B). Therefore, the remaining symmetries $(SU(3)_L \otimes SU(3)_R \otimes U(1)_V)$ do not ensure that the resulting physical masses will be real and positive.

1.5.4 Neutrino Digression

In the Standard Model, the neutrinos are only left-handed and massless. Yet, there is now convincing evidence that the neutrinos oscillate and are thus massive, albeit very light. This observation calls for an extension of the Standard Model. Here are some of them.

Dirac Right-handed Neutrinos The most immediate scenario is to add three right-handed neutrinos and to write a Dirac mass (1.51) for

[12] A Majorana mass is not invariant under $U(1)$ and would break the lepton number.

the neutrinos. All the fermions are then on an equal footing. This is the case we have considered so far although this is not the simplest one. Moreover, this extension assumes that the same mechanism is at work for all fermion families. The huge mass scale between neutrinos and the other fermions needs then to be explained.

Majorana Left-handed Neutrinos The neutrinos are neutral with respect to QED and QCD. Their simplest mass term is thus a Majorana mass. Keeping only the left-handed neutrinos of the Standard Model, we are allowed to give them a Majorana mass:

$$\mathcal{L} \ni -\tfrac{1}{2}\left(\overline{\nu_L^c}\, M_L\, \nu_L + \overline{\nu_L}\, M_L\, \nu_L^c\right) \quad . \tag{1.52}$$

However, in the Standard Model, the left-handed neutrino and electron transform as a doublet of $SU(2)_L$ and their weak-isospin charge is different from zero. Such a mass term would then be an explicit violation of the gauge symmetry. This issue might be solved via a Higgs mechanism (see Sec. 1.6) involving, for instance, a triplet [13] of $SU(2)_L$ [13]. In particular, the extension with the usual Higgs scalar doublet and a scalar triplet is sometimes referred as Type II seesaw. In that case, the resulting masses for the neutrinos are inversely proportional to the mass of the triplet, allowing for naturally small masses [14]. The existence of a scalar triplet is however strongly constrained by the ρ parameter (cf. Sec. 1.6.1).

Majorana Right-handed Neutrinos Compared with the other families, it seems natural to consider the existence of right-handed neutrinos. In the Standard Model, the right-handed fields transform as a singlet of $SU(2)_L$. Their charge being exactly zero for all gauge symmetries, an explicit Majorana mass term is allowed:

$$\mathcal{L} \ni -\tfrac{1}{2}\left(\overline{\nu_R^c}\, M_R\, \nu_R + \overline{\nu_R}\, M_R\, \nu_R^c\right) \quad . \tag{1.53}$$

Such a mass term violates the lepton number. Since this number was associated to an accidental symmetry this is not really an issue. However, if for some reason we would like to preserve this global symmetry, this term must be disregarded.

[13] Charge conservation rules out the possibility of a singlet.

Type I seesaw With right-handed neutrinos in the theory, there is no reason to privilege either a Dirac or a Majorana mass. Both are indeed allowed by the gauge symmetries. In the general case [15], the neutrino mass matrix is effectively described by a six-by-six matrix $\mathcal{M}$:

$$\begin{aligned}
\mathcal{L} \ni & -\overline{\nu_L}\ m\ \nu_R - \tfrac{1}{2}\overline{\nu_R^c}\ M_R\ \nu_R + \text{H.c.} \\
&= -\overline{\nu_L}\ m\ \nu_R - \overline{\nu_R}\ m\ \nu_L - \tfrac{1}{2}\overline{\nu_R^c}\ M_R\ \nu_R - \tfrac{1}{2}\overline{\nu_R}\ M_R\ \nu_R^c \\
&= -\tfrac{1}{2}\begin{pmatrix}\overline{\nu_L} & \overline{\nu_R^c}\end{pmatrix}\begin{pmatrix}0 & m \\ m^T & M_R\end{pmatrix}\begin{pmatrix}\nu_L^c \\ \nu_R\end{pmatrix} + \text{H.c.} \\
&= -\tfrac{1}{2}\overline{n^c}\ \mathcal{M}\ n + \text{H.c.}
\end{aligned} \tag{1.54}$$

since $\overline{\nu_R^c} m^T \nu_L^c = \overline{\nu_R} m \nu_L$. In (1.54), we have defined

$$n = \begin{pmatrix}\nu_L^c \\ \nu_R\end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix}0 & m \\ m^T & M_R\end{pmatrix} \quad . \tag{1.55}$$

With such a rewriting, the resulting fields n appears to have a Majorana mass.

It seems natural to assume that the neutrino Dirac mass m has the same order of magnitude as the other fermions. On the contrary, the scale associated with the Majorana mass M_R is completely free and could be very high. Within this assumption ($m \ll M_R$), the diagonalization of $\mathcal{M}$ leads naturally to very small neutrino masses:

$$U = \begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(mM_R^{-1}) \\ \mathcal{O}(mM_R^{-1}) & \mathcal{O}(1)\end{pmatrix} \tag{1.56}$$

$$U^T \mathcal{M} U = \begin{pmatrix}-m^T M_R^{-1} m & 0 \\ 0 & M_R\end{pmatrix} \quad . \tag{1.57}$$

Consequently, the light neutrinos are to a good approximation mixtures of the left-handed neutrinos only while the right-handed ones are extremely massive. A naive estimate for their mass yields a new scale:

$$m_\nu = m^2/M_R \sim v^2/M_R \sim 0.1 \text{ eV} \Rightarrow M_R \sim 10^{14} \text{ GeV} \tag{1.58}$$

which is close to the Grand Unification scale. It is worth mentioning that the heavy right-handed neutrinos can also play an important role in baryogenesis [16].

1.5.5 Weak Interactions

Let's come back to the construction of the Standard Model Lagrangian. For simplicity, we shall first assume that the electric charge is equivalent to the hypercharge: $Q = \frac{Y}{2}$. The inclusion of the weak interactions in the scheme used above is not straightforward. A first issue comes from the fermion mass terms which are not invariant under $SU(2)_L$. For the moment, we tolerate an explicit breaking of the symmetry and do not care with the problem of non-renormalizability. This problem can indeed be solved in various ways, most notably through the so-called Higgs mechanism (cf. Sec. 1.6). The second issue is related to the fact that the weak interactions connect particles which do not belong to the same field ψ as defined in (1.42).

So far, the gauge interactions were accidentally invariant under a global $[U(3)_L \otimes U(3)_R]^4$ symmetry, if we neglect the $U(1)_A$ problem. Therefore, there is no reason to assume that a $W^\pm$ would couple directly to ψ_{UL} and ψ_{DL} but it would rather couple to some transformed fields:

$$\begin{aligned} \psi'_{UL} &= U_L^U \psi_{UL} \\ \psi'_{DL} &= V_L^D \psi_{DL} \end{aligned} \tag{1.59}$$

where $U_L^{U,D}$ are unitary matrices.

The neutral currents are clearly invariant under such transformation while the charged currents require a suitable transformation of both fields involved in order to become diagonal. The relevant parts of the Lagrangian read indeed:

$$\begin{aligned} \mathcal{L} \ni &- \overline{\psi_U}\, igW_\mu^+ \gamma^\mu \tfrac{(1-\gamma_5)}{2}\, V\, \psi_D + \text{H.c.} \\ &- \overline{\psi_U}\, igZ_\mu \gamma^\mu \tfrac{(1-\gamma_5)}{2}\, \psi_U - \overline{\psi_D}\, igZ_\mu \gamma^\mu \tfrac{(1-\gamma_5)}{2}\, \psi_D \\ &- \overline{\psi_U}\, M_U\, \psi_U - \overline{\psi_D}\, M_D\, \psi_D \end{aligned} \tag{1.60}$$

where $V \equiv U_L^{U\dagger}\, U_L^D$ is, in general, not diagonal.

It should be noted that the transformations (1.59) are also used for the diagonalization of the mass matrices. A priori these new transformations are independent from the former ones. One should not expect that the same set of matrices U_L^U and U_L^D allows for both diagonal masses and diagonal weak currents. In the most general case, the weak charged currents have thus no reason to be diagonal and we observe a physical mixing between the states.

1.6 Higgs Mechanism

When considering weak interactions, the fermion mass terms create an explicit breaking of the gauge symmetry. Moreover, the masses for the weak gauge bosons are strictly forbidden by the local invariance. The Standard Model's solution to these issues is the Brout-Englert-Higgs mechanism [17, 18] which requires scalar fields.

1.6.1 Electroweak Symmetry Breaking

We should first relax the assumption about the hypercharge. The third generator of $SU(2)_L$ being diagonal, it is allow to mix with the generator of $U(1)_Y$. The electric charge arises as a linear combination of the charges associated to these two generators: $Q = T_3^L + \frac{Y}{2}$.

We would like to break the $SU(2)_L \otimes U(1)_Y$ group down to the $U(1)_Q$ subgroup while keeping $SU(3)_c$ unbroken. The Higgs field should thus be colourless but needs some non-zero weak isospin *and* hypercharge. In order to become massive, the three gauge bosons require three Goldstone bosons. In the simplest realization of the Higgs mechanism, the scalar field is thus a complex doublet of $SU(2)_L$. Since we want to keep electromagnetism unbroken, we chose a doublet of hypercharge $+1$:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{1.61}$$

while its $SU(2)_L$ conjugate doublet is

$$\tilde{\phi} \equiv i\tau_2 \phi^{\dagger^T} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \quad . \tag{1.62}$$

The Higgs part of the Lagrangian then reads

$$\mathcal{L}_{\text{Higgs}} = D_\mu \phi^\dagger D^\mu \phi - V(\phi) \tag{1.63}$$

where the last term represents the potential of the scalar field:

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 \quad . \tag{1.64}$$

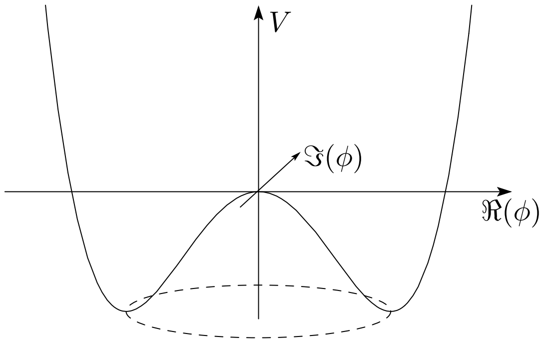


Figure 1.2: *Higgs Potential with a non-zero vacuum expectation value.*

The covariant derivative in (1.63) is given by

$$D_\mu = \partial_\mu + igW^i_\mu \frac{\tau_i}{2} + ig' B_\mu \tfrac{1}{2} \tag{1.65}$$

where g and g' are the coupling constants of $SU(2)_L$ and $U(1)_Y$, respectively, while the W's and the B are the associated gauge bosons.

The Higgs mechanism is based on the assumption that the vacuum expectation value of the scalar field breaks spontaneously the gauge symmetry. This is possible if both μ^2 and λ are real and positive. In that case, the potential takes a Mexican-hat shape (Fig. 1.2) whose minimum is reached for

$$\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \tag{1.66}$$

Thanks to the gauge invariance, the minimum of the potential can always be rotated in order to keep the electromagnetism unbroken:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.67}$$

where $v = \sqrt{\frac{\mu^2}{\lambda}}$. The fluctuations of the Higgs field around this minimum have this form:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1 + i\pi_2 \\ H + v + i\pi_3 \end{pmatrix} \quad . \tag{1.68}$$

The $\pi_{1,2,3}$ modes are the would-be Goldstone bosons "eaten" by the gauge bosons, while H will become a physical massive scalar, the Higgs boson. After insertion of (1.68) into the Lagrangian (1.63), the mass of the Higgs bosons is indeed readily found:

$$m_H = \frac{\mu^2}{2} \tag{1.69}$$

The gauge boson mass spectrum is determined by $|D_\mu \langle\phi\rangle|^2$ where

$$D_\mu \langle\phi\rangle = \frac{iv}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} g v W_\mu^+ \\ (g' B_\mu - g W_\mu^3) v \end{pmatrix} \tag{1.70}$$

with

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left[W_\mu^1 \mp W_\mu^2 \right] \quad . \tag{1.71}$$

It is readily seen that the $W_\mu^\pm$ bosons become massive:

$$m_W = \frac{1}{2} g v \tag{1.72}$$

while only the combination $(g' B_\mu - g W_\mu^3)$ get a mass in (1.70). The orthogonal combination $(g' B_\mu + g W_\mu^3)$ remains thus massless and is identified with the photon. Defining

$$\tan\theta_W = \frac{g'}{g} \tag{1.73}$$

the physical fields are then given by

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W_\mu^3 \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W_\mu^3 \end{aligned} \tag{1.74}$$

while their mass is

$$\begin{aligned} m_A &= 0 \\ m_Z &= \frac{1}{2} \frac{g v}{\cos\theta_W} = \frac{m_W}{\cos\theta_W} \end{aligned} \tag{1.75}$$

As simple analysis of the neutral electroweak currents gives back the QED coupling constant:

$$e = g \sin\theta_W = g' \cos\theta_W \tag{1.76}$$

where $\sin\theta_W \approx \frac{1}{2}$ [12]. The vacuum expectation value of the Higgs field can then be computed from (1.72): $v \approx 246$ GeV.

It is noteworthy that the ratio

$$\rho \equiv \frac{m_W}{m_Z \cos\theta_W} \tag{1.77}$$

is exactly one in the Standard Model [14]. This prediction is specific to the choice of a scalar doublet. Other representations do not lead naturally to $\rho \approx 1$. The accidental $SU(2)$ symmetry protecting this value is called custodial symmetry.

It is also worth mentioning that the physical Higgs boson is required for the mathematical consistency of the theory but has not yet been observed. Direct searches at LEP have excluded a Higgs lighter than 114.4 GeV [19]. Its mass can also be bounded from above. For instance, the unitarity of the theory would be broken by the $W-W$ scattering at around 1 TeV [20]. The occurrence of a scalar particle would cure this issue. A more stringent indirect constraint comes from the radiative corrections to $\rho = 1$: $m_H < 182$ GeV [19]. The remaining admissible window will be thoroughly searched at LHC.

1.6.2 Yukawa Couplings

In the simplest realization of the Higgs mechanism, the scalar doublet is also responsible for the fermion masses. In this section we shall use the usual representations for the fermions: left-handed doublets and right-handed singlets of $SU(2)_L$ (see Tab. 1.1). In the Standard Model, the right-handed neutrinos are absent and the most general gauge invariant and renormalizable interaction between the fermions and the scalar field, called Yukawa interaction, reads:

$$\begin{aligned}
\mathcal{L}_Y &= -\left(\overline{Q_L}\, Y^n \phi\, n_R + \overline{Q_L}\, Y^p \tilde{\phi} p_R + \overline{L_L}\, Y^l \phi l_R\right) + \text{H.c.} && (1.78)\\
&= -\left[\begin{pmatrix}\overline{p_L} & \overline{n_L}\end{pmatrix} Y^n \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} n_R + \begin{pmatrix}\overline{p_L} & \overline{n_L}\end{pmatrix} Y^p \begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix} p_R\right.\\
&\quad \left. + \begin{pmatrix}\overline{v_L} & \overline{l_L}\end{pmatrix} Y^l \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} l_R\right] + \text{H.c.} && (1.79)
\end{aligned}$$

where the Yukawa couplings, $Y^{p,n,l}$, are arbitrary complex three-by-three matrices.

[14] At least at tree level. One-loop radiative corrections are however very small. The biggest contributions come from the top quark ($\sim m_t^2$) and the scalar sector ($\sim \log m_H^2$).

	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$Q^i_L = \begin{pmatrix} p^i_L \\ n^i_L \end{pmatrix}$	3	2	$\frac{1}{3}$
p^i_R	3	1	$\frac{4}{3}$
n^i_R	3	1	$-\frac{2}{3}$
$L^i_L = \begin{pmatrix} v^i_L \\ l^i_L \end{pmatrix}$	1	2	-1
v^i_R	1	1	0
l^i_R	1	1	-2

Table 1.1: *The fermion content of the standard model and their gauge quantum numbers ($Q = T_3 + \frac{Y}{2}$). We use different names to avoid confusion with the mass eigenstates. p^i and n^i are the up- and down-like quarks, respectively; while v^i and l^i correspond to the neutrinos and charged leptons, respectively.*

After the spontaneous symmetry breaking, the Higgs field is replaced by its vacuum expectation value (1.67) leading to fermion mass terms:

$$\mathcal{L}_Y = -\left[\overline{n_L} M^n n_R + \overline{p_L} M^p p_R + \overline{l_L} M^l l_R\right] + \text{H.c.} \qquad (1.80)$$

where the mass matrices are proportional to the Yukawa couplings:

$$M^x = Y^x \, \frac{v}{\sqrt{2}} \qquad (x = n, p, l). \qquad (1.81)$$

Their diagonalization will be detailed in the next chapter.

Since the vacuum expectation is $v \approx 246$ GeV, a naive estimate ($Y = \mathcal{O}(1)$) of the order of magnitude for the fermion masses yields 174 GeV. If the top quark is right at the expected mass, all others fermions are much lighter. The Yukawa couplings require thus some fine-tuning in order to reproduce the fermion mass spectrum.

1.7 Grand Unification Theories

The running of the three gauge coupling constants displays a convergence just below the Planck scale. This unexpected behaviour suggests that the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry of the Standard Model could originate from a bigger local symmetry unifying all interactions. The minimal anomaly free Lie group which contains the Standard Model as a subgroup is $SU(5)$ [21], whereas $SO(10)$ is the smallest group with right-handed neutrinos on an equal footing.

The unification of the gauge couplings at high energy effectively reduces the number of free parameters. For instance in $SU(5)$ at, or above, the unification scale ($\sim 10^{15}$ GeV), the weak angle, defined in (1.73), is readily found to be

$$\sin^2\theta_W = \tfrac{3}{8}\ . \tag{1.82}$$

At lower energy, this angle changes along with the running of the coupling constants and becomes quite close to its observed value at the Fermi scale.

In theories of Grand Unification, quarks and leptons are gathered together in irreducible representations. In $SU(5)$, the τ and the b quarks must have the same mass at the unification scale [22]. At lower energy, they evolve differently and their ratio become comparable to the experimental value. The same kind of relation can also be obtained for the lighter generations, but with less success [23, 24].

One of the main drawback of grand unification theories is the introduction of new gauge bosons which, in general, break the lepton and baryon numbers and therefore allow the proton to decay. The current measurement settles firm constraints on these models and has ruled out many of them, including the original $SU(5)$. Another drawback, aesthetic this time, is related to the breaking of the Grand Unification group. Usually, this breaking requires several Higgs representations which imply a rather complex scalar sector with many physical scalar particles.

