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Non-perturbative Quantum Electrodynamics in low dimensions

Doctoral dissertation presented by
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in fulfilment of the requirements for the degree of Doctor in Sciences

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Contents

CHAPTER 1

Introduction

A quest for an enhanced understanding, an endeavour for the simplest explanation. This is a major theme in Physics at the beginning of the twenty-first century. In this long and arduous search, the successes of high energy physics are among the most significant contributions to Science. This is especially true with the discovery of the Brout-Englert-Higgs boson at the Large Hadron Collider. Beyond the technological developements followed by progress in the fundamental sciences, advances in physics seem to contribute to discoveries in mathematics, and conversely. Namely, this virtuous circle between physics and mathematics appears to bring significant improvement in both fields since Newton's work.

Up to now, quantum field theories are the relevant frameworks to describe the physics of the smallest constituents of matter that we know and of their non-gravitational interactions. However, from the intuitive point of view, these theories are particularly complicated to present in a non expert language. Furthermore, the interpretation of quantum mechanics is still subjected to a controversial debate in order to reach a

broad consensus, against or in favour of its orthodox version. On the other hand, a fully consistent mathematical formulation of all the aspects of quantum field theories has still to be completely defined.
Actually, an essential cornerstone to the modern approach towards the unification of quantum interactions is the gauge invariance principle. The dominant framework available for the study of gauge theories remains a specific approximation: the perturbative approach, which is at the origin of the major phenomenological successes of gauge theories. Nonetheless important questions cannot be answered in the perturbative framework. While extremely elegant and powerful techniques were developed in the context of supersymmetric Yang-Mills theory and M-theory, in the absence of supersymmetry the understanding of non-perturbative effects still requires the developement and the improvement of alternative techniques. This is why efforts are pursued in lattice gauge theories, or based on functional equations.
At first sight, the title of the thesis could raise an understandable question: Isn't Quantum ElectroDynamics (QED) a well-known theory? Indeed, quantum electrodynamics is the quantum field theory which is the best verified experimentally, among all gauge theories, so that it may appear as thoroughly understood. To put it into perspective, this impression is only justified if the perturbative behaviour of the theory is considered in $3+1$ dimensions. Incidentaly, non-perturbative questions remain to be answered in presence of strong electromagnetic fields, such as Schwinger pair production in an electric field and vacuum birefringence in a strong magnetic field.
The present work intends to explore non-perturbative aspects of low dimensional formulations of quantum electrodynamics. Concerning the $1+1$ and $2+1$ dimensional situations, even tough they may not be of direct relevance for the phenomenology of high energy physics, these low dimensional versions of QED can still excite the curiosity of theoreticians, as well as condensed matter physicists. Although interesting for their own sake, these theories provide also valuable playgrounds to study more realistic quantum field theories, as for example quantum chromodynamics. Besides their formal relationship with high energy physics, the theories considered here share many features with the effective models of

quasi-particles in some two-dimensional materials of interest. Noteworthy examples of such behaviours are graphene or specific strong topological insulators.
The outline of the thesis is:

- Chapter 2 gives an overview of both classical and quantum electrodynamics in $1+1$ dimensions, which sheds light on the important features that will be emphasized in the solution presented in Chapter 3. After the review of the role of topological degrees of freedom in the pure quantum electrodynamics, the exact solution of massless QED_{1+1}, namely the Schwinger model, is briefly described.

- Chapter 3 gives an account of the solution for the Schwinger model on the manifold $\mathbb{R}\times S^1$, with a specific emphasis on the role of large gauge transformations rendered manifest by the compactification of space into a circle. The consequence is that the topological gauge degree of freedom is singled out and its role in the dynamics is displayed, especially in relation with the axial anomaly. Furthermore, it is possible to apply in the case of this model a quantisation free of gauge fixing. A fermion field dressed by a photon cloud is introduced in line with a suggestion by Dirac. Thanks to the factorization of the local gauge transformations and gauge degrees of freedom, the description concentrates on the dynamics of "composite" fermion fields in interplay with the topological gauge degree of freedom. Finally, the exact solution of the model is recovered providing some new understanding of its non-perturbative properties.

- Chapter 4 is dedicated to the study of massless quantum electrodynamics in $2+1$ dimensions on the manifold $\mathbb{R}\times\mathbb{R}^2$ with one electron species. After a review of the main features of the theory, the factorization of the local gauge symmetry and the gauge degrees of freedom is performed in parallel with the technique exposed in Chapter 3, and the dynamics of dressed fermion fields is considered. We explore the structure of the vacuum state, using a variational procedure. An ansatz for the lowest energy state is suggested, inspired by the BCS (Bardeen-Cooper-Schrieffer) theory of superconductivity. Its wave function is determined by solving a non linear integral equation. Subsequently, the dynamics of the pseudo-particles propagating in the condensate is described and an argument in favour of their confinement is given. Eventually, the effect of the condensate on the propagation of the physical electromagnetic degrees of freedom is examined.

- Chapter 5 gives an account of the case of massless relativistic fermions on a plane in a constant perpendicular magnetic field, which is shown to be a non trivial problem. The question of a vacuum condensation due to the magnetic field is addressed.

- Chapter 6 includes a conclusion and discusses the perspectives of the work.

Finally, some useful technical informations are gathered in the appendices A and B, while the reader can find a concise exposition of the other research work done in the appendix C.

Throughout the document an implicit choice of units is made such that $\hbar = c = 1$.

This work is based on the following publications:

- M. Fanuel and J. Govaerts. Dressed fermions, modular transformations and bosonization in the compactified Schwinger model. *J.Phys.*, A **45**:035401, 2012
- M. Fanuel and J. Govaerts. Non-Perturbative Dynamics, Pair Condensation, Confinement and Dynamical Masses in Massless QED_{2+1}. Submitted to *J.Phys.* A, hep-th:1405.7230

while the research presented in the publications

- M. Fanuel and S. Zonetti. Affine quantization and the initial cosmological singularity. *Europhys.Lett.*, **101**:10001, 2013
- M. Fanuel, J. Govaerts, G. Y. H. Avossevou and A. F. Dossa. The $\mathcal{N} = 1$ Supersymmetric Wong Equations and the Non-Abelian Landau Problem, Submitted to *J.Phys.* A, hep-th:1405.5335

is summarized in appendix C.

CHAPTER 2

General features of QED_{1+1}

This chapter is dedicated to the study of massless quantum electrodynamics in one space and one time dimensions. After a succint glimpse of the classical dynamics in section 2.1, sections 2.2 and 2.3 deal with the quantum dynamics of QED_{1+1} in the absence and in the presence of dynamical matter, respectively.

2.1 Brief overview of classical electrodynamics on the line

As an instructive preamble, we review the typical features of classical electrodynamics in $1+1$ in the absence of dynamical matter, where the space topology is the one of a line. The language of this preliminary section is intended to be "heuristic" and follows references [1, 2]. The intuition gained from the classical physics will provide clues to interpret better the more rigorous study of the quantum theory with and without interactions, that will follow.

Notational conventions include the Levi-Civita tensor $\epsilon^{\mu\nu}$ being defined by $\epsilon^{01} = +1$, and the Minkowski metric taken to be $\eta_{\mu\nu} = \text{diag}\,(1, -1)$. In $D = 2$ space-time dimensions, the gauge coupling constant e has dimension $[e] = M^1$ in units of mass, while the gauge and matter fields have mass dimensions $[A_\mu] = M^0$ and $[\psi] = M^{1/2}$. The case of fermionic dynamical matter will be considered in section 2.3, while the following introductory analysis will include only static point-like matter sources.

The classical action of electrodynamics on the line, in the presence of a pointlike charge is

$$S_{\text{class}} = \int \mathrm{d}t \int \mathrm{d}x \{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \frac{e\theta}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} - e j^0 A_0 \}, \tag{2.1}$$

where $j^0(x) = \rho(x) = \delta(x - x_0)$ is the charge density of a static charge at the point $x = x_0$. The θ-term in 1+1 dimensions $\propto \epsilon_{\mu\nu} F^{\mu\nu}$, which is reminiscent of the QCD θ-term, plays a subtle role in the classical and quantum formulations of this simple model, as will be made clear in the sequel. The electric field $E^1 = F_{01}$ has to satisfy the Maxwell equations

$$\partial_1 E^1(x) = e\delta(x - x_0), \quad \partial_0 E^1 = 0, \tag{2.2}$$

and, consequently, has to be constant in space and in time, away from the point-like charge, as illustrated in Figure 2.1. The presence of the charge is responsible for a jump in the electric field, as a consequence of the integration of the Gauss law : $E^1(x) = e\theta(x - x_0) + \text{const.}$ In a sense, the discontinuity of the electric field accross the charge tells us that $\theta_E = 2\pi E(x)/e$ may be intuitively considered as a constant angle, jumping by a multiple of 2π across a charge. Hence, we understand that the θ angle in the classical action can be interpreted instinctively as the analogue of a "background electric field". A first comment about this simple classical theory is that the Coulomb potential behaves $\propto |x|$, so that the electrostatic energy increases with the distance. It is often claimed that this feature is a strong argument in favour of confinement of static electric charges at the classical level. Nevertheless, the same question in the quantum theory, with massive or massless dynamical matter, remains intriguing from this point of vue. A second comment

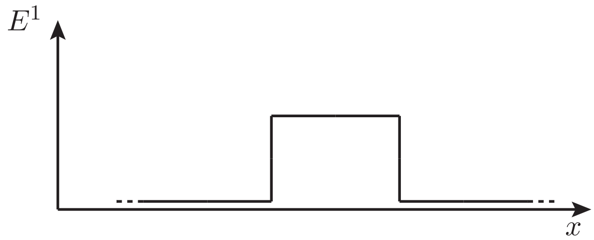


Figure 2.1: A typical electric field configuration due to the presence of two opposite point electric charges is schematically illustrated.

concerns the energy density of classical configurations of the electric field, as we shall clarify shortly. The energy density associated to a solution of the equations of motion $E^1(x)$ is

$$\mathcal{H} = \frac{1}{2}(E^1(x) - \frac{e\theta}{2\pi})^2. \tag{2.3}$$

Allowing for a finite number of pointlike charges, we find that, in order to minimize the energy density, the asymptotic behaviour of the electric field should be $E^1(x) \to e\theta/2\pi$ as $x \to \pm\infty$. Because opposite charges produce opposite jumps in the electric field, this means that the total charge has to vanish. This is an heuristic argument in favour of confinement, as long as $\theta \neq \pi$. In the peculiar situation $\theta = \pi$, the electric field may behave as $E^1(x) \to e$ or $E^1(x) \to 0$ as $x \to \pm\infty$, so that a single point-like charge can minimize the energy density.
After these simple considerations and cursory glance at the classical situation, we can proceed to the study of pure quantum electrodynamics, in a more rigourous language.

2.2 Pure quantum electrodynamics on the circle

Before treating the interacting case, we will review, as a first study of the quantum theory, the case of pure gauge QED on $S^1 \times \mathbb{R}$ where the length

of the spatial circle is $L > 0$. The features that we want to emphasize are made manifest by the compactification and the study will follow the Hamiltonian constraint analysis as advocated by Dirac [3].
As a matter of fact, this system provides a very simple example where the dynamics resides essentially in the "topological" sector, as will be shortly demonstrated. We take here the opportunity to study the quantum theory on the circle, rather than on a line, in order to highlight the role of this topological sector, which will be of decisive importance when we will investigate the interacting case in Chapter 3.
The crucial step is the decomposition of the spatial component of the gauge field into the sum of its Fourier zero-mode and k-modes $A_1(t,x) = a_1(t) + \partial_1\phi(t,x)$. The Wilson loop degree of freedom

$$a_1(t) = \frac{1}{L}\int_0^L \mathrm{d}x A_1(t,x), \tag{2.4}$$

will play a predominant role, as can be guessed from the Gauss law which requires in $1+1$ dimensions that the electric field is constant in space. The pure gauge classical action in presence of a topological θ term has the simple expression

$$S_0 = \int \mathrm{d}t \int_0^L \mathrm{d}x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\frac{e\theta}{2\pi})\epsilon_{\mu\nu}F^{\mu\nu}\}, \tag{2.5}$$

where the gauge field obeys periodic boundary conditions. The presence of the topological density proportional to θ can be interpreted as the effect of an homogeneous electric field.

The associated Lagrangian

$$L_0 = \frac{L}{2}(\dot{a}_1^2 + 2\frac{e\theta}{2\pi}\dot{a}_1) + \int_0^L \mathrm{d}x\frac{1}{2}(\partial_1\dot{\phi} - \partial_1 A_0)^2 \tag{2.6}$$

mainly describes the dynamics of the electric field on the circle. Under a gauge transformation, the gauge potential transforms as $A'_\mu(t,x) = A_\mu(t,x) + \frac{1}{e}\partial_\mu\alpha(t,x)$. The gauge parameter $\alpha(t,x) = \alpha_0(t,x) + 2\pi x\ell/L$ can be decomposed in terms of a periodic function $\alpha_0(t,x) = \alpha_0(t,x+L)$ and an integer $\ell \in \mathbb{Z}$, corresponding to the winding number. In the

Hamiltonian formalism, the conjugate momenta are

$$\frac{\partial L_0}{\partial \dot{a}_1} = L(\dot{a}_1 + \frac{e\theta}{2\pi}) = p^1, \tag{2.7}$$

$$\frac{\partial L_0}{\partial \dot{\phi}} = -\partial_1(\partial_1\dot{\phi} - \partial_1 A_0) = \pi_\phi. \tag{2.8}$$

The classical observation that $\dot{A}^0$ is absent from the Lagrangian gives us a primary constraint $\pi_0 = 0$, with $\{A^0(x), \pi_0(y)\} = \delta_{S^1}(x-y)$. It reminds us that A_0 plays the role of Lagrange multiplier for the Gauss law $\partial_1 E^1 = 0$, where the electric field is $E^1 = F_{01}$. Poisson brackets are given by $\{a_1; p^1\} = 1$, $\{\phi(t,x); \pi_\phi(t,y)\} = \delta_{S^1}(x-y)$, where the variable $p^1(t)$ may be considered as an electric field constant in space. Hence, the classical Hamiltonian is easily obtained,

$$H_0 = \frac{1}{2L}(p^1 - \frac{e\theta L}{2\pi})^2 + \int_0^L \mathrm{d}x\{-\frac{1}{2}\pi_\phi \Delta^{-1}\pi_\phi + A_0\pi_\phi\}. \tag{2.9}$$

A consistent time evolution of the constraint $\pi_0 = 0$, requires $\{H_0, \pi_0\} = 0$. This consistency requirement results in a secondary constraint which is nothing else than the Gauss law $\pi_\phi = 0$. The natural conclusion is that the Gauss law, generating the local gauge transformations, helps us to eliminate from the formulation the k-modes of $A_1(t,x)$. The relevant quantities are therefore $a_1(t)$ and its conjugate momentum $p^1(t)$, which are "global", i.e. independent of the coordinate x. The gauge symmetry transforms the zero-mode by a shift

$$a_1(t) \to a_1'(t) = a_1(t) + \frac{2\pi\ell}{eL}, \qquad \ell \in \mathbb{Z}. \tag{2.10}$$

Such a transformation is not infinitesimally generated and is called a "large gauge transformation".

Reducing the dynamics to the non-trivial degrees of freedom, we find the curious result for the Hamiltonian

$$H_{\mathrm{red}} = \frac{1}{2L}(p^1 - \frac{e\theta L}{2\pi})^2, \tag{2.11}$$

which is effectively the Hamiltonian of a classical particle on a "dual circle" of length $2\pi/eL$, and where the length L somehow plays the role of a mass.

It is a simple exercise to proceed to the quantisation of the system, by introducing operators $\hat{a}_1$ and $\hat{p}^1$ verifying the Heisenberg algebra $[\hat{a}_1;\hat{p}^1] = \mathrm{i}$. The eigenstates of the canonical operators are $|a_1\rangle$ and $|p^1_n\rangle$, with $\hat{a}_1|a_1\rangle = a_1|a_1\rangle$ and $\hat{p}^1|p^1_n\rangle = p^1_n|p^1_n\rangle$, $n \in \mathbb{Z}$. Their overlap has to take the form

$$\langle a_1|p^1_n\rangle = \sqrt{\frac{eL}{2\pi}}\exp\,\mathrm{i}neLa_1, \qquad p^1_n = eL(n+\frac{\theta_0}{2\pi}) \tag{2.12}$$

where $\theta_0/2\pi \in [0,1[$ is an holonomy parametrizing the inequivalent representations of the Heisenberg algebra on the circle[1]. The appearance of this parameter is a pure quantum mechanical effect.

The "momentum" eigenstates provide an orthonormal basis $\langle p^1_m|p^1_n\rangle = \delta_{n,m}$. In "position" space, the momentum operator can be represented by $-\mathrm{i}\partial_{a_1} + eL\theta_0/2\pi$. Hence the unitary operator, associated to the winding number $\ell \in \mathbb{Z}$, realizing the corresponding large gauge transformation, is given by

$$\hat{U}(\ell) = \exp\,\mathrm{i}\frac{2\pi}{eL}\ell(\hat{p}^1 - \frac{\theta_0}{2\pi}eL), \tag{2.13}$$

and verifies $\hat{U}(\ell)\hat{a}_1\hat{U}^\dagger(\ell) = \hat{a}_1 + 2\pi\ell/(eL)$, while the composition law $\hat{U}(\ell)\hat{U}(k) = \hat{U}(\ell+k)$ is natural. For that reason, one may interpret a gauge transformation in a "topological context" as being a shift in the coordinate of a point, sending a point to an equivalent one on the line. Nevertheless, the "momentum" eigenstates are left invariant by a large gauge transformation $\hat{U}(\ell)|p^1_n\rangle = |p^1_n\rangle$. The quantum Hamiltonian is straightforwardly given by

$$\hat{H}_{\mathrm{red}} = \frac{1}{2L}(\hat{p}^1 - \frac{e\theta L}{2\pi})^2. \tag{2.14}$$

Therefore the gauge invariant states $|p^1_n\rangle$ are also the energy eigenstates, corresponding to states with a quantised value for the electric field. The spectrum of the Hamilton operator is readily obtained,

$$\hat{H}_{\mathrm{red}}|p^1_n\rangle = \frac{1}{2L}\Big(neL - \frac{(\theta-\theta_0)}{2\pi}eL\Big)^2|p^1_n\rangle. \tag{2.15}$$

[1] A more general discussion can be found in [4].

Undoubtedly the relevance of the appearance of $\theta_0/2\pi$ in the spectrum of the quantum Hamiltonian has to be addressed. We may observe that setting $\theta_0 = \theta$ removes the θ angle from the spectrum. The interpretation of the gauge invariant eigenstates $|p^1_n\rangle$ is that they correspond to states where the electric field is constant in space, but takes a quantised value. The minimal absolute value of the electric field is reached in the state $|p^1_0\rangle$ and corresponds to $p^1_0 = eL\theta/2\pi$, that is to say the value of the homogeneous electric field associated to the θ angle.

These conclusions are valid in absence of dynamical matter.

2.3 The Schwinger model on a line

After the overview of the properties of pure QED_{1+1}, our attention is inevitably drawn to the interacting case. The purpose of this preliminary study is to review the exact solution of massless QED_{1+1} on $\mathbb{R} \times \mathbb{R}$ [5], while the next chapter will be focused on the more elaborate solution of the same theory on the manifold $\mathbb{R} \times S^1$, with a particular emphasis on the role of large gauge transformations[2]. The solution outlined here relies on the bosonization method, which seems to be one of the most natural formulations. The discussion is inspired by the article [6] and the book [7], while the conventions follow reference [8]. Besides these references, many inspiring research works have been carried out, for example, in various gauges [9, 10], or at finite temperature [11], while the Schwinger model may also be used as a test-bed to investigate techniques relevant to QCD_{3+1} as for instance in [12].

In the expression hereafter for the Lagrangian density of the massive or massless Schwinger model, a possible choice for the Clifford-Dirac algebra of γ^μ matrices ($\mu = 0, 1$) is taken to be given by $\gamma^0 = \sigma^1$ and $\gamma^1 = \mathrm{i}\sigma^2$, the chirality matrix then being $\gamma_5 = \gamma^0\gamma^1 = -\sigma^3$, while the σ^i ($i = 1, 2, 3$) stand of course for the usual Pauli matrices. The complex

[2]Chapter 3 will provide a favourable framework in order to study the influence of a θ angle.

coordinates and the (anti-)holomorphic derivatives

$$z = \quad -\mathrm{i}(x-t), \qquad \partial_z = -\frac{\mathrm{i}}{2}(\partial_t - \partial_x), \tag{2.16}$$

$$\bar{z} = \quad \mathrm{i}(x+t), \qquad \partial_{\bar{z}} = -\frac{\mathrm{i}}{2}(\partial_t + \partial_x), \tag{2.17}$$

with $t = x^0$ and $x = x^1$, will prove themselves useful.
Because it is a common thread of our work, the analysis of the model will follow the Hamiltonian approach, starting from the Lagrangian of massless QED_{1+1}

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}\mathrm{i}\partial_\mu\bar{\psi}\gamma^\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi. \tag{2.18}$$

At first sight it would appear that the classical theory will lead to an interacting quantum field theory for the dynamics of photons and electrons. It is however surprising that the exact solution describes a free massive pseudo-scalar. Before pursuing further, it is interesting to mention that, although a fermion can be bosonized in $1+1$ dimensions, the associated boson does not correspond to the massive pseudo-scalar of the Schwinger model.
Quantisation requires first to determine the constraints of the classical formulation. After a straightforward Hamiltonian analysis, the Hamiltonian action readily follows[3]

$$S = \int \mathrm{d}t \int \mathrm{d}x \Big\{ \quad \partial_0 A^1\pi_1 + \frac{1}{2}\mathrm{i}\psi^\dagger\partial_0\psi - \frac{1}{2}\mathrm{i}\partial_0\psi^\dagger\psi - \mathcal{H} \\ -A^0\phi + \partial_1(A^0\pi_1)\Big\} \tag{2.19}$$

where $\pi_1 = F_{01} = -E$ is the momentum conjugate to A^1, while A^0 has to be considered as a Lagrange multiplier for the first-class constraint

$$\phi = \partial_1\pi_1 + e\psi^\dagger\psi. \tag{2.20}$$

Furthermore, the Hamitonian density is

$$\mathcal{H} = \frac{1}{2}\pi_1^2 - \frac{1}{2}\mathrm{i}\psi^\dagger\gamma_5\partial_1\psi + \frac{1}{2}\mathrm{i}\partial_1\psi^\dagger\gamma_5\psi - eA^1\psi^\dagger\gamma_5\psi. \tag{2.21}$$

[3]All the indices are euclidian.