2

A Peculiar Mixing

The occurrence of mass matrices in the Lagrangian is intimately connected to flavour mixing in the charged weak currents. Within the Higgs mechanism, they both have the same origin: the Yukawa couplings. In this chapter, we proceed with their diagonalization and explore the properties and representations of the mixing matrix. It is known that nine different parametrizations exist. We show that one of them has some unique features. A simple texture for the mass matrices is presented. Their diagonalization allows us to compute quite accurately the quark mixing angles as function of mass ratios.

2.1 Diagonalization

The Yukawa couplings, and thus the mass matrices, are in general arbitrary complex three-by-three matrix and their diagonalization requires a bi-unitary transformation. In the quark sector, one has

$$\begin{aligned} p_L &= U_L^p u_L \\ p_R &= U_R^p u_R \\ n_L &= U_L^n d_L \\ n_R &= U_R^n d_R \end{aligned} \tag{2.1}$$

where $u_{L,R}$ and $d_{L,R}$ represent the chiral components of the quarks mass eigenstates, i.e. the physical quarks. The three-by-three unitary matrices U_L^p and U_R^p are chosen in order to bi-diagonalize M_p, while U_L^n and U_R^n

bi-diagonalize M_n.

$$\begin{aligned} U_L^{p\dagger} M_p U_R^p &= \mathrm{diag}(m_u, m_c, m_t) \equiv M_u \\ U_L^{n\dagger} M_n U_R^n &= \mathrm{diag}(m_d, m_s, m_b) \equiv M_d \quad . \end{aligned} \tag{2.2}$$

The matrices M_u and M_d are, by definition, diagonal and their elements are real and positive.

The charged currents, which were diagonal in terms of the electroweak eigenstates, may now be written in terms of the mass eigenstates:

$$\mathcal{J}_W \sim \overline{p_L}\gamma^\mu n_L = \overline{u_L} U_L^{p\dagger} \gamma^\mu U_L^n d_L = \overline{u_L}\gamma^\mu V_{\mathrm{ckm}} d_L \quad . \tag{2.3}$$

In general, they are no more diagonal and the mixing matrix V_{ckm} — named after Cabibbo [25], Kobayashi and Maskawa [26] — is defined as

$$V_{\mathrm{ckm}} \equiv U_L^{p\dagger} U_L^n \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad . \tag{2.4}$$

On the contrary the neutral currents remain diagonal:

$$\mathcal{J}_Z \sim \overline{p_L}\gamma^\mu p_L = \overline{u_L} U_L^{p\dagger} \gamma^\mu U_L^p u_L = \overline{u_L}\gamma^\mu u_L \quad . \tag{2.5}$$

This important feature is possible thanks to the unitarity of $U_L^{p,n}$, ensuring the non-existence of Flavour Changing Neutrial Current (FCNC) at tree-level.

The same reasoning is valid for the lepton sector. M_l is bi-diagonalized thanks to unitary transformations of the fields. However, if we assume massless neutrinos, it is always possible to find a neutrino transformation in order to cancel the mixing matrix of the lepton sector:

$$\begin{aligned} v_L &= U_L^l \nu_L \\ l_L &= U_L^l e_L \\ l_R &= U_R^l e_R \end{aligned} \tag{2.6}$$

where e and ν represent lepton mass eigenstates. The same transformation U_L^l has been used for l_L and v_L, cancelling thus any flavour mixing. Furthermore, the unitary matrices U_L^l and U_R^l are chosen such that

$$U_L^{l\,\dagger} M_l U_R^l = \mathrm{diag}(m_e, m_\mu, m_\tau) \equiv M_e \quad . \tag{2.7}$$

In the lepton sector of the Standard Model, the electroweak currents are diagonal in terms of both the electroweak and mass eigenstates. However, we know that neutrinos can oscillate and are therefore massive, albeit very light. The usual extensions, presented in Sec. 1.5.4, can be summarized in two categories: Dirac or Majorana neutrinos.

Dirac Neutrinos

In this extension right-handed neutrinos are present in the theory and lepton number is conserved. All families are on a equal footing and the procedure is the same as for the quarks. The two mass matrices are bi-diagonalized thanks to unitary transformations of the fields:

$$\begin{aligned} v_L &= U_L^v \nu_L \\ v_R &= U_R^v \nu_R \\ l_L &= U_L^l e_L \\ l_R &= U_R^l e_R \end{aligned} \tag{2.8}$$

where $U_{L,R}^{v,l}$ are chosen in order to have

$$\begin{aligned} {U_L^v}^\dagger M_v U_R^v &= \mathrm{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}) \equiv M_\nu \\ {U_L^l}^\dagger M_l U_R^l &= \mathrm{diag}(m_e, m_\mu, m_\tau) \equiv M_e \quad . \end{aligned} \tag{2.9}$$

Exactly as for the quarks, a mixing matrix — named after Pontecorvo [27], Maki, Nakagawa and Sakata [28] — appears in the charged currents:

$$V_{\text{pmns}} \equiv U_L^{l\dagger} U_L^v \equiv \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \quad . \tag{2.10}$$

For historical reason, the definition of the lepton mixing matrix is different from the quark's one. It is indeed conventionally considered that the up-type quarks and the charged leptons are diagonal, i.e. $U_L^l = U_L^p = \mathbb{1}$, while the mixing occurs only for the down-type quarks and the neutrinos, i.e. $V_{\text{ckm}} = U_L^n$ and $V_{\text{pmns}} = U_L^v$. We emphasize here that this basis fixing choice is purely conventional and that the "real" way how the mixing is split among the species may only be known, if ever, from extensions to the Standard Model.

Majorana Neutrinos

We have already seen that they are different ways of realizing Majorana neutrinos. Let us first consider the case of a mass term involving only the right-handed neutrinos, i.e. with no Dirac mass nor seesaw:

$$\mathcal{L} \sim -\tfrac{1}{2}\overline{\nu^c_R}\, M_R\, \nu_R + \text{H.c.} \tag{2.11}$$

The diagonalization of M_R requires a unitary transformation of the right-handed neutrinos only. The left-handed neutrinos can be freely transformed in order to cancel the part of the mixing coming from the charged-leptons,

The observed mixing in the lepton sector requires thus a mass term involving the left-handed neutrinos. This condition is met either with left-handed neutrinos only or with a seesaw mechanism. In this latter case (see (1.54)), we have

$$\begin{aligned} v_L &= U^v_L \nu_L \\ v_R &= U^v_R \nu_R \end{aligned} \tag{2.12}$$

such that

$$\begin{aligned} U^{v\,\dagger}_R M_R U^v_R &= \text{diag}(m_{N_{R1}}, m_{N_{R2}}, m_{N_{R3}}) \equiv M_{N_R} \\ U^{v\,\dagger}_L \; (M^T_v M^{-1}_R M_v)\; U^v_L &= \text{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}) \equiv M_\nu \quad . \end{aligned} \tag{2.13}$$

The mixing matrix V_{pmns} is again given by (2.10). From now on, when speaking of Majorana neutrinos, the type I seesaw is understood.

2.2 Properties of Mixing Matrices

2.2.1 Parameters

From a mathematical point of view, a n_g by n_g complex mixing matrix [1] contains $2n_g^2$ degrees of freedom. Fortunately, only a few of them turn out to be physical.

First, the unitarity of the mixing matrices (2.4) and (2.10) imposes n_g^2 constraints, leaving n_g^2 free parameters for each one. Secondly, the

[1] n_g is the number of generations.

Lagrangian is invariant under rephasing of the fields thanks to the accidental $U(1)$ symmetries.

In the quark sector, this ensures the cancellation of $2n_g - 1$ unphysical relative phases between the quarks. Consequently, the number of physical parameters of the quark mixing matrix is

$$N_{\text{param}} = n_g^2 - (2n_g - 1) = (n_g - 1)^2 \tag{2.14}$$

i.e. $N_{\text{param}} = 1$ (the Cabibbo mixing angle) for two generations while $N_{\text{param}} = 4$ for three generations.

In the lepton sector one should be more careful. Indeed, a Majorana mass term is not invariant under $U(1)$. As a consequence, the number of phases which can be cancelled is only n_g, yielding

$$N_{\text{param}} = n_g^2 - n_g = n_g(n_g - 1) \tag{2.15}$$

i.e. $N_{\text{param}} = 2$ for two generations while $N_{\text{param}} = 6$ for three generations. With Dirac neutrinos, the number of parameters is obviously the same as for the quarks.

2.2.2 Representation

A unitary matrix is a complex extension of an orthogonal matrix. A natural way to parametrize the mixing matrix is to identify some of these parameters as mixing angles, the remaining ones being phases. A n_g by n_g orthogonal matrix can be represented by a limited number of rotations[2]:

$$N_{\text{angle}} = \tfrac{1}{2} n_g (n_g - 1) \quad . \tag{2.16}$$

For quarks and Dirac neutrinos, the number of physical phases is

$$N_{\text{phase}} = N_{\text{param}} - N_{\text{angle}} = \tfrac{1}{2}(n_g - 1)(n_g - 2) \quad . \tag{2.17}$$

At least three generations are thus required in order to have some phases. If $n_g = 3$, this unique phase is sometimes called Dirac phase.

In case of Majorana neutrinos, the number of angles is of course the same as before (2.16), but the number of physical phases is bigger:

$$N_{\text{phase}} = N_{\text{param}} - N_{\text{angle}} = \tfrac{1}{2} n_g (n_g - 1) \tag{2.18}$$

[2]This number corresponds to the dimension of $SO(n_g)$.

giving $N_{\text{phase}} = 3$ for three generations. These two additional phases — called Majorana phases — can actually be "factorized out" (see Sec. 2.3):

$$V_{\text{pmns}} = V \cdot \Phi \tag{2.19}$$

where V is an unitary matrix with the same number of parameters as for the quarks and Φ a diagonal matrix with two phases.

It is worth mentioning that these Majorana phases do not affect neutrino oscillations but would influence other processes like neutrinoless double-beta decay, see [12, 29] for a review.

2.2.3 CP Violation

A pure gauge Lagrangian is necessarily [3] invariant under CP. In the Standard Model, with one Higgs doublet, the scalar potential is also CP conserving. Consequently, a CP violation can only arise from the simultaneous occurrence of Yukawa and gauge interactions.

The CP transformation of the fields are detailed in App. A. If we impose CP conservation for the $SU(2)_L$ gauge interaction terms involving the scalar fields ($\sim W^+\varphi^- - W^-\varphi^+$), it is readily seen that

$$\begin{aligned}(\mathcal{CP})\ W^{+\mu}(t,\vec{r})\ (\mathcal{CP})^\dagger &= -e^{i\xi_W} W^-_\mu(t,-\vec{r})\\ (\mathcal{CP})\ W^{-\mu}(t,\vec{r})\ (\mathcal{CP})^\dagger &= -e^{-i\xi_W} W^+_\mu(t,-\vec{r}) \quad .\end{aligned} \tag{2.20}$$

Similarly, the $\mathcal{CP}$ conservation of QED gives for the quarks

$$\begin{aligned}(\mathcal{CP})\ u_\alpha(t,\vec{r})\ (\mathcal{CP})^\dagger &= e^{i\xi_\alpha}\gamma^0 C\,\overline{u}^T_\alpha(t,-\vec{r})\\ (\mathcal{CP})\ \overline{u}_\alpha(t,\vec{r})\ (\mathcal{CP})^\dagger &= -e^{-i\xi_\alpha} u^T_\alpha(t,-\vec{r})\,C^{-1}\gamma^0\\ (\mathcal{CP})\ d_k(t,\vec{r})\ (\mathcal{CP})^\dagger &= e^{i\xi_k}\gamma^0 C\,\overline{d}^T_k(t,-\vec{r})\\ (\mathcal{CP})\ \overline{d}_k(t,\vec{r})\ (\mathcal{CP})^\dagger &= -e^{-i\xi_k} d^T_k(t,-\vec{r})\,C^{-1}\gamma^0 \quad .\end{aligned} \tag{2.21}$$

In terms of the mass eigenstates, the charged-current interaction transforms under CP like

$$(\mathcal{CP})\ \ W^+_\mu\,\overline{u}_{\alpha L}\,\gamma^\mu V_{\alpha k}\,d_{kL}\ \ (\mathcal{CP})^\dagger \ \ = \ \ e^{i(\xi_W+\xi_k-\xi_\alpha)} W^-_\mu\,\overline{d}_{kL}\,\gamma^\mu V_{\alpha k}\,u_{\alpha L}$$

[3] We neglect here the possible parity violation by the vacuum of the theory (Cf. App. B). This violation has never been observed and must be extremely small to accommodate with the constraint on the neutron electric dipole moment.

which has to be compared to $W_\mu^-\,\overline{d}_{kL}\,\gamma^\mu V_{\alpha k}^*\,u_{\alpha L}$. As a result, the weak interactions are CP invariant only if

$$V_{\alpha k}^* = e^{i(\xi_W+\xi_k-\xi_\alpha)} V_{\alpha k} \quad . \tag{2.22}$$

Since the phases ξ_W, ξ_k and ξ_α are arbitrary, this latter condition can always be fulfilled if we consider only one element of the matrix. However, when considering several matrix elements [4], this condition implies that they all should be real. The CP violation is therefore equivalent to the existence of a physical phase in the flavour mixing matrix. The strength of the CP violation is however not directly proportional to this physical phase. This fact will become clearer once explicit representations of the mixing will be introduced.

It can be shown [30] that, for three generations, there is only one CP violating invariant which arise from the determinant of the commutator of the mass matrices: $\det[M_p, M_n]$. If the mass matrices are not Hermitian, $\det[M_pM_p^\dagger, M_nM_n^\dagger]$ has to be used instead. To simplify the reasoning, we shall work in a specific basis:

$$U_L^p = \mathbb{1} \qquad \text{and} \qquad U_L^n = V_{\text{ckm}} \quad . \tag{2.23}$$

We then define

$$\begin{aligned} H_u &= M_pM_p^\dagger = M_u^2 \\ H_d &= M_nM_n^\dagger = V_{\text{ckm}}^\dagger M_d^2 V_{\text{ckm}} \quad . \end{aligned} \tag{2.24}$$

If V_{ckm} is real, $H_uH_d = (H_dH_u)^T$ and the commutator is skew-symmetric. It follows:

$$\begin{aligned} \det[H_u, H_d] &= \det[H_u, H_d]^T = \det(-[H_u, H_d]) \\ &= (-1)^3 \det[H_u, H_d] \\ &= 0 \quad . \end{aligned} \tag{2.25}$$

On the contrary, if V_{ckm} is complex, $H_uH_d = (H_dH_u)^\dagger$ and the commutator is skew-hermitian, yielding:

$$\begin{aligned} \det[H_u, H_d] &= \det[H_u, H_d]^T = \det(-[H_u, H_d]^*) \\ &= (-1)^3 (\det[H_u, H_d])^* \\ &= iK \end{aligned} \tag{2.26}$$

[4] This is a simple exercise to check that the relation (2.22) cannot be automatically fulfilled for more than seven matrix elements at the same time in the case of three generations.

where K is real. An explicit computation gives

$$\det[H_u, H_d] = 2iJ\, \Delta^2_{uc}\, \Delta^2_{ct}\, \Delta^2_{tu}\, \Delta^2_{ds}\, \Delta^2_{sb}\, \Delta^2_{bd} \tag{2.27}$$

with

$$\Delta^2_{ij} = m_i^2 - m_j^2 \tag{2.28}$$

and

$$J \sum_{\gamma,l} (\epsilon_{\alpha\beta\gamma}\ \epsilon_{jkl}) = \Im(V_{\alpha j} V_{\beta k} V^*_{\alpha k} V^*_{\beta j}) \quad . \tag{2.29}$$

As a consequence, CP violation only occurs if the flavour mixing matrix is complex *and* if all the up-type (down-type) masses are different.

From the definition (2.29), it is obvious that the choice of the four matrix elements in the right-hand side must be made by selecting a two-by-two submatrix. Manifestly, there exist nine different choices.

2.3 Parametrizations

The mixing matrix can be represented via three rotations and one phase. A general three dimensional rotation matrix can indeed be constructed from three simple rotations. The choice of the rotation axes is not unique and we have no clue on which ones Nature could have chosen. For simplicity, we shall use a simple basis :

$$\begin{aligned}
R_{12}(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R_{23}(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\
R_{31}(\theta) &= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad .
\end{aligned} \tag{2.30}$$

Two rotations around two different axes do not commute, on the contrary two rotations around the same axis can be regarded as only one.

Therefore, there exist only twelve different combinations of these three rotations:

$$\begin{array}{rl}
(1) & R_{12}(\theta_1)\, R_{23}(\theta_2)\, R_{12}(\theta_3) \\
(2) & R_{12}(\theta_1)\, R_{31}(\theta_2)\, R_{12}(\theta_3) \\
(3) & R_{23}(\theta_1)\, R_{12}(\theta_2)\, R_{23}(\theta_3) \\
(4) & R_{23}(\theta_1)\, R_{31}(\theta_2)\, R_{23}(\theta_3) \\
(5) & R_{31}(\theta_1)\, R_{12}(\theta_2)\, R_{31}(\theta_3) \\
(6) & R_{31}(\theta_1)\, R_{23}(\theta_2)\, R_{31}(\theta_3) \\
(7) & R_{12}(\theta_1)\, R_{23}(\theta_2)\, R_{31}(\theta_3) \\
(8) & R_{12}(\theta_1)\, R_{31}(\theta_2)\, R_{23(}\theta_3) \\
(9) & R_{23}(\theta_1)\, R_{12}(\theta_2)\, R_{31(}\theta_3) \\
(10) & R_{23}(\theta_1)\, R_{31}(\theta_2)\, R_{12}(\theta_3) \\
(11) & R_{31}(\theta_1)\, R_{12}(\theta_2)\, R_{23}(\theta_3) \\
(12) & R_{31}(\theta_1)\, R_{23}(\theta_2)\, R_{12}(\theta_3)
\end{array}$$

The six first contain two times a rotation around the same axis while the last six consist of rotations around the three different axes. However only nine of them are structurally different. The products $R_{ij}R_{kl}R_{ij}$ and $R_{ij}R_{mn}R_{ij}$ $(ij \neq kl \neq mn)$ are indeed correlated with each other [31]. For instance

$$R_{12}(\theta_1)\, R_{23}(\theta_2)\, R_{12}(\theta_3) = R_{12}(\theta_1 + \frac{\pi}{2})\, R_{31}(\theta_2)\, R_{12}(\theta_3 - \frac{\pi}{2}) \quad . \tag{2.31}$$

We have now to introduce the fourth parameter, the phase ϕ. The easiest way is to modify one of the rotations. Since the resultant matrix should remain unitary, there are three different possibilities. For instance for R_{12}:

$$\begin{aligned}
R_{12} \rightsquigarrow \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta & 0 \\ -e^{-i\phi}\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad &= P_1(-\phi)\cdot R_{12}(\theta)\cdot P_1(\phi) \\
&= P_2(\phi)\cdot R_{12}(\theta)\cdot P_2(-\phi)
\end{aligned}$$

or

$$R_{12} \rightsquigarrow \begin{pmatrix} e^{i\phi}\cos\theta & \sin\theta & 0 \\ -\sin\theta & e^{-i\phi}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} &= P_1(-\phi)\cdot R_{12}(\theta)\cdot P_2(\phi) \\ &= P_2(\phi)\cdot R_{12}(\theta)\cdot P_1(-\phi) \end{aligned}$$

or

$$R_{12} \rightsquigarrow \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix} \qquad \begin{aligned} &= R_{12}(\theta)\cdot P_3(\phi) \\ &= P_3(\phi)\cdot R_{12}(\theta) \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} P_1(\phi) &= \begin{pmatrix} e^{-i\phi} & & \\ & 1 & \\ & & 1 \end{pmatrix} \\ P_2(\phi) &= \begin{pmatrix} 1 & & \\ & e^{-i\phi} & \\ & & 1 \end{pmatrix} \\ P_3(\phi) &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{-i\phi} \end{pmatrix} \quad . \end{aligned} \tag{2.33}$$

The two other rotations R_{23} and R_{31} can also be modified in the same way. From now on, we shall use the third modification (2.32) as it is the simplest one. This choice is however not really important since the location of the phase in the flavour mixing matrix can be easily modified via rephasing of the individual fields. It is moreover always possible to locate the phase in a two-by-two submatrix, the remaining five elements being real. The nine possibilities of such an arrangement correspond to the nine different parametrizations [31] of Tab. 2.1.