The dynamics is encoded in the Poisson and Dirac brackets, respectively given by

$$\{A^1(x,t), \pi_1(y,t)\} = \delta(x-y), \tag{2.22}$$

$$\{\psi_\alpha(t,x), \psi^\dagger_\beta(,y)\} = -\mathrm{i}\delta_{\alpha\beta}\delta(x-y), \tag{2.23}$$

so that the constraint ϕ, namely the Gauss law, is first class. In the quantum theory, Poisson brackets are replaced by commutators and Dirac brackets by anticommutators, at the reference time $t=0$,

$$[A^1(x), \pi_1(y)] = \mathrm{i}\delta(x-y), \tag{2.24}$$

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y). \tag{2.25}$$

In order to formulate the bosonized form of the quantum Hamiltonian, still to be defined, we introduce the chiral fermions verifying $\gamma_5\psi_\pm = \pm\psi_\pm$, with

$$\psi_\pm(x,t) = \frac{1\pm\gamma_5}{2}\psi(x,t). \tag{2.26}$$

Namely, these chiral fermions may be understood as coherent states of chiral bosons, with the help of the bosonization formulas,

$$\psi_\pm(x,t) \;\; = \big(\frac{\tilde{\mu}}{2\pi}\big)^{1/2} : e^{\pm\mathrm{i}\sqrt{4\pi}\varphi_\pm(x,t)} :, \tag{2.27}$$

$$\psi^\dagger_\pm(x,t) \;\; = \big(\frac{\tilde{\mu}}{2\pi}\big)^{1/2} : e^{\mp\mathrm{i}\sqrt{4\pi}\varphi_\pm(x,t)} :, \tag{2.28}$$

where $\tilde{\mu} > 0$ is a mass scale needed for dimensional reasons, since the theory is not defined on a circle. The normal ordering prescription will be defined below. The chiral bosons are imagined as left and right propagating waves, namely as functions of z and $\bar{z}$, and may be expanded in plane waves as follows[4]

$$\varphi_+(x,t) \;\; = \varphi(z) = \int_{k>0} \frac{\mathrm{d}k}{2\pi}\frac{1}{2k}[b_+(k)e^{-kz} + b^\dagger_+(k)e^{kz}], \tag{2.29}$$

$$\varphi_-(x,t) \;\; = \bar{\varphi}(\bar{z}) = \int_{k>0} \frac{\mathrm{d}k}{2\pi}\frac{1}{2k}[b_-(k)e^{-k\bar{z}} + b^\dagger_-(k)e^{k\bar{z}}], \tag{2.30}$$

[4]The operator $\bar{\varphi}$ is not the hermitian conjugate of φ.

where we have defined $b^\dagger_\pm(k) = b_\pm(-k)$, and with the only non vanishing commutators $[b_\pm(k), b^\dagger_\pm(k')] = 2\pi\delta(k-k')$. Hence, the Fock vacuum is defined as verifying $b_\pm(k)|0\rangle = 0$ and, as a consequence, the prescription $: \, :$ will merely order creators to the left of annihilators.
As a result, chiral fermions can be associated to the following holomorphic and anti-holomorphic fields $\psi_+(x,t) = \psi(z)$ and $\psi_-(x,t) = \bar{\psi}(\bar{z})$, where $\bar{\psi}$ is not the Dirac conjugate of ψ.
It is customary to choose to normalize the expectation values in the Fock vacuum of the chiral fields as

$$\langle\varphi(z)\varphi(z')\rangle = -\frac{1}{4\pi}\ln[\tilde{\mu}(z-z')], \tag{2.31}$$

$$\langle\bar{\varphi}(\bar{z})\bar{\varphi}(\bar{z}')\rangle = -\frac{1}{4\pi}\ln[\tilde{\mu}(\bar{z}-\bar{z}')], \tag{2.32}$$

where the scale $\tilde{\mu}$ has been introduced for dimensional consistency. Because the quantum Hamiltonian includes products of operators defined at the same points, it may be divergent and, therefore has to be defined by preserving gauge invariance. To do so, the point-splitting technique is particularly appropriate in order regularise the short distance divergencies in the fermionic bilinears. Hence we need to evaluate, for instance, the operator $\psi^\dagger_\pm(x,0)\psi_\pm(x,0)$ at the reference time $t=0$, by the insertion of a Wilson line

$$\psi^\dagger_\pm(x+\epsilon,0)e^{\mathrm{i}e\int_x^{x+\epsilon}A^1(y)\mathrm{d}y}\;\psi_\pm(x,0) \tag{2.33}$$

in the limit where $\epsilon \to 0$. The calculation can be performed, following reference [8], with the help of the property

$$:e^{\mathrm{i}\alpha\varphi(z)}::e^{\mathrm{i}\beta\varphi(z')}:=:e^{\mathrm{i}\alpha\varphi(z)+\mathrm{i}\beta\varphi(z')}:e^{-\alpha\beta\langle\varphi(z)\varphi(z')\rangle}. \tag{2.34}$$

The limit of the bilinears in (2.33) at the reference time $t=0$, when $\epsilon \to 0$, is taken after the subtraction of the singular terms in ϵ, so that we obtain

$$N[\psi^\dagger(x,0)\gamma_5\psi(x,0)] = -\frac{eA^1(x)}{\pi} + \frac{1}{\sqrt{\pi}}\partial_1[\varphi_+(x,0)-\varphi_-(x,0)]. \tag{2.35}$$

Similarly, the fermionic kinetic term can be defined thanks to the same technique, after the subtraction of the divergent small distance terms, yielding the result

$$\begin{aligned}&\tfrac{1}{2}N[-\mathrm{i}\psi^\dagger(x)\gamma_5\partial_1\psi(x)+\mathrm{h.c.}]\\&=\big(\partial_1\varphi_+(x)\big)^2+\big(\partial_1\varphi_-(x)\big)^2-\tfrac{1}{2\pi}\big(eA^1(x)\big)^2.\end{aligned}\tag{2.36}$$

Hence, the Hamiltonian operator is ordered in a bosonic formulation by a gauge invariant procedure, that is to say,

$$\hat{\mathcal{H}}=\frac{1}{2}\pi_1^2+\big(\partial_1\varphi_+\big)^2+\big(\partial_1\varphi_-\big)^2+\frac{(eA^1)^2}{2\pi}-\frac{eA^1}{\sqrt{\pi}}\partial_1[\varphi_+-\varphi_-],\tag{2.37}$$

while the Gauss constraint in the bosonic form reads

$$\hat{\phi}=\partial_1\pi_1+\frac{e}{\sqrt{\pi}}\partial_1[\varphi_++\varphi_-].\tag{2.38}$$

For convenience, it is useful to define the rescaled fields $\tilde{\varphi}_\pm=\sqrt{4\pi}\varphi_\pm$, and to introduce the mass parameter $\mu=|e|/\sqrt{\pi}$. After the completion of a square, the quantum Hamiltonian can be recast in the form

$$\hat{\mathcal{H}}=\frac{1}{2}\pi_1^2+\frac{1}{2}\big(\frac{\partial_1\pi_1}{\mu}\big)^2+\frac{1}{2}\mu^2\big(A^1-\frac{1}{2e}\partial_1[\tilde{\varphi}_+-\tilde{\varphi}_-]\big)^2+\phi',\tag{2.39}$$

where $\phi'=(\phi/2\mu)^2-\phi\partial_1\pi_1/\mu^2$ is a pure constraint. Hence, it is natural to introduce the following definition of a Bose field and its conjugate momentum:

$$\Phi=-\frac{1}{\mu}\pi_1,\quad \Pi_\Phi=\mu\big(A^1-\frac{1}{2e}\partial_1[\tilde{\varphi}_+-\tilde{\varphi}_-]\big).\tag{2.40}$$

As a result, the Hamiltonian becomes the one of a free boson of mass $\mu=|e|/\sqrt{\pi}$,

$$\hat{\mathcal{H}}=\frac{1}{2}\pi_\Phi^2+\frac{1}{2}\big(\partial_1\Phi\big)^2+\frac{1}{2}\mu^2\Phi^2+\phi'.\tag{2.41}$$

Finally, with the help of the definitions (2.40), the canonical commutator may be easily checked

$$[\Phi(x),\Pi_\Phi(y)]=\mathrm{i}\delta(x-y),\tag{2.42}$$

while the other non vanishing commutators are obtained straightforwardly. Since the equivalence between the Hamiltonian of the original formulation and the Hamiltonian of the massive pseudo-scalar is now self-evident, canonical quantisation can be pursued using, for example, the projector approach of reference [6], in order to make use of the constraint ϕ to project the dynamics onto the physical Hilbert space.

2.4 Conclusions

Summarising, although the examples of the pure classical and quantum QED_{1+1} are admitedly elementary, they can provide a valuable intuition for the themes exposed in the next chapter. On the contrary, the less trivial dynamics of QED_{1+1} already teaches us a lesson. As a matter of fact, in the Schwinger model, the perturbative intuition is misleading. Namely, it is one of the few models (in the absence of supersymmetry) whose non-perturbative solution is known. As announced in the preamble, the dynamics of massless QED_{1+1} is that of a free massive boson, which intuitively describes the propagation of a bound state of the electric field in interaction with the fermions, rather than a massive photon. Nevertheless, in the presence of massive fermions, no exact solution has been formulated. To be specific, it seems that some features of the massive theory can be described by treating the fermion mass term in perturbation theory, or by considering semi-classical solutions of the bosonized formulation [2].
Our interest in the Schwinger model will be focused on its massless version in order to take advantage of its solution. The main purpose of the next chapter will be to uncover the role in the interacting theory of the "Wilson loop degrees of freedom" as discussed in section 2.2, which are essential in the absence of dynamical matter. As long as space-time has the topology $\mathbb{R} \times \mathbb{R}$, the study of the role of the gauge zero-mode (2.4) would give the feeling of looking for a needle in a haystack. This is why we shall compactify space into a circle, in order to single out the gauge zero-mode from the other "quantum field degrees of freedom". Eventually, the compactification of space will be advantageous in order to address the question of the role of a θ-term[5] in the Lagrangian, whose form is analogous to the θ-term of QCD.

[5] Actually, in the massless model, the parameter θ should not modify the physical content of the quantised system.

CHAPTER 3

Topology and the exact solution of massless QED_{1+1}

3.1 Introductory aspects and the Dirac dressed electron field

Important aspects of the approach described in this chapter have to be emphasized: to begin with, the originality of the research presented here lies in the description of the interplay between the fermionic field and the gauge field. Fermions are not completely integrated out of the theory to leave an effective action for the gauge field accounting for the presence of the fermionic excitations in a way similar to the Euler-Heisenberg action. On the contrary, the gauge and matter sectors are quantised and treated on an equal footing. Our attention is focused on a particular type of gauge transformations called: "large gauge transformations". Because these gauge transformations cannot be generated from infinitesimal transformations, they have an intrinsic non-local character. In the literature, many investigations focus on the properties of interesting non-local observables, named Wilson loops. More precisely, the quantity associated to a closed path C

$$\exp \mathrm{i}e \int_C A_\mu \mathrm{d}x^\mu, \tag{3.1}$$

namely an abelian Wilson loop, furnishes a gauge invariant[1] extended observable which provides useful information on the non-perturbative features of the gauge theory. However arguments based on the calculation of Wilson loops very often consider a "pure gauge" theory in the absence of matter fields [13], or include matter as pointlike "static external" particles which are considered as being heavy.

Because of the low dimensional character of the Schwinger model and the compactification of space into a circle, it is possible to remove from the gauge sector the space dependent part of the gauge potential to keep a time dependent global degree of freedom. This global "quantum mechanical" degree of freedom is the analogue of the Wilson loop of the gauge field calculated on the circle. This mode, constant in space, is called the "zero-mode" and has a peculiar transformation law under a large gauge transformation. We will emphasize the fundamental importance of this dynamical zero-mode, in interplay with the fermionic sector of the theory.

Incidentaly, the qualifier "topological" is associated to the "zero-mode", as defined in (2.4) in the case of a circle, because of its relevance in the description of topological field theories, in canonical quantisation. In particular, the Hamiltonian formulation of the pure Chern-Simons theory on a three-manifold $\mathbb{R} \times \Sigma$, where Σ is compact, gives an illustration of a theory formulated as an equivalent quantum mechanical problem for the "zero-modes" [14].

Furthermore, the role of the zero-mode dynamics was often considered in models of spontaneous supersymmetry breaking [15, 16]. In this context, the "QFT degrees of freedom" are factorized in the so-called Born-Oppenheimer approximation. The resulting effective action only accounts for the dynamics of a finite number of degrees of freedom. However, in this work, we interest ourselves in the interrelationship between

[1] The exponential in (3.1) is taken in order to make the expression gauge invariant. In a non-abelian gauge theory, a trace over the matrices of the representation is needed to have a gauge invariant expression.

the fermionic field and the "gauge variant" Wilson loop $\int_C eA_\mu \mathrm{d}x^\mu$.

The second important aspect of the analysis performed concerns the way fermions are treated. As is well known, in QED the quantum excitations of the Dirac spinor field are associated to the electrons and positrons. However, the asymptotic electron states used in perturbation theory are considered as free electrons in the absence of the electromagnetic interaction. The identification of the physical asymptotic electrons with the fields appearing in the QED Lagrangian is not straightforward. Indeed, this fact is puzzling because we know from Nature that the electrons always carry with them a Coulomb electric field, which can be considered as a "dressing" of the electron appearing in the free Dirac equation. A single electron wave function as it appears in the free Dirac equation is certainly not gauge invariant, because these transformations act as $A'_\mu(t,x) = A_\mu(t,x) + \frac{1}{e}\partial_\mu \alpha(t,x)$ and $\psi'(t,x) = \exp\,(-\mathrm{i}\alpha(t,x))\psi(t,x)$. The necessity to design a way to create simultaneously an electron and its surrounding "Coulomb cloud" was first studied by Dirac in [17]. A first attempt to conceive a gauge invariant formulation of a single electron is to attach a infinite tail (or string) to the electron. To be more precise, we can define the dressed fermion in 3 + 1 dimensions by

$$\psi_\gamma(x) = \exp\left[\mathrm{ie}\int_{-\infty}^{x} \mathrm{d}\ell^i A^i\right]\psi(x), \tag{3.2}$$

where γ is a contour going from x to infinity[2]. The construction may be understood as an electron-positron pair, linked by a Wilson line, and where the positron has been pushed to infinity, as represented in figure 3.1.

Hence the newly defined field is gauge invariant, if the gauge transformation reduces to the identity at infinity. The contour γ inevitably introduces an arbitrariness in the definition. By decomposing the gauge potential into the sum of its longitudinal and transverse components $A^i = A^i_L + A^i_T$ with $\partial_i A^i_T = 0$, it is possible to remove the dependence on the path γ. We find that the longitudinal component is a gradiant

[2]This discussion follows reference [18].

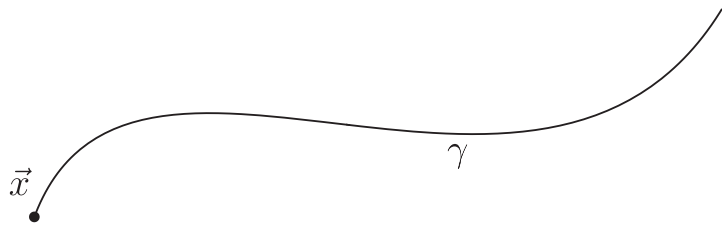


Figure 3.1: An intuitive view of a string of electric field linking an electron to an anti-electron positioned at infinity is pictured.

$A^i_L = \partial_i \alpha$, with $\alpha = \Delta^{-1}(\partial_j A^j)$. The result of the decomposition is the factorization of the dependence on the path in the dressing factor

$$\exp\left[\mathrm{i}e\int_{-\infty}^{x} \mathrm{d}\ell^i A^i\right] = \exp\left[\mathrm{i}e\int_{-\infty}^{x} \mathrm{d}\ell^i A^i_T\right]\exp\left[\mathrm{i}e\alpha(x)\right], \qquad (3.3)$$

which suggests a way to define the dressing in a path independent manner, *i.e.* by omitting the first factor on the *rhs* of (3.3). Hence the dressed fermion field is defined by the non local and manifestly gauge invariant expression

$$\chi(\vec{x}) = \exp\left[\mathrm{i}e\frac{\partial_i A^i}{\Delta}\right]\psi(\vec{x}), \qquad (3.4)$$

where $1/\Delta$ denotes the Green function of the Laplacian in three space dimensions, as given by

$$(\frac{1}{\Delta}f)(\vec{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3y^i \frac{f(\vec{y})}{|\vec{x}-\vec{y}|}. \qquad (3.5)$$

As a consequence of this definition, motivated by the requirement of gauge invariance, we can show that an electric field has been attached to the fermion field. Although heuristic, a simple and eluminating argument [18] shows that the electric field associated with such a dressed electron is a Coulomb field in $3+1$ dimensions. The extended and long ranged nature of the Coulomb field is reflected in the non-local feature of the dressing. In a canonically quantised electrodynamics, it is is straightforward to calculate the commutator

$$[E^i(\vec{y}), \chi(x)] = \frac{e}{4\pi}\frac{x^i - y^i}{|\vec{x}-\vec{y}|^3}\chi(x), \qquad (3.6)$$

where we used the fact that the electric field is the momentum conjugate to the gauge potential: $[A^i(\vec{x}); E^j(\vec{y})] = \delta^{ij}\delta^{(3)}(\vec{x} - \vec{y})$. This result can be interpreted as follows. Let us imagine that we have an eigenstate of the electric field $E^i(\vec{y})|\mathcal{E}\rangle = \mathcal{E}^i(\vec{y})|\mathcal{E}\rangle$. Therefore we find that the state $\chi(\vec{x})|\mathcal{E}\rangle$ is also an eigenstate of the electric field operator,

$$E^i(\vec{y})\chi(\vec{x})|\mathcal{E}\rangle = \Big[\mathcal{E}^i(\vec{y}) + \frac{e}{4\pi}\frac{x^i - y^i}{|\vec{x} - \vec{y}|^3}\Big]\chi(\vec{x})|\mathcal{E}\rangle, \tag{3.7}$$

which means that the electric field, at the position $\vec{y}$, associated to the dressed fermion field is the Coulomb field of a pointlike charge at rest at the position $\vec{x}$.

The solution to the Schwinger model presented in this work will treat the fermion field as the dressed field (3.4) as advocated by Dirac. It constitutes an important feature of the approach.

The outline of the present chapter is as follows. In the next section, the Hamiltonian formulation of the model is reviewed. Section 3.3 considers its canonical quantisation in careful detail in the fermionic formulation, by paying due attention in particular to large gauge transformations, namely the topological modular symmetries of the dynamics, an issue which to the best of the author's knowledge is new in the literature as well as the new understanding while our approach provides for some of the non-perturbative properties of the Schwinger model. These physical consequences are addressed in the following sections 3.4 to 3.5. Some concluding remarks are provided in section 3.8, with other useful considerations being detailed also in Appendix A.

The conclusions of this chapter have been published in the paper [19], which has been selected by the editors of Journal of Physics A for inclusion in the "Highlights of 2012" collection.

3.2 Hamiltonian formulation

The starting point of the analysis is the QED Lagrangian density in its explicitly self-adjoint form,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+\frac{1}{2}\mathrm{i}\overline{\psi}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu)\psi - \frac{1}{2}\mathrm{i}\overline{(\partial_\mu + \mathrm{i}eA_\mu)\psi}\gamma^\mu\psi - \mu\overline{\psi}\psi,\end{aligned} \tag{3.8}$$

with $\overline{\psi} = \psi^\dagger\gamma^0$, where ψ, A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mu \geq 0$ denote the Dirac spinor field, the gauge field, the field strength tensor, and a fermionic mass term, respectively. This theory having a coupling constant of strictly positive mass dimension is perturbatively superrenormalizable. Infrared divergencies inherent to such a theory are regularised in our case by having compactified space into a circle of circumference L, with the further consequence of a discretization of momentum space implying a countable set of quantum modes for the fields. Given the cylindrical spacetime topology which breaks the symmetry under Lorentz boosts but not under spacetime translations, the boundary conditions of the fields in the spatial circular direction are taken to be

$$A_\mu(t, x + L) = A_\mu(t, x), \qquad \psi(t, x + L) = \exp\,(-2\mathrm{i}\pi\lambda)\psi(t, x), \tag{3.9}$$

where $\lambda \in [0, 1[$ is a fermionic holonomy parameter.