All these parametrizations are mathematically equivalent. The interest of choosing one or another depends mainly on the context or the habit. The usual parametrization is the so-called standard or PDG convention [12] corresponding to P3. In addition to this most-used parametrization, we should also mention the original mixing proposed by Kobayashi and Maskawa [26] corresponding to P2; and the Fritzsch convention [32] P1.

Param.	Decomposition
P1	$V = R_{12}(\theta_1)\, R_{23}(\theta_2)\, P_1(\phi)\, R_{12}(-\theta_3)$
P2	$V = R_{23}(\theta_1)\, R_{12}(\theta_2)\, P_3(\phi)\, R_{23}(-\theta_3)$
P3	$V = R_{23}(\theta_1)\, R_{31}(\theta_2)\, P_2(\phi)\, R_{12}(\theta_3)$
P4	$V = R_{12}(\theta_1)\, R_{31}(\theta_2)\, P_2(\phi)\, R_{23}(-\theta_3)$
P5	$V = R_{31}(\theta_1)\, R_{12}(\theta_2)\, P_3(\phi)\, R_{31}(-\theta_3)$
P6	$V = R_{12}(\theta_1)\, R_{23}(\theta_2)\, P_1(\phi)\, R_{31}(\theta_3)$
P7	$V = R_{23}(\theta_1)\, R_{12}(\theta_2)\, P_3(\phi)\, R_{31}(-\theta_3)$
P8	$V = R_{31}(\theta_1)\, R_{12}(\theta_2)\, P_3(\phi)\, R_{23}(-\theta_3)$
P9	$V = R_{31}(\theta_1)\, R_{23}(\theta_2)\, P_1(\phi)\, R_{12}(-\theta_3)$

Table 2.1: *The nine independent parametrizations of a mixing matrix in terms of rotation matrices. See* (2.30) *and* (2.33) *for definitions*

These two latter ones, mostly because of their symmetric construction, have already been studied in various contexts [26, 32, 33, 34, 35, 36]. Indeed, their first and third rotations involve the same axes and allow for a symmetric decomposition of the mixing. To our knowledge, the others conventions did not receive any in-depth studies.

Thanks to these parametrizations, (2.19) is easily demonstrated. For instance, let us consider P1 but with three independent phases. Using the permutation relations:

$$P_i(\phi) \cdot R_{ij}(\theta) \cdot P_j(-\phi) = P_j(-\phi) \cdot R_{ij}(\theta) \cdot P_i(\phi) \tag{2.34}$$

one finds successively

$$\begin{aligned} V_{\text{pmns}} &= R_{12}\, R_{23}\, P_1(\phi_1)\, P_2(\phi_2)\, P_3(\phi_3)\, R_{12} \\ &= R_{12}\, R_{23}\, P_1(\phi_1)\, P_2(\phi_2)\, R_{12}\, P_3(\phi_3) \\ &= R_{12}\, R_{23}\, P_1(\phi_1)\, P_1(-\phi_2)\, R_{12}\, P_2(\phi_2)\, P_1(\phi_2)\, P_3(\phi_3) \\ &= R_{12}\, R_{23}\, P_1(\phi_1 - \phi_2)\, R_{12}\, P_2(\phi_2)\, P_1(\phi_2)\, P_3(\phi_3) \quad . \end{aligned} \tag{2.35}$$

Finally, a global rephasing of the charged leptons allows to cancel one of the outer phases, for instance P_1, leading to

$$V_{\text{pmns}} = V \cdot P_2(\phi_2)\, P_3(\phi_3) \quad . \tag{2.36}$$

Consequently, the two Majorana phases can always be factorized out.

Param.	θ_1	θ_2	θ_3	ϕ
P1	$(5.36\pm0.13)^\circ$	$(2.43\pm0.03)^\circ$	$(11.07\pm0.87)^\circ$	$(101.04\pm10.28)^\circ$
P2	$(2.05\pm0.16)^\circ$	$(13.14\pm0.03)^\circ$	$(1.00\pm0.02)^\circ$	$(100.00\pm10.39)^\circ$
P3	$(2.42\pm0.03)^\circ$	$(0.23\pm0.01)^\circ$	$(13.13\pm0.03)^\circ$	$(56.70\pm9.56)^\circ$
P4	$(13.13\pm0.03)^\circ$	$(0.47\pm0.04)^\circ$	$(2.39\pm0.03)^\circ$	$(24.37\pm1.17)^\circ$
P5	$(10.38\pm0.12)^\circ$	$(13.36\pm0.03)^\circ$	$(10.53\pm0.11)^\circ$	$(1.07\pm0.12)^\circ$
P6	$(13.14\pm0.03)^\circ$	$(2.39\pm0.03)^\circ$	$(0.47\pm0.04)^\circ$	$(155.67\pm1.17)^\circ$
P7	$(2.45\pm0.03)^\circ$	$(13.13\pm0.03)^\circ$	$(0.23\pm0.01)^\circ$	$(124.34\pm9.44)^\circ$
P8	$(0.48\pm0.04)^\circ$	$(13.13\pm0.03)^\circ$	$(2.48\pm0.03)^\circ$	$(156.67\pm1.07)^\circ$
P9	$(0.23\pm0.01)^\circ$	$(2.42\pm0.03)^\circ$	$(13.14\pm0.03)^\circ$	$(123.33\pm9.55)^\circ$

Table 2.2: *Angles and phases in the nine different parametrizations of the quark mixing matrix V_{ckm} in terms of rotation matrices. Errors are at 1σ.*

2.4 Quarks

In addition to the inherent advantages of a particular decomposition we may legitimately wonder if the values of the mixing angles, extracted from the data, would single out one of these parametrizations.

The computation of the parameters has been performed via a maximum likelihood. The input parameters consist of the absolute value of the mixing matrix elements. The analysis has been performed on two different sets. First we used the latest [12] values for $|V_{ij}|$ plus the well-measured angle $\beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right)$. The second set was composed of the mixing matrix elements already constrained by a global fit as found in [37]. We have moreover assumed that the error on each input was Gaussian. The differences between the results from the two input sets were not significant. The CP violating parameter J has been computed afterwards for each parametrization and compared to its experimental value ($J = (3.05 \pm 0.2) \times 10^{-5}$) as a cross-check.

The angles can always be chosen to lie in the first quadrant while the phase remains in the first two quadrants. It is worth mentioning that the phase can also be shifted ($\phi \rightsquigarrow \pi - \phi$), for instance by changing the sign of one of the angles. Numerical results are summarized in Tab. 2.2.

In view of these results, two classes of parametrization are readily spotted. The six parametrizations which have rotations around the three axes (P3, P4, P6, P7, P8 and P9) present indeed the same pattern. They all have approximately the same angle for each identical rotation. In other words, the angles for the rotation R_{12} in all these parametrizations are approximately the same, let's call it $\theta_{12} \sim 13.13°$; and equivalently for R_{23} ($\theta_{23} \sim 2.45°$) and R_{31} ($\theta_{31} \sim 0.35°$). The only difference between these parametrizations is the order in which the individual rotations appear. Obviously, rotations around two different axes do not commute. However, because of the hierarchy among the three angles, the commutators are very small, bringing only minor corrections to the angles. For the same reason, the phases are also roughly equal (up to a 180° difference). Consequently, at this level, there is no particular difference among these six parametrizations.

The three remaining parametrizations (P1, P2, and P5) were constructed in a symmetric way with rotations involving only two different axes. Their angles and phase are quite different from the previous ones, except for the middle rotation. θ_2 is indeed almost equal to the corresponding angle coming from the preceding case, i.e. $\theta_2 \simeq \theta_{23}$ for P1 and $\theta_2 \simeq \theta_{12}$ for P2 and P5.

The absolute value of the quark mixing matrix is almost symmetric. Since these three latter parametrizations are also constructed symmetrically, this particularity implies that θ_1 and θ_3 should be of the same order of magnitude. It is worth mentioning that, in P5, the two outer angles will never be equal although the error bars seem to allow it. A plot of the admissible zone for θ_1 and θ_3 (Fig.2.1) shows that their error bars are indeed highly correlated.

Among all these parametrizations, P5 is the most peculiar. This is the only one where all the angles have the same order of magnitude, corresponding roughly to the Cabibbo angle. Moreover, the most striking feature is probably the smallness of its phase: $\phi \sim 1°$, while the other parametrizations have all a much bigger phase: $\phi \sim 60°$. This can be easily explained after an expansion à la Wolfenstein [38]. If we make the further assumption that the sine of all angles is equal to λ,

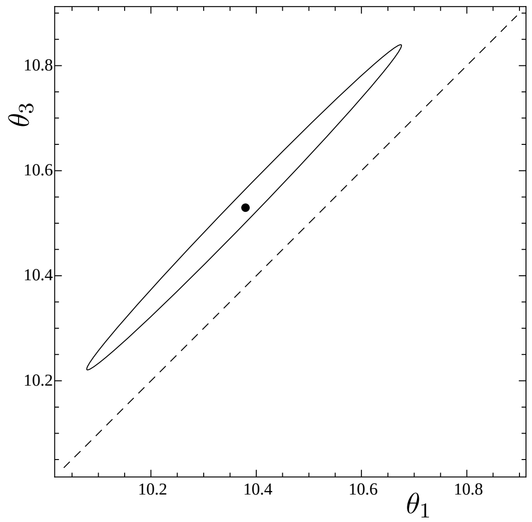


Figure 2.1: *θ_3 with respect to θ_1 (in degree) in the P5 convention. The admissible zone corresponds to a 3σ deviation. The dash line represents $\theta_1 = \theta_3$.*

one finds

$$V_5 = \begin{pmatrix} c_1c_2c_3 + s_1s_3e^{-i\phi} & c_1s_2 & -c_1c_2s_3 + s_1c_3e^{-i\phi} \\ -s_2c_3 & c_2 & s_2s_3 \\ -s_1c_2c_3 + c_1s_3e^{-i\phi} & -s_1s_2 & s_1c_2s_3 + c_1c_3e^{-i\phi} \end{pmatrix} \tag{2.37}$$

$$\approx \begin{pmatrix} 1 + \lambda^2(e^{-i\phi} - \frac{3}{2}) & \lambda & \lambda(e^{-i\phi} - 1) \\ -\lambda & 1 - \frac{\lambda^2}{2} & \lambda^2 \\ \lambda(e^{-i\phi} - 1) & -\lambda^2 & \lambda^2 + e^{-i\phi}(1 - \lambda^2) \end{pmatrix} \quad . \tag{2.38}$$

The smallness of V_{ub} and V_{td} can only be explained by a very small phase. Let us remark that the difference between θ_1 and θ_3 can be seen as a small perturbation to the limit case where the absolute value of the mixing matrix is symmetric. In the same idea, the phase introduces a very small perturbation only visible on the (3,1) and (1,3) elements.

The smallness of this phase revives the old idea of explaining all the mixing parameters only in terms of mass ratios [5], in a way similar to the famous

$$\theta_{\text{Cabibbo}} \approx \sqrt{\frac{m_s}{m_d}} = (13.5 \pm 1.9)^\circ \quad . \tag{2.39}$$

[5] Hereafter, unless specified, all quark masses are the running masses at the Z mass (Cf. Tab. 3.2)

In all the other parametrizations, including the standard one, the phase is of the order of 1 radian. This latter value cannot be easily obtained via a ratio because of the mass hierarchy. However, in P5, such a relation can be straightforwardly found. For instance:

$$\phi \approx \frac{m_s}{m_b} = (1.08 \pm 0.22)^\circ$$

or

$$\phi \approx \frac{m_b}{m_t} = (0.96 \pm 0.02)^\circ$$

are really close to the extracted value: $\phi = (1.07 \pm 0.12)^\circ$.

CP violation

Whatever the representation, the J parameter has only two forms [31]:

$$J = \begin{cases} s_1 c_1 s_2 c_2^2 s_3 c_3 \sin\phi & \text{for P3, P4, P6, P7, P8, P9} \\ s_1 c_1 s_2^2 c_2 s_3 c_3 \sin\phi & \text{for P1, P2, P5} \end{cases} \tag{2.40}$$

Its maximal theoretical value is reached for $\theta_1 = \theta_3 = 45^\circ$ and

$$\theta_2 = \begin{cases} \arcsin 1/\sqrt{3} \approx 36.2^\circ \\ \arccos 1/\sqrt{3} \approx 54.7^\circ \end{cases} \tag{2.41}$$

yielding:

$$J_{\max} = \frac{\sin\phi}{6\sqrt{3}} \quad . \tag{2.42}$$

For given angles, the maximum CP violation is therefore related to a maximum phase: $\phi = 90^\circ$. However, in general a bigger phase does not necessarily implies a bigger violation since the angles and phase are all linked within the mixing matrix.

In particular, in terms of the P5 parameters, the J parameter reads

$$J = s_1 c_1 s_2^2 c_2 s_3 c_3 \sin\phi \tag{2.43}$$

J being fixed and P5 having three angles of the same order and relatively big compared to the others, its phase is minimal.

2.5 Leptons

It would be interesting to study the parametrizations of the lepton mixing matrix in the same spirit of the quarks. Unfortunately, from the four (or six with Majorana neutrinos) parameters, only two angles are measured. They are usually quoted within the standard parametrization P3 (at 1σ) [39, 40, 41]:

$$\sin^2\theta_{\rm sun} \approx \sin^2\theta_{12} = 0.32 \pm 0.023 \quad \Leftrightarrow \theta_{12} = (34.45 \pm 1.41)^\circ \tag{2.44}$$

$$\sin^2\theta_{\rm atm} \approx \sin^2\theta_{23} = 0.50 \pm 0.055 \quad \Leftrightarrow \theta_{23} = (45.00 \pm 3.15)^\circ \tag{2.45}$$

The third angle is only weakly constrained (at 95% CL):

$$\sin^2\theta_{\rm ch} \approx \sin^2\theta_{31} < 0.033 \qquad \Leftrightarrow \theta_{31} < 10.5^\circ \tag{2.46}$$

The Dirac phase is completely unknown and may even be irrelevant with vanishing θ_{31}. In that case, a useful ansatz is the so-called tri-bi-maximal mixing matrix [42] which is compatible with all current data:

$$|V| = |R_{23}(\theta_{23})\, R_{12}(\theta_{12})| \tag{2.47}$$

$$= \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad . \tag{2.48}$$

The values $\theta_{23} = \arcsin(1/\sqrt{2}) = 45^\circ$ and $\theta_{12} = \arcsin(1/\sqrt{3}) \simeq 35.26^\circ$ reflect a maximal mixing between two and three species, respectively. It is worth mentioning that only five from the nine parametrizations (P1, P2, P3, P7 and P9) are compatible with this ansatz (Cf. Tab. 2.3), that is to say the ones where $R_{23}R_{12}$ appear in that order in their decomposition. These are also the only ones in which the apparent quark-lepton complementarity [43, 44] may be realized:

$$\theta_{12}^{\rm quarks} + \theta_{12}^{\rm leptons} \approx \frac{\pi}{4} \quad . \tag{2.49}$$

Instead of using the tri-bi-maximal ansatz, we can can choose a reasonable lepton mixing matrix. The central values of $\theta_{\rm sun}$ and $\theta_{\rm atm}$ have been used together with $\theta_{\rm ch} = 5^\circ$ and $\phi = 45^\circ$. The results are presented in Tab. 2.4. In this case, P5 is noteworthy again. It displays

Param.	θ_1	θ_2	θ_3	ϕ
P1	$0°$	$45°$	$35.26°$	$\Re$
P2	$45°$	$35.26°$	$0°$	$\Re$
P3	$45°$	$0°$	$35.26°$	$\Re$
P4	$26.56°$	$24.09°$	$39.23°$	$0°$
P5	$45°$	$54.73°$	$60°$	$0°$
P6	$45°$	$35.26°$	$30°$	$180°$
P7	$45°$	$35.26°$	$0°$	$\Re$
P8	$26.56°$	$24.09°$	$50.77°$	$0°$
P9	$0°$	$45°$	$35.26°$	$\Re$

Table 2.3: *Mixing parameters for the tri-bi-maximal mixing matrix. If one of the mixing angle is vanishing, the phase is unphysical and can be any reals ($\Re$).*

Param.	θ_1	θ_2	θ_3	ϕ
P1	$7.1°$	$45.2°$	$31.0°$	$131.7°$
P2	$39.9°$	$34.8°$	$8.8°$	$134.0°$
P3	$45°$	$5°$	$34.5°$	$45°$
P4	$28.0°$	$21.5°$	$40.8°$	$12.6°$
P5	$47.1°$	$56.0°$	$58.2°$	$9.6°$
P6	$45.2°$	$37.5°$	$27.4°$	$169.5°$
P7	$47.4°$	$34.3°$	$6.1°$	$134.8°$
P8	$24.0°$	$25.9°$	$51.6°$	$192.8°$
P9	$7.1°$	$44.8°$	$38.0°$	$137.2°$

Table 2.4: *Mixing parameters for the lepton mixing matrix. P3 is used as input.*

the smallest phase of all parametrizations and its three angles are of the same order of magnitude, exactly as for the quarks. The choice of the input values is not restrictive. The computed angles and phases are indeed only slightly sensitive to the input chooz angle. On the contrary, the computed phases are highly dependent on the input phase. However, we have checked that the hierarchy between the phases in the various parametrizations is almost conserved for $\phi = 0°$ to $180°$ as input : P5 presents the smallest phase for the biggest part of the parameter space.