The $U(1)$ gauge symmetry of the model acts through the transformations $A'_\mu(t, x) = A_\mu(t, x) + \frac{1}{e}\partial_\mu\alpha(t, x)$ and $\psi'(t, x) = \exp\,(-\mathrm{i}\alpha(t, x))\psi(t, x)$, where $\alpha(t, x)$ is an arbitrary spacetime dependent continuous rotation angle (defined mod 2π). In addition to the infinitesimally generated "small gauge transformations" continuously connected to the identity transformation with $\alpha(t, x) = 0$, spatial compactification brings to the fore the topologically non trivial group of "large gauge transformations". The distinction between these classes of transformations is made explicit by expressing the arbitrary function $\alpha(t, x)$ through the decomposition $\alpha(t, x) = \alpha_0(t, x) + 2\pi x\ell/L$ in terms of a periodic function $\alpha_0(t, x) = \alpha_0(t, x + L)$ and an integer $\ell \in \mathbb{Z}$, the so-called (additive)

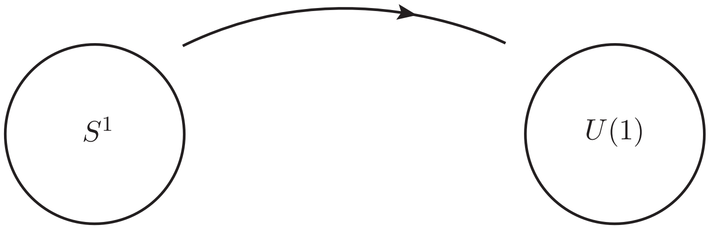


Figure 3.2: The winding number represents the number of times the spatial circle is wound onto the gauge group.

winding number of the "large gauge transformation". This group of integers is the fundamental or first homotopy group $\pi_1(S^1)$ which classifies the mappings $S^1 \to U(1)$, as illustrated in figure 3.2.

"Small gauge transformations" form the local gauge group, *i.e.*, they are connected to the identity. These transformations are generated by exponentiation of the parameter $\alpha(t,x) = \alpha_0(t,x)$ with $\ell = 0$. If the holonomy of the gauge transformation around the circle, namely $\ell \in \mathbb{Z}$, does not vanish, we are dealing with a large gauge transformation. One of the purposes of this work is to emphasize the topological difference between these two classes of gauge transformations and especially the consequences of large gauge transformations. This is done by considering the "modular group", namely the quotient of the full gauge group by the local gauge group. For the present system the modular group is isomorphic to the additive group $\mathbb{Z}$ of the winding number ℓ. It will be shown that complete gauge invariance under all gauge transformations may conveniently be enforced by requiring separately invariance under the local gauge group and the modular group.

One may take advantage of these considerations to distinguish the various sectors on which these gauge transformations act. From the point of view of the spatial S^1 which is a compact manifold, let us apply a Hodge decomposition of the gauge field of which the time component is a 0-form and the space component a 1-form. Hence,

$$A_0(t,x) = a_0(t) + \partial_1\omega_1(t,x), \tag{3.10}$$

$$A_1(t,x) = a_1(t) + \partial_1\phi(t,x), \tag{3.11}$$

where the periodic functions $\omega_1(t,x)$ and $\phi(t,x)$ do not include a spatial zero-mode, *i.e.*, these 1- and 0-form fields do not include a space independent component, while $a_0(t)$ and $a_1(t)$ are the corresponding harmonic forms. Similarly a Hodge decomposition also applies to the gauge parameter 0-form,

$$\alpha_0(t,x) = \beta_0(t) + \partial_1\beta_1(t,x),$$

where once again the 1-form $\beta_1(t,x)$ does not include a (spatial) zero-mode. In terms of this separation of variables, gauge transformations of winding number ℓ and parameter $\alpha(t,x) = \alpha_0(t,x) + 2\pi x\ell/L$ act as follows on the Hodge components of $A_0(t,x)$,

$$\begin{cases} a_0'(t) = a_0(t) + \frac{1}{e}\partial_0\beta_0(t), \\ \omega_1'(t,x) = \omega_1(t,x) + \frac{1}{e}\partial_0\beta_1(t,x), \end{cases}$$

while for $A_1(t,x)$,

$$\begin{cases} a_1'(t) = a_1(t) + \frac{2\pi\ell}{eL}, \\ \phi'(t,x) = \phi(t,x) + \frac{1}{e}\partial_1\beta_1(t,x). \end{cases}$$

A noticeable fact is that the modular transformation of winding number ℓ is found to act in the gauge sector only as a shift in the zero-mode $a_1(t)$ which is itself invariant under any local gauge transformation. Furthermore the Hodge decomposition in (3.11) allows one to "dress" the fermionic field with the longitudinal gauge field as follows

$$\chi(t,x) = \exp\,(\mathrm{i}e\phi(t,x))\psi(t,x). \tag{3.12}$$

This redefinition of the Dirac spinor is reminiscent of Dirac's construction [17] of a "physical electron" carrying its own "photon cloud" so that this composite object be gauge invariant. The boundary condition for the dressed fermion is still given by the holonomy condition of parameter λ, $\chi(t, x+L) = \exp\,(-2\mathrm{i}\pi\lambda)\,\chi(t, x)$. However gauge transformations of the redefined spinor simplify as,

$$\chi(t,x)' = \exp\,(-\mathrm{i}\beta_0(t))\,\exp\,(-2\mathrm{i}\pi\ell\frac{x}{L})\,\chi(t,x), \tag{3.13}$$

showing that a local gauge transformation induces only a time dependent but space independent phase change $\exp\,(-\mathrm{i}\beta_0(t))$ of the "composite" fermionic field. A space dependent gauge transformation of $\chi(t,x)$ is associated now to the modular group only, whose topologically non trivial action multiplies $\chi(t,x)$ by $\exp\,(-2\mathrm{i}\pi\ell x/L)$. In other words modular transformations, which account for the topological features of the compactified theory and its gauge symmetries, act only on the following degrees of freedom,

$$\chi'(t,x) = \exp\,(-2\mathrm{i}\pi\ell\frac{x}{L})\,\chi(t,x), \qquad a_1'(t) = a_1(t) + \frac{2\pi\ell}{eL}, \quad \ell \in \mathbb{Z}.$$

These different field redefinitions making manifest a separation of the gauge degrees of freedom into local and topological ones, imply the following expression for the action of the theory,

$$\begin{aligned} S = \textstyle\int dt \quad & \Big\{\frac{1}{2}L\dot{a}_1^2 - ea_0\int_{S^1} dx\chi^\dagger\chi - ea_1\int_{S^1} dx\overline{\chi}\gamma^1\chi \\ & + \int_{S^1} dx\Big(\frac{1}{2}\mathrm{i}\chi^\dagger\partial_0\chi - \frac{1}{2}\mathrm{i}\partial_0\chi^\dagger\chi + \frac{1}{2}\mathrm{i}\overline{\chi}\gamma^1\partial_1\chi - \frac{1}{2}\mathrm{i}\partial_1\overline{\chi}\gamma^1\chi \\ & -\mu\overline{\chi}\chi - \frac{1}{2}(\partial_0\phi - \partial_1\omega_1)\partial_1^2(\partial_0\phi - \partial_1\omega_1) \\ & +e(\partial_0\phi - \partial_1\omega_1)(\chi^\dagger\chi)'\Big)\Big\}, \end{aligned}$$

where the notation $(\chi^\dagger\chi)'$ stands for the quantity shown in parenthesis but with its spatial zero-mode subtracted (and where as usual a dot above a quantity stands for the time derivative of that quantity).

Given the existence of gauge symmetries, the identification of the Hamiltonian formulation of this system must rely on the methods of constrained dynamics [3]. The momenta canonically conjugate to all degrees of freedom are (here Grassmann odd derivatives for the spinor components are left-derivatives, while L_0 is the total quantity in curly brackets in the above expression for the action),

$$\begin{aligned}
p^0 &= \frac{\partial L_0}{\partial \dot{a}_0} = 0, \\
\pi^1 &= \frac{\partial L_0}{\partial \dot{\omega}_1} = 0, \\
p^1 &= \frac{\partial L_0}{\partial \dot{a}_1} = L\dot{a}_1, \\
\pi_\phi &= \frac{\partial L_0}{\partial \dot{\phi}} = - \bigtriangleup (\partial_0\phi - \partial_1\omega_1) + e(\chi^\dagger\chi)', \\
\xi_1 &= \frac{\partial L_0}{\partial \dot{\chi}} = -\frac{1}{2}\mathrm{i}\chi^\dagger, \\
\xi_2 &= \frac{\partial L_0}{\partial \dot{\chi}^\dagger} = -\frac{1}{2}\mathrm{i}\chi,
\end{aligned}$$

with $\xi_1^\dagger(t,x) = -\xi_2(t,x)$. For two of the degrees of freedom one may express their velocity in terms of their conjugate momentum, namely $\dot{a}_1(t) = p^1(t)/L$ and $\partial_0\phi(t,x) = \partial_1\omega_1(t,x) - \bigtriangleup^{-1}(\pi_\phi(t,x) - e(\chi^\dagger\chi)'(t,x))$. Here the symbol $\bigtriangleup^{-1}$ denotes the Green function of the spatial Laplacian on the circle, $\Delta = \partial_1^2$, again not including the spatial zero-mode. Since π_ϕ does not include a zero-mode the action of $\bigtriangleup^{-1}$ in the previous expression for $\partial_0\phi$ is well defined. However since the Hessian of the Lagrange function for the other degrees of freedom possesses null eigenvectors, there exist primary phase space constraints. Clearly these primary constraints are $p^0(t) = 0$, $\pi^1(t,x) = 0$, $\xi_1(t,x) + \mathrm{i}\chi^\dagger(t,x)/2 = 0$ and $\xi_2(t,x) + \mathrm{i}\chi(t,x)/2 = 0$.

Since the canonical Hamiltonian is readily identified to be given as,

$$\begin{aligned}
H_0 &= \frac{1}{2L}(p^1)^2 + ea_0\int_{S^1} dx\chi^\dagger\chi + ea_1\int_{S^1} dx\overline{\chi}\gamma^1\chi + \\
&+ \int_{S^1} dx\Big\{ - \frac{1}{2}\mathrm{i}\overline{\chi}\gamma^1\partial_1\chi + \frac{1}{2}\mathrm{i}\partial_1\overline{\chi}\gamma^1\chi + \mu\overline{\chi}\chi \\
&+\partial_1\omega_1\pi_\phi - \frac{1}{2}(\pi_\phi - e(\chi^\dagger\chi))' \bigtriangleup^{-1} (\pi_\phi - e(\chi^\dagger\chi))'\Big\}.
\end{aligned}$$

a consistent time evolution of the primary constraints must consider as primary Hamiltonian the following total quantity

$$H_1 = H_0 + \lambda_0 p^0 + \int_{S^1} dx\Big[\lambda_1\pi^1 + (\xi_1 + \frac{1}{2}\mathrm{i}\chi^\dagger)\tilde{\lambda}_1 + \tilde{\lambda}_2(\xi_2 + \frac{1}{2}\mathrm{i}\chi)\Big], \quad (3.14)$$

where ($\lambda_0(t)$, $\lambda_1(t,x)$) and ($\tilde{\lambda}_1(t,x)$, $\tilde{\lambda}_2(t,x)$) are Grassmann even and Grassmann odd would-be Lagrange multipliers, respectively. Requiring a consistent time evolution of the primary constraints generated through the (Grassmann graded) Poisson brackets by this primary Hamiltonian implies the following further conditions,

$$\begin{aligned}
\{p^0, H_1\} &= -e\int_{S^1} dx\chi^\dagger\chi = 0, \\
\{\pi^1, H_1\} &= \partial_1\pi_\phi = 0, \\
\{\xi_1 + \frac{1}{2}\mathrm{i}\chi^\dagger, H_1\} &= 0, \\
\{\xi_2 + \frac{1}{2}\mathrm{i}\chi, H_1\} &= 0.
\end{aligned}$$

In actual fact, the last two conditions imply equations for the Grassmann odd multipliers $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ which are thereby uniquely determined. The other two conditions however, define secondary constraints, the first of which, namely $e\int_{S^1} dx\chi^\dagger\chi = 0$, is the zero-mode of the ordinary Gauss law. A consistent time evolution of these new constraints requires to include them in a secondary Hamiltonian which is to generate time evolution,

$$H_2 = H_1 + e\lambda_3\int_{S^1} dx\chi^\dagger\chi + \int_{S^1} dx\lambda_3^1\partial_1\pi_\phi, \tag{3.15}$$

where $\lambda_3(t)$ and $\lambda_3^1(t,x)$ are would-be Lagrange multipliers enforcing the secondary constraints. It is readily checked that no further constraints are then generated from H_2. A consistent time evolution of physical states is ensured.

According to Dirac's classification the set of constraints decomposes into first and second class constraints. In the case under study, $p^0 = 0$ and $e\int_{S^1} dx\chi^\dagger\chi = 0$ are first class while $\xi_1 + \frac{1}{2}\mathrm{i}\chi^\dagger = 0$ and $\xi_2 + \frac{1}{2}\mathrm{i}\chi = 0$ are second class constraints. First class constraints always generate gauge symmetries. Second class constraints on the other hand, indicate that some degrees of freedom are unnecessary and may be reduced through the introduction of the associated Dirac brackets. In the present case Dirac brackets act in the fermionic sector only, and are given as,

$$\left\{\chi_\alpha(t,x), \chi_\beta^\dagger(t,y)\right\}_D = -\mathrm{i}\delta_{\alpha,\beta}\delta_{S^1}(x-y)\exp\left(-2\mathrm{i}\pi\frac{(x-y)}{L}\lambda\right), \tag{3.16}$$

where λ is the fermionic holonomy while $\delta_{S^1}(x-y)$ stands for the Dirac δ-function defined over the spatial circle S^1, and $\alpha, \beta = 1, 2$ are spinor indices.

The first-order action associated with the Hamiltonian formulation is thus defined by the first-order Lagrange functional

$$\begin{aligned} L = \;& \dot{a}_0 p^0 + \dot{a}_1 p^1 - \lambda_0 p^0 - e a_1 \int_{S^1} dx \overline{\chi} \gamma^1 \chi - e(a_0 + \lambda_3) \int_{S^1} dx \chi^\dagger \chi \\ & -\frac{p_1^2}{2L} + \int_{S^1} dx \Big\{ \partial_0 \omega_1 \pi^1 + \partial_0 \phi \pi_\phi - \partial_1 \omega_1 \pi_\phi - \lambda_3^1 \partial_1 \pi_\phi - \lambda_1 \pi^1 \\ & + \frac{1}{2} \mathrm{i} \chi^\dagger \partial_0 \chi - \frac{1}{2} \mathrm{i} \partial_0 \chi^\dagger \chi + \frac{1}{2} \mathrm{i} \overline{\chi} \gamma^1 \partial_1 \chi - \frac{1}{2} \mathrm{i} \partial_1 \overline{\chi} \gamma^1 \chi - \mu \overline{\chi} \chi \\ & + \frac{1}{2} (\pi_\phi - e(\chi^\dagger \chi)') \, \triangle^{-1} \, (\pi_\phi - e(\chi^\dagger \chi))' \Big\}. \end{aligned}$$

However some of the first class constraints, namely $p^0 = 0$ and $\pi^1 = 0$, appear because some of the degrees of freedom are in actual fact already Lagrange multipliers for some of the other first class constraints, namely in the present case $A_0(t,x) = a_0(t) + \partial_1 \omega_1(t,x)$ is the Lagrange multiplier for Gauss' law which is the first class constraint generating small gauge transformations of parameter $\alpha_0(t,x)$. In such a situation one may use the freedom in choosing the Lagrange multipliers for such superfluous first class constraints without affecting the actual gauge invariances of the system, and thereby determine a more "fundamental" or basic Hamiltonian formulation [3]. First let us make the choice $\lambda_0(t) = \dot{a}_0(t)$ and then replace $a_0(t) + \lambda_3(t)$ by $a_0(t)$. Consequently the sector (a_0, p^0) decouples altogether from the dynamics, with the new variable $a_0(t)$ being the Lagrange multiplier for the first class constraint $e \int_{S^1} dx \chi^\dagger \chi = 0$. Likewise the choice $\lambda_1(t,x) = \partial_0 \omega_1(t,x)$ and then applying the redefinition $-\lambda_3^1(t,x) + \omega_1(t,x) \to \lambda^1(t,x)$ shows that the sector (ω_1, π^1) decouples as well, with the new quantity $\lambda^1(t,x)$ being the Lagrange multiplier for the first class constraint $\partial_1 \pi_\phi = 0$. Given these redefinitions the

Hamiltonian formulation is specified by the first order Lagrangian

$$\begin{aligned} L = \;& \dot{a}_1 p^1 - \frac{1}{2L}(p^1)^2 - e a_0 \int_{S^1} dx \chi^\dagger \chi - e a_1 \int_{S^1} dx \overline{\chi}\gamma^1 \chi \\ & + \int_{S^1} dx \Big\{ \partial_0 \phi \pi_\phi + \frac{1}{2}\mathrm{i}\chi^\dagger \partial_0 \chi - \frac{1}{2}\mathrm{i}\partial_0 \chi^\dagger \chi + \frac{1}{2}\mathrm{i}\overline{\chi}\gamma^1 \partial_1 \chi - \frac{1}{2}\mathrm{i}\partial_1 \overline{\chi}\gamma^1 \chi \\ & + \lambda^1 \partial_1 \pi_\phi - \mu \overline{\chi}\chi + \frac{1}{2}(\pi_\phi - e(\chi^\dagger\chi))' \triangle^{-1} (\pi_\phi - e(\chi^\dagger\chi))' \Big\}. \end{aligned}$$

However, since the sector (ϕ, π_ϕ) contributes only linearly and quadratically to this action, it may easily be reduced as well through its equations of motion, which read,

$$\begin{cases} \partial_0 \phi = - \triangle^{-1} (\pi_\phi - e(\chi^\dagger\chi)') + \partial_1 \lambda^1, \\ \partial_0 \pi_\phi = 0, \end{cases}$$

with the constraint $\partial_1 \pi_\phi = 0$, and where π_ϕ does not include a zero-mode. Hence one has $\pi_\phi(t,x) = 0$ while the pure gauge degree of freedom $\phi(t,x)$ is determined from $\partial_0 \phi = e\Delta^{-1}(\chi^\dagger\chi)' + \partial_1 \lambda^1$.

Upon this final reduction, the Hamiltonian formulation of the system consists of the phase space variables $(a_1(t), p^1(t); \chi(t,x), \chi^\dagger(t,x))$ with the Poisson-Dirac brackets

$$\begin{aligned} \{a_1(t), p^1(t)\} \;&=\; 1, \\ \left\{\chi_\alpha(t,x), \chi^\dagger_\beta(t,y)\right\}_D \;&=\; -\mathrm{i}\delta_{\alpha,\beta}\delta_{S^1}(x-y)\exp\,(-2\mathrm{i}\pi\frac{x-y}{L}\lambda), \end{aligned}$$

subjected to the single first class constraint $e\int_{S^1} dx \chi^\dagger\chi = 0$ of which the Lagrange multiplier is $a_0(t)$, and a dynamics deriving from the Hamiltonian first-order action

$$\begin{aligned} S = \;& \int dt \Big\{ \dot{a}_1 p^1 + \textstyle\int_{S^1} dx \Big(\frac{1}{2}\mathrm{i}\chi^\dagger\partial_0\chi - \frac{1}{2}\mathrm{i}\partial_0\chi^\dagger\chi \Big) \\ & \qquad - H \,-\, e a_0 \textstyle\int_{S^1} dx \chi^\dagger \chi \Big\}, \end{aligned}$$

where the first class Hamiltonian H is given by,

$$\begin{aligned} H \;&= \frac{(p^1)^2}{2L} + \int_{S^1} dx \{ \overline{\chi}\gamma^1(-\mathrm{i}\partial_1 + e a_1)\chi + \mu\overline{\chi}\chi \\ & \qquad - \frac{1}{2}e^2 (\chi^\dagger\chi)' \triangle^{-1} (\chi^\dagger\chi)' \}. \end{aligned} \tag{3.17}$$

Note how the very last four-fermion contribution to H stands for the instantaneous Coulomb interaction, even though no gauge fixing procedure has been enforced, but rather a parametrization of the degrees of freedom which factorizes the physical from the gauge dependent degrees of freedom. The remaining gauge invariances of the system in the present formulation consist of the space independent small gauge transformations with parameter $\alpha_0(t,x) = \beta_0(t)$ which are generated by the single remaining first class constraint, $e\int_{S^1} dx\chi^\dagger\chi = 0$, as well as the modular transformations of winding numbers $\ell \in \mathbb{Z}$, acting as follows on the phase space variables,

$$\begin{aligned} a_1'(t) &= a_1(t) + \frac{2\pi\ell}{eL}, \\ p^{1'}(t) &= p^1(t), \\ \chi'(t,x) &= \exp\left(-\mathrm{i}\beta_0(t)\right)\,\exp\left(-2\mathrm{i}\pi\ell\frac{x}{L}\right)\,\chi(t,x). \end{aligned} \tag{3.18}$$

In particular the first class constraint, merely the space integrated Gauss law, requires physical states to carry a vanishing net electric charge. In addition however, physical states need also to be modular invariant, a restriction which is intrinsically of a purely topological character involving the gauge harmonic form $a_1(t)$ as well as the winding numbers of the gauge symmetry group.