2.6 Mass Textures

Within the Standard Model for electroweak interactions, adopting a particular parametrization of flavour mixing is arbitrary and not a physical issue. Nevertheless, it is quite likely that the actual values of flavour mixing parameters, including the strength of the CP violation, would give interesting information about the physics beyond Standard Model. In many models, the generation of quark masses is intimately related to the phenomenon of flavour mixing. A particular structure of the underlying mass matrices calls for a particular choice of parametrization. Moreover, if we assume that the masses and mixing can be computed within a new flavour theory, this theory will probably pick *one* of the nine parametrizations.

The angles of the P1 parametrization have already received some physical interpretation (see [36] and references therein). In the limit of a heavy third generation, i.e. $m_t \gg m_c > m_u$ and $m_b \gg m_s > m_d$, the angle θ_2 would be vanishing. In this decoupling limit, the angle θ_1 describes a mixing between the up and charm quarks, whereas θ_3 connects the down and strange quarks. A simple pattern of mass matrix representing this limit case would be:

$$M = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix} \quad . \tag{2.50}$$

The mass hierarchy implies $d \gg c, b > a \approx 0$ and leads to

$$\begin{aligned} \theta_1 &\approx \sqrt{\frac{m_u}{m_c}} \approx 3.0° \\ \theta_3 &\approx \sqrt{\frac{m_d}{m_s}} \approx 13.5° \end{aligned} \tag{2.51}$$

which is quite close to the observed values for P1.

Unlike P1, the P5 parametrization has never attracted much attention. One would like to know if the peculiarities of this parametrization could also be reflected in a simple mass matrix.

The symmetric construction of P5 calls for a symmetric separation of the mixings coming from the up- and down-type quark sectors. Ideally, one would like the flavour mixing matrix to be described by

$$\begin{aligned} V_{\text{ckm}} &= U_L^{p\dagger} U_L^n \\ &= \Big(R_{31}(\theta_3^u)\, R_{12}^T(\theta_2^u)\Big)\ \Big(R_{12}(\theta_2^d) R_{31}^T(\theta_3^d)\Big) \quad . \end{aligned} \tag{2.52}$$

For the moment, we do not consider CP violation and the inclusion of a phase in this decomposition will be done later.

Let us first imagine that all quark masses are equal but different from zero. The symmetry in the Lagrangian is increased and the mixings are unphysical, yielding $V_{\text{ckm}} = \mathbb{1}$.

The deviation from this symmetric situation, could occur in a sort of cascade breaking. Some mechanism will first modify the heavy quark mass while the lighter ones keep degenerated. In each sector, the mixing is then merely a rotation in the (1,3) plane. In the limit of $\theta_1 = \theta_3$, which is very close to the real world, the flavour mixing is again disappearing: $V_{\text{ckm}} = \mathbb{1}$.

The flavour mixing will only be present once a splitting between the first and second generations is introduced. This new hierarchy will generate a mixing in the (1,2) plane, yielding (2.52).

For the first stage of this scheme, one may be tempted to use the same kind of mass matrix successfully used with P1, but with a (1,3) mixing:

$$M = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & d \end{pmatrix} \quad . \tag{2.53}$$

Unfortunately, this naive assumption leads to completely wrong angles for P5:

$$\begin{aligned} \theta_3^u &\approx \sqrt{\frac{m_u}{m_t}} \approx 0.18^\circ \\ \theta_3^d &\approx \sqrt{\frac{m_d}{m_b}} \approx 1.8^\circ \quad . \end{aligned} \tag{2.54}$$

This situation is however not hopeless. We note indeed that

$$\begin{aligned} \theta_3^u &\approx \sqrt[4]{\frac{m_u}{m_t}} \approx 3.2^\circ \\ \theta_3^d &\approx \sqrt[4]{\frac{m_d}{m_b}} \approx 10.3^\circ \quad . \end{aligned} \tag{2.55}$$

Whereas the first ratio is a bit too small, the second one fits perfectly the observed value. Moreover, the rotation in the (1,2) plane is also recovered:

$$\theta_2 = -\theta_2^u + \theta_2^d \approx -\sqrt[4]{\frac{m_u}{m_c}} + \sqrt[4]{\frac{m_d}{m_s}} \approx -13.1^\circ + 27.8^\circ \approx 14.7^\circ \quad . \tag{2.56}$$

These results are quite promising and seem to indicate that the flavour mixing could arise from a simple texture, not for the mass matrix, but for its square root. From this perspective, we should build the mass matrix M from a more fundamental matrix S which would play the role of its square root:

$$M = SS^\dagger \quad . \tag{2.57}$$

It is worth mentioning that, within this context, the mass matrices are always Hermitian and the θ-strong problem might get a natural solution (Cf. App. B).

The mixing angles are now related to the bi-diagonalization of S and the ratios (2.55) are readily found from a very simple matrix:

$$S = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & c \end{pmatrix} \quad . \tag{2.58}$$

For simplicity, we assume that the parameters a, b and c, are real. In this case, they can be computed from

$$\begin{aligned} \mathrm{Tr}S = b + c &= \sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3} \\ \mathrm{Det}S = -a^2 b &= \sqrt{m_1}\sqrt{m_2}\sqrt{m_3} \\ \mathrm{Tr}(SS^\dagger) = 2a^2 + b^2 + c^2 &= m_1 + m_2 + m_3 \end{aligned} \tag{2.59}$$

where the index on the masses corresponds to the generation. This system has a solution only if one of the square roots is actually negative [6],

[6] This is not an issue since the mass will still be positive.

for instance $\sqrt{m_1} \rightsquigarrow -\sqrt{m_1}$. This gives

$$\begin{aligned} a &= \sqrt[4]{m_1 m_3} \\ b &= \sqrt{m_2} \\ c &= \sqrt{m_3} - \sqrt{m_1} \quad . \end{aligned} \tag{2.60}$$

The matrix S is diagonalized thanks to an orthogonal matrix corresponding to a rotation in the (1,3) plane with an angle θ given by

$$\tan(2\theta) = \frac{2a}{c} \Leftrightarrow \theta = \arctan \sqrt[4]{\frac{m_1}{m_3}} \quad . \tag{2.61}$$

Starting from

$$S = \begin{pmatrix} 0 & a' & 0 \\ a' & b' & 0 \\ 0 & 0 & c' \end{pmatrix} \quad . \tag{2.62}$$

The ratios in (2.56) are equivalently found. Indeed, with

$$\begin{aligned} a' &= \sqrt[4]{m_1 m_2} \\ b' &= \sqrt{m_2} - \sqrt{m_1} \\ c' &= \sqrt{m_3} \end{aligned} \tag{2.63}$$

the mixing angle is

$$\tan(2\theta') = \frac{2a'}{c'} \Leftrightarrow \theta' = \arctan \sqrt[4]{\frac{m_1}{m_2}} \quad . \tag{2.64}$$

The challenge is now to find a way to combine (2.58) and (2.62) such that their individual properties are kept almost intact. This goal can be easily achieved, however, if the number of parameters is bigger than three, we will loose any power of prediction since the remaining ones can always be fine-tuned in order to fit the observations. Instead of combining directly the matrices — which would lead to at least four parameters — another way would be to combine their results (2.60) and (2.63). The eigenvalues of the resulting matrix will no more be the observed masses. Nevertheless, we can hope that the difference will remain small such that the matrix represents a good approximation to the real world.

Our first guess is the mere superposition of the results (2.60) and (2.63):

$$S = \begin{pmatrix} 0 & \sqrt[4]{m_1 m_2} & \sqrt[4]{m_1 m_3} \\ \sqrt[4]{m_1 m_2} & \sqrt{m_2} - \sqrt{m_1} & 0 \\ \sqrt[4]{m_1 m_3} & 0 & \sqrt{m_3} - \sqrt{m_1} \end{pmatrix} \quad . \tag{2.65}$$

This texture is close to the ones used in [34, 45]. However, its eigenvalues have changed significantly. This scheme needs thus some modification, for instance:

$$S = \begin{pmatrix} \sqrt{m_1} & \sqrt[4]{m_1 m_2} & \sqrt[4]{m_1 m_3} \\ \sqrt[4]{m_1 m_2} & \sqrt{m_2} - \sqrt{m_1} & 0 \\ \sqrt[4]{m_1 m_3} & 0 & \sqrt{m_3} - \sqrt{m_1} \end{pmatrix} \tag{2.66}$$

which induces only a difference of order $\mathcal{O}(m_1)$ in m_2 and m_3.

The diagonalization of (2.66) requires in general three rotations but one of them will be very small. Analytical expressions are rather involved and, from now on, we shall proceed numerically. In terms of P5, the mixing matrices for the up and down sectors are described by the angles:

$$\begin{array}{lll} \theta_1^u = 3.4^\circ & \theta_2^u = 12.9^\circ & \theta_3^u = 0.2^\circ \\ \theta_1^d = 12.2^\circ & \theta_2^d = 25.6^\circ & \theta_3^d = 1.8^\circ \end{array} \tag{2.67}$$

This is exactly what we were expecting for, up to a big difference: the rotations appear in the wrong order with respect to (2.52).

The correct orthogonal matrix is found by tackling the problem from the opposite side. The matrix we are looking for is indeed

$$S = R_{12}(\theta') R_{31}(-\theta) \begin{pmatrix} -\sqrt{m_1} & & \\ & \sqrt{m_2} & \\ & & \sqrt{m_3} \end{pmatrix} R_{31}(\theta) R_{12}(-\theta') \tag{2.68}$$

where θ and θ' are given in (2.61) and (2.64). The hierarchical structure of the masses leads to a simple approximation:

$$S \simeq \begin{pmatrix} \sqrt{m_1} & \sqrt[4]{m_1 m_2} & -\sqrt[4]{m_1 m_3} \\ \sqrt[4]{m_1 m_2} & \sqrt{m_2} - \sqrt{m_1} & \sqrt[4]{\frac{m_3}{m_2}} \sqrt{m_1} \\ -\sqrt[4]{m_1 m_3} & \sqrt[4]{\frac{m_3}{m_2}} \sqrt{m_1} & \sqrt{m_3} - \sqrt{m_1} \end{pmatrix} \quad . \tag{2.69}$$

This matrix is very similar to (2.66) except the non-vanishing elements $(2,3)$ and $(3,2)$.

For each sector, the angles in the P5 parametrization are now exactly the ones we were expecting for

$$\begin{array}{lll} \theta_1^u = -0.16^\circ & \theta_2^u = 13.5^\circ & \theta_3^u = 3.1^\circ \\ \theta_1^d = -1.8^\circ & \theta_2^d = 29.6^\circ & \theta_3^d = 9.6^\circ \end{array} \tag{2.70}$$

The combination of these two mixings leads to the flavour mixing matrix with:

$$\theta_1 = 0.22^\circ \quad \theta_2 = 16.1^\circ \quad \theta_3 = 8.2^\circ \tag{2.71}$$

The second and third mixing angles corresponds roughly to the observed values. Unfortunately, the first one is still too small. This situation can however be improved.

So far, we did not consider CP violation. In the above scheme, a phase can appear as a mismatch between the mixings coming from the up- and down-sector [45, 46]. It turns out that the best results are achieved with a phase $\phi = 180^\circ$ which acts on the third generation:

$$V_{\text{ckm}} = U_L^{P\dagger}\ P_3(\phi)\ U_L^n \quad . \tag{2.72}$$

At the level of the S matrix, this modification corresponds to a rephasing of the elements (1,3), (3,1), (2,3) and (3,2), i.e. the elements that would generate the mixings involving the heavy quarks:

$$S \rightsquigarrow \begin{pmatrix} X & X & Xe^{i\phi} \\ X & X & Xe^{i\phi} \\ Xe^{-i\phi} & Xe^{-i\phi} & X \end{pmatrix} \quad . \tag{2.73}$$

With this modification, the angles of the new flavour mixing matrix, in terms of P5, are

$$\theta_1 = 6.6^\circ \quad \theta_2 = 16.1^\circ \quad \theta_3 = 7.9^\circ \quad \phi = 180^\circ \tag{2.74}$$

Thanks to the inclusion of a maximal mismatch between the phases of the up and down sectors, the two outer angles are now roughly the same. Although the phase is not correct, the diagonalization of (2.69) leads to a flavour mixing which is still close to V_{ckm}. These results are quite encouraging since the matrix texture (2.69) is rather simple. It would be interesting to know whether these mass matrices can be implemented beyond the Standard Model.

Unfortunately, the same texture (2.69) applied to the leptons leads to a wrong flavour mixing matrix. This issue arises probably from the neutrino sector. The mechanism which generates the neutrino masses could be different from the one responsible for the other fermions. It is indeed quite unlikely that the neutrinos have only a Dirac mass since it would require an additional symmetry. In particular, if the neutrino masses arise from a seesaw mechanism, the Dirac texture (2.69) will be modified by the right-handed neutrino Majorana mass matrix.

In this chapter, we have shown that the P5 parametrization of the flavour mixing matrix presents some unique features for both the quarks and the leptons. We have postulated that these peculiarities are an insight on what could be the physics beyond Standard Model. If the mass matrices are actually built by squaring a more fundamental matrix, the mixing angles seem a priori computable from simple ratios. We have proposed a simple texture for this "square root" matrix which leads to a quark flavour mixing similar to the observed one.

It appears that the square roots of the masses present some physical interest for flavour mixing. If they are related to some new physics, these square roots should also appear in other contexts. In particular, in the next chapter, we study an extremely successful empirical relation connecting the square roots of the charged lepton masses.

3

A Mass Relation

In the Standard Model of electroweak and strong interactions, the Yukawa couplings contribute to most of the free parameters of the theory. The gauge couplings and the Higgs potential represent 5 free parameters [1], while the fermion content requires 12 masses and at least 8 mixings parameters. Since the advent of the Standard Model, physicists have aimed at reducing these numbers. There were many attempts to get further insights on how these constants arise and whether one might calculate them. The Yukawa textures presented in the previous chapter are one of them, the grand unification theories are another. These two techniques are typically representative of the two possible approaches that have been used.

The top-down approach tries to define a more general point of view and to observe its consequences on a testable sector. On the contrary, in the bottom-up approach, one starts from a particular situation and tries to clarify the properties that a general theory should meet.

The early success of $SU(5)$ and $SO(10)$ has motivated a lot of interests in the top-down approaches. However, the simplest grand unification groups are strongly constrained by the proton decay while their supersymmetric variants introduce a bunch of new free parameters.

On the other side, the Yukawa textures and the horizontal symmetries, i.e. symmetries between generations, are able to provide good estimates of the mixing angles. A number of patterns and symmetries have been proposed. However, no simple pattern or symmetry has ever conducted to the precise computation of masses or angles.

[1] To be complete, we should also take into account the θ strong parameter.

In a colourful description, the top-down approach is like looking for a needle in a haystack. The number of models is virtually infinite and one has to try them one by one before finding the good one. This way of doing cannot reasonably work without a first selection to limit the possibilities. Making this selection is the role of the bottom-up approach. For instance, in this picture, this could be the discovery that the needle is ferromagnetic.

Currently, this latter approach (bottom-up) seems to us the most promising. In our opinion, before trying to build a model able to explain the fermion mass spectrum, one should first have a better understanding of the mass hierarchy. In this chapter, we shall analyze an extremely successful empirical relation among the masses. If this relation is indeed realized in Nature, its properties should give us some interesting hints of what could be the flavour physics beyond Standard Model.

3.1 Koide Mass Relation

In 1982, while looking for an empirical relation giving the Cabibbo angle in terms of the masses of the fermions, Koide proposed a relation that bound together all the charged-leptons masses [47]. At that time, the τ mass was only badly known and the formula allowed him to give a good estimate of it. This is maybe the reason why the relation has been neglected for some years until the precision of the masses increases. The formula got recently a new lease of life once its accuracy could be tested and impressively checked [48].

In its original form, the relation is written:

$$m_e + m_\mu + m_\tau = q^l \left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2 \tag{3.1}$$

where q^l is supposed to be $\frac{2}{3}$. The latest charged-leptons masses [10, 12] are:

$$\begin{aligned} m_e &= 0.510998910 \pm 0.000000013 \text{ MeV} \\ m_\mu &= 105.6583663 \pm 0.0000038 \text{ MeV} \\ m_\tau &= 1776.99 \pm 0.29 \text{ MeV}. \end{aligned} \tag{3.2}$$

Since the muon mass is measured with respect to the electron's one, their errors are highly correlated. The same references give their Pearson correlation coefficient: $r_{e\mu} = 0.7006$. A first order propagation of the errors gives

$$q^l = 0.666667 \pm 0.000016 \tag{3.3}$$

which is indeed compatible with $\frac{2}{3}$ with a 5-digit precision.

This relation can of course be inverted in order to give a prediction for the τ mass — which is the least precise mass — after imposing $q^l = \frac{2}{3}$:

$$m_\tau = 1776.968874 \pm 0.000061 \text{ MeV} \quad . \tag{3.4}$$

The accuracy of the relation (3.1) has no equivalent in any other proposed model or pattern. The legitimate question is then to wonder how such a relation could emerge from a theory. These efforts will be detailed later. We should first focus on the properties of the relation and their implications.

3.1.1 Features and Issues

Democracy

The relation (3.1) is democratic, in the sense that all generations are on an equal footing. The relation is indeed invariant under a change of generation: $m_i \leftrightarrow m_j$ (i and j stand for e, μ or τ), i.e. a $S(3)$ permutation symmetry. In particular, this interesting feature suggests that a democratic behaviour must exist at some point in the mechanism generating the masses.

Moreover, the challenging middle value of $q^l = \frac{2}{3}$ turns out to be an extremely efficient measure of the mass splitting inside the charged lepton family (Cf. Tab. 3.1). Its maximal value ($q = 1$) would correspond to a full hierarchy ($m_1, m_2 \ll m_3$) while its minimal value ($q = \frac{1}{3}$) would occur in the case of complete degeneracy ($m_1 = m_2 = m_3$). Let us remark that Nature comes close to these boundary values with the up quark and the neutrino families, respectively.

This property has received a nice geometrical interpretation [49]. Let us first indeed define a three dimensional space where each axis is associated to a generation ($\vec{e}$, $\vec{\mu}$ and $\vec{\tau}$). In this generation space, the

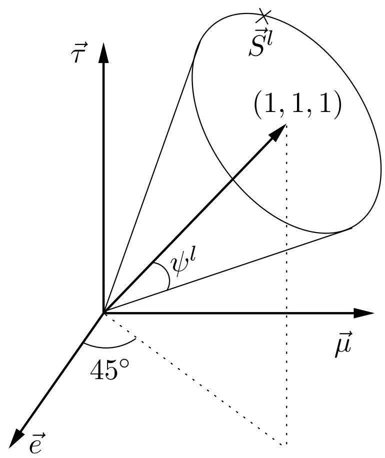


Figure 3.1: *Geometric representation of the Koide mass relation. The cross represents the vector containing the square roots of the lepton masses* $\vec{S}^l = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$. *It lies on a cone with aperture* $\psi^l = 45°$ *aligned with the vector representing degeneracy:* $(1,1,1)$. *For illustration purposes, the cone is not drawn at scale.*

relation (3.1) states that the vector $\vec{S}^l$ containing the square-roots of the charged-lepton masses lies at $45°$ of the vector $(1,1,1)$ representing the degeneracy among the masses (Cf. Fig. 3.1):

$$\cos\psi^l = \frac{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}) \cdot (1,1,1)}{|(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})||(1,1,1)|} = \frac{1}{\sqrt{3q^l}} \tag{3.5}$$

giving

$$\psi^l = (45.00005 \pm 0.00070)° \quad . \tag{3.6}$$

This democratic behaviour is really impressive. For instance, if we switch off the small electron mass, we obtain a completely wrong tau-to-muon mass ratio: 0.072, to be compared with the experimental value: $m_\mu/m_\tau \approx 0.059$.

We may hope that the parameter q actually arises from some flavour symmetries broken above the Fermi scale. In that case, its value could be related in one way or another to the number of generations or to some quantum numbers. In particular, we may assume that a mass relation should be possible order by order with an increasing number of

	q	ψ
Degeneracy	$\frac{1}{3}$	$\arccos\sqrt{1/1} = 0$
Charged leptons	$\frac{2}{3}$	$\arccos\sqrt{1/2} = 45°$
Strong hierarchy	1	$\arccos\sqrt{1/3} = 54.73°$

Table 3.1: *Special values of the splitting parameters*

generations. With only one generation, the relation is indeed trivially satisfied for $q^l_1 = 1$. However, with two generations, the electron and the muon, its value does not seem particular $q^l_2 = 0.8784$. We can nevertheless do it all the same with a fourth generation. The most stringent limit on the mass of an heavy charged lepton comes from direct search ($W \to \nu l^{\pm}$) [12]:

$$m_4 > 100.8 \text{ GeV} \quad . \tag{3.7}$$

This latter constraint is equivalent to

$$q^l_4 > 0.7474 \quad . \tag{3.8}$$

The interesting possibility $q^l_4 = \frac{3}{4}$ leads to $m_4 = 103.5$ GeV.

Square Roots

The occurrence of square roots of masses in (3.1) is not real surprise for us. We have seen that they naturally appear during the diagonalization of a mass matrix (Cf. Sec. 2.6). However, here, the square roots of the masses seem to play a special role in their *own* hierarchy, independently of any mixing angles.

If one wants to explain the mass relation from some symmetries, this mere fact constitutes an important issue. Indeed, square roots of masses do not appear so easily from the Lagrangian. It is also worth reminding that the precision achieved does not allow to neglect the small electron mass. The mass relation should therefore arise as an exact expression.

In the geometric description (3.5), we have also pointed the importance of the square roots of masses. This impression will be confirmed all along this chapter.

Pole Masses

Equation (3.1) provides a wonderful numerical relation among the observed masses. So far, we have implicitly, and naturally, used the physical mass of the charged-leptons, corresponding to their pole mass. However, we have seen that, in the Standard Model, this choice is not unique. Depending on the context, one usually refers to one of the bare, running or pole mass (Cf. Sec. 1.2.1). At tree level all these definitions are of course equivalent and this is quite sufficient for many mass relations. However, the precision achieved by (3.1) legitimates the question: which masses?

The question might first seem irrelevant for the charged leptons whose masses are quite stable under renormalization. Indeed, at one loop, the evolution of their running mass reads [50]:

$$\mu \frac{\mathrm{d}m(\mu)}{\mathrm{d}\mu} = -6\alpha(\mu) \quad . \tag{3.9}$$

In this leading log approximation [2] all the masses evolve in the same way and one has to go beyond this approximation to observe a tiny variation in their ratios. The mass relation is therefore virtually constant under renormalization albeit energy scale dependent.

However, the key point is the transition from the pole masses to the running masses. The renormalization group equations for QED give the relation connecting the running mass $m(\mu)$ at a given scale μ and the pole masses m. At one loop, one has for the fermions [50]:

$$m(\mu) = m\left[1 - \frac{\alpha(\mu)}{\pi}\left(1 + \frac{3}{2}\log\frac{\mu}{m(\mu)}\right)\right] \quad . \tag{3.10}$$

Clearly, the mass ratios are not constant under such a transformation. Although the variation is small, the effect is still important enough with respect to the precision achieved [51]. For instance, the charged lepton masses at the Z scale [52] are:

$$\begin{aligned} m_e(M_Z) &= 0.486570161(42)\ \text{MeV} \\ m_\mu(M_Z) &= 102.7181359(92)\ \text{MeV} \\ m_\tau(M_Z) &= 1746.24(20)\ \text{MeV} \end{aligned} \tag{3.11}$$

[2] In the leading-log approximation, one keeps only the dominant contribution coming from the gauge bosons, here photons. Beyond this approximation, one takes also into account the contribution of fermions loops

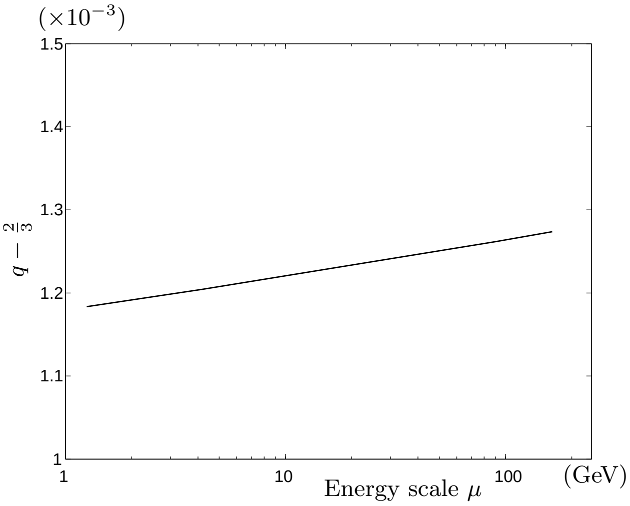


Figure 3.2: *Running q^l for the charged leptons*

giving

$$
\begin{aligned}
q^l(M_Z) &= 0.667928(11) \\
\psi^l(M_Z) &= 45.0541(5)^\circ
\end{aligned}
\tag{3.12}
$$

which are no longer compatible with $\frac{2}{3}$ or 45°. These numbers can be slightly improved when going at lower energy where the (running) mass ratios are bigger, i.e. where the hierarchy is weaker (Cf. Fig. 3.2).

If we assume that the relation (3.1) should be valid — as it seems — for the pole masses, we are probably facing the biggest problem raised by the relation. The first obvious issue concerns the quarks. Indeed, quarks are never free. Consequently, their propagator as well as their pole mass are ill-defined. Any generalization should then use their running mass leading to only approximate relations. The second issue of the pole masses arises when trying to build a model explaining the relation. As we have seen, the Standard Model way to derive relations is to impose some extra symmetries to the Lagrangian. However, these symmetries are actually applied to the bare masses which are not related to the pole masses in a simple way. So, we do not see how Lagrangian symmetries could gather together pole masses in such a concise form and such a precise way.

$m_u(M_Z) = 1.27 \pm 0.5$ MeV	$m_c(M_Z) = 0.619 \pm 0.084$ GeV
$m_d(M_Z) = 2.9 \pm 1.24$ MeV	$m_b(M_Z) = 2.89 \pm 0.09$ GeV
$m_s(M_Z) = 55 \pm 16$ MeV	$m_t(M_Z) = 171.7 \pm 3$ GeV

Table 3.2: *Running quark masses computed at the Z mass [52].*

Other Families

When looking at (3.1), the first idea is to check if the same relation applies to the other families [53]. However, as explained before, the relation might be valid only for the pole masses whereas the running masses of the quarks should only give a rough equality.

In principle, for the up quark family, it could be possible to work with pole masses although the errors for the charm quark might be big [52]. In that case, we find that the pole mass of the up quark should be heavier than the charm one to satisfy $q^u = \frac{2}{3}$, which is excluded.

A better solution could be to use the running masses computed at the same scale. For instance, at the Z scale, (Cf. Tab. 3.2), we get:

$$q^u(M_Z) \approx 0.888 \tag{3.13}$$

$$q^d(M_Z) \approx 0.746 \quad . \tag{3.14}$$

The running of the mass ratio is again not enough to recover the relation at a lower scale. The direct transposition of (3.1) to the quark sector seems hopeless since the observed hierarchies are here too strong.

On the contrary, the neutrinos appear almost degenerate. While their mass differences are well-measured [12], their global scale is still unknown:

$$\begin{aligned} \Delta m_{21}^2 =& m_2^2 - m_1^2 = (8.0 \pm 0.3) \times 10^{-5}\ \text{eV}^2 \\ |\Delta m_{32}^2| =& |m_3^2 - m_2^2| = (2.5 \pm 0.2) \times 10^{-3}\ \text{eV}^2 \quad . \end{aligned} \tag{3.15}$$

Moreover, there is still an arbitrariness in their hierarchy: normal (N) or inverted (I), respectively when ν_3 is the heaviest or the lightest neutrino. In both case, when ν_1 is getting heavier, the mass differences are becoming relatively smaller and q^ν tends to the limit value for the degeneracy [54]: $q^\nu \to \frac{1}{3}$. Conversely, the hierarchy is maximal when m_1 has its lowest value: $m_1^N = 0$ or $m_1^I = \sqrt{\Delta m_{32}^2 - \Delta m_{21}^2} \approx 49.19$ meV.

In both case, one of the mass is thus vanishing and, at first order, the relation involves only two almost degenerated species. The maximal value for the parameter q^ν is therefore close to $\frac{1}{2}$:

$$\begin{aligned} q^\nu_N &\approx 0.583 \\ q^\nu_I &\approx 0.500 \quad . \end{aligned} \tag{3.16}$$

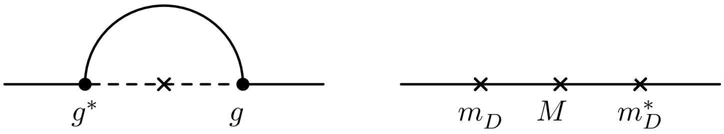
These results could at first sight be disappointing. However, since (3.1) is not trivially fulfilled by the other families of fermions, it may be a chance to incorporate in the formula a connection with the mixing angles or with the other families. This generalization then could be the missing link towards a viable dynamical model.

Let us also stress that a generalization to the neutrino and quark sectors makes only sense if the mass generation mechanism is the same as for the charged-leptons. In particular, if the small neutrino mass is generated via a see-saw mechanism, the mass relation (3.1) will certainly be spoiled by the Majorana mass term.

3.1.2 Models

A number of dynamical models have been proposed to explain the mass relation (3.1). It is worth mentioning that almost all of them have been proposed by Koide (see [48] and references therein). The major reason of the lack of proposals in this domain resides probably in the above mentioned problems. Apart from the pole masses, the biggest one is to find a dynamical model which ensures the appearance of square roots of the masses. Moreover, none of these models seems particularly convincing, mainly because they need a lot of new particles or they do not work for all the fermion's families at the same time. This fact motivates us for a bottom-up approach to the problem: to focus first on a better understanding of the properties and to find a generalization valid for the other families.

In broad outlines, all models implement a democratic behaviour in one way or another, for instance a democratic mass matrix [55] or a $S(3)$ symmetry on the scalar sector [56]. Furthermore, the masses in these models are generated via one of these mechanisms [48]: radiative masses or universal seesaw. These correspond to the two main ways of dealing



(a) *Radiative mass generation* **(b)** *Universal seesaw*

Figure 3.3: *Two proposed topologies for mass generation.*

with square roots of mass in a Lagrangian. Fig 3.3a is representative of a radiative mass generation:

$$m \sim |g|^2$$

while Fig 3.3b corresponds to a universal see-saw mechanism:

$$m \sim m_D M^{-1} m_D^* \quad .$$

Obviously, these two possibilities imply an extension of the particle content of the Standard Model, namely new heavy fermions and scalars.

It should be noted that the first idea developed by Koide [47] was based on a preon model. In this model, the fermions are supposed to be composite of more fundamental particles, preons, which are massless but can form bound states with a dynamical mass. This is a property shared by all technicolour-like models. The originality lies in the fact that there are two types of interactions. The first one has a coupling proportional to a charge z_i which is different for each generation i. The second one is identical for all generations and has a coupling proportional to the mean of the square of the generation charges:

$$z_0^2 = \tfrac{1}{3}\sum_i z_i^2 \quad . \tag{3.17}$$

Finally the charged-lepton masses are proportional to the sum of these two interactions:

$$m_i = E(z_i + z_0)^2 \tag{3.18}$$

where E is a common scale. With these hypothesis, the mass relation (3.1) is equivalent to

$$\sum_i z_i = 0 \quad . \tag{3.19}$$

This latter relation is quite interesting because it is very similar to anomaly cancellation constraints in an horizontal gauge symmetry.

This particularity deserves a deeper analysis. The parameters of this model are easily computed:

$$z_i = \frac{2\sqrt{m_i} - \sqrt{m_j} - \sqrt{m_k}}{3\sqrt{E}} \qquad (i \neq j \neq k) \tag{3.20}$$

$$z_0 = \frac{\sqrt{m_i} + \sqrt{m_j} + \sqrt{m_k}}{3\sqrt{E}} \tag{3.21}$$

while E is completely arbitrary allowing to choose a normalization for the z_i's. An interesting choice consists in seeing the z_i's as forming a unit vector in a Cartesian generation space: $\sum z_i^2 = 1$. Within this assumption, we find:

$$\begin{aligned} E &= 941.58(17) \\ z_1 &= -0.554053(12) \\ z_2 &= -0.242366(16) \\ z_3 &= 0.796419(4) \\ z_0 &= \tfrac{1}{\sqrt{3}} \quad . \end{aligned} \tag{3.22}$$

Defining

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad \text{and} \quad Z_0 = \begin{pmatrix} z_0 \\ z_0 \\ z_0 \end{pmatrix}$$

it is clear that Z_0 is also a unit vector, pointing in the direction of degeneracy, or democracy, in the generation space, i.e. it has no preferred direction. Moreover, because of (3.19), Z is perpendicular to Z_0. The vector containing the square root of the masses is proportional to the sum of these two unit vectors, see (3.18). One recovers then the geometrical interpretation of (3.5). In other words, the formula can now be regarded as the maximal breaking from a degeneracy state towards the direction defined by Z. This idea will be developed later in Sec. 3.2.4.

Interestingly, the same kind of relations involving square roots of mass have also been derived in another (supersymmetric) composite model [57, 58]:

$$\begin{aligned} m_e m_\tau^2 &= m_\mu^3 \\ \frac{m_c}{m_u} - \frac{m_s}{m_d} &= \frac{m_\mu}{m_e} \\ \frac{m_t}{m_c} - \frac{m_b}{m_s} &= \frac{m_\tau}{m_\mu} \end{aligned} \tag{3.23}$$

Although these relations are only roughly satisfied, composite models seem to provide a suitable framework for generating square root type relations.

3.2 Generalizations

3.2.1 Matrix Form

The occurrence of square root of masses constitutes a problem for model building. We can legitimately wonder if is it possible to remove them [3]. The answer is yes. First, we rewrite (3.1) by expanding the parenthesis ($q^l = \frac{2}{3}$) :

$$m_1 + m_2 + m_3 = 4\,(\sqrt{m_1 m_2} + \sqrt{m_2 m_3} + \sqrt{m_1 m_3}) \quad .$$

Then, taking its square and keeping all the remaining square-roots on the right-hand side gives

$$\begin{aligned}(m_1 + m_2 + m_3)^2 - 16(m_1 m_2 + m_2 m_3 + m_1 m_3) = \\ 32\sqrt{m_1 m_2 m_3}\,(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}) \quad .\end{aligned}$$

Once again, taking the square, yields

$$\begin{aligned}\Big((m_1 + m_2 + m_3)^2 - 16(m_1 m_2 + m_2 m_3 + m_1 m_3)\Big)^2 = \\ 32^2 m_1 m_2 m_3\,(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2 \quad .\end{aligned}$$

At this point, we note that the remaining square-roots are exactly the ones in (3.1). A simple replacement gives then

$$\begin{aligned}\Big((m_1 + m_2 + m_3)^2 - 16(m_1 m_2 + m_2 m_3 + m_1 m_3)\Big)^2 = \\ \tfrac{3}{2}\,32^2 m_1 m_2 m_3 (m_1 + m_2 + m_3) \quad . \qquad (3.24)\end{aligned}$$

[3] Let us remark that (3.1) can also be rewritten in terms of only square-roots of masses:

$$(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2 = 6\,(\sqrt{m_1 m_2} + \sqrt{m_1 m_3} + \sqrt{m_2 m_3}) \quad .$$

From a mathematical point of view, this last relation is not equivalent to the original one. However, the relation (3.1) implies the relation (3.24). To be precise, if we suppose that two masses are fixed, the original equation (3.1) has only two solutions for the third mass:

$$\begin{aligned} m_3 = 7(m_1+m_2) + 20\sqrt{m_1 m_2} \\ \pm 4\sqrt{3}\sqrt{\left(\sqrt{m_1}+\sqrt{m_2}\right)^2\left(m_1 + 4\sqrt{m_1 m_2} + m_2\right)} \quad . \end{aligned} \tag{3.25}$$

If $m_2 > m_1$, the solution with the minus sign is always smaller than m_2. Therefore, if we assume a normal hierarchy, we are left with only one physical solution. The new equation (3.24) has two more solutions:

$$\begin{aligned} m_3 = 7(m_1+m_2) - 20\sqrt{m_1 m_2} \\ \pm 4\sqrt{3}\sqrt{\left(\sqrt{m_1}-\sqrt{m_2}\right)^2\left(m_1 - 4\sqrt{m_1 m_2} + m_2\right)} \quad . \end{aligned} \tag{3.26}$$

These two solutions are real only for

$$0 \leq m_2 \leq (7-4\sqrt{3})m_1 < m_1 \tag{3.27}$$

or

$$m_2 \geq (7+4\sqrt{3})m_1 \simeq 14m_1 \quad . \tag{3.28}$$

Only this last possibility is allowed if we suppose normal hierarchy. In that case, only the solution with a plus sign in (3.26) is always greater than m_2. The special case where $m_1 = m_2$ gives $m_3 = -6m_1$ should also be discarded. In conclusion, with the new relation (3.24), we have two admissible solutions, instead of one.