3.3 Canonical quantisation

Canonical quantisation of the system in the Schrödinger picture (at $t = 0$) proceeds from its basic Hamiltonian formulation of the previous section. It is necessary to consider a mode expansion of the dressed spinor $\chi(t = 0, x)$, which is taken in the form,

$$\chi(x) = \sqrt{\frac{\hbar}{L}}\sum_{m\in\mathbb{Z}} \begin{pmatrix} d^\dagger_{-m} \\ b_m \end{pmatrix} \exp\left(2\mathrm{i}\pi\frac{x}{L}(m-\lambda)\right), \tag{3.19}$$

with the anti-commutation relations $\{d_{-m}, d^\dagger_{-n}\} = \delta_{m,n} = \{b_m, b^\dagger_n\}$. Note that the mode indices $m, n \in \mathbb{Z}$ also label the momentum eigenvalues $2\pi m/L$ of the fermion total momentum operator. For example

b_m and $d^\dagger_{-m}$ both carry momentum $(-2\pi m/L)$. A particle and anti-particle interpretation of the sectors $(b_m, b^\dagger_m)$ and $(d_m, d^\dagger_m)$, respectively, is warranted by considering the mode expansion of the total electric charge, $Q = \int_{S^1} dx \chi^\dagger(x)\chi(x)$ (the specific definition and expression of this composite operator is provided below). This choice of mode expansion translates also into the following anti-commutation relations for the spinor field,

$$\{\chi_\alpha(x), \chi^\dagger_\beta(y)\} = \delta_{\alpha,\beta} \frac{\hbar}{L} \sum_m \exp\,(2\mathrm{i}\pi \frac{x-y}{L}(m-\lambda)),$$

which are in direct correspondence with their classical Dirac bracket counterparts. Similarly, the zero-mode of the gauge sector, (a_1, p^1), is quantised by the Heisenberg algebra, $\left[\hat{a}_1, \hat{p}^1\right] = \mathrm{i}\hbar$, $\hat{a}_1$ and $\hat{p}^1$ needing to be self-adjoint operators as well.

In terms of the above mode expansion the fermionic bilinear contribution to the first class Hamiltonian (3.17), namely $H = (p^1)^2/(2L) + H_0 + H_C$, takes the form

$$\begin{aligned} H_0 &= \\ &= \int_{S^1} dx \overline{\chi} \gamma^1 (-\mathrm{i}\partial_1 + e\hat{a}_1)\chi \\ &= \sum_m \Big[(2\pi \frac{m-\lambda}{L} + e\hat{a}_1)(b^\dagger_m b_m - d_{-m} d^\dagger_{-m}) + \mu(d_{-m} b_m + b^\dagger_m d^\dagger_{-m})\Big], \end{aligned}$$

while the instantaneous Coulomb interaction energy becomes,

$$H_C = \kappa \sum_{\ell \neq 0} \frac{1}{\ell^2} \sum_{m,n} (d_{-n} d^\dagger_{-m} + b^\dagger_n b_m)\delta_{m,n+\ell} \sum_{p,q} (d_{-q} d^\dagger_{-p} + b^\dagger_q b_p)\delta_{p,q-\ell},$$

with $\kappa = e^2 L/(2(2\pi)^2)$. To establish the last expression the following representation of the Green function of the spatial Laplacian is used,

$$(\triangle^{-1} g)(x) = \frac{-1}{L} \int_{S^1} dy \sum_{\ell \neq 0} \frac{\exp\,(2\mathrm{i}\pi(x-y)\frac{\ell}{L})}{(\frac{2\pi\ell}{L})^2} g(y).$$

Note that a specific ordering prescription for these composite operators H_0 and H_C is implicit at this stage. An explicit ordering prescription and complete definition of composite operators is to be given hereafter.

A consistent quantisation should also implement the action of all remaining gauge transformations, in correspondence with the classical transformations (3.18), through the adjoint action of specific quantum operators. The action of the modular transformation of winding number ℓ is

$$\begin{aligned} \hat{U}(\ell)\,\hat{a}_1\,\hat{U}^\dagger(\ell) &= \hat{a}_1 + \tfrac{2\pi}{eL}\ell, & \hat{U}(\ell)\,\hat{p}^1\,\hat{U}^\dagger(\ell) &= \hat{p}^1, \\ \hat{U}(\ell)\,b_m\,\hat{U}^\dagger(\ell) &= b_{m+\ell}, & \hat{U}(\ell)\,d_{-m}\,\hat{U}^\dagger(\ell) &= d_{-m-\ell}, \end{aligned} \tag{3.20}$$

with the corresponding quantum modular operator of winding number $\ell \in \mathbb{Z}$ given as,

$$\hat{U}(\ell) = \exp\left\{2\mathrm{i}\pi\ell\Big(\frac{1}{e}\frac{\hat{p}_1}{L} - \frac{\theta_0}{2\pi} + \frac{1}{L}\int_{S_1} dx : x\chi^\dagger(x)\chi(x) :\Big)\right\}. \tag{3.21}$$

The actual meaning of the ordering prescription, ": :", is specified below. The arbitrary new constant parameter θ_0, which is defined mod 2π, arises as follows. The quantum unitary operators, $\hat{U}(\ell)$, realising modular transformations involve *a priori* an arbitrary phase factor that may be winding number dependent. However since the modular group is additive in the winding number, the choice of phase should be consistent with the group composition law, $\hat{U}(\ell_1)\,\hat{U}(\ell_2) = \hat{U}(\ell_1 + \ell_2)$. The general solution to this requirement implies that the phase factor be linear in the winding number, hence the θ_0 parameter as the arbitrary linear factor in ℓ. In actual fact, θ_0 may be viewed as defining a purely quantum mechanical degree of freedom [4, 20], and is the analogue for the present model of the θ vacuum angle in QCD.

Similarly small gauge transformations act as follows

$$b_m \;\to \exp\,(-\mathrm{i}\beta_0)b_m, \qquad d_{-m} \;\to \exp\,(\mathrm{i}\beta_0)d_{-m},$$

while the corresponding quantum generator, namely the total electric charge Q which is the first class constraint for these local symmetries and of which the exponential, when multiplied by a factor proportional to β_0, determines the unitary operator of which the adjoint action induces these finite transformations, is defined hereafter.

What is most remarkable indeed about these modular transformations is that in the fermionic sector they map spinor modes of a given electric charge and of all possible momentum values into one another. In other words, modular symmetries, which are characteristic of the topological properties of a gauge invariant system, induce transformations connecting the infrared and the ultraviolet, namely the large and the small distance properties of a gauge invariant dynamics. This observation remains totally relevant in the context of non-abelian Yang-Mills theories as well, coupled to charge matter fields. Physical consequences of such modular symmetries are presumably far reaching, and deserve to be fully explored especially since they are intrinsically of a topological hence non-perturbative character.

Obviously composite quantum operators need to be carefully defined in order to preserve the modular gauge symmetry in a manifest way (see (3.20); that a regularisation prescription also preserves in a manifest way gauge invariance under local small transformations is readily checked). Let us first consider the bilinear fermion contributions to the first class Hamiltonian H, which need to be properly defined to ensure both finite matrix elements and a ground state of finite energy, given that b_m and d_m are taken to be annihilators of a fermionic Fock vacuum, with $b_m^\dagger$ and $d_m^\dagger$ acting as creators. Making the choice[3] of a gaussian regularisation with energy cut-off Λ, the bilinear fermion contributions to the first class Hamiltonian become,

$$\sum_m \left\{ \left(\tfrac{2\pi}{L}(m-\lambda)+e\hat{a}_1\right)\left(b_m^\dagger b_m - d_{-m}d_{-m}^\dagger\right) + \mu\left(d_{-m}b_m + b_m^\dagger d_{-m}^\dagger\right)\right\}$$
$$\times\exp\left(-\tfrac{1}{\Lambda^2}\left(\tfrac{2\pi}{L}(m-\lambda)+e\hat{a}_1\right)^2\right).$$

This choice of regularisation prescription ensures that this bilinear operator has finite matrix elements while it remains manifestly invariant under all modular gauge transformations (3.20). A further subtraction to be discussed hereafter, still needs to be applied to this expression, in order that eventually the regulator may be removed while leaving a well defined composite operator H_0. Let us note that the mass term couples left- and right-moving modes. This fact will make possible to smoothly redefine what will be the creators and annihilators of left- and right-moving particles.

[3]Other regularisation choices have been considered, and shown to lead to the same final conclusions.

In order to diagonalize this regularised operator, let us consider the sector of modes $(b_m, b_m^\dagger) \equiv (b, b^\dagger)$ and $(d_{-m}, d_{-m}^\dagger) \equiv (d, d^\dagger)$ for any given $m \in \mathbb{Z}$. For definiteness the corresponding fermionic Fock space is spanned by the Fock vacuum $|0,0\rangle$ and the states $|1,0\rangle = b^\dagger|0,0\rangle$, $|0,1\rangle = d^\dagger|0,0\rangle$ and $|1,1\rangle = d^\dagger b^\dagger|0,0\rangle$. The contribution of that sector to the above bilinear operator is thus of the following form,

$$h = \beta(b^\dagger b - d d^\dagger) + \alpha(b^\dagger d^\dagger + db),$$

with $\beta = (\frac{2\pi}{L}(m-\lambda) + e\hat{a}_1)\mathrm{exp}\,\{-(\frac{2\pi}{L}(m-\lambda) + e\hat{a}_1)^2/\Lambda^2\}$ and $\alpha = \mu\mathrm{exp}\,\{-(\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1)^2/\Lambda^2\}$. This operator h has 4 orthonormalized eigenstates listed in Table 3.1, in which $\psi_\mp = -(\beta \pm \sqrt{\beta^2+\alpha^2})/\alpha$ so that $\psi_+\psi_- = -1$.

Table 3.1: Eigenstates and eigenvalues of h.

State	Eigenvalue
$\lvert\psi_+\rangle = \frac{\lvert 0,0\rangle + \psi_+\lvert 1,1\rangle}{\sqrt{1+\psi_+^2}}$	$\sqrt{\alpha^2+\beta^2}$
$\lvert 1,0\rangle$ and $\lvert 0,1\rangle$	0
$\lvert\psi_-\rangle = \frac{\lvert 0,0\rangle + \psi_-\lvert 1,1\rangle}{\sqrt{1+\psi_-^2}}$	$-\sqrt{\alpha^2+\beta^2}$

In any given m sector, the state $|\psi_-\rangle$ is thus the minimal energy eigenstate. One may consider two pairs of fermionic creators and annihilators defined by

$$B_\pm^\dagger = \frac{b^\dagger + \psi_\pm d}{\sqrt{1+\psi_\pm^2}}, \qquad D_\pm = \frac{d - \psi_\pm b^\dagger}{\sqrt{1+\psi_\pm^2}},$$

whether for the index "+" or the index "−". These B and D operators and their adjoints obey two separate fermionic Fock algebras whether for the index "+" or the index "−", namely $\{B_{\pm,m}^\dagger, B_{\pm,n}\} = \delta_{m,n}$ and $\{D_{\pm,-m}^\dagger, D_{\pm,-n}\} = \delta_{m,n}$. The operators B_+ and D_+ (resp., B_- and D_-) annihilate the state $|\psi_+\rangle$ (resp., $|\psi_-\rangle$). Given these definitions, h acquires two separate though equivalent expressions,

$$h = -\sqrt{\alpha^2+\beta^2}\big(B_+^\dagger B_+ - D_+ D_+^\dagger\big) = \sqrt{\alpha^2+\beta^2}\big(B_-^\dagger B_- - D_- D_-^\dagger\big).$$

Among these two possibilities, in the sequel let us choose to work with the operators defined with the "$-$" index, of which B_- and D_- thus annihilate the ground state in the fermionic sector m, $|\psi_-\rangle$,

$$B_-|\psi_-\rangle = 0 = D_-|\psi_-\rangle.$$

Henceforth the index "$-$" will thus be suppressed, with $(B_m, B_m^\dagger)$ and $(D_{-m}, D_{-m}^\dagger)$ acting truly as annihilators and creators of fermionic Fock algebras of which the Fock vacuum is the state $|\psi_-\rangle$. Note however that all these quantities involve also the gauge zero-mode operator $\hat{a}_1$.

We may now rewrite all the quantities of interest in terms of the original variables,

$$\sqrt{\alpha^2+\beta^2} = \left[(\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1)^2+\mu^2\right]^{1/2} e^{-\frac{1}{\Lambda^2}(\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1)^2},$$

and

$$\psi_- = -\frac{\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1}{\mu} + \frac{1}{\mu}\sqrt{(\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1)^2+\mu^2}.$$

It is convenient to introduce a rotation angle, so that $\cos\phi_- = 1/\sqrt{1+\psi_-^2}$ and $\sin\phi_- = \psi_-/\sqrt{1+\psi_-^2}$.

Consider now the limit where μ tends to zero. First, if $\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1 \neq 0$ the limit $\mu \to 0$ implies

$$\lim_{\mu\to 0}\begin{pmatrix} b_m^\dagger \\ d_{-m} \end{pmatrix} = \begin{pmatrix} \cos\phi_- & \sin\phi_- \\ -\sin\phi_- & \cos\phi_- \end{pmatrix}\begin{pmatrix} B_m^\dagger \\ D_{-m} \end{pmatrix},$$

with the following specific values,

$$\cos\phi_- = \begin{cases} 1 & \text{if } \frac{2\pi}{L}(m-\lambda)+e\hat{a}_1 > 0; \\ 0 & \text{if } \frac{2\pi}{L}(m-\lambda)+e\hat{a}_1 < 0; \end{cases}$$

$$\sin\phi_- = \begin{cases} 0 & \text{if } \frac{2\pi}{L}(m-\lambda)+e\hat{a}_1 > 0; \\ 1 & \text{if } \frac{2\pi}{L}(m-\lambda)+e\hat{a}_1 < 0. \end{cases}$$

If however $\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1=0$ the "mixing angle" is of $\pi/4$ radians in the massless limit,

$$\lim_{\mu\to 0}\begin{pmatrix} b^\dagger_m \\ d_{-m}\end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix} B^\dagger_m \\ D_{-m}\end{pmatrix}.$$

It is rather obvious that one may readily express all these results in terms of the Heaviside step function, $\Theta(x)$, with the value $\Theta(0)=1/2$ as it turns out to be convenient for our purposes. However care needs to be exercised, as the sequel will illustrate. It is also useful to note that

$$\Theta\big(\frac{2\pi}{L}(m-\lambda)+e\hat{a}_1\big)=\Theta(m+\hat{a}),$$

with the notation $\hat{a}=e\hat{a}_1L/(2\pi)-\lambda$. Under a large gauge transformation of winding number ℓ, $\hat{a}$ transforms as $\hat{a}\to\hat{a}+\ell$. Finally we are in the position to make the following crucial identifications,

$$\lim_{\mu\to 0}\cos\phi_-=\sqrt{\Theta(m+\hat{a})},\qquad \lim_{\mu\to 0}\sin\phi_-=\sqrt{\Theta(-m-\hat{a})}.$$

The above transformations "à la Bogoliubov" redefine creators and annihilators for Fock algebras through linear transformations. By construction this definition behaves "covariantly" under modular transformations, and may be written in a compact way as,

$$b^\dagger_m=B^\dagger_m\sqrt{\Theta(m+\hat{a})}+D_{-m}\sqrt{\Theta(-m-\hat{a})},\tag{3.22}$$

$$d_{-m}=D_{-m}\sqrt{\Theta(m+\hat{a})}-B^\dagger_m\sqrt{\Theta(-m-\hat{a})}.\tag{3.23}$$

while $d^\dagger_{-m}$ and b_m are the adjoint operators of the previous expressions. It is recalled also that $B^{(\dagger)}_m$ and $D^{(\dagger)}_{-m}$ involve an implicit dependence on $\hat{a}$. The dependence on $\hat{a}$ of these definitions, with a spectral flow in the eigenvalues of that operator, may be interpreted as a dynamical "Fermi surface" in one dimension.

With the help of this definition, the electric charge operator reads,

$$Q=\sum_{m=-\infty}^{+\infty}\big(b^\dagger_m b_m+d_{-m}d^\dagger_{-m}\big)=\sum_{m=-\infty}^{+\infty}\big(B^\dagger_m B_m+D_{-m}D^\dagger_{-m}\big).$$

An ordered expression of the gauge invariant regularised charge operator is, with $\tilde{\alpha} = 2\pi/(L\Lambda^2)$,

$$: Q : \overset{\tilde{\alpha}\to 0}{=} \sum_{m=-\infty}^{+\infty} (B_m^\dagger B_m - D_{-m}^\dagger D_{-m} + 1) e^{-\tilde{\alpha}(m+\hat{a})^2},$$

where the divergent contribution independent of $\hat{a}$ may be subtracted while no further finite contribution in $\hat{a}$ arises. The reader will find a detailed discussion of the technical result concerning the subtraction of infinities in (A.1) of the Appendix. In order to prove that no additional term depending on $\hat{a}$ is generated by the normal ordering procedure the Poisson resummation formula is used, leading to

$$\sum_{m=-\infty}^{+\infty} \Theta(m+\hat{a}) e^{-\tilde{\alpha}(m+\hat{a})^2} \overset{\tilde{\alpha}\to 0}{=} \frac{1}{2}\sqrt{\frac{\pi}{\tilde{\alpha}}} + \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n\hat{a}}}{2\pi \mathrm{i} n}, \qquad (3.24)$$

$$\sum_{m=-\infty}^{+\infty} \Theta(-m-\hat{a}) e^{-\tilde{\alpha}(m+\hat{a})^2} \overset{\tilde{\alpha}\to 0}{=} \frac{1}{2}\sqrt{\frac{\pi}{\tilde{\alpha}}} + \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n\hat{a}}}{-2\pi \mathrm{i} n}. \qquad (3.25)$$

The subtraction consists in removing the contribution in $\frac{1}{2}\sqrt{\pi/\tilde{\alpha}}$ while no other infinite term remains. Eventually the normal ordered expression is given by

$$: \hat{Q} :_{\hat{a}} = \sum_{m=-\infty}^{+\infty} (B_m^\dagger B_m - D_{-m}^\dagger D_{-m}), \qquad (3.26)$$

which is the definition of the quantum U(1) charge operator. The normal ordering prescription, $: :_{\hat{a}}$, depends on $\hat{a}$ in such a manner that this operation respects all gauge symmetries including modular transformations. The regulator has safely been removed. As expected this operator is the generator of the $U(1)$ local gauge transformation,

$$B_m^\dagger \to \exp\,(\mathrm{i}\beta) B_m^\dagger, \qquad D_{-m}^\dagger \to \exp\,(-\mathrm{i}\beta) D_{-m}^\dagger.$$

We may follow a similar analysis towards a quantum definition of the fermion bilinear contributions to the first class Hamiltonian in the massless limit[4],

$$H_{bil} = \frac{2\pi}{L} \sum_{m=-\infty}^{+\infty} (m+\hat{a})(b^\dagger_m b_m - d_{-m} d^\dagger_{-m}).$$

With the help of the relations (3.22) and (3.23), the regularised normal ordered expression is,

$$: H_{bil} :_{\hat{a}} = \frac{2\pi}{L} \sum_{m=-\infty}^{+\infty} |m+\hat{a}|(B^\dagger_m B_m + D^\dagger_{-m} D_{-m} - 1) e^{-\tilde{\alpha}(m+\hat{a})^2}.$$

Given the normal ordering contribution, the spectrum of $: H_{bil} :_{\hat{a}}$ includes an infinite contribution when the regulator is removed. However we are not allowed to simply subtract this (regularised) contribution since it also involves a dependence on $\hat{a}$, which is brought about by the choice of a modular invariant regularisation. The finite $\hat{a}$ dependent part may be computed after careful subtraction of the divergent contribution for $\hat{a} = 0$. Once again the Poisson resummation formula is used to isolate and extract the $\hat{a}$ dependent finite contribution. Given (A.7) in the Appendix, one finds

$$- \sum_{m=-\infty}^{+\infty} |m+\hat{a}| e^{-\tilde{\alpha}(m+\hat{a})^2} \overset{\tilde{\alpha}\to 0}{=} -\Big[\frac{2}{2\tilde{\alpha}} - 2 \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2i\pi n\hat{a}}}{(2\pi n)^2}\Big].$$

The only divergence in $2/(2\tilde{\alpha})$ and which is independent of $\hat{a}$, is subtracted before removing the gaussian regulator. Thus finally the definition of this gauge invariant operator is,

$$\begin{aligned} : \hat{H}_{bil} :_{\hat{a}} &= \frac{2\pi}{L}(\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2})^2 - \frac{\pi}{6L} \\ &+ \frac{2\pi}{L} \sum_{m=-\infty}^{+\infty} |m+\hat{a}|(B^\dagger_m B_m + D^\dagger_{-m} D_{-m}), \end{aligned} \tag{3.27}$$

where it is noted that the additional $\hat{a}$ dependent part is the Fourier series of a periodic potential given by

$$\sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2i\pi n a}}{(2\pi n)^2} = \frac{1}{2}(a - \lfloor a \rfloor - \frac{1}{2})^2 - \frac{1}{24}, \tag{3.28}$$

[4] A discussion of the modular invariant definition of this specific operator, in the context of the Schwinger model in the limit $e \to \infty$, is available in [21].