Interestingly, (3.24) can easily be transformed in a matrix form. In the Standard Model, the masses are not a priori the eigenvalues of the mass matrix. Without any more assumption, the masses are indeed given by the positive square root of the eigenvalues of $MM^\dagger$ or $M^\dagger M$. For simplicity, we shall assume that the mass matrix M is hermitian. Its characteristic polynomial is then :

$$P(\lambda) = -\lambda^3 + c_2\lambda^2 - c_1\lambda + c_0 \tag{3.29}$$

where

$$\begin{aligned} c_0 &= \mathrm{Det}(M) = m_1 m_2 m_3 \\ c_1 &= \frac{1}{2}\left[\mathrm{Tr}(M)^2 - \mathrm{Tr}(M^2)\right] = (m_1 m_2 + m_2 m_3 + m_1 m_3) \\ c_2 &= \mathrm{Tr}(M) = m_1 + m_2 + m_3 \quad . \end{aligned}$$

This shows that (3.24) can be written in a matrix form :

$$\left[7(\mathrm{Tr}(M))^2 - 8\mathrm{Tr}(M^2)\right]^2 = \tfrac{3}{2}\, 32^2\mathrm{Det}(M)\mathrm{Tr}(M) \quad . \tag{3.30}$$

Although the goal of removing the square roots has been achieved, the new relations (3.24) and (3.30) are not particularly attracting since the simplicity of the original relation has been lost in the process. If the mass matrix is not Hermitian, the resulting relation would be even more complex.

3.2.2 Pseudo-Masses

The empirical mass relation (3.1) is now precise at the level of 10^{-5}. Yet, at first sight this remarkable but mysterious mass relation seems to be a false trail. Any attempt to apply it to the quarks or neutrinos is indeed doomed to failure.

Interestingly, the relation is democratic, in the sense that all generations are on an equal footing and the q parameter turns out to be an extremely efficient measure of the mass splitting. The strong hierarchy observed in the quark sector ($q \to 1$) should be opposed to the almost degeneracy between the neutrinos ($q \to \frac{1}{3}$).

We argue that the flavour mixing which also displays quite different patterns, from the small angles of the CKM matrix for the quarks to the large ones of the PMNS matrix for the leptons, could hold the key of this puzzle. In models with flavour symmetries, small angles are closely linked to mass hierarchy and large ones to mass degeneracy (Cf. Sec. 2.6). This points us the way towards a universal mass/angle relation.

We would like to take advantage of the recent experimental progress in the neutrino sector to generalize the mass relation (3.1) for the lepton families. A lepton-quark connection beyond the Standard Model is then called upon to validate it also for the quark families.

Lifting the near degeneracy in the neutrino family

A first attempt to apply the original mass relation (3.1) to the neutrinos was unsuccessful in both schemes [54]. The reason simply lies in the mild splitting of the neutrino masses: the strongest hierarchy, ensured for $m_1 = 0$, always implies $q^\nu < 0.6$. One way out is to amplify the mass hierarchy with the help of the well-measured neutrino mixing matrix elements $(U_L^\nu)^{ij}$. We thus propose [35] the following minimal extension of relation (3.1):

$$\sum \tilde{m}_i = \frac{2}{3}\left(\sum \sqrt{\tilde{m}_i}\right)^2 \tag{3.31}$$

which involves the "pseudo-masses" $\tilde{m}_i$ defined as

$$\tilde{m}_i \equiv |\sum_j U_L^{ij}\ m_j| \tag{3.32}$$

rather than the physical masses m_j. In our convention, $U_L^\dagger\ MU_R \equiv \mathrm{diag}(m_1, m_2, m_3)$ such that if $U_R = \mathbb{1}$, these Dirac pseudo-masses are simply related to the Yukawa couplings of a single Higgs doublet.

The latest results from neutrino experiments at 1σ [41] are in good agreement with $\theta_{\mathrm{ch}} \approx \theta_{31} = 0$, since

$$\sin^2\theta_{31} < 0.033 \quad (95\%\ \mathrm{CL}) \quad .$$

So, let us therefore assume the following flavour mixing matrix

$$\begin{aligned} V_{\mathrm{pmns}} &\equiv U_L^{e\dagger} U_L^\nu = R_{23}(\theta_{\mathrm{atm}}) R_{12}^T(\theta_{\mathrm{sun}}) \\ &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{\mathrm{atm}} & \sin\theta_{\mathrm{atm}} \\ 0 & -\sin\theta_{\mathrm{atm}} & \cos\theta_{\mathrm{atm}} \end{pmatrix} \begin{pmatrix} \cos\theta_{\mathrm{sun}} & -\sin\theta_{\mathrm{sun}} & 0 \\ \sin\theta_{\mathrm{sun}} & \cos\theta_{\mathrm{sun}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{3.33}$$

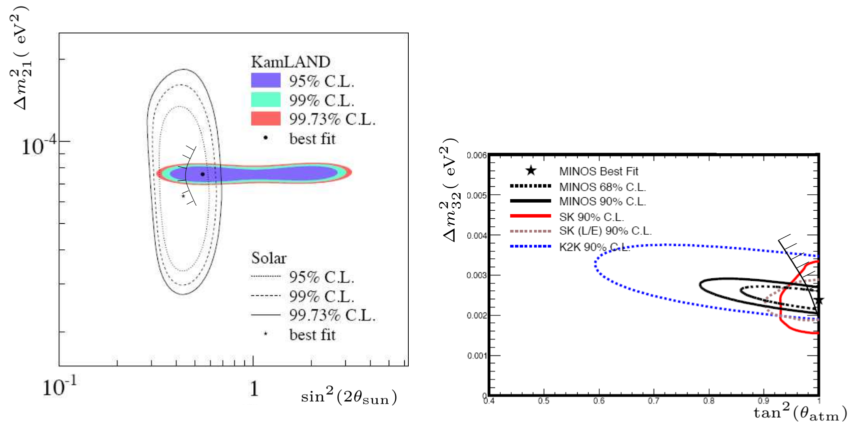
with the experimental values for the mixing angles at 1σ [12]

$$\begin{aligned} \sin^2\theta_{\mathrm{sun}} &= 0.32 \pm 0.023 \\ \sin^2\theta_{\mathrm{atm}} &= 0.50 \pm 0.055 \end{aligned}$$

i.e.

$$\begin{aligned} \theta_{\mathrm{sun}} &= (34.42 \pm 1.41)^\circ \\ \theta_{\mathrm{atm}} &= (45 \pm 3.15)^\circ \quad . \end{aligned}$$

There are three natural solutions in this limit [59]. Either the large solar mixing angle θ_{sun} comes from M^ν and the large atmospheric angle θ_{atm}



(a) *Solar Neutrinos* **(b)** *Atmospheric Neutrinos*

Figure 3.4: *Exclusion domains (3σ) derived from relation* (3.31)*. The experimental results have been taken from the latest global fit [39, 40]*

from M^e, or both come from M^e or M^ν. The remarkable accuracy of the relation (3.1) requires that any successful extension involving mixing angles should reduce to this form for the charged leptons. Consequently, we focus on the last possibility, namely

$$U_L^e = \mathbb{1} \quad , \quad U_L^\nu = R_{23}(\theta_{\rm atm})R_{12}^T(\theta_{\rm sun}) \quad . \tag{3.34}$$

Numerical computations provide us with a continuous set of solutions satisfying (3.31) and compatible with the present data (see Fig. 3.4). All these solutions correspond to the normal hierarchy for $\{m_i\}$, the inverted one being excluded.

It appears that the predicted range for the Dirac mass of the lightest neutrino m_1 is

$$1.5\ 10^{-2}\ \text{eV} < m_1 < 4.1\ 10^{-2}\ \text{eV} \tag{3.35}$$

at 3σ. Notice that the normal hierarchy for $\{m_i\}$ is turned into an inverted one for $\{\tilde{m}_i\}$, i.e. $0 \approx \tilde{m}_3 \ll \tilde{m}_1 < \tilde{m}_2$. On the other hand, the solar $\theta_{\rm sun}$ and the atmospheric $\theta_{\rm atm}$ angles are bounded from below at 3σ by

$$\begin{aligned} \theta_{\rm sun} &> 33.4^\circ \\ \theta_{\rm atm} &> 46.4^\circ \end{aligned} \tag{3.36}$$

such that the so-called maximal mixing solution ($\theta_{\rm atm} = \pi/4$) is ruled out. These two extrema cannot be reached simultaneously. The allowed region spanned actually mainly over the 3σ region (Cf. Fig. 3.5)

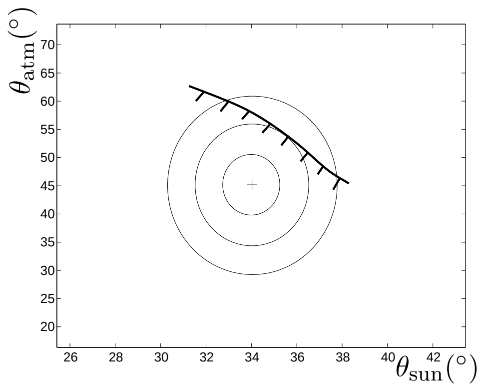


Figure 3.5: *Exclusion domains (*3σ*) for the neutrino mixing angles derived from relation* (3.31)*. Circles correspond to 1, 2 and 3 sigmas experimental values [12]. In the limit of vanishing* $\theta_{\rm ch}$*, these angles are independently found by considering a two generation mixing. Their error bars are then uncorrelated.*

Pure Dirac masses have been assumed for the neutrinos to put all the leptons on an equal footing. Needless to say that the introduction of Majorana masses to implement the seesaw mechanism would imply less stringent constraints on the masses and mixing angles. In particular, we have checked that this is already the case for degenerate Majorana masses.

Taming the strong hierarchy in the quark families

As already mentioned, a naive estimate shows that relation (3.1) is certainly not valid for the up and down quark families, because of the large top and bottom masses. However, if a quark-lepton connection exists beyond the Standard Model, $\theta_{31} = 0$ could be a property shared by all elementary fermions. Viewing $U_L^e = \mathbb{1}$ as a basis fixing choice, one may then write

$$U_L^u = \Phi(\varphi_u) R_{23}(\theta_t) R_{12}^T(\theta_u) \quad , \quad U_L^d = \Phi(\varphi_d) R_{23}(\theta_b) R_{12}^T(\theta_d) \qquad (3.37)$$

where

$$\Phi(\varphi) \equiv \mathrm{diag}(e^{-i\varphi_1}, e^{-i\varphi_2}, e^{-i\varphi_3}) \quad . \qquad (3.38)$$

If $U_R = \mathbb{1}$, the number of arbitrary phases could be reduced ($\varphi_u = -\varphi_d$) by imposing the auxiliary condition $\arg\det(M^u M^d) = 0$ from the conspicuous time-reversal invariance of the strong interactions, see App. B. But anyhow, the pseudo-masses defined in (3.32) only depend on the small rotation angles, not on the phases. So, let us try to extract these angles from the data.

A suitable re-phasing of the quark fields leads to the CKM mixing matrix

$$V_{CKM} \equiv U_L^{u\dagger} U_L^d = R_{12}(\theta_u)\ \mathrm{diag}(e^{-i\varphi}, 1, 1)\ R_{23}(\theta)\ R_{12}^T(\theta_d) \qquad (3.39)$$

where $\theta \equiv \theta_b - \theta_t$. Here, the CP-violating phase φ is only linked to the first and second families. This parametrization (called P1 in Chapter 2) coincides with the one convincingly advocated in [31, 60] on the basis of the hierarchical structure of the quark mass spectrum. We find it quite interesting to reach the same description of the CKM mixing matrix from two different approaches. The parameters can be computed at the 1σ level using the latest experimental data (Cf. Tab. 2.2):

$$\begin{aligned} \theta_u &= (5.36 \pm 0.13)^\circ \\ \theta &= (2.43 \pm 0.03)^\circ \\ \theta_d &= (11.07 \pm 0.87)^\circ \\ \varphi &= (101.04 \pm 10.28)^\circ \quad . \end{aligned} \qquad (3.40)$$

So, we are just left with the freedom on one mixing angle (say θ_t) to test the pseudo-mass relation (3.31) for the up and down quark families. Contrary to the leptons, the quark masses are not directly measurable quantities. If the quark masses are chosen at a common energy scale M_Z, both relations can be reasonably satisfied (i.e. $\tilde{q}^u = \tilde{q}^d \simeq 0.69$) for $\theta_t = -2.18^\circ$. Conversely, imposing $\tilde{q}^{u,d} = \frac{2}{3}$ gives $\theta_t = -3.53^\circ$ together with $m_s(M_Z) = 133 \pm 15$ MeV, if the rather stable ratios of the light quarks are used.

These theoretical results are quite encouraging once one keeps in mind that relation (3.31) is energy scale dependent. The heavy quark mixing (θ) and masses ($m_{t,b}$) are indeed subject to strong renormalization-group effects. In particular, the running of masses beyond the leading log approximation in QCD flattens the hierarchy such that $\tilde{q}$ decreases with increasing energy. Moreover, the original mass relation (3.1) is only roughly satisfied by running masses and one should not expect such precise results for quarks. We should also comment that, if the ratio of the light quarks masses are well under control, the absolute value of the strange mass has drastically decreased during the last decade. For all these reasons, we argue that relation (3.31) might provide us with some clues for the understanding of the lepton *and* quark mass spectrum.

3.2.3 Family Interconnection

The original formula cannot be applied directly on the different families of particles. There seems to be no reason that the charged leptons have a different behaviour than the others. Maybe (3.1) is an approximation to a more complete relation. But then, we may wonder how an approximation can be so precise. The concept of pseudo-masses has shown some interesting possibilities. Here we shall explore another path.

In the electroweak theory, charged-leptons and neutrinos are bound together in a left-handed doublet; the same occurs for the up and down-type quarks. The mass generation is intimately connected with the $SU(2)_L$ gauge symmetry and its breaking. Instead of working with the four families separately, we can thus assume that each of the two fermion sectors (leptons and quarks) has to be considered as a whole. The relation (3.1) should then be applied, not on the charge-lepton family, but on the whole lepton sector, the sums being now on six particles instead of three. We therefore assume that the neutrino mass is generated in the same way as the charged lepton one and they are thus of Dirac type. The smallness of the neutrino masses legitimizes this approach for the lepton sector.

$$\frac{m_e + m_\mu + m_\tau + m_{\nu_1} + m_{\nu_2} + m_{\nu_3}}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} + \sqrt{m_{\nu_1}} + \sqrt{m_{\nu_2}} + \sqrt{m_{\nu_3}}\right)^2} \equiv q^L = \frac{2}{3} \tag{3.41}$$

Using (3.15), this new relation allows to constrain at 1σ the unknown mass of the lightest neutrino and the badly-known tau mass. For normal hierarchy, there is a solution when

$$1777.11 \text{ GeV} < m_\tau < 1777.28 \text{ GeV} \tag{3.42}$$

yielding

$$0 < m_{\nu_1} < 46 \text{ meV} \tag{3.43}$$

while the inverted hierarchy is admissible when

$$1777.16 \text{ GeV} < m_\tau < 1777.28 \text{ GeV} \quad . \tag{3.44}$$

leading to

$$0 < m_{\nu_3} < 40 \text{ meV} \tag{3.45}$$

The preferred value, $m_{\nu_1} = 0$ and $m_{\nu_3} = 0$ respectively, gives the lowest bound for m_τ. It is worth mentioning that this generalization changes the geometrical interpretation (3.5) where the angle ψ is no more 45° but 60°.

Because of the smallness of the neutrinos masses, this generalization is almost trivial unless being also valid for the quarks:

$$\frac{m_d + m_s + m_b + m_u + m_c + m_t}{\left(\sqrt{m_d} + \sqrt{m_s} + \sqrt{m_b} + \sqrt{m_u} + \sqrt{m_c} + \sqrt{m_t}\right)^2} \equiv q^Q \quad . \tag{3.46}$$

Assuming correlated errors, the running masses of the quarks at the Z mass give indeed:

$$q^Q = \tfrac{2}{3}(1.038 \pm 0.019) \tag{3.47}$$

i.e. only a 2σ deviation! One has to remember that the relation is not energy scale invariant and that the running masses could only give an approximation. These results seem therefore quite encouraging. A model in the same spirit will be developed later on for the leptons (Cf. p. 82).

3.2.4 Cascade Breaking

Independently of the mechanism at work to generate the mass, assumptions can be made on the way it works. For instance, the masses could be generated via a cascade of breakings [61, 62, 63, 64, 46]. These models were motivated by the democratic mass matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad . \tag{3.48}$$

On a first stage, M is diagonalized by a unitary matrix U and only one particle gets a mass while the two others remain massless:

$$U^\dagger M U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad . \tag{3.49}$$

They would eventually acquired a mass during a second and third stage as correction to this scheme.

We propose here a slightly different behaviour where, at first, all masses are equal but different from zero. More precisely, the square root of them are all equal. On the first breaking, one of them is set apart and become heavier than the two others which keep degenerated. Their differentiation occurs in a second stage.

So let us start from a symmetric situation where all the masses are degenerated. In regard of the geometric interpretation (3.5), the vector containing the square root of the masses, $\vec{S}^l_{123}$, is therefore aligned with the normalized symmetric direction

$$\vec{x}_1 = \tfrac{1}{\sqrt{3}}(1, 1, 1) \quad . \tag{3.50}$$

Instead of using the natural coordinates ($\vec{e}$, $\vec{\mu}$, $\vec{\tau}$) of the three dimensional generation space, we would like to define a new orthogonal basis with $\vec{x}_1$. Obviously, the two other vectors have to lie on the plane $P \equiv e + \mu + \tau = 0$.

In the first step of the cascade breaking, the two first generations should remain degenerated. The second basis vector is given by the intersection of P and $e = \mu$, namely

$$\vec{x}_2 = \tfrac{1}{\sqrt{6}}(-1, -1, 2) \quad . \tag{3.51}$$

The third basis vector [4] which will set apart the first two generations is then the intersection of P and $e = -\mu$, namely

$$\vec{x}_3 = \tfrac{1}{\sqrt{2}}(-1, 1, 0) \quad . \tag{3.52}$$

Starting from the degenerate direction $\vec{S}^l_{123} \propto \vec{x}_1$, the first breaking should occur along the $\vec{x}_2$ direction, giving a new vector $\vec{S}^l_{12}$ (Cf. Fig. 3.6). As shown in (3.5), this breaking has to be maximal: $\psi = 45°$. The Koide relation, via its geometrical interpretation, is now viewed as a maximal breaking from a degenerate situation ($m_1 = m_2 = m_3$) to a partially degenerate one ($m_1 = m_2 < m_3$).

[4] These three vectors constitutes also the tri-bi-maximal mixing U matrix in (3.49) which diagonalizes a democratic mass matrix.