and where $\lfloor a \rfloor$ denotes the "integer part" of a, *i.e.*, the largest integer less or equal to a. The quantum operator is bounded from below and is manifestly invariant under small as well as modular gauge transformations. It is also relevant to address a well-known feature of the massless classical theory, namely its invariance under global chiral transformations,

$$b_m^\dagger \;\to \exp\,(\mathrm{i}\beta) b_m^\dagger, \qquad d_{-m}^\dagger \;\to \exp\,(\mathrm{i}\beta) d_{-m}^\dagger,$$

a symmetry which implies that the dynamics does not couple the left- and right-moving modes. The corresponding classical conserved charge is the axial charge, which in the quantised theory takes the form,

$$\begin{aligned} Q_5 &= \sum_{m=-\infty}^{+\infty} (b_m^\dagger b_m - d_{-m} d_{-m}^\dagger) \\ &= \sum_{m=-\infty}^{+\infty} \Big\{ sign(m+\hat{a})(B_m^\dagger B_m - D_{-m} D_{-m}^\dagger) \\ &\qquad\qquad + \delta_{m+\hat{a},0}(B_m^\dagger D_{-m}^\dagger + D_{-m} B_m) \Big\}. \end{aligned}$$

The last expression uses the identity $\Theta(m+a) - \Theta(-m-a) = sign(m+a)$ where "$sign$" is the sign function whose value in 0 is taken to be $sign(0) = 0$. Furthermore the notation $\delta_{m+\hat{a},0}$ stands for a generalized Kronecker symbol of which the indices may take continuous values, such that its value vanishes unless the two indices are equal in which case the symbol takes the value unity. Once again the normal ordered form for the regularised operator Q_5 needs to be considered. The Poisson resummation formula allows to isolate divergent contributions in (3.24) and (3.25), leading to,

$$\begin{aligned} &\sum_{m=-\infty}^{+\infty} (\Theta(m+a) - \Theta(-m-a)) e^{-\tilde{\alpha}(m+a)^2} \\ &\quad \overset{\tilde{\alpha}\to 0}{=} \frac{1}{2}\Big(\sqrt{\frac{\pi}{\tilde{\alpha}}} - \sqrt{\frac{\pi}{\tilde{\alpha}}}\Big) + \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n a}}{\mathrm{i}\pi n}. \end{aligned}$$

Furthermore the series corresponds to the following Fourier expansion, provided a is non integer (see appendix A),

$$\sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n a}}{\mathrm{i}\pi n} = 1 - 2(a - \lfloor a \rfloor). \tag{3.29}$$

However one needs to specify what the *rhs* of (3.29) means when a is an integer. If the series in the *lhs* of (3.29) is summed symmetrically, its value vanishes. Hence for the sake of consistency, the final and complete expression for (3.29) reads,

$$\sum_{n=-\infty,n\neq 0}^{+\infty}\frac{e^{2\mathrm{i}\pi na}}{\mathrm{i}\pi n}=1-2(a-\lfloor a\rfloor+\frac{1}{2}I(a)),$$

where $I(a)$ stands for the discontinuous function which vanishes for all real values of a except when a is an integer, $a\in\mathbb{Z}$, in which case $I(a)$ takes the value unity. It is also useful to keep in mind the property $\lfloor -a\rfloor=-\lfloor a\rfloor-1+I(a)$. Thus finally the fully gauge invariant expression of the axial charge, which remains now well defined in the absence of a regulator, is

$$\begin{aligned} :\hat{Q}_5:_{\hat{a}} &= 2(\hat{a}-\lfloor\hat{a}\rfloor)-1+I(\hat{a}) \\ &\quad+\sum_{m=-\infty}^{+\infty}\big[sign(m+\hat{a})(B^\dagger_m B_m+D^\dagger_{-m}D_{-m}) \\ &\qquad+\delta_{m+\hat{a},0}(B^\dagger_m D^\dagger_{-m}+D_{-m}B_m)\big]. \end{aligned} \tag{3.30}$$

This operator indeed generates global axial $U(1)_A$ transformations, for $m+\hat{a}\neq 0$,

$$B^\dagger_m\ \to \exp\,(\mathrm{i}\beta)B^\dagger_m,\qquad D^\dagger_{-m}\ \to \exp\,(\mathrm{i}\beta)D^\dagger_{-m}$$

(for $m+\hat{a}=0$ an additional contribution arises because of the spectral flow properties in $\hat{a}$ of these operators).

Finally, let us point out that even though the Coulomb interaction contribution to the first class Hamiltonian has not been considered explicitly so far, the reason for this is that a simple consideration of the expression (3.20) for that operator $\hat{H}_C$ in terms of the fermionic modes readily shows that in the given form, it does not suffer quantum ordering ambiguities nor divergences since no contribution with $\ell=0$ is involved in either of the two factors being multiplied in the sum over ℓ.

3.4 Modular invariant operators and the axial anomaly

All potential divergences in the operators of interest having been subtracted consistently and in a manifestly modular invariant manner, let us first now focus our attention on the global symmetry of the massless classical theory, namely its axial symmetry. As is well-known these transformations are no longer a symmetry of the quantised dynamics because of a mechanism that involves the "topological" zero-mode sector which, in the present formulation, is clearly identified. The gauge invariant composite operators having been constructed so far include (the Casimir vacuum energy $(-\pi/(6L))$ is henceforth ignored in the total first class Hamiltonian),

$$\begin{aligned} :\hat{H}:_{\hat{a}} &= \frac{(\hat{p}^1)^2}{2L} + \frac{2\pi}{L}(\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2})^2 \\ &+ \frac{2\pi}{L}\sum_m |m+\hat{a}|(B^\dagger_m B_m + D^\dagger_{-m} D_{-m}) + :\hat{H}_C:_{\hat{a}}, \end{aligned} \quad (3.31)$$

$$:\hat{Q}_5:_{\hat{a}} = 2(\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2}) + I(\hat{a}) + q_5, \quad (3.32)$$

$$:\hat{Q}:_{\hat{a}} = \sum_m (B^\dagger_m B_m - D^\dagger_{-m} D_{-m}), \quad (3.33)$$

where

$$\begin{aligned} q_5 = \sum_m \Big[\quad & sign(m+\hat{a})(B^\dagger_m B_m + D^\dagger_{-m} D_{-m}) \\ & + \delta_{m+\hat{a},0}(B^\dagger_m D^\dagger_{-m} + D_{-m} B_m) \Big], \end{aligned} \quad (3.34)$$

while the gauge invariant total momentum operator of the system may be shown to be given as,

$$:\hat{P}:_{\hat{a}} = \sum_m \frac{2\pi}{L}(m+\hat{a})(B^\dagger_m B_m - D^\dagger_{-m} D_{-m}). \quad (3.35)$$

Since the B and D operators and their adjoints depend on the operator $\hat{a}_1$ through the operator $\hat{a} = \frac{e\hat{a}_1 L}{2\pi} - \lambda$, the B and D's do not commute with the conjugate momentum of $\hat{a}_1$, namely $\hat{p}^1$. A direct calculation finds,

$$[\hat{p}^1, B^\dagger_m] = -\mathrm{i}\frac{eL}{2\pi}\delta(m+\hat{a})D_{-m}, \quad [\hat{p}^1, D_{-m}] = \mathrm{i}\frac{eL}{2\pi}\delta(m+\hat{a})B^\dagger_m, \tag{3.36}$$

as well as the corresponding adjoint relations (here, $\delta(m+\hat{a})$ stands for the usual Dirac δ function). These results use the definitions (3.22) and (3.23) and the identity between distributions, $\partial_x\sqrt{\Theta(x)} = \delta(x)/\sqrt{2}$, given the choice $\Theta(0) = 1/2$. From these commutation relations it easily follows that $:\hat{H}:_{\hat{a}}$ commutes with $:\hat{Q}:_{\hat{a}}$. However, the same is not true for the axial charge operator for which the calculation requires the evaluation of the commutator $\left[\hat{p}^1, :\hat{Q}_5:_{\hat{a}}\right]$. By differentiation of (3.29) and making use of (3.36), one finds,

$$:\left[\hat{p}^1, :\hat{Q}_5:_{\hat{a}}\right]:_{\hat{a}} = -2\mathrm{i}\frac{eL}{2\pi}, \tag{3.37}$$

and in turn finally,

$$:\left[:\hat{H}:_{\hat{a}}, :\hat{Q}_5:_{\hat{a}}\right]:_{\hat{a}} = :\left[\frac{(\hat{p}^1)^2}{2L}, :\hat{Q}_5:_{\hat{a}}\right]:_{\hat{a}} = -2\mathrm{i}\frac{e\hat{p}^1}{2\pi}. \tag{3.38}$$

Since this relation expresses the quantum equation of motion for the axial charge in the Heisenberg picture, one observes that this charge is no longer conserved, hence suffers a "quantum anomaly". It is noticeable that this anomaly finds its origin only in the topological sector $(\hat{a}_1, \hat{p}^1)$. The physical interpretation and consequences of this result have been discussed in the literature [2, 22]. Namely, the equation (3.38) is the analogue of the non-conservation of the axial current

$$\partial_\mu j^\mu_5 = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}, \tag{3.39}$$

derived in the covariant formalism.

3.5 Modular invariant bosonization

Rather than wanting to diagonalize the gauge invariant Hamiltonian for physical states, it is possible to show that the theory describes in fact the dynamics of a free massive (pseudo)scalar boson of mass $m > 0$ on the physical space, in the form,

$$\begin{aligned} :\hat{H}:_{\hat{a}} &= \frac{1}{2} : \Pi(0)^{\dagger}\Pi(0) :_{\hat{a}} + \frac{1}{2} m^2 : \Phi^{\dagger}(0)\Phi(0) :_{\hat{a}} + \\ &+ \frac{1}{2} \sum_{k \neq 0} : \Big\{ \Pi^{\dagger}(k)\Pi(k) + (m^2 + (\frac{2\pi k}{L})^2)\Phi^{\dagger}(k)\Phi(k) \Big\} :_{\hat{a}} . \end{aligned}$$

The normal ordering prescription, $: :_{\hat{a}}$, for the fields $(\Phi(k), \Pi(k))$ will be specified hereafter. As usual the scalar bosonic theory is defined by

$$H = \int_{S^1} dx \frac{1}{2} \big\{ \Pi^{\dagger}(x)\Pi(x) + \Phi^{\dagger}(x)(-\partial_1^2 + m^2)\Phi(x) \big\},$$

with $\Phi(x) = 1/\sqrt{L} \sum_k \Phi(k) e^{\mathrm{i}\frac{2\pi k x}{L}}$ and $\Pi(x) = 1/\sqrt{L} \sum_k \Pi(k) e^{\mathrm{i}\frac{2\pi k x}{L}}$, $\Pi(x)$ being the momentum canonically conjugate to $\Phi(x)$ and $k \in \mathbb{Z}$. Let us now define the Fourier k-modes $(k \neq 0)$ for the boson and its conjugate momentum in terms of the fermionic modes as [9, 10],

$$\begin{aligned} \Phi(k) &= \tfrac{1}{\sqrt{2}\mathrm{i}k} \sqrt{\tfrac{L}{2\pi}} : (j_1(k) + j_2(k)) :_{\hat{a}} \\ \Pi(k) &= \tfrac{1}{\sqrt{2}} \sqrt{\tfrac{2\pi}{L}} : (j_1(k) - j_2(k)) :_{\hat{a}}, \end{aligned}$$

where $j_1(k) = \sum_m b^{\dagger}_{m+k} b_m$ and $j_2(k) = \sum_m d_{-(m+k)} d^{\dagger}_{-m}$. Note that for $k \neq 0$ these operators are involved in the contributions to the Coulomb interaction energy.

These definitions ensure that the k-modes $\Phi(k)$ and $\Pi(k)$ fulfil the following necessary properties, $\Phi^{\dagger}(k) = \Phi(-k)$ and $\Pi^{\dagger}(k) = \Pi(-k)$. For $k \neq 0$ the operators $j_1(k)$ and $j_2(k)$ may be expressed in terms of the B and D operators and their adjoints. Actually normal ordering of $j_j(k)$ $(j = 1, 2)$ is only required for $k = 0$. As long as $k \neq 0$, no ordering ambiguity arises. By extension of the ordering procedure described in the previous sections, henceforth the normal ordered form, denoted $: \hat{O} :_{\hat{a}}$, of an operator $\hat{O}$ made of a product of $b^{(\dagger)}$'s and $d^{(\dagger)}$'s is given by the normal ordered form with respect to the $B^{(\dagger)}$ and $D^{(\dagger)}$ operators upon the appropriate substitutions. However since intermediate steps in calculations or partial contributions to quantities may produce divergent quantities, it should be wise to regularise expressions before performing computations.

It being understood that the operators $j_j(k)$ are defined as has just been described, namely $j_j(k) \equiv: j_j(k) :_{\hat{a}}$, an explicit evaluation finds that these operators obey the following closed algebra,

$$\begin{aligned} :[j_1(k), j_1(\ell)]:_{\hat{a}} &= \ell\delta_{k+\ell,0}, &(3.40)\\ :[j_2(k), j_2(\ell)]:_{\hat{a}} &= -\ell\delta_{k+\ell,0}, &(3.41)\\ :[j_1(k), j_2(\ell)]:_{\hat{a}} &= 0. &(3.42)\end{aligned}$$

Let us establish here the first of these results. To compute the commutator (3.40) consider the case when k and ℓ have opposite signs (if they have the same sign it is easy to prove that the commutator vanishes), and introduce the gaussian regularisation procedure to handle potential divergences,

$$[j_1(k), j_1(-\ell)] = \big[\sum_m b^\dagger_{m+k} b_m e^{-\tilde{\alpha}(m+\hat{a})^2}, \sum_n b^\dagger_n b_{n+\ell} e^{-\tilde{\alpha}(n+\hat{a})^2}\big], \quad (3.43)$$

for $k, \ell > 0$. Using the anti-commutation relations, in normal ordered form (3.43) becomes,

$$\sum_{m,n} (: b^\dagger_{m+k} b_{n+\ell} :_{\hat{a}} \delta_{m,n} - : b^\dagger_m b_n :_{\hat{a}} \delta_{m+k,n+\ell}) e^{-\tilde{\alpha}(m+\hat{a})^2} e^{-\tilde{\alpha}(n+\hat{a})^2}.$$

When substituted in terms of the B, D operators, in the limit $\tilde{\alpha} \to 0$ this last expression reduces to,

$$\sum_n \exp\left[-2\tilde{\alpha}(n+\hat{a})^2\right](\Theta(-n-k-\hat{a}) - \Theta(-n-\hat{a}))\delta_{k,\ell} \overset{\tilde{\alpha}\to 0}{=} -k\delta_{k,\ell},$$

which is indeed the result in (3.40). And from the commutation relations (3.40) to (3.42), it readily follows that bosonic k-modes $(\Phi(k), \Pi(k))$ $(k \neq 0)$ do indeed obey the Heisenberg algebra as it should,

$$[\Phi(k), \Pi(\ell)] = \mathrm{i}\delta_{k+\ell,0}, \qquad k, \ell \neq 0. \quad (3.44)$$

Let us now tackle the bosonized version of the Hamiltonian, by showing that it indeed reproduces the expression (3.31). The k-mode part of the bosonic Hamiltonian is

$$\begin{aligned} &\frac{1}{2}\sum_{k\neq 0} : \left\{\Pi^\dagger(k)\Pi(k) + (\frac{2\pi k}{L})^2 \Phi^\dagger(k)\Phi(k)\right\} :_{\hat{a}} \\ &= \frac{1}{2}\frac{2\pi}{L}\sum_{k\neq 0} : \Big(j_1^\dagger(k) j_1(k) + j_2^\dagger(k) j_2(k)\Big) :_{\hat{a}} .\end{aligned}$$

Using the commutation relations (3.40) and (3.41) one finds,

$$\begin{aligned} &\tfrac{1}{2}\textstyle\sum_{k\neq 0} :\{\Pi^\dagger(k)\Pi(k) + (\tfrac{2\pi k}{L})^2\Phi^\dagger(k)\Phi(k)\}:_{\hat{a}} \qquad (3.45)\\ &= \tfrac{2\pi}{L}\textstyle\sum_{k>0}\sum_{m,n} :\{b^\dagger_{m+k}b_m b^\dagger_{n-k}b_n + d_{-(m-k)}d^\dagger_{-m}d_{-(n+k)}d^\dagger_{-n}\}:_{\hat{a}}\\ &= \tfrac{2\pi}{L}\textstyle\sum_{k>0}\sum_{m,n} :\{b^\dagger_{m+k}b_m b^\dagger_{n}b_{n+k} + d^\dagger_{-(m+k)}d_{-m}d^\dagger_{-n}d_{-(n+k)}\}:_{\hat{a}}\ . \end{aligned}$$

A little algebra shows that the sum over the range of values when $m\neq n$ vanishes on account of the anti-commutation properties of the $b^{(\dagger)}_m$ and $d^{(\dagger)}_m$ operators. Only the diagonal $m=n$ terms remain and provide the normal ordered expression,

$$\sum_{k>0}\sum_m : \left(b^\dagger_{m+k}b_m b^\dagger_m b_{m+k} + d^\dagger_{-(m+k)}d_{-m}d^\dagger_{-m}d_{-(m+k)}\right) :_{\hat{a}}\ . \qquad (3.46)$$

Substituting now for the $B^{(\dagger)}_m$ and $D^{(\dagger)}_m$ operators and using their anti-commutation relations, (3.46) becomes in an explicitly normal ordered form,

$$\begin{aligned} &\sum_{k>0}\sum_m\\ &\Big[(B^\dagger_{m+k}B_{m+k}D^\dagger_{-m}D_{-m} + D^\dagger_{-(m+k)}D_{-(m+k)}B^\dagger_m B_m)\times\\ &\qquad\times\Big(\Theta(m+k+\hat{a})\Theta(-m-\hat{a})\Big) \qquad (3.47)\\ &-(B^\dagger_{m+k}B_{m+k}B^\dagger_m B_m + D^\dagger_{-(m+k)}D_{-(m+k)}D^\dagger_{-m}D_{-m})\times\\ &\times\Big(\Theta(m+k+\hat{a})\Theta(m+\hat{a}) + \Theta(-m-k-\hat{a})\Theta(-m-\hat{a})\Big) \qquad (3.48)\\ &-(B^\dagger_{m+k}B_{m+k} + D^\dagger_{-(m+k)}D_{-(m+k)})(B^\dagger_m D^\dagger_{-m} + D_{-m}B_m)\times\\ &\qquad\times\tfrac{1}{2}\Theta(m+\hat{a}+k)\delta_{m+\hat{a},0} \qquad (3.49)\\ &+(B^\dagger_m B_m + D^\dagger_{-m}D_{-m})(B^\dagger_{m+k}D^\dagger_{-(m+k)} + D_{-(m+k)}B_{m+k})\times\\ &\qquad\times\tfrac{1}{2}\Theta(-m-\hat{a})\delta_{m+k+\hat{a},0} \qquad (3.50)\\ &+(B^\dagger_{m+k}B_{m+k} + D^\dagger_{-(m+k)}D_{-(m+k)})\Theta(m+k+\hat{a})\Theta(m+\hat{a}) \qquad (3.51)\\ &+(B^\dagger_m B_m + D^\dagger_{-m}D_{-m})\Theta(-m-k-\hat{a})\Theta(-m-\hat{a})\Big]. \qquad (3.52) \end{aligned}$$

The first eight lines (3.47) to (3.50) are quadrilinear in the $B^{(\dagger)}$ and $D^{(\dagger)}$ operators while the last two lines (3.51) and (3.52) are bilinear. They need to be handled differently.

The quadrilinear terms combine to give

$$-\frac{1}{4}(\hat{Q}^2 + q_5^2) + \sum_m \frac{1-\delta_{m+\hat{a},0}}{2}(B_m^\dagger B_m + D_{-m}^\dagger D_{-m}) + \frac{1}{4}I(\hat{a}), \quad (3.53)$$

with the help of (3.33) and (3.34), as may be checked by writing out (3.53) explicitly.

Factorizing the sum over the index k, the bilinear terms in (3.51) and (3.52) may be written as,

$$\begin{aligned}&\sum_m [N_{m+k}\Theta(m+k+\hat{a})\Theta(m+\hat{a}) + N_m\Theta(-m-k-\hat{a})\Theta(-m-\hat{a})] \\ &\quad = \sum_m N_m[\Theta(m+\hat{a})\Theta(m+\hat{a}-k) + \Theta(-m-\hat{a})\Theta(-m-\hat{a}-k)],\end{aligned} \quad (3.54)$$

where $N_m = (B_m^\dagger B_m + D_{-m}^\dagger D_{-m})$. Let us focus on any one of the terms in the series in curly brackets for any specific value of $m \in \mathbb{Z}$, in which N_m is multiplied by the following series,

$$\sum_{k>0}[\Theta(m+a)\Theta(m+a-k) + \Theta(-m-a)\Theta(-m-a-k)]. \quad (3.55)$$

If $m + a = 0$ this latter quantity vanishes explicitly since $\Theta(-k) = 0$ for $k > 0$. Consider then the case when $m + a \neq 0$. Making use of the identity

$$\sum_{k=1}^{+\infty}\theta(x-k) = \lfloor x \rfloor - \frac{1}{2}I(x), \quad (3.56)$$

which applies only for $x > 0$, one finds,

$$\Theta(m+a)\sum_{k>0}\Theta(m+a-k) = \Theta(m+a)\left(\lfloor m+a \rfloor - \frac{1}{2}I(a)\right),$$

and

$$\Theta(-m-a)\sum_{k>0}\Theta(-m-a-k) = \Theta(-m-a)\left(\lfloor -m-a \rfloor - \frac{1}{2}I(a)\right).$$

However since one has,

$$\lfloor m+a \rfloor = m + \lfloor a \rfloor, \qquad \lfloor -(m+a) \rfloor = -\lfloor m+a \rfloor - 1 + I(a),$$

the series (3.55) takes the form,

$$\begin{aligned} &\Theta(m+a)\left(m + \lfloor a \rfloor - \tfrac{1}{2}I(a)\right) \\ &+\Theta(-m-a)\left(-m - \lfloor a \rfloor - 1 + I(a) - \tfrac{1}{2}I(a)]\right), \end{aligned}$$

or equivalently,

$$\begin{aligned} &\Theta(m+a)\left(m + a - a + \lfloor a \rfloor - \tfrac{1}{2}I(a)\right) \\ &+\Theta(-m-a)\left(-m - a + a - \lfloor a \rfloor + \tfrac{1}{2}I(a)\right) - \theta(-m-a). \end{aligned}$$

Using now the fact that $\Theta(-m-a) = (1 - sign(m+a))/2$ the series (3.55) finally takes the following expression when $m + a \neq 0$,

$$\begin{aligned} &|m+a| - sign(m+a)\left(a - \lfloor a \rfloor + \tfrac{1}{2}I(a)\right) - \tfrac{1}{2}\left(1 - sign(m+a)\right) \\ &= |m+a| - \tfrac{1}{2} - sign(m+a)\left(a - \lfloor a \rfloor - \tfrac{1}{2} + \tfrac{1}{2}I(a)\right). \end{aligned} \qquad (3.57)$$

Since the series (3.55) vanishes when $m+a=0$, the complete expression may be written by subtracting from the above result its value when $m+a=0$, producing the final expression for the series (3.55),

$$|m+a| - \frac{1}{2} - sign(m+a)\left(a - \lfloor a \rfloor - \frac{1}{2} + \frac{1}{2}I(a)\right) + \frac{1}{2}\delta_{m+a,0}, \quad (3.58)$$

valid for any $m \in \mathbb{Z}$ and any $a \in \mathbb{R}$.

Substituting this identity in (3.54), one finally obtains for the sum of (3.51) and (3.52),

$$\begin{aligned} &\frac{1}{2}\sum_m \delta_{m+\hat{a},0} N_m + \sum_m (|m+\hat{a}| - \frac{1}{2}) N_m \\ &-(\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2} + \frac{1}{2}I(\hat{a})) \sum_m sign(m+\hat{a}) N_m. \end{aligned} \qquad (3.59)$$

Then the sum of (3.59) and (3.53) leads to the following expression for the k-mode contribution ($k \neq 0$) to the bosonic Hamiltonian,

$$\begin{aligned} \frac{2\pi}{L}\Big(&\sum_m |m+\hat{a}| N_m - (\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2} + \frac{1}{2}I(\hat{a})) \sum_m sign(m+\hat{a}) N_m \\ &- \frac{1}{4}(\hat{Q}^2 + q_5^2) + \frac{1}{4}I(\hat{a})\Big). \end{aligned} \qquad (3.60)$$

Obviously this last expression includes the fermionic bilinear contribution to the Hamiltonian in (3.31). Furthermore (3.60) gives also a clue for the zero-mode part of the bosonized Hamiltonian. Let us complete a square as follows,

$$\frac{2\pi}{L}\Big(\sum_m |m+\hat{a}|N_m - \Big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}+\frac{1}{2}I(\hat{a})+\frac{1}{2}q_5\Big)^2 + \big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}+\frac{1}{2}I(\hat{a})\big)^2 - \frac{1}{4}\hat{Q}^2+\frac{1}{4}I(\hat{a})\Big), \qquad (3.61)$$

with q_5 given in (3.34) and where the contribution in $\hat{Q}^2$ vanishes for the physical states. Indeed this last relation applies since one has the property

$$\begin{aligned}&(a-\lfloor a\rfloor-\frac{1}{2}+\frac{1}{2}I(\hat{a}))q_5\\ &\quad=(a-\lfloor a\rfloor-\tfrac{1}{2}+\tfrac{1}{2}I(\hat{a}))\textstyle\sum_m sign(m+a)N_m,\end{aligned}$$

given the expression in (3.29) and the fact that the product of $\delta_{m+\hat{a},0}$ with the first factor in this last expression vanishes identically. Likewise by direct expansion, one finds,

$$\begin{aligned}&\big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}+\frac{1}{2}I(\hat{a})\big)^2+\frac{1}{4}I(\hat{a})\\ &= \big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}\big)^2+\big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}\big)I(\hat{a})+\frac{1}{4}I(\hat{a})+\frac{1}{4}I(\hat{a})\\ &= \big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}\big)^2.\end{aligned}$$

We may now complete the bosonization procedure and define the missing pieces in the bosonized formulation. One needs to identify the bosonic conjugate momentum zero-mode, $\Pi(0)$. The result (3.61) provides this identification through,

$$\frac{1}{2}\Pi(0)^\dagger\Pi(0)=\frac{2\pi}{L}\Big(\hat{a}-\lfloor\hat{a}\rfloor-\frac{1}{2}+\frac{1}{2}I(\hat{a})+\frac{1}{2}q_5\Big)^2=\frac{\pi}{2L}\Big(:\hat{Q}_5:_{\hat{a}}\Big)^2,$$

hence one defines,

$$\Pi(0)=\pm\sqrt{\frac{\pi}{L}}:\hat{Q}_5:_{\hat{a}}\,. \qquad (3.62)$$

To sum up we have established the following identity, which is valid for physical states only with $\hat{Q}=0$,

$$\begin{aligned}&\tfrac{1}{2}:\Pi(0)^\dagger\Pi(0):_{\hat{a}}+\tfrac{1}{2}\textstyle\sum_{k\neq 0}:\big(\Pi^\dagger(k)\Pi(k)+(\tfrac{2\pi k}{L})^2\Phi^\dagger(k)\Phi(k)\big):_{\hat{a}}\\ &\quad=\tfrac{2\pi}{L}\big(\hat{a}-\lfloor\hat{a}\rfloor-\tfrac{1}{2}\big)^2+\tfrac{2\pi}{L}\textstyle\sum_m|m+\hat{a}|(B_m^\dagger B_m+D_{-m}^\dagger D_{-m}).\end{aligned}$$

Finally the Coulomb interaction Hamiltonian provides the mass term for the boson,

$$\begin{aligned}&\frac{1}{2}\sum_{k\neq 0}m^2:\Phi(k)^\dagger\Phi(k):_{\hat{a}}\\ &\quad=\tfrac{e^2L}{2(2\pi)^2}\textstyle\sum_{k\neq 0}:\frac{(:j_1^\dagger(k):_{\hat{a}}+:j_2^\dagger(k):_{\hat{a}})(:j_1(k):_{\hat{a}}+:j_2(k):_{\hat{a}})}{k^2}:_{\hat{a}},\end{aligned}$$

hence the identification $m^2=e^2/\pi$. And the very last piece of the puzzle is the zero-mode of the boson, $\Phi(0)$, provided by,

$$\frac{1}{2}m^2\Phi^\dagger(0)\Phi(0)=\frac{(\hat{p}^1)^2}{2L},$$

which leads to $\Phi(0)=\sqrt{\pi}\hat{p}^1/(e\sqrt{L})$. The choice of sign for this quantity is correlated to that of the conjugate momentum zero mode, $\Pi(0)$. By choosing the minus sign for the square root in (3.62), one then also obtains the proper Heisenberg algebra for the boson zero-modes,

$$:\big[\Phi(0)\,,\,\Pi(0)\big]:_{\hat{a}}=\frac{\pi}{eL}:\big[\hat{p}^1\,,\,-:\hat{Q}_5:_{\hat{a}}\big]:_{\hat{a}}=\mathrm{i}. \tag{3.63}$$

The axial anomaly thus proves to be central in establishing the correct commutation relation in the zero-mode sector of the bosonized fermion.

In conclusion, when restricted to the space of physical quantum states of total vanishing electric charge, $\hat{Q}=0$, the total first class Hamiltonian, whether expressed in terms of the original fermion modes or the bosonic ones given by

$$\begin{aligned}&\Phi(0)=\sqrt{\pi}\tfrac{\hat{p}^1}{e\sqrt{L}}, &&\Phi(k\neq 0)=\frac{1}{\sqrt{2}\mathrm{i}k}\sqrt{\frac{L}{2\pi}}:(j_1(k)+j_2(k)):_{\hat{a}},\\ &\Pi(0)=-\sqrt{\tfrac{\pi}{L}}:\hat{Q}_5:_{\hat{a}}, &&\Pi(k\neq 0)=\frac{1}{\sqrt{2}}\sqrt{\frac{2\pi}{L}}:(j_1(k)-j_2(k)):_{\hat{a}},\end{aligned}$$

determines the same quantum theory and physical content.

3.6 Vacuum state of the interacting theory

For the sake of completeness, the vacuum structure of the Schwinger model is succintly examined. This question was addressed previously by Azakov [23], comparing the functional to the Hamiltonian approach at finite temperature, while the Hamiltonian formulation was more recently reviewed in [24, 25] at zero temperature.
For convenience, the bosonic zero-mode sector will be represented on wave functions with $\hat{a}_1$ acting as a multiplicative operator and $\hat{p}^1$ as a derivative operator.

Until now, the fermionic operators were not represented on a Hilbert space. In order to understand the complete vacuum structure of the Schwinger model, the first step is to specify how the fermionic Fock vacuum is constructed. It is defined customarily by the condition

$$b_m|0\rangle = 0 = d_{-m}|0\rangle \tag{3.64}$$

for any integer m. The Bogoliubov transformation introduced previously in (3.22) and (3.23) can be implemented thanks to the adjoint action of a unitary operator

$$\mathcal{U}(a)b_m\mathcal{U}^\dagger(a) = B_m, \qquad \mathcal{U}(a)b_m^\dagger\mathcal{U}^\dagger(a) = B_m^\dagger \tag{3.65}$$

$$\mathcal{U}(a)d_{-m}\mathcal{U}^\dagger(a) = D_{-m}, \qquad \mathcal{U}(a)d_{-m}^\dagger\mathcal{U}^\dagger(a) = D_{-m}^\dagger \tag{3.66}$$

where the operators $B_m = B_m(m+a)$, $D_{-m} = D_{-m}(m+a)$ and their adjoints are implicitly functions of the combination $m+a$. The transformation is explicitly given by

$$\mathcal{U}(a) = \exp\Big\{ -\sum_m \frac{\pi}{2}\Theta(-m-a)[b_m^\dagger d_{-m}^\dagger - d_{-m}b_m]\Big\}. \tag{3.67}$$

By construction, the Bogoliubov operator (3.67) is invariant under the gauge transformations $\hat{U}(\ell)\mathcal{U}(a)\hat{U}^\dagger(\ell) = \mathcal{U}(a)$ for $\ell \in \mathbb{Z}$. In consequence, the state annihilated by $B_m = B_m(m+a)$ and $D_{-m} = D_{-m}(m+a)$ is the modular invariant vacuum

$$\mathcal{U}(a)|0\rangle. \tag{3.68}$$

The fermionic vacuum being specified, it may be related to the exact vacuum of the Schwinger model. To do so, it is necessary to identify the creators and annihilators of the (pseudo)scalar boson in terms of the fermionic operators. The Fourier decompositions of the (pseudo)scalar boson and its conjugate momentum

$$\Phi(x) = \frac{1}{\sqrt{L}}\Phi(0) + \frac{1}{\sqrt{L}}\sum_{k\neq 0}\Phi(k)e^{2\mathrm{i}\pi kx/L}, \tag{3.69}$$

$$\Pi(x) = \frac{1}{\sqrt{L}}\Pi(0) + \frac{1}{\sqrt{L}}\sum_{k\neq 0}\Pi(k)e^{2\mathrm{i}\pi kx/L}, \tag{3.70}$$

have to be written in terms of creators and annihilators of a boson with dispersion relation $\omega_n = \sqrt{|\frac{2\pi n}{L}|^2 + m^2}$, so that their Fock state can be defined in the zero-mode and k-mode sector. Considering first the k-mode sector, the commutation relations of the Fourier modes of the boson $[\Phi(k);\Pi(\ell)] = \mathrm{i}\delta_{k+\ell,0}$ for $k,\ell \in \mathbb{Z}$, suggest to define the gauge invariant bosonic operators, for $n > 0$,

$$A_n = \tfrac{1}{\sqrt{2\omega_n^0}}[\Pi(-n) - \mathrm{i}\omega_n^0\Phi(-n)], \quad A_n^\dagger = \frac{1}{\sqrt{2\omega_n^0}}[\Pi(n) + \mathrm{i}\omega_n^0\Phi(n)],$$
$$A_{-n} = \tfrac{-1}{\sqrt{2\omega_n^0}}[\Pi(n) - \mathrm{i}\omega_n^0\Phi(n)], \quad A_{-n}^\dagger = \frac{-1}{\sqrt{2\omega_n^0}}[\Pi(-n) + \mathrm{i}\omega_n^0\Phi(-n)],$$

with $\omega_n^0 = |\frac{2\pi n}{L}|$, verifying the Fock algebra

$$[A_n, A_m^\dagger] = \delta_{n,m} = [A_{-n}, A_{-m}^\dagger].$$

The construction of the above bosonic Fock operators is a first step towards the definition of the creators and annihilators of the massive (pseudo)scalar. The reason for this intermediate definition is that it allows to relate the fermionic Fock state to the complete vacuum state. Indeed, in terms of the fermionic operators, we have

$$A_n = \tfrac{j_1(-n)}{\sqrt{n}}, \quad A_n^\dagger = \frac{j_1(n)}{\sqrt{n}},$$
$$A_{-n} = \tfrac{j_2(n)}{\sqrt{n}}, \quad A_{-n}^\dagger = \frac{j_2(-n)}{\sqrt{n}},$$

for $n > 0$. Writing the bosonic oscillators in terms of $B_m(m+a)$, $D_{-m}(m+a)$ and their adjoints, it is straightforward to show

$$A_n \mathcal{U}(a)|0\rangle = 0 = A_{-n}\mathcal{U}(a)|0\rangle, \tag{3.71}$$

that is to say, the A_n's annihilate the fermionic vacuum (3.68). Since the above bosonic Fock operators are associated to creators and annihilators of a massless bosons, a bosonic Bogoliubov transformation is still necessary to introduce the creators and annihilators of the massive boson. The creators and annihilators of the boson of mass $m = e/\sqrt{\pi}$ are given by

$$\begin{aligned} a_n &= \tfrac{1}{\sqrt{2\omega_n}}[\Pi(-n) - \mathrm{i}\omega_n \Phi(-n)], \quad & a_n^\dagger &= \frac{1}{\sqrt{2\omega_n}}[\Pi(n) + \mathrm{i}\omega_n \Phi(n)], \\ a_{-n} &= \tfrac{-1}{\sqrt{2\omega_n}}[\Pi(n) - \mathrm{i}\omega_n^0 \Phi(n)], \quad & a_{-n}^\dagger &= \frac{-1}{\sqrt{2\omega_n}}[\Pi(-n) + \mathrm{i}\omega_n \Phi(-n)], \end{aligned}$$

with the dispersion relation $\omega_n = \sqrt{|\frac{2\pi n}{L}|^2 + m^2}$. The transformation between the massless and massive oscillators is given by the adjoint action of a unitary operator

$$\begin{aligned} a_n &= \mathcal{B}^\dagger A_n \mathcal{B}, \qquad & a_n^\dagger &= \mathcal{B}^\dagger A_n^\dagger \mathcal{B}, \\ a_{-n} &= \mathcal{B}^\dagger A_{-n} \mathcal{B}, \qquad & a_{-n}^\dagger &= \mathcal{B}^\dagger A_{-n}^\dagger \mathcal{B}, \end{aligned}$$

while the unitary operator $\mathcal{B}$ is

$$\mathcal{B} = \exp\Big\{ -\sum_{n>0} \eta_n (A_{-n}^\dagger A_n^\dagger - A_{-n} A_n) \Big\}, \tag{3.72}$$

with the parameters η_n obtained by solving the implicit equation

$$\tanh \eta_n = \frac{\sqrt{\omega_n^0/\omega_n} - \sqrt{\omega_n/\omega_n^0}}{\sqrt{\omega_n^0/\omega_n} + \sqrt{\omega_n/\omega_n^0}}. \tag{3.73}$$

Hence, if $|\Omega\rangle$ is a general state annihilated by A_n and A_{-n} for $n > 0$, then a state annihilated by a_n and a_{-n} for $n > 0$ is given by

$$\mathcal{B}^\dagger |\Omega\rangle. \tag{3.74}$$

The vacuum state is not yet completely determined since the zero momentum modes have been considered. In the zero-mode sector, the creator and annihilator of the massive (pseudo)scalar are

$$a_0 = \frac{1}{\sqrt{2m}}\Big(\Pi(0) - \mathrm{i}m\Phi(0)\Big), \quad a_0^\dagger = \frac{1}{\sqrt{2m}}\Big(\Pi(0) + \mathrm{i}m\Phi(0)\Big), \quad (3.75)$$

so that the quantum Hamiltonian (3.40) takes the familiar form

$$:\hat{H}:_a = m(a_0^\dagger a_0 + \frac{1}{2}) + \sum_{k\neq 0}\omega(k)(a_k^\dagger a_k + \frac{1}{2}). \quad (3.76)$$

The state, denoted by $|\Omega\rangle$, can be determined by considering a suitable linear combination of the states $\mathcal{U}(a)|0\rangle$ for a fixed, which are annihilated by A_n and A_{-n} for $n > 0$. The condition determining $|\Omega\rangle$ is furnished by the zero-mode sector, because $\mathcal{B}^\dagger$ and $\mathcal{B}$ commute with $\Phi(0)$ and $\Pi(0)$. Indeed, the state $|\Omega\rangle$ has to satisfy

$$\Big(\Pi(0) - \mathrm{i}m\Phi(0)\Big)|\Omega\rangle = 0 = \Big(\hat{Q}_5 + \frac{\mathrm{i}}{\sqrt{\pi}}\hat{p}^1\Big)|\Omega\rangle. \quad (3.77)$$

In order to solve this differential equation, we first introduce the "fiducial" state defined by the limit

$$\mathcal{U}(0_+)|0\rangle = \lim_{\epsilon\to 0^+}\mathcal{U}(\epsilon)|0\rangle = \prod_{m<0} d_{-m}^\dagger b_m^\dagger|0\rangle, \quad (3.78)$$

as well as the mutually orthogonal states, obtained by the action of a modular transformation $\hat{U}(\ell)$ given in (3.20) and (3.21),

$$\hat{U}(\ell)\mathcal{U}(0_+)|0\rangle, \quad \ell \in \mathbb{Z}, \quad (3.79)$$

which have the explicit expression,

$$\hat{U}(n)\mathcal{U}(0_+)|0\rangle \quad = e^{-\mathrm{i}n\theta_0}\prod_{0\leq m<n} d_{-m}^\dagger b_m^\dagger \mathcal{U}(0_+)|0\rangle, \quad n > 0, (3.80)$$

$$\hat{U}(-n)\mathcal{U}(0_+)|0\rangle \quad = e^{\mathrm{i}n\theta_0}\prod_{-n\leq m<0} b_m d_{-m}\mathcal{U}(0_+)|0\rangle, \quad n > 0. (3.81)$$

Then, fixing one sector $\ell \in \mathbb{Z}$ and a generic wave function $f_\ell(a)$, it is possible to solve

$$\Big(\hat{Q}_5 + \frac{\mathrm{i}}{\sqrt{\pi}}\hat{p}^1\Big) f_\ell(a)\hat{U}(\ell)\mathcal{U}(0_+)|0\rangle, \quad (3.82)$$

for $0 < a < 1$, using the conjugation relation (3.63). Indeed, the solutions of (3.77) in each sector $\ell \in \mathbb{Z}$ may be glued in order to preserve the modular symmetry. Hence, a gauge invariant solution to the condition (3.77) is given by the superposition

$$\sum_{\ell} f_\ell(a) \hat{U}(\ell) \mathcal{U}(0_+)|0\rangle, \tag{3.83}$$

provided that the wave functions obey $f_\ell(a+1) = f_{\ell+1}(a)$, so that the solution in each sector ℓ has to satisfy the boundary condition $f_\ell(1) = f_{\ell+1}(0)$ as claimed in [10]. Using the property

$$:\hat{Q}_5:_a \mathcal{U}(0_+)|0\rangle = 2\Big(a - 1/2\Big)\mathcal{U}(0_+)|0\rangle, \quad a \in \mathbb{R} \setminus \mathbb{Z}, \tag{3.84}$$

the equation (3.82) may be solved in the domain $0 < a < 1$. Since the boundary condition is satisfied by $f_\ell(a) = f(a+\ell)$, the solution of the differential equation is the Gaussian

$$f(a) = N e^{-\frac{2\pi}{2eL}\sqrt{\pi}(a-1/2)^2}, \tag{3.85}$$

where N is a normalization. Consequently, the gauge invariant solution to (3.77) is

$$|\Omega\rangle = N \sum_{\ell \in \mathbb{Z}} e^{-\frac{2\pi}{2eL}\sqrt{\pi}(a+\ell-1/2)^2} e^{-\mathrm{i}\ell\theta_0} \hat{U}_f(\ell) \mathcal{U}(0_+)|0\rangle, \tag{3.86}$$

where we have factorized the quantum modular operator (3.21) as follows

$$\hat{U}(\ell) = \exp\Big\{2\mathrm{i}\pi\ell\Big(\frac{1}{e}\frac{\hat{p}_1}{L} - \frac{\theta_0}{2\pi}\Big)\Big\}\hat{U}_f(\ell). \tag{3.87}$$

As a consequence, the lowest energy state of the interacting theory is given by

$$\mathcal{B}^\dagger|\Omega\rangle. \tag{3.88}$$

The norm squared of the vacuum state is obtained by computing

$$||\mathcal{B}^\dagger|\Omega\rangle||^2 = |N|^2 \int_0^1 \mathrm{d}a \sum_{\ell} e^{-\frac{2\pi}{eL}\sqrt{\pi}(a+\ell-1/2)^2}. \tag{3.89}$$

In conclusion, the vacuum state of the Schwinger model is found to have a "periodic" structure, which is a consequence of the invariance under the modular gauge transformations. The ground state of the pseudo-scalar boson is non trivially expressed in the fermionic state space.