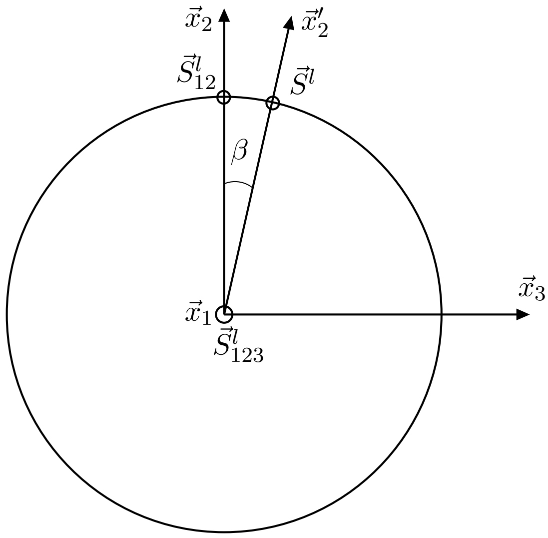


Figure 3.6: *Schematic representation of the cascade breaking. For drawing reasons, the vector containing the square roots of the masses, $\vec{S}^l$, has been unitarized. The circle represents the cone of aperture $\psi = 45°$ aligned with $\vec{x}_1$.*

The splitting between the first two generations is generated by a new breaking. As the charged leptons (vector $\vec{S}^l$) are lying on the cone with aperture 45°, this new breaking has to be a rotation around $\vec{x}_1$. Finally, the vector containing the square root of the masses is given by (E is a global scale);

$$\vec{S}^l = E\left(\cos\psi\ \vec{x}_1 + \sin\psi\left(\cos\beta\ \vec{x}_2 + \sin\beta\ \vec{x}_3\right)\right) \tag{3.53}$$

or, if the index k refers to the generation,

$$S^l_k = \frac{E}{\sqrt{3}}\left(\cos\psi + \sqrt{2}\sin\psi\ \cos\left(\beta + \frac{2k\pi}{3}\right)\right) \quad . \tag{3.54}$$

The computation of the parameters is straightforward. For instance, the angle β is computed as the angle between $\vec{x}_2$ and $\vec{x}_2'$ which is the projection of $\vec{S}^l$ on the plane $P \equiv e + \mu + \tau = 0$.

$$\begin{aligned} E &= \sqrt{m_e + m_\mu + m_\tau} \\ \cos\psi &= \frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{3}\sqrt{m_e + m_\mu + m_\tau}} \\ \cos\beta &= \frac{-\sqrt{m_e} - \sqrt{m_\mu} + 2\sqrt{m_\tau}}{2\sqrt{m_e + m_\mu + m_\tau - \sqrt{m_e m_\mu} - \sqrt{m_e m_\tau} - \sqrt{m_\mu m_\tau}}} \quad . \end{aligned} \tag{3.55}$$

The angle ψ has already been computed in (3.6) and its value is compatible with a maximal breaking. On the contrary the angle β is surprisingly close to $\frac{2}{9}$:

$$\beta = 0.222221(16) \quad . \tag{3.56}$$

Unfortunately, these two angles are highly correlated and it is not possible to assume both $\beta = \frac{2}{9}$ and $\psi = \frac{\pi}{4}$ at the same time. In particular, we can compute one angle while imposing the second one:

$$\psi = 45^\circ \qquad \Rightarrow \quad \beta = 0.2222220470(2) \tag{3.57}$$

$$\beta = \tfrac{2}{9} \qquad \Rightarrow \quad \psi = 44.99999389(7)^\circ \tag{3.58}$$

Strangely, the correlation line is *exactly* aligned with this ideal case (Cf. Fig. 3.7). It can be shown that this correlation line corresponds to a rescaling of the third parameter: the global scale E. Consequently, the peculiar alignment seems to indicate that an exact $\beta = \frac{2}{9}$ and $\psi = \frac{\pi}{4}$ would be possible if, for some reason, the global scale E is modified.

However, assuming both $\beta = \frac{2}{9}$ and $\psi = \frac{\pi}{4}$ at this stage gives already a very good estimate of the charged lepton mass ratios:

$$\begin{aligned} \frac{\sqrt{m_\mu}}{\sqrt{m_e}} &= 14.3795103 \qquad \text{vs.} \quad 14.37943957(83)\ \text{(exp)} \\ \frac{\sqrt{m_\tau}}{\sqrt{m_\mu}} &= 4.10098 \qquad \text{vs.} \quad 4.10101(34)\ \text{(exp)} \end{aligned} \tag{3.59}$$

Although the discrepancy is relatively big ($\sim 85\sigma$), the precision is still impressive (up to 5 digits). We are then encouraged to look for how small deviations to this scheme can occurred, in particular, a change in E. An idea could be the very small mass of the neutrinos. Indeed the observed deviation is exactly of the order of their expected mass.

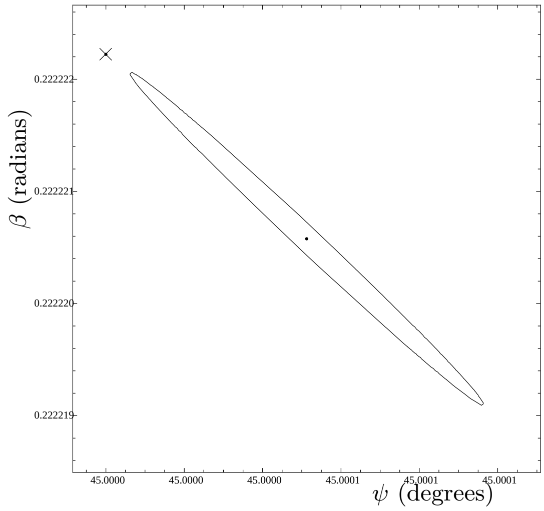


Figure 3.7: *Admissible domain for the parameters ψ and β from the cascade breaking. For legibility purposes, the ellipse has been enlarged and corresponds to a 300σ deviation! The cross represents the ideal case $\psi = \frac{\pi}{4}$ and $\beta = \frac{2}{9}$.*

Neutrinos

If there exists some connection between the charged lepton and the neutrino mass [5], we can imagine a mass generation mechanism which consists of two parts and where each of them is proportional to some charge.

For instance, the first part of the mass could be proportional to the third component of the weak-isospin (T_3), leading to opposite contributions for the charged leptons and the neutrinos. While the second part could be proportional to the electric charge, contributing only to the charged-leptons. How to implement such a mechanism is beyond the scope of this work but anyway, if it exists, the second contribution can be computed from the sum of the charged lepton and neutrino masses.

This is exactly the kind of slight perturbation we where looking for in the previous section:

$$S_k^l + S_k^\nu = \frac{E}{\sqrt{3}} \left(\cos\psi + \sqrt{2} \sin\psi \ \cos\left(\beta + \frac{2k\pi}{3}\right)\right) \tag{3.60}$$

[5] We therefore assume Dirac neutrinos.

where β is exactly $\frac{2}{9}$ and ψ is exactly $\frac{\pi}{4}$. A maximum likelihood is again used to compute the mass of the neutrinos. The inverted scheme is excluded at 11σ while normal hierarchy is allowed, leading to the constraints:

$$\begin{aligned} m_\tau &= 1776.9810 \pm 0.0005 \text{ MeV} \\ m_{\nu_1} &= 0.0094 \pm 0.0004 \text{ meV} \\ m_{\nu_2} &= 8.9 \pm 0.2 \text{ meV} \\ m_{\nu_3} &= 50.8 \pm 2.0 \text{ meV}. \end{aligned} \tag{3.61}$$

These results confirm our assumption that the small Dirac neutrino masses could be responsible for the small discrepancy with respect to the ideal case $\theta = \frac{\pi}{4}$ and $\beta = \frac{2}{9}$.

3.3 A Toy Model

As a conclusion of this chapter, we would like to show a realization of the cascade breaking scheme presented before.

We first consider a democratic mixing, i.e. a unitary mixing matrix whose elements have all the same modulus. For n_g generations, this matrix is readily found to be (up to columns and rows permutations):

$$U^{(j,k)} = \frac{1}{\sqrt{n_g}} \omega^{j \cdot k} \tag{3.62}$$

where ω is the n_gth-root of unity:

$$\omega = e^{i\frac{2\pi}{n_g}} \quad . \tag{3.63}$$

Let us remark that, in this form, the inverse of U is equal to its conjugate. The mixing matrix is always complex for $n_g > 2$. The most general matrix diagonalized by (3.62) is

$$S = \begin{pmatrix} a & b* & c* & \cdots & d & c & b \\ b & a & b* & \cdots & e & d & c \\ c & b & a & \cdots & f & e & d \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \\ b* & c* & d* & \cdots & c & b & a \end{pmatrix} \quad . \tag{3.64}$$

For instance, with three generations, powers of $\omega = \exp(i\frac{2}{3}\pi)$ arise :

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.65}$$

which diagonalized

$$S = \begin{pmatrix} a & b^* & b \\ b & a & b^* \\ b^* & b & a \end{pmatrix} \quad . \tag{3.66}$$

The corresponding eigenvalues are

$$\begin{cases} s_1 = a + b\omega + b^*\omega^2 & = a + 2|b|\cos(\beta + \frac{2\pi}{3}) \\ s_2 = a + b\omega^2 + b^*\omega & = a + 2|b|\cos(\beta - \frac{2\pi}{3}) \\ s_3 = a + b + b^* & = a + 2|b|\cos(\beta) \end{cases} \tag{3.67}$$

or simply

$$s_k = a + 2|b|\cos(\beta + k\tfrac{2\pi}{3}) \tag{3.68}$$

where $\beta = \arg b$ and k is the generation index. This clearly shows that β is responsible for the splitting between s_1 and s_2 while the ratio $|b|/a$ controls the global hierarchy between the three eigenvalues:

$$\begin{array}{lr} 0 < s_1 = s_2 < s_3 \longleftrightarrow b^* = b & s_1 - s_2 \text{ degeneracy} \\ 0 = s_1 = s_2 < s_3 \longleftrightarrow b^* = b = a & \text{max hierarchy} \\ 0 < s_1 = s_2 = s_3 \longleftrightarrow b^* = b = 0 & \text{degeneracy} \end{array}$$

Let us stress that the democratic mixing (3.62) leads to a maximal CP violation. In terms of the standard parametrization the mixing matrix U can be decomposed as

$$|U| = |R_{23}(\theta_1)\, R_{31}(\theta_2)\, P_2(\phi)\, R_{12}(\theta_3)| \quad . \tag{3.69}$$

The angles are

$$\begin{aligned} \theta_1 &= 45^\circ \\ \theta_2 &= \arcsin(1/\sqrt{3}) \simeq 35.26^\circ \\ \theta_3 &= 45^\circ \\ \phi &= 90^\circ \end{aligned} \tag{3.70}$$

and the maximal value of J (Cf. Sec. 2.4) is indeed reached for these angles.

The matrix

$$M = SS^\dagger = \begin{pmatrix} a^2+2|b|^2 & b^2+2ab^* & (b^*)^2+2ab \\ (b^*)^2+2ab & a^2+2|b|^2 & b^2+2ab^* \\ b^2+2ab^* & (b^*)^2+2ab & a^2+2|b|^2 \end{pmatrix} \tag{3.71}$$

$$= \begin{pmatrix} A & B^* & B \\ B & A & B^* \\ B^* & B & A \end{pmatrix} \tag{3.72}$$

is also diagonalized by the same matrix U and the eigenvalues of M are the square of the S ones. Now, if M is the mass matrix, the matrix S is its "square root".

We can easily compute the parameters of the model. The invariants of a 3 by 3 matrix are

$$\begin{aligned} \mathrm{Det} S &= a^3 - 3b^2 a + 2b^3\cos(3\beta) && \equiv \sqrt{m_1 m_2 m_3} \\ \mathrm{Tr} S &= 3a && \equiv \sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3} \\ \mathrm{Tr} M &= 3a^2 + 6|b|^2 && \equiv m_1 + m_2 + m_3 \end{aligned} \tag{3.73}$$

This system can be easily solved

$$\begin{aligned} a &= \frac{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}}{3} \\ |b| &= \frac{1}{3}\sqrt{m_1 + m_2 + m_3 - \sqrt{m_1 m_2} - \sqrt{m_1 m_3} - \sqrt{m_2 m_3}} \\ \cos(3\beta) &= \frac{(2\sqrt{m_1} - \sqrt{m_2} - \sqrt{m_3})(-\sqrt{m_1} + 2\sqrt{m_2} - \sqrt{m_3})(-\sqrt{m_1} - \sqrt{m_2} + 2\sqrt{m_3})}{2\left[m_1 + m_2 + m_3 - \sqrt{m_1 m_2} - \sqrt{m_1 m_3} - \sqrt{m_2 m_3}\right]^{3/2}} \end{aligned} \tag{3.74}$$

The ratio $|b|/a$ can also be parametrized via a new angle ψ:

$$\frac{|b|}{a} = \frac{\tan\psi}{\sqrt{2}} \quad . \tag{3.75}$$

Then, the cosine of this angle is exactly the geometric interpretation (3.5) of the Koide relation

$$\cos\psi = \frac{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}}{\sqrt{3}\sqrt{m_1 + m_2 + m_3}} \quad . \tag{3.76}$$

In other word, for the charged leptons, ψ turns out to be $\frac{\pi}{4}$ to a great precision, leading thus to assume that

$$\frac{|b|}{a} = \frac{1}{\sqrt{2}} \quad . \tag{3.77}$$

Within this hypothesis, (3.74) can be simplified into

$$\cos(3\beta) = \sqrt{2}\left(\frac{\sqrt{m_1 m_2 m_3}}{\left[3(\sqrt{m_1}+\sqrt{m_2}+\sqrt{m_3})\right]^3} + 1/2\right) \quad . \tag{3.78}$$

In the particular case of the charged leptons, we find:

$$a = 17.7161 \pm 0.0011 \quad \text{MeV}^{1/2} \tag{3.79}$$

$$|b|^2/a^2 = 0.5000017 \pm 0.0000245 \tag{3.80}$$

$$\beta = 0.222220 \pm 0.000020 \quad . \tag{3.81}$$

The angle β is very close to the value (3.59). This toy model is indeed mathematically equivalent to the cascade breaking scheme. A comparison of (3.54) and (3.68) yields the equivalence of the angles β in both scheme and

$$\begin{aligned} \frac{|b|}{a} &= \frac{\tan\psi}{\sqrt{2}} \\ a &= E\frac{\cos\psi}{\sqrt{3}} \quad . \end{aligned} \tag{3.82}$$

In this chapter, we have shown that an accurate relation connecting the charged lepton masses could be used as a window on physics beyond Standard Model. We have first proposed a generalization of this relation which is valid, not only for the charged leptons, but for all families. This generalization has been possible thanks to an interplay between masses and mixings, called pseudo-mass. We have also studied the properties of the original relation and have shown some interesting features. In particular, it appears that the square root of masses could be of the utmost importance in new physics. For instance, a geometric description of the square root of masses leads to much interesting albeit intriguing angles. The observation of a small deviation from an ideal case has been successfully interpreted as a neutrino effect. Finally a toy model realizing our observations has been presented.

4

Conclusion and Outlook

A need for new physics

The origin of mass constitutes a long-standing problem. Since the 17th century the questioning has inexorably evolved from “What is the mass?” to “Why this particular mass?”. The Higgs mechanism has indeed answered the first question but requires ad hoc values for the Yukawa couplings to account for the observed mass spectrum of the elementary particles. In particular, with massive Dirac neutrinos, the Standard Model of electroweak and strong interactions contains 26 free parameters. Among them, 20 are coming from the Yukawa interactions and are thus directly related to mass matrices. Nowadays, there is convincing evidence that the Standard Model is not the end of the story but only a low energy effective theory. In particular, the tiny neutrino masses call for an extension of the matter content with, for instance, right-handed neutrinos or a scalar triplet. We should also mention the lack of CP violation in the Standard Model to account for the observed matter-antimatter asymmetry of the universe.

Any modification of the Standard Model should aim at reducing the number of free parameters. However, among all the already proposed extensions, none has ever conducted to the precise computation of masses or mixing parameters. We may hope that the LHC or some future colliders will help to clarify the situation by discovering new particles or spotting some unexpected events. In the meantime, more precise measurements of masses and mixing parameters could also play an important role. In this work we have used a bottom-up approach which consists in identifying some interesting properties coming from these data in order to speculate on the characteristics that any extension to the Standard Model should share.

New physics and the choice of a parametrization

In the standard electroweak theory, the phenomenon of flavour mixing is described by a three-by-three unitary matrix. This matrix can be expressed in terms of four parameters which are usually taken as three rotation angles and one phase (or three in the case of Majorana neutrinos). There are only nine independent parametrizations in terms of these parameters. Adopting a particular parametrization of flavour mixing is arbitrary and not directly a physical issue. Nevertheless, it is quite likely that the actual values of flavour mixing parameters, including the strength of the CP violation, already provide us with interesting information about the physics beyond Standard Model. It is indeed highly probable that one of the parametrizations will be selected by new flavour physics.

It turns out that the analysis of the various decompositions of the quark mixing matrix singles out one of them, called P5. In this peculiar parametrization, the three mixing angles have all roughly the magnitude of the Cabibbo angle while the phase is remarkably small. These features shed some new light on the CKM matrix. It is indeed generally believed that only the quark mixing angles are linked to the mass hierarchy, not the phase. However, the P5 parametrization has a small phase which can also be expressed as a simple mass ratio. Furthermore, this parametrization presents an almost symmetric decomposition, allowing us to split the flavour mixing matrix into two very similar mixings coming from the up and down sector, respectively. For all these reasons, the P5 parametrization looks quite promising for model building.

The connection between masses and mixings

Throughout this work, it has been assumed that the mechanism responsible for the generation of masses is at the same time responsible for the flavour mixing, i.e., any change of the eigenvalues of masses would in general also lead to a change of the flavour mixing parameters. In many models based on flavour symmetries which go beyond the standard electroweak theory, the flavour mixing parameters are indeed functions of mass ratios. This hypothesis has been checked in two different contexts.

First, we have shown that a simple pattern for the quark mass matrix can lead to quite an accurate computation of the mixing angles. Then, starting from a mass relation which is amazingly precise for the charged leptons, we have proposed a generalization which is valid for all families. For that purpose, a suitable modification of the masses has been performed thanks to the concept of pseudo-mass which binds together the masses and mixings. The mixing angles in a particular decomposition were used in order to lift the almost degeneracy among the neutrinos and to tame the strong hierarchy among the quarks. This generalization has furthermore given some constraints for the neutrino masses.

The primacy of the square root of mass

In several widely different contexts, we have argued that the square root of a mass seems to play a important role, suggesting that it could be more fundamental than the mass itself.

In the Standard Model, all parametrizations of the flavour mixing matrix are mathematically equivalent. However, a particular parametrization could be more useful than the others. It is indeed reasonable to assume that any physics beyond Standard Model would select one of them. We have presented toy models where mass matrices are built out of more fundamental ones playing the role of their square root. We have applied some simple textures to this “square root” matrix. In particular, the angles in the P5 parametrization could be individually computed from simple ratios related to square roots of masses. A more general matrix has also been proposed in order to combine favourably the former results. This simple model has proved to lead to a mixing matrix rather similar to the observed CKM one.

In a quite different context, we have analyzed the properties of an accurate relation involving the charged lepton masses. These facts have convinced us that a geometrical description of the observed hierarchy is possible if the square roots of masses are used instead of the masses.