3.7 Adding a theta term

A natural extension of this low dimensional model is the inclusion of a "theta" term, which is the analogue of the topological θ term in four dimensional QCD, by adding the following contribution to the original Lagrangian density of the Schwinger model,

$$\mathcal{L}_\theta = \frac{e}{2}\frac{\theta}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}, \tag{3.90}$$

where the parameter θ has mass dimension M^0. The entire analysis of constraints can be carried through once again in a manner similar to what has been done previously, leading to the following first class quantum Hamiltonian corresponding to the one in (3.31),

$$\begin{aligned} :\hat{H}:_{\hat{a}} &= \frac{1}{2L}(\hat{p}^1 - eL\frac{\theta}{2\pi})^2 + \frac{2\pi}{L}(\hat{a} - \lfloor \hat{a} \rfloor - \frac{1}{2})^2 \\ &\quad + \sum_m \frac{2\pi}{L}|m+\hat{a}|(B^\dagger_m B_m + D^\dagger_{-m}D_{-m}) + :\hat{H}_C:_{\hat{a}} \,. \end{aligned} \tag{3.91}$$

The shift by a term proportional to θ in the contribution of the gauge zero-mode conjugate momentum $\hat{p}^1$ is also observed in the axial anomaly,

$$:\big[\, :\hat{H}:_{\hat{a}} , :\hat{Q}_5:_{\hat{a}} \,\big]:_{\hat{a}} \;=\; :\big[\frac{(\hat{p}^1 - eL\theta/2\pi)^2}{2L} , :\hat{Q}_5:_{\hat{a}} \,\big]:_{\hat{a}} \tag{3.92}$$

$$= -\mathrm{i}\frac{e^2}{\pi}L\left(\frac{\hat{p}^1}{eL} - \theta/2\pi\right). \tag{3.93}$$

Given this observation which applies to the model with a massless fermion, it should be clear that all previous considerations remain valid in terms of the shifted conjugate momentum, $(\hat{p}^1 - eL\theta/(2\pi))$, which still defines a Heisenberg algebra with the gauge zero mode $\hat{a}_1$. Note that the introduction of the shifted variable affects the modular transformation operators $\hat{U}(\ell)$ only by a redefinition of the arbitrary phase factor θ_0 as $\theta_0 \to \theta_0 - \theta$, with no further consequence. Hence, in the massless fermion model, the introduction of the θ term does not lead to a modified gauge invariant physical content of the quantised system. It still is equivalent to a theory of a free (pseudo)scalar bosonic field of mass $m = |e|/\sqrt{\pi} > 0$.

3.8 Conclusions

In order to better understand the relevance and physical consequences of the topological sectors of gauge invariant dynamics, the present work developed a careful analysis of the Schwinger model in its fermionic formulation on a compactified spacetime with the cylindrical topology, within a manifestly gauge invariant formulation without resorting to any gauge fixing procedure. Among different reasons for considering a spatial compactification, one feature proves to be central to the discussion, namely that of large gauge or modular transformations which capture the topologically non trivial characteristics of the dynamics. Through proper regularisation a quantisation that remains manifestly invariant under modular transformations is feasible, and allows at the same time a clear separation between locally gauge variant and invariant degrees of freedom and globally gauge variant and invariant degrees of freedom, the latter being acted on by modular transformations only. Spatial compactification brings to the fore all the subtle aspects related to the topological sectors and their dynamics of the model.

What proves to be a most remarkable fact indeed, which remains relevant more generally for any non-abelian Yang-Mills theory coupled to charged matter fields in higher spacetime dimensions as well, is that the topologically non trivial modular gauge transformations act by mixing the small and large distance and energy scales of the dynamics, a feature which is intrinsically non-perturbative as well and thus cannot be captured through any perturbation theory that includes gauge invariance under small gauge transformations only.

To the author's best knowledge such an analysis of the Schwinger model has not been available in the literature so far. Besides recovering the well known result that as soon as the gauge coupling constant of the electromagnetic interaction is turned on this theory is in actual fact that of a free spin zero massive particle in two dimensions, rather than a theory of electrons and positrons coupled to photons, the analysis provides an original insight into the role played by topology and modular invariance in a mechanism leading to the confinement of charged particles in an abelian gauge theory. The fact that the chiral anomaly also finds its sole origin in the purely topological gauge sector is clearly made manifest through the considered separation of variables which is devoid of any gauge fixing procedure whatsoever. And finally the bosonization of the massless fermion is done at the operator level in terms of the fermionic modes rather than through vertex operators of the boson, by paying due care and attention to the contributions of the topological sector which again are crucial for the quantum equivalence between the two theories. In particular a manifestly modular invariant bosonization of the fermion degrees of freedom has been achieved.

CHAPTER 4

Fermion condensation in QED_{2+1}

4.1 Brief overview and motivations

In Chapter 3 we studied the non-perturbative aspects of massless QED_{1+1}. The same technique of factorization of the local gauge symmetries, the gauge degrees of freedom and the dressing of the electron field can be applied to QED_{2+1}. Nevertheless, the dynamics in the gauge sector is richer and more complex so that we do not expect to find an exact solution to the quantum field theory in interaction. Notwithstanding, an approximation will be developed in this chapter based on techniques similar to those one explained in Chapter 3. Before describing the obtained results, this section aims at emphasing various features attracting interest in QED_3, from the high energy physics point of view as well as from the condensed matter perspective.
Being a toy model of more realistic high energy physics models, quantum electrodynamics in two space dimensions has attracted interest for many years. Among the different aspects of this quantum field theory,

a striking property of pure QED_{2+1} is that the electromagnetic degrees of freedom can be compactly expressed in terms of a scalar. This procedure is called dualization of a $U(1)$ gauge field and is often used in the context of supersymmetric gauge theory in 3 spacetime dimensions. The classical action

$$S_{\text{class}} = \int \mathrm{d}t \int \mathrm{d}^2 x^i \{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\}, \tag{4.1}$$

gives rise to the equations of motion $\partial_\mu F^{\mu\nu} = 0$, while the Bianchi identity $\partial_\mu \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0$ is trivially satisfied. The peculiar expression of the Bianchi identity is the source of the dualization, which introduces a scalar degree of freedom thanks to the relation

$$\partial_\mu \phi = \epsilon_{\mu\nu\rho} F^{\nu\rho}. \tag{4.2}$$

The dualization is a relativistic covariant procedure. Therefore the dual scalar obeys the equation of motion of a free massless scalar $\Box\phi = 0$, where $\Box = \partial_t^2 - \Delta$. To be more precise we should mention that the general principle of the abelian duality is to write the Bianchi identity as the equation of motion of the dual theory. Incidentally, a similar procedure is available in $1+1$ dimensions where the duality relation involves two scalar fields, $\epsilon_{\mu\nu}\partial^\mu \phi = \partial_\nu \tilde{\phi}$. Similarly, in 3+1 dimensions, the duality exchanges the fields strength tensor $F_{\mu\nu}$ by ${}^*F_{\mu\nu}$. Although it might seem an elegant mathematical procedure, duality renders the coupling to matter, $A_\mu J^\mu$, non-local, when expressed in terms of the dual scalar. Apparently the dual formulation is therefore not advantageous to study the theory coupled to matter. However the presence of a defect in the classical gauge field configuration is "felt" by the dual scalar. For this reason the dual scalar is considered when studying in QED_3 the analogues of the instantons, which are known to play potentially a fundamental role in the non-perturbative regime QCD_4.

Singular configurations of A_μ are central to the proof by Polyakov of the confinement property of a specific version of QED_3 in which the dual scalar plays a role in the description of the low energy dynamics. In the seventies, Alexander Polyakov [26] studied QED_3 in euclidian space in the absence of fermionic matter, as a limit of a Georgi-Glashow model.

The crucial feature of his approach was that the gauge group $U(1)$ was considered as subgroup of a spontaneously broken $SU(2)$ gauge symmetry. Consequently, this version of QED was called "compact", because it emphasizes that the gauge group has a compact topology[1], so that the gauge potential has to be understood as an angular variable. Polyakov's achievement was to prove confinement of static charges due to the presence of topological defects in the gauge field. He calculated, in a weakly coupled regime, the contribution to the euclidian partition function of a dilute "monopole-instanton" plasma and proved the area law for the Wilson loop. As a confirmation, the lattice formulation of compact QED_3 was also studied by Gopfert and Mack [27], which provided a detailed proof of the confinement. Later Kogan and Kovner studied the so-called vortex operator in compact QED_{2+1} in a variational setup, in the absence of matter and obtained parallel conclusions. As emphasized by the latter authors (see [28]), the crucial feature of compact QED_{2+1} is the presence of singular gauge transformations of the type

$$\exp \frac{\mathrm{i}}{e} \int \mathrm{d}^2 x^i \{\partial_j \alpha(\vec{x}) E_j(\vec{x}) + \alpha(\vec{x})\rho(\vec{x})\}, \tag{4.3}$$

where one define the planar angle[2] $\alpha(\vec{x}) = \mathrm{Atan}(x_1/x_2)$. Let us explain why these transformations are admissible in the compact theory and how they are related to singular configurations in the gauge field. When one goes through the cut discontinuity in $\alpha(\vec{x})$, the gauge parameter jumps by 2π. However, in compact QED_{2+1}, a particle has a quantised charge $q = ne$ and, hence, will not feel this 2π jump, as may be easily seen in (4.3). In [29], it was made clear that the 't Hooft operator associated to the transformation (4.3)

$$V(\vec{x}) = \exp \frac{\mathrm{i}}{e} \int \mathrm{d}^2 y^i \epsilon_{jk} \frac{(x-y)_k}{(\vec{x}-\vec{y})^2} E_j(\vec{y}), \tag{4.4}$$

corresponds to the creation of a singular magnetic vortex, which can not be distinguished from a unit operator. This is convincingly shown by

[1] When the gauge group is the Lie group $U(1)$, which is compact, the quantisation of the electric charge is automatic, i.e. all the charges are commensurate. This is not true if the gauge group is $\mathbb{R}$, which is not compact. The lattice formulations of the two versions are different.

[2] The position of the cut is irrelevant. This is a consequence of charge quantisation.

the identity

$$V(\vec{x})^\dagger B(\vec{y}) V(\vec{x}) = B(\vec{y}) + \frac{2\pi}{e}\delta^{(2)}(\vec{x}-\vec{y}). \tag{4.5}$$

Consequently, the magnetic field is not by itself gauge invariant in compact QED. In the compact theory, because all the electric charges are integer multiples of e, the effect of the presence such a magnetic vortex cannot be measured. Thanks to the presence of such "defects", the confinement mechanism established by Polyakov is indeed similar to the dual superconductivity mechanism, i.e. a dual Meissner effect. In the presence of fermionic dynamical matter, it does not seem to be possible to study the dynamics with similar technical tools.

Undeniably, important features render this theory an interesting laboratory in order to develop techniques addressing non-perturbative dynamics. Namely, the excellent ultraviolet behaviour of perturbative QED_{2+1} is remarkable. Among the primary divergent diagrams of QED_{3+1}, only the electron self-energy and the vacuum polarisation of QED_{2+1} are superficially one-loop divergent. Following from gauge invariance and a symmetric integration of the loop, both diagrams are actually finite in dimensional regularisation. In a renowned paper [30], Jackiw and Templeton analysed the infrared divergences occuring in perturbation theory in QED_{2+1} with massless fermions, while the excellent behaviour of the theory in the UV is emphazised. Using a toy model treated non-perturbatively, these authors explain how the perturbative expansion in the coupling constant has to be completed by an expansion in logarithms of the coupling constant, while they expect also contributions which remain beyond the reach of perturbation theory.

In analogy with QCD_{3+1}, the question of spontaneous chiral symmetry breaking was also raised in the context of QED_{2+1} with N flavours. Chirality may be defined in $2+1$ dimensions by considering 4-spinors, in a reducible representation of the Lorentz group, as it is briefly summarized in sections B.6, B.7 and B.8. The analysis of the Schwinger-Dyson equations with various truncation schemes lead to a critical number of flavours, varying slightly according to the different authors.

On the other hand, QED_{2+1} unexpectedly arised as an effective theory of recently discovered condensed matter models. Remarkably a two-flavour version of massless QED_{2+1} has been shown to describe well the low energy dynamics of graphene. Due to its cristalline structure, the valence and conduction bands of graphene meet in two inequivalent conical points in the fundamental cell. The conical shape of the valence band at these "Dirac points" allows to linearize the dispersion relation so that the quasi-particles in the material are Dirac fermions [31]. The appearance of the two flavours is a consequence of fermion doubling, as a result of their definition on a lattice. This could be understood as a consequence of the Nielsen-Ninomiya theorem [32, 33], which requires that the lattice fermions are always expected to come in pairs. The two possible spins and "valleys" give rise to a set of four relativistic fermions, rotated into one another by a flavour $SU(4)$ symmetry. This picture was confirmed by the observation of the quantum Hall plateaus. Because of the smallness of the Fermi velocity compared to the speed of light, the effective coupling constant in graphene is approximately 300 times larger than in QED. The upshot is that the traditional approach based on perturbation theory has to be questioned.

Strikingly, solid state physics can also effectively reproduce the dynamics of a "undoubled" Dirac fermion in $2 + 1$ dimensions. For instance, a model discussed in [34] by Haldane is an example of a continuum limit of a condensed matter model in a periodic magnetic field where excitations correspond to a single Dirac fermion[3]. More recently, the discovery of a new class of materials called "topological insulators" [35] has opened a new age in condensed matter physics. Indeed, the surface of a 3D strong topological insulator [36] exihibits a peculiar behaviour, since it is possible to tune the Fermi energy to intersect a single "Dirac point". For this reason, the constraint from the Nielsen-Ninomiya theorem is eluded. The result is that the effective quasi-particle dynamics can be described by a single Dirac field. Hence, the peculiarity of such a material would be to realize a novel quantisation of conductance, called half-integer Quantum Hall Effect, due to the relativistic Landau level structure of the spectrum.

[3]This model breaks time reversal symmetry.

4.1.1 Considerations about the running of the coupling constant in massless $\mathbf{QED}_{2+1}$

Is QED_{2+1} with massless fermions asymptotically free? The question of asymptotic freedom in QED_{2+1} is particularly subtle, since the coupling constant has $M^{1/2}$ dimension. From the perturbative point of view, only two diagrams are superficially one-loop divergent. The one-loop fermion self-energy and vacuum polarization are respectively logarithmically and linearly divergent in power counting. However, both are finite in dimensional regularisation. As a consequence, no infinite subtraction has to be performed and therefore the bare parameters do not aquire a scale dependence by this mechanism, as it is the case for instance in QED_{3+1}. The question of writing a β-function in the sense of Gell-mann and Low, analogous to QED_{3+1} and QCD_{3+1} β-functions, is therefore delicate since the MS or $\overline{MS}$ scheme can not be applied. As a consequence, an orthodox answer to the question of the existence of a β-function would be that it simply vanishes: $\beta = 0$ in these schemes.
Nevertheless, the issue of the qualitative running of the coupling constant can still be addressed in a more heuristic way. Namely, the coupling constant being dimensional, its flow under scale transformations will be non-trivial. Furthermore, the behaviour of the coupling constant as the scale of the process varies can still be described thanks to the construction of an effective running coupling constant, at least at one-loop in perturbation theory. A definition for the non-perturbative running of the coupling constant is discussed in [37].

As a preliminary remark and before addressing the quantum theory, the classical evolution of the coupling constant can be considered from a purely dimensional point of view. Since the coupling constant has a positive mass dimension, scale transformations are not symmetries of the classical action. Therefore, the flow of the coupling constant may be simply apprehended from an elementary analysis of the classical dimensions of the fields.
To be more precise, we shall examine the classical dynamics as given by

the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{i}\overline{\psi}\gamma^{\mu}(\partial_{\mu} + \mathrm{i}eA_{\mu})\psi - \frac{1}{2}\mathrm{i}(\overline{\partial_{\mu} + \mathrm{i}eA_{\mu})\psi}\gamma^{\mu}\psi. \quad (4.6)$$

In order to understand the flow of the coupling constant, we shall introduced the scale transformations: $x \to e^{\sigma}x$ acting on a generic field as follows

$$\phi(x) \to e^{\sigma d}\phi(e^{\sigma}x), \quad (4.7)$$

where d depends on the field considered. Therefore, the action of an infinitesimal scale transformation on the gauge and fermion fields is

$$\delta A^{\mu} = \sigma(\frac{1}{2} + x^{\lambda}\partial_{\lambda})A^{\mu}, \quad (4.8)$$

$$\delta\psi = \sigma(1 + x^{\lambda}\partial_{\lambda})\psi. \quad (4.9)$$

As a consequence, the transformation of the Lagrangian density is, up to a total derivative,

$$\delta\mathcal{L} = -\sigma e(\frac{5}{2} + x^{\lambda}\partial_{\lambda})\bar{\psi}\gamma^{\mu}A_{\mu}\psi. \quad (4.10)$$

The transformation of the action follows thanks to an integration by parts

$$\delta\int \mathrm{d}^3x\mathcal{L} = \sigma\int \mathrm{d}^3x\frac{1}{2}e\bar{\psi}\gamma^{\mu}A_{\mu}\psi = \sigma\int \mathrm{d}^3x\Delta. \quad (4.11)$$

Defining the scale current $\partial_{\mu}s^{\mu} = \Delta$, related to the classical energy momentum tensor by the relation $s_{\mu} = x_{\nu}T^{\mu\nu}$, we recover the well-known relationship between the violation of scale invariance and the trace of the energy momentum tensor

$$\partial_{\mu}s^{\mu} = T^{\nu}_{\ \nu} = \Delta. \quad (4.12)$$

Next, we shall proceed in analogy with QCD, where the coupling constant is scale dependent $\alpha_S = \alpha_S(\mu)$. Under an infinitesimal scale transformation of parameter σ, the strong coupling constant is modified: $g_s \to g_s + \sigma\beta(g_s)$. The variation of the Lagrangian is [38]

$$\delta\mathcal{L} = \sigma\beta(g_S)\frac{\partial\mathcal{L}}{\partial g_s} = \sigma T^{\nu}_{S\ \nu}, \quad (4.13)$$

which is named the trace anomaly of QCD_{3+1}. By inspection of this formula, we can identify the analogue of the β-function in classical QED_{2+1}, so that we find the classical β-function of the coupling constant

$$\beta(e) = -\frac{1}{2}e. \tag{4.14}$$

Indeed, because the value of the coupling changes under scale transformations, we should study the flow of the adimensional quantity $\bar{e}(\mu) = e/\mu^{1/2}$, where μ is a reference scale [39]. The β-function of $\bar{e}$ is simply given by the logarithmic derivative

$$\beta(\bar{e}) = \mu\frac{d}{d\mu}\Big(\frac{e}{\mu^{1/2}}\Big) = -\frac{1}{2}\Big(\frac{e}{\mu^{1/2}}\Big) = -\frac{1}{2}\bar{e}(\mu). \tag{4.15}$$

In conclusion, we observed that at the classical level, a β-function can be associated to the coupling constant, as a measure of its flow under classical scale transformations.