Intriguing results

Physics is the art of confronting mathematics with nature. The natural path of physics has always been to conceive some model, then to make numerical predictions and finally to confront them with observations. However, there are some famous examples where the things happen in the opposite order, for instance with the Balmer's formula. With humility, we would like to point out the numerical peculiarities we have found in a mass relation connecting the different families ($\psi = \frac{\pi}{3}$) and in the cascade breaking scheme we have presented ($\beta = \frac{2}{9}$). It is noteworthy that both numbers arise in models combining in some way the various families. The accuracy of these numbers is intriguing but, without any further explanation, they remain currently at the level of curiosities.

Symmetries and pole masses

In quantum field theories, the mass parameter appearing in a Lagrangian is subjected to the self-interaction of the particle in its own field. The corrections induced are logarithmically divergent and lead to infinities. The solution consists in a redefinition of the bare mass in order to cancel these infinities. This renormalization procedure is not unique. The propagator of a charged lepton is well-defined [1] and its renormalization leads to the definition of pole mass, corresponding to the physical mass. Because of the confinement, the propagator of a quark is infrared divergent. Another renormalization scheme is then used, leading to a well-defined running mass which is energy scale dependent.

The relation discovered by Koide is extremely accurate if the masses involved are the charged-lepton pole masses. This mere observation is however disturbing. Since the success of gauge theories and the advent of the Standard Model, most of the new flavour symmetries are based on horizontal groups. These symmetries are naturally implemented at the Lagrangian level, i.e. on the bare masses. We do not see any obvious way to impose a symmetry directly on a pole mass. Consequently, the Koide mass relation cannot be explained on basis of symmetry arguments, at least at the level of the Standard Model.

[1] At least if we neglect its decay.

Perspectives

All the observations we have done plead for a deeper modification of the Standard Model than just adding new symmetries or particles to the Lagrangian. In regard to all these results, our guess is that the masses do not result from a coupling to an elementary Higgs field. If we speculate on the mechanism responsible for the electroweak symmetry breaking, we would say that preon models could fulfil most of the properties presented here. On the one hand, a dynamical symmetry breaking could in principle lead to some relations between the pole masses. On the other hand, preons constitute a suitable framework where square roots of masses may appear. The practical way to implement such a dynamical model is beyond the scope of this work but constitutes its natural outcome.

Appendices

A
CP Violation

The parity symmetry $\mathcal{P}$ consists in the invariance of physics under a discrete transformation which changes the sign of the space coordinates, i.e. $\vec{r} \rightsquigarrow -\vec{r}$. The charge conjugation symmetry $\mathcal{C}$ transforms a particle into its antiparticle. The $\mathcal{C}$ symmetry asserts that it is a mere matter of convention which of them we call particles and which we call antiparticles. Naively, these symmetries are expected to be realized in Nature. However, the weak interactions maximally violate the $\mathcal{P}$ and $\mathcal{C}$ symmetry.

How to define then these operators? We start [65] from the realization that electromagnetic interactions are $\mathcal{C}$ and $\mathcal{P}$ invariant. We then assume that the strong interactions are $\mathcal{C}$ and $\mathcal{P}$ invariant too [1] Once these operators are defined, they can be used as a probe to check the invariance, or not, of the weak and strong interactions.

A.1 C and P Invariance of QED

A.1.1 Photon Field

By analogy with classical mechanics, the four-vector representing position must transform under $\mathcal{P}$ as $x^\mu = (t, \vec{r}) \rightsquigarrow x_\mu = (t, -\vec{r})$ and thus $\partial^\mu \rightsquigarrow \partial_\mu$. Similarly, the invariance of classical electromagnetism

[1] All available experimental data warrant this assumption although the θ-strong term could induce a $\mathcal{P}$ violation (Cf. App.B).

leads to :

$$\begin{aligned} \mathcal{P}\, A^{\mu}\, \mathcal{P}^{\dagger} &= A_{\mu} \\ \mathcal{P}\, F^{\mu\nu}\, \mathcal{P}^{\dagger} &= F_{\mu\nu} \quad . \end{aligned} \tag{A.1}$$

Consequently, the Lagrangian for the photon field

$$\begin{aligned} \mathcal{L}_{\mathrm{A}} &= -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\tfrac{1}{2}(\partial_{\mu} A_{\nu})(\partial^{\mu} A^{\nu}) + \tfrac{1}{2}(\partial_{\mu} A_{\nu})(\partial^{\nu} A^{\mu}) \end{aligned} \tag{A.2}$$

is invariant under P:

$$\mathcal{P}\, \mathcal{L}_{\mathrm{A}}\, \mathcal{P}^{\dagger} = \mathcal{L}_{\mathrm{A}} \quad . \tag{A.3}$$

It is readily seen that the charged current $j^{\mu} = (\rho, \vec{j})$ must change its sign under $\mathcal{C}$. In order for the electromagnetic interaction $A_{\mu} j^{\mu}$ to be $\mathcal{C}$ invariant, one postulates:

$$\mathcal{C}\, A_{\mu}\, \mathcal{C}^{\dagger} = -A_{\mu} \tag{A.4}$$

such that the Lagrangian is also invariant under $\mathcal{C}$

$$\mathcal{C}\, \mathcal{L}_{\mathrm{A}}\, \mathcal{C}^{\dagger} = \mathcal{L}_{\mathrm{A}} \quad . \tag{A.5}$$

Within these conventions, the $\mathcal{CP}$ transformation of the photon field reads:

$$(\mathcal{CP})\, A^{\mu}\, (\mathcal{CP})^{\dagger} = -A_{\mu} \quad . \tag{A.6}$$

A.1.2 Scalar field

We consider a massive charged scalar field, described by the Klein-Gordon Lagrangian

$$\mathcal{L} = (\partial_{\mu}\phi^{\dagger} - iqA_{\mu}\phi^{\dagger})(\partial^{\mu}\phi + iqA^{\mu}\phi) - m^2 \phi^{\dagger}\phi \quad . \tag{A.7}$$

In order for this Lagrangian to be invariant under parity, it is readily seen that ϕ must transform like

$$\begin{aligned} \mathcal{P}\phi\mathcal{P}^{\dagger} &= e^{i\alpha_p}\phi \\ \mathcal{P}\phi^{\dagger}\mathcal{P}^{\dagger} &= e^{-i\alpha_p}\phi^{\dagger} \end{aligned} \tag{A.8}$$

where the phase α_p is arbitrary.

The $\mathcal{C}$ transformation must link ϕ to its adjoint $\phi^\dagger$. Thanks to the transformation rules for the photon field, we find

$$\begin{aligned} \mathcal{C}\phi\mathcal{C}^\dagger &= e^{i\alpha_c}\phi^\dagger \\ \mathcal{C}\phi^\dagger\mathcal{C}^\dagger &= e^{-i\alpha_p}\phi \end{aligned} \tag{A.9}$$

with α_c an arbitrary phase.

The $\mathcal{CP}$ transformation of a complex scalar field is then

$$(\mathcal{CP})\,\phi\,(\mathcal{CP})^\dagger = e^{i\alpha}\phi^\dagger \quad . \tag{A.10}$$

A.1.3 Fermions

Parity

Let us first look at a generic Lorentz transformation

$$x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \tag{A.11}$$

where Λ is a 4×4 matrix such that

$$g_{\alpha\beta} = \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, g_{\mu\nu} \quad . \tag{A.12}$$

Under a Lorentz transformation, a Dirac spinor transforms as

$$\psi(x) \rightsquigarrow \psi'(x') = S(\Lambda)\,\psi(x') \tag{A.13}$$

where $S(\Lambda)$ is a 4×4 matrix such that

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\, \gamma^\nu \quad . \tag{A.14}$$

From the Dirac algebra we know that

$$\gamma^0\gamma_\mu\gamma^0 = \gamma^\mu \quad . \tag{A.15}$$

Since the parity transformation corresponds to a Lorentz transformation $\Lambda^\mu{}_\nu = \mathrm{diag}(1,-1,-1,-1)$, a comparison between (A.15) and (A.14) suggests:

$$\begin{aligned} \mathcal{P}\,\psi\,\mathcal{P}^\dagger &= e^{i\beta_p}\gamma^0\psi \\ \mathcal{P}\,\overline{\psi}\,\mathcal{P}^\dagger &= e^{-i\beta_p}\overline{\psi}\gamma^0 \quad . \end{aligned} \tag{A.16}$$

Consequently, the Dirac action is invariant under $\mathcal{P}$:

$$\begin{aligned}\mathcal{P}\,\mathcal{L}_{\mathrm{D}}\,\mathcal{P}^{\dagger} &= \overline{\psi}\gamma^0\{\gamma_\mu[i\partial^\mu - qA_\mu] - m\}\gamma^0\psi \\ &= \overline{\psi}\{\gamma^\mu[i\partial^\mu - qA_\mu] - m\}\psi \\ &= \mathcal{L}_{\mathrm{D}} \quad .\end{aligned}$$

Charge Conjugation

It is readily found that the action of charge conjugation reads:

$$\begin{aligned}\mathcal{C}\,\psi\,\mathcal{C}^{\dagger} &= e^{i\beta_c}\psi^c \\ \mathcal{C}\,\overline{\psi}\,\mathcal{C}^{\dagger} &= e^{-i\beta_c}\overline{\psi^c} \quad .\end{aligned} \tag{A.17}$$

The Dirac Lagrangian is then invariant under $\mathcal{C}$

$$\begin{aligned}\mathcal{C}\,\mathcal{L}_{\mathrm{D}}\,\mathcal{C}^{\dagger} &= -\psi^T C^{-1}\left[\gamma_\mu\left(i\partial^\mu + qA^\mu\right) - m\right] C\overline{\psi}^T \\ &= \psi^T\left[\gamma_\mu^T\left(i\partial^\mu + qA^\mu\right) + m\right]\overline{\psi}^T \\ &= -\overline{\psi}\left[\gamma_\mu\left(i\overleftarrow{\partial^\mu} + qA^\mu\right) + m\right]\psi \\ &= \mathcal{L}_{\mathrm{D}}\end{aligned} \tag{A.18}$$

where we have used the fact that ψ and $\psi^\dagger$ are anticommuting.

CP

Finally, combining all these results, one finds the action of the $\mathcal{CP}$ transformation on a spinor ψ:

$$(\mathcal{CP})\ \psi\ (\mathcal{CP})^{\dagger} = e^{i\xi_\psi}\gamma^0 C\overline{\psi}^T \tag{A.19}$$

$$(\mathcal{CP})\ \overline{\psi}\ (\mathcal{CP})^{\dagger} = -e^{-i\xi_\psi}\psi^T C^{-1}\gamma^0 \quad . \tag{A.20}$$

A.2 C and P invariance of QCD

Up to now, there is no evidence of a $\mathcal{C}$ or $\mathcal{P}$ violation by the strong interactions. The QCD Lagrangian reads

$$\mathcal{L}_{\mathrm{QCD}} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{q_x}\left[\delta_{xy}(i\gamma^\mu\partial_\mu - m_q) + g_s\gamma^\mu G^a_\mu\frac{\lambda^a_{xy}}{2}\right] q_y \tag{A.21}$$

Here, g_s is the strong coupling constant; x and y are colour indices; G^a_μ ($a = 1$ to 8) are the gluon fields; and the λ^a are the Gell-Mann matrices. The field-strength tensor is given by

$$F^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \tag{A.22}$$

where f^{abc} are the structure constant of $SU(3)$. Notice that the Gell-Mann matrices are either symmetric or anti-symmetric:

$$\lambda^{aT} = \lambda^{a*} = s^a \lambda^a \tag{A.23}$$

where $s^a = +1$ for $a = 1$, 3, 4, 6 and 8; and $s^a = -1$ for $a = 2$, 5 and 7.

Taking into account the transformations of the quark fields (A.16) and (A.17), the $\mathcal{P}$ and $\mathcal{C}$ invariance of the Lagrangian (A.21) gives the following transformations for the gluon fields:

$$\begin{aligned} \mathcal{P}\ G^a_\mu\ \mathcal{P}^\dagger &= G^{a\mu} \\ \mathcal{C}\ G^a_\mu\ \mathcal{C}^\dagger &= -s^a G^a_\mu \end{aligned} \tag{A.24}$$

while the field-strength tensor transforms as:

$$\begin{aligned} \mathcal{P}\ F^a_{\mu\nu}\ \mathcal{P}^\dagger &= F^{a\mu\nu} \\ \mathcal{C}\ F^a_{\mu\nu}\ \mathcal{C}^\dagger &= -s^a F^a_{\mu\nu} \quad . \end{aligned} \tag{A.25}$$

The kinematic part of the Lagrangian (A.21) is finally invariant under both transformations:

$$\begin{aligned} \mathcal{P}\ F^a_{\mu\nu} F^{a\mu\nu}\ \mathcal{P}^\dagger &= +F^{a\mu\nu} F^a_{\mu\nu} \\ \mathcal{C}\ F^a_{\mu\nu} F^{a\mu\nu}\ \mathcal{C}^\dagger &= +F^a_{\mu\nu} F^{a\mu\nu} \quad . \end{aligned} \tag{A.26}$$

It is also interesting to look at the transformation of $F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$ where we have defined the dual field $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. As $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$, one finds

$$\mathcal{P}\ F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\ \mathcal{P}^\dagger = -F^{a\mu\nu}\tilde{F}^a_{\mu\nu} \tag{A.27}$$

$$\mathcal{C}\ F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\ \mathcal{C}^\dagger = +F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \quad . \tag{A.28}$$

The presence of a term proportional to $F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$ would imply a violation of the parity. Since the charge conjugation is conserved, this term would break the $\mathcal{CP}$ symmetry.

B
Strong CP Problem

At first sight, the QCD Lagrangian (A.21) is invariant under $\mathcal{C}$ and $\mathcal{P}$. However, in principle, it should also contain a so-called θ-strong term proportional to $F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$ which violates parity while preserving charge conjugation. The presence of this $\mathcal{CP}$ violating term is closely related to the $U(1)_A$ problem and its solution as proposed by 't Hooft [66].

B.1 Axial U(1) problem

In the limit where the u, d and s quarks are massless, we have shown in Sec. 1.5.2 that the Lagrangian is invariant under a global $U(3)_L \otimes U(3)_R$ symmetry. However, the axial part of this approximate symmetry is spontaneously broken by quark condensates

$$\langle s\overline{s}\rangle = \langle d\overline{d}\rangle = \langle u\overline{u}\rangle \neq 0 \quad . \tag{B.1}$$

The vector symmetries $SU(3)_V \otimes U(1)_V$ are unbroken and correspond to the $SU(3)$ of flavour and to the baryon number, respectively. One expects then to find nine Goldstone bosons associated with the broken generators. Eight of them have effectively been observed. The $\pi^{\pm}$, π^0, $K^{\pm}, K^0, \overline{K^0}$ and η are indeed massless in first approximation. The spontaneous breakdown of the $U(1)_A$ symmetry should lead to a ninth Goldstone bosons. The η' has the right quantum number (a pseudoscalar singlet) to play this role but its mass is too close to the QCD confining scale (~ 1 GeV) to be considered as light [1]. This is the $U(1)_A$ problem.

[1] It can however mix with the η.

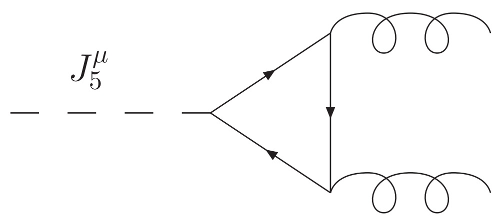


Figure B.1: *Chiral anomaly.*

The solution to the missing Goldstone boson came from 't Hooft who showed that the QCD vaccum has a non trivial structure. The complex nature of the QCD vacuum makes $U(1)_A$ not a true symmetry of QCD even though it is an apparent symmetry of the classical Lagrangian in the limit of vanishing quarks masses. The chiral anomaly for the axial currents is induced at the one-loop level (Cf. Fig. B.1) and leads indeed to a non-vanishing divergence:

$$\partial_\mu J_5^\mu = N \frac{g_s^2}{32\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu} \tag{B.2}$$

where N is the number of flavour. It can be shown [66, 67] that the chiral anomaly leads to the breakdown of $U(1)_A$ without producing a Goldstone boson. In return, an extra term should be added the QCD Lagrangian:

$$\mathcal{L}_\theta = \theta_{\rm QCD} \frac{g_s^2}{32\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu} \quad . \tag{B.3}$$

This new term violates the parity [2] but conserves the charge conjugation invariance, so it violates $\mathcal{CP}$ (Cf. App. A). Therefore, this term should induce a neutron electric dipole moment. The existing strong bound ($|d_n| < 3\times10^{-26} e$ cm [68]) requires $\theta_{\rm QCD}$ to be very small. In particular, chiral perturbation theory gives $|d_n| \approx 3.6\times10^{-16}\theta_{\rm QCD}\, e$ cm [69] leading to $\theta_{\rm QCD} < 3 \times 10^{-10}$. This unexpected small value implies some fine-tuning and is known as the strong CP problem.

One could try to avoid this problem by imposing the $\mathcal{P}$ or $\mathcal{CP}$ invariance of the QCD Lagrangian, thereby setting $\theta_{\rm QCD} = 0$ by hand. However, in the Standard Model for electroweak interactions, the quark

[2] It also breaks the time reversal invariance.

mass matrices are in general arbitrary complex matrices which are bi-diagonalized by bi-unitary transformations. These transformations include in particular the chiral transformation necessary to make the masses real:

$$\theta_{\text{mass}} = \arg\det(M^p M^n) \quad . \tag{B.4}$$

Under a chiral transformation of the quarks fields both θ_{QCD} and θ_{mass} change, but their sum θ_s remains constant:

$$\theta_s = \theta_{\text{QCD}} + \theta_{\text{mass}} \quad . \tag{B.5}$$

Consequently, the solution $\theta_{\text{QCD}} = 0$ cannot solve alone the strong CP problem since, in general, the mass matrices are completely arbitrary and complex.

Various solutions to this issue have been proposed. In particular, the problem can be solved thanks to an additional chiral symmetry. For instance, if one of the quarks has a vanishing mass, θ_{mass} would be arbitrary since $\det(M^p M^n) = 0$ and we can set $\theta_s = 0$ without loss of generality. This most simple solution with a vanishing quark mass is however excluded [70]. Another elegant solution consists in the introduction of a global $U(1)_{\text{PQ}}$ symmetry under which both the quarks and two Higgs doublets transform non-trivially [71]. The parameter θ_s becomes then a dynamical variable and can be relaxed dynamically to zero. Since the $U(1)_{\text{PQ}}$ symmetry is spontaneously broken by the vacuum, there should exist a Goldstone boson, named axion, with a very small mass. Such an axion linked to the Fermi scale has however been ruled out (see [72] for an argument based on $K^+ \to \pi^+ a$).

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