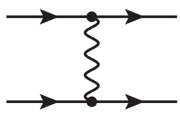
An alternative answer to the question of the running of the coupling constant follows from the analysis of the behaviour of the effective coupling constant in perturbation theory. At one-loop, the photon progator reads

$$D_{\mu\nu}(p) = -\mathrm{i}\frac{P_{\mu\nu}(p)}{p^2 - \Pi(p^2)} - \mathrm{i}\xi\frac{p_\mu p_\nu}{p^4}, \tag{4.16}$$

with the projector $P_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu/p^2$, and the one-loop polarisation $\Pi(p^2) = e^2\sqrt{-p^2}/16$. In the Landau gauge $\xi = 0$, which plays a privileged role [40], the propagator can be recast in the form

$$D_{\mu\nu}(p) = -\mathrm{i}\frac{P_{\mu\nu}(p)}{p^2}\frac{1}{1+\frac{e^2}{16\sqrt{-p^2}}}, \tag{4.17}$$

which is suitable to guess the definition of the effective coupling constant. Let us consider a scattering process in the t-channel or u-channel,



with $-p^2 > 0$, and where the exchanged photon propagator is given by (4.17). Then, the one-loop effective coupling can be defined thanks to

$$e_{\text{eff}}^2 = e^2 \frac{1}{1 + \frac{e^2}{16\sqrt{-p^2}}}. \tag{4.18}$$

The consequences of this definition have to be discussed in further details[4]. Going to euclidian signature for convenience, we define the euclidian propagator

$$D_{\mu\nu}^E(p_E) = \frac{P_{\mu\nu}^E(p_E)}{p_E^2 + \Pi(p_E)}, \quad \Pi(p_E) = e^2 p_E/16, \tag{4.19}$$

with $p_E = \sqrt{p_E^2}$. Notice that the infrared behaviour of the propagator is $\propto 1/p_E$ rather than $\propto 1/p_E^2$. However, as underlined in [30, 42], the perturbative expansion is actually a power series in the adimensional parameter e^2/p_E. This is seen for instance in (4.17). Hence, the results of the expansion should be trusted at best if $p_E \gg e^2$.
As a result, what really matters is the value of the ratio of the coupling constant squared with respect to the momentum scale of the process p_E. This is a guiding principle in order to build an effective coupling constant [43, 44]. Therefore, relying on the definition of the effective constant in QED_{3+1} and QCD_{3+1}, whose relevance was emphasised in [41], we introduce the adimensional effective structure constant at one-loop in perturbation theory

$$\bar{\alpha}(p_E) = \frac{\alpha}{p_E} \frac{1}{1 + \Pi(p_E)/p_E^2} = \frac{\alpha}{p_E + \alpha/16}, \tag{4.20}$$

where $\alpha = e^2$. In consequence, the flow of the adimensional structure constant in the UV leads to an asymptotically free theory. On the contrary, in the IR, the effective coupling goes to a constant $\bar{\alpha}(p_E) \to 16$. In addition, examining the effective running structure constant (4.20), we notice that its value is indeed small when $p_E > e^2/16$, so that the scale $e^2/16$ plays the role of an effective "strong coupling scale". The result is

[4] This definition is very useful in QED_{3+1} and provides the same understanding of the running of coupling as the Gell-mann and Low definition [38, 41].

that the perturbative expansion can be trusted in the large momentum limit, while the effective coupling grows in the IR. Since the ordinary perturbation theory is incomplete [30, 42] and due to the qualitative growth of the effective coupling constant, a non-perturbative behaviour of the theory is expected at a scale roughly estimated by a perturbative argument to be close to $e^2/16$. This simple line of reasoning motivates the study of the low momentum regime of massless QED_{2+1}.

To conclude this introductory section, here is a brief summary of the results presented in this chapter. Section 4.2 deals with the classical formulation of the theory. Working with a factorized gauge symmetry, we are facing the particular case of the logarithmic confining electrostatic potential. The Fourier transform of the x-space potential is found to be a distribution. The relationship between this distribution and the massless limit of the Fourier transform of the electrostatic potential given a massive photon is explained section B.1 of Appendix B. Within the Hamitonian framework, section 4.3 deals with the quantisation of the theory and the construction of a non-perturbative approximation. In order to look for a stable ground state, a fermionic coherent state, similar to the BCS superconducting vacuum state, is constructed, inspired by previous works in QED_{3+1} and QCD_{3+1}. In section 4.4, we formulate an integral equation for the vacuum wave function from the requirement of the minimization of the energy. This equation is a truncation of a Schwinger-Dyson equation. An approximate solution to the integral equation is found, inclusive of the effects of an infinite number of photon exchanges. The energy density of this condensate is lower than the energy density of the Fock state, so that the Fock state is expected to be unstable. There is a spontaneous parity violation with only one fermion flavour, supporting a similar conclusion by Hoshino and Matsuyama [45, 46]. Incidentally, the question of spontaneous parity violation has also been studied in the context of multi-flavour QED_3 (see for example [47, 48]).

By analysing in section 4.5 the dynamics of the fermions in the condensate, quasi-particles interpreted as constituent fermions are identified.

The effective energy of a quasi-particle in the condensate is impacted by its non-perturbative interactions with pairs in the new vacuum. We observe that a state with a single charged particle is not gauge invariant. The gauge dependence originates from the choice of zero value in the electrostatic potential energy. However the energy of a particle/antiparticle pair is not gauge dependent. The divergence of the energy at zero momentum is a signature for the confinement of dynamical charges, as confirmed in section 4.6. Subsequently, a Green function interpretation of the results of the variational analysis is presented in section 4.7. Treating the residual interactions as perturbations, the analysis is in a favour of a dynamical mass for the fermions.
By the way, in recent years, the confining property, the dynamical mass, and related aspects of QED_{2+1} have been investigated with success by Y. Hoshino within another framework relying on the study of the position space fermion propagator [49].
Finally the effect of the condensate on the electromagnetic sector is addressed in section 4.8. In the approximation considered in the present work, because of spontaneous parity violation due to pair condensation, the "magnetic mode excitation" – related to the transverse electromagnetic polarization – initially massless, appears to have a dynamically generated mass, which is calculated within an approximation scheme and in a perturbative setting.
At the very end, section 4.9 is devoted to conclusions.

The conclusions of the next sections have been presented in [50].

4.2 Classical Hamiltonian QED_{2+1}

The analysis starts with the statement of the conventions chosen. In order to appropriately describe a single fermion flavour, Dirac matrices are chosen in terms of the Pauli matrices as follows: $\gamma^0 = \sigma_3$ and $\gamma^i = \mathrm{i}\sigma_i$ for $i = 1, 2$, and satisfy the useful properties

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 2\eta_{\mu\nu}, \quad \mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho) = -2\mathrm{i}\epsilon_{\mu\nu\rho}, \tag{4.21}$$

where the totally anti-symmetric symbol is chosen so that $\epsilon_{012} = \epsilon^{012} = 1$. The mostly minus signature is chosen for the Minkowski metric, while an implicit choice of units is done such that $\hbar = c = 1$. As for the dimensional specificities, in $D = 3$ space-time dimensions, and in units of mass M the gauge coupling constant e has dimension $[e] = M^{1/2}$, while the gauge and matter fields have dimensions $[A_\mu] = M^{1/2}$ and $[\psi] = M^1$.

4.2.1 Classical Hamiltonian and the Green function

The classical dynamics is given by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{i}\overline{\psi}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu)\psi - \frac{1}{2}\mathrm{i}\overline{(\partial_\mu + \mathrm{i}eA_\mu)\psi}\gamma^\mu\psi. \quad (4.22)$$

We shall apply here a factorization of the local gauge transformations and gauge degrees of freedom, following closely the techniques explained in [19] for the case of the Schwinger model.

In two space dimensions, the spatial gauge potential can be written[5] as the sum of a longitudinal and a transverse component

$$A_i(t,\vec{x}) = \partial_i\phi(t,\vec{x}) + \epsilon_{ij}\partial_j\Phi(t,\vec{x}), \quad (4.23)$$

where the scalar Φ is related the magnetic field through $\Delta\Phi = B$, so that Φ will be referred to as the "magnetic mode". Similarly, we also introduce the decomposition

$$A_0(t,\vec{x}) = a_0(t) + \partial_i\omega_i(t,\vec{x}). \quad (4.24)$$

The local gauge parameter may also be decomposed as the sum of its "global" (by which we mean throughout a space independent but yet possibly a time dependent gauge transformation parameter) and local components, $\alpha(t,\vec{x}) = \beta_0(t) + \partial_i\beta_i(t,\vec{x})$. In order to factorize these local gauge transformations, the fermion field is "dressed", in a way completely analogous to that of reference [17],

$$\chi(t,\vec{x}) = e^{\mathrm{i}e\phi(t,\vec{x})}\psi(t,\vec{x}), \quad (4.25)$$

[5] Henceforth, all latin indices are euclidian.

so that the dressed fermion transforms, under gauge transformations of general parameter $\alpha(t,\vec{x}) = \beta_0(t) + \partial_i\beta_i(t,\vec{x})$, only by a global (time dependent) phase change

$$\chi(t,\vec{x}) \to e^{-\mathrm{i}\beta_0(t)}\chi(t,\vec{x}). \tag{4.26}$$

Following the study of the Hamiltonian dynamics of constrained systems, as advocated by Dirac (see for example [3]), we give only a few details of the constrained analysis which is analogous to the one given in [19]. From the previous definitions, we obtain the Lagrangian action as a function of the new configuration space variables

$$\begin{aligned} S = \int dt \quad & \Big\{ - ea_0(t)\int_{S^1} \mathrm{d}x^i \chi^\dagger\chi + \int \mathrm{d}x^i \Big(\frac{1}{2}\mathrm{i}\chi^\dagger\partial_0\chi - \frac{1}{2}\mathrm{i}\partial_0\chi^\dagger\chi \\ & + \frac{1}{2}\mathrm{i}\overline{\chi}\gamma^i\partial_i\chi - \frac{1}{2}\mathrm{i}\partial_i\overline{\chi}\gamma^i\chi - \frac{1}{2}(\partial_0\phi - \partial_i\omega_i)\Delta(\partial_0\phi - \partial_i\omega_i) \\ & + e(\partial_0\phi - \partial_i\omega_i)\chi^\dagger\chi - \frac{1}{2}\partial_0\Phi\Delta\partial_0\Phi - \frac{1}{2}\Phi\Delta^2\Phi \\ & - e\epsilon^{ij}\partial_j\Phi\bar{\chi}\gamma^i\chi\Big)\Big\}. \end{aligned}$$

In order to study the Hamiltonian structure, we identify the conjugate momenta

$$\begin{aligned} \pi_\Phi &= \frac{\partial L_0}{\partial\dot{\Phi}} \\ p^0 &= \frac{\partial L_0}{\partial\dot{a}_0} = 0, \\ \pi^i &= \frac{\partial L_0}{\partial\dot{\omega}_i} = 0 \\ \pi_\phi &= \frac{\partial L_0}{\partial\dot{\phi}} = - \triangle\,(\partial_0\phi - \partial_i\omega_i) + e(\chi^\dagger\chi), \\ \xi_1 &= \frac{\partial L_0}{\partial\dot{\chi}} = -\frac{1}{2}\mathrm{i}\chi^\dagger, \\ \xi_2 &= \frac{\partial L_0}{\partial\dot{\chi}^\dagger} = -\frac{1}{2}\mathrm{i}\chi, \end{aligned}$$

where we observe that the fermion field is already in Hamiltonian form. Subsequently, the constraint analysis can be performed in close analogy with [19], while the first class constraints $p^0 = 0$ and $\pi^i = 0$ can be solved. After this straightforward analysis, the equations of motion of the sector $(\phi,\,\pi_\phi)$ can be used to reduce these phase space variables from the dynamics. Finally, we obtain the following Hamiltonian action

$$S=\int \mathrm{d}t\Big\{\int \mathrm{d}^2x^i\Big[\partial_0\Phi\pi_\Phi+\frac{1}{2}\mathrm{i}\chi^\dagger\partial_0\chi-\frac{1}{2}\mathrm{i}\partial_0\chi^\dagger\chi]-H\Big\} \tag{4.27}$$

where the classical expression of the Hamitonian is

$$H=\int \mathrm{d}^2x^i\Big\{\mathcal{H}_F+\mathcal{H}_\Phi+\mathcal{H}_{\Phi\chi}\Big\}, \tag{4.28}$$

with the Hamiltonian densities

$$\begin{aligned}\mathcal{H}_F\quad &=\frac{1}{2}\bar\chi(t,\vec x)\gamma^i(-\mathrm{i}\partial_i)\chi(t,\vec x)+\frac{1}{2}\mathrm{i}\partial_i\bar\chi(t,\vec x)\gamma^i\chi(t,\vec x)\\ &-\frac{e^2}{2}(\chi^\dagger\chi)(t,\vec x)\big[\Delta^{-1}(\chi^\dagger\chi)\big](t,\vec x), \qquad (4.29)\\ \mathcal{H}_\Phi\quad &=-\frac{1}{2}\pi_\Phi(t,\vec x)\big[\Delta^{-1}\pi_\Phi](t,\vec x)+\frac{1}{2}(\Delta\Phi)^2(t,\vec x), \qquad (4.30)\\ \mathcal{H}_{\Phi\chi}\quad &=e\epsilon^{ij}\partial_j\Phi(t,\vec x)(\bar\chi\gamma^i\chi)(t,\vec x). \qquad (4.31)\end{aligned}$$

On account of the factorisation of local gauge transformations and gauge degrees of freedom, the dynamics is still constrained by the condition stemming from the time-dependent “global” gauge transformations with $\alpha(t)=\beta_0(t)$ which is analogous to the spatially integrated Gauss law,

$$\int \mathrm{d}^2x^i\chi^\dagger(t,\vec x)\chi(t,\vec x)=0, \tag{4.32}$$

which is first class and generates the remaining global gauge transformations. Examining more closely the terms in (4.28), we observe that the Hamiltonian density $\mathcal{H}_F$ describes the dynamics of the fermion with its Coulomb interaction, while $\mathcal{H}_\Phi$ characterizes the dynamics of the magnetic mode sector. The Hamiltonian density $\mathcal{H}_{\Phi\chi}$ accounts for the interaction between the fermion current and the magnetic mode.

In order to understand the quantum theory, we first need to study the peculiarities of the Green function of the Laplacian in two spatial dimensions. A peculiarity of this $2+1$-dimensional theory is that the Green function of the spatial Laplacian, conveniently expressed in x-space and verifying $\Delta G(\vec x,\vec y)=\delta^{(2)}(\vec x-\vec y)$, is the tempered distribution defined by

$$G(\vec x,\vec y)=\frac{1}{2\pi}\ln(\mu|\vec x-\vec y|), \tag{4.33}$$

where the mass scale $\mu > 0$ is introduced for dimensional consistency. In classical electrostatics, this Green function is proportional to the electrostatic potential of a pointlike particle in two space dimensions. In three space dimensions, the electrostatic potential of an infinite charged wire would have a similar expression. The scale μ is therefore understood as parametrizing the possible choices for a "zero of the potential", and will be kept arbitrary in the sequel. When the potential tends to a constant at spatial infinity, it is allowed to choose this constant to be zero. On the contrary, because the logarithmic Coulomb potential is confining, the remaining gauge freedom μ has to be considered at all steps of the calculation. In p-space, the presence of μ can be interpreted as an infrared regulator, as we shall see.

Because the Green function is divergent at large as well as at small distances, we may expect to encounter also infrared divergences in the quantum formulation of the theory. We will pay special attention to the classical large distance divergence of the Green function. The inverse of the Laplacian is obtained by the convolution integral

$$(\Delta^{-1}f)(\vec{x}) = \langle G(\vec{x},\cdot), f(\cdot)\rangle = \int \mathrm{d}y^i \frac{1}{2\pi}\ln(\mu|\vec{x}-\vec{y}|)f(\vec{y}). \tag{4.34}$$

Adding a constant to (4.33), amounts to redefining μ by a multiplicative constant. For technical reasons, we should like to express the Green function in Fourier space. However the Fourier transform of the Green function is not a function, but rather a distribution. The naive expression for the Fourier transform, namely $\propto 1/|\vec{p}|^2$, would indeed fail to converge in the infrared region. After a careful integration, one finds the identity

$$\frac{1}{2\pi}\ln(\frac{e^\gamma}{2}\mu|\vec{x}-\vec{y}|) = G_\epsilon(\vec{x},\vec{y}) - \frac{1}{2\pi}\ln(\epsilon/\mu), \tag{4.35}$$

where γ is the Euler constant[6]. Here we have defined

$$\begin{aligned} G_\epsilon(\vec{x},\vec{y}) = \; & \int_{|\vec{p}|<\epsilon} \frac{\mathrm{d}^2p^i}{(2\pi)^2}\frac{-1}{|\vec{p}|^2}(e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})}-1) \\ & + \int_{|\vec{p}|>\epsilon} \frac{\mathrm{d}^2p^i}{(2\pi)^2}\frac{-1}{|\vec{p}|^2}e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})}, \end{aligned} \tag{4.36}$$

[6]This integration is performed with the help of $\int_0^\infty \ln y J_1(ay)\mathrm{d}y = (-1/a)(\ln(a/2)+\gamma)$, where J_1 is a Bessel function of the first kind.

where $\epsilon > 0$ can take any value. The above is an exact result involving the arbitrary parameter ϵ playing the role of a cut-off which makes the integral convergent close to the infrared singularity at $p = 0$. The last definition (4.36) depends on the free parameter ϵ because we have $2\pi\partial_\epsilon G_\epsilon(\vec{x}, \vec{y}) = -1/\epsilon$. This dependence is, however, cancelled by the logarithmic term in (4.35).

4.2.2 The Hadamard finite part

In order to relate the discussion of the previous section to the mathematical theory of distributions, we will use here variables without physical dimensions. Restoring physical dimensions is straightforward.

In a renowned work [51], Hadamard introduced very useful generalized functions, among them the so-called Hadamard finite part $\mathcal{P}\frac{1}{x^2}$, which is related to the more popular Cauchy principal value $\mathcal{P}\frac{1}{x}$ by the "weak" derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{P}\frac{1}{x} = -\mathcal{P}\frac{1}{x^2}. \tag{4.37}$$

This definition of the finite part is valid for functions of one variable, but it may be generalized to functions of two variables.
Following [52], it is interesting to introduce here a two-dimensional version of the finite part of $1/x^2$, by defining its action on a test function ϕ,

$$(\mathcal{P}\frac{1}{|\vec{p}|^2}, \phi) = \int_{|\vec{p}|<1} \mathrm{d}^2p^i \frac{\phi(\vec{p}) - \phi(\vec{0})}{|\vec{p}|^2} + \int_{|p|>1} \mathrm{d}^2p^i \frac{\phi(\vec{p})}{|\vec{p}|^2}, \tag{4.38}$$

where the presence of the value 1 in the bounds of the integration domain is conventional. Let us denote the Fourier transform of the Green function of the Laplacian as $\mathcal{F}[G](\vec{p})$. We can now show that the generalized function $-\mathcal{P}\frac{1}{|\vec{p}|^2}$ is the "generalized" Fourier transform of the Green function, by proving that the Hadamard finite part solves $-|\vec{p}|^2\mathcal{F}[G](\vec{p}) = 1$. To do so we calculate

$$
\begin{aligned}
(|\vec{p}|^2\mathcal{P}\tfrac{1}{|\vec{p}|^2},\phi) &= (\mathcal{P}\frac{1}{|\vec{p}|^2},|\vec{p}|^2\phi) \\
&= \int_{|\vec{p}|<1} \mathrm{d}^2p^i \frac{|\vec{p}|^2\phi(\vec{p}) - [|\vec{p}|^2\phi(\vec{p})]|_0}{|\vec{p}|^2} + \int_{|p|>1} \mathrm{d}^2p^i \frac{|\vec{p}|^2\phi(\vec{p})}{|\vec{p}|^2} \\
&= \int \mathrm{d}^2p^i\phi(\vec{p}) = (1,\phi) \qquad (4.39)
\end{aligned}
$$

giving the solution $\mathcal{F}[G](\vec{p}) = -\mathcal{P}\frac{1}{|\vec{p}|^2}$. This relation rephrases the results found in (4.35) and (4.36). Hence, the upshot is that the apparent IR divergent "Coulomb" propagator in p-space, proportional to $\frac{1}{|\vec{p}|^2}$ has not to be considered as a function. On the contrary, it should be understood as a generalized function, that is to say the Hadamard finite part $\mathcal{P}\frac{1}{|\vec{p}|^2}$. In the sequel we will see that in the absence of IR divergences, this last prescription reduces to the usual multiplication by the function $\frac{1}{|\vec{p}|^2}$.

Although instructive, the previous mathematical treatment could obscure one's physical intuition. It may be enlightening to relate the Hadamard finite part representation of the Fourier space Green function to a more usual treatment of the infrared singularities. As is often done, a "ad hoc" mass term could be included for the photon to consider then the p-space Green function $\frac{1}{|\vec{p}|^2+\mu^2}$. The massless limit of the massive Green function could provide a more intuitive picture. B.1 explains how the Hadamard representation is recovered from the zero-mass limit of the massive Green function.

4.3 Quantum Hamiltonian and ordering prescription

The careful and detailed definition of the Coulomb Green function will prove to be most relevant to the understanding of singularities in the quantum theory. Given the classical formulation, a quantum version can be formulated. Following the correspondence principle, classical (graded) Poisson brackets are replaced by quantum commutators or anti-commutators. This formal quantisation should be performed in both the fermionic and the bosonic sectors of the theory.

4.3.1 Magnetic sector

As pointed out previously, the field $\Phi(t,\vec{x})$ is related to the magnetic field by the identity $\Delta\Phi = B$. In order to quantise this sector, we decide to expand the magnetic mode and its momentum conjugate in terms of the plane wave Fock modes as follows, at the reference time $t=0$,

$$\Phi(0,\vec{x}) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2}} \frac{-\mathrm{i}}{|\vec{k}|^{3/2}} \left[\phi(\vec{k})e^{\mathrm{i}\vec{k}.\vec{x}} - \phi^\dagger(\vec{k})e^{-\mathrm{i}\vec{k}.\vec{x}}\right], \qquad (4.40)$$

$$\pi_\Phi(0,\vec{x}) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2}} (-|\vec{k}|^{3/2}) \left[\phi(\vec{k})e^{\mathrm{i}\vec{k}.\vec{x}} + \phi^\dagger(\vec{k})e^{-\mathrm{i}\vec{k}.\vec{x}}\right], \quad (4.41)$$

where the creators and annihilators satisfy $[\phi(\vec{\ell}),\phi^\dagger(\vec{k})] = \delta^{(2)}(\vec{\ell}-\vec{k})$, in order that fields obey the Heisenberg algebra $[\Phi(0,\vec{x}),\Pi_\Phi(0,\vec{y})] = \mathrm{i}\delta^{(2)}(\vec{x}-\vec{y})$. In a familiar way, the bosonic Fock algebra is represented in a Fock space, with the annihilators satisfying $\phi(\vec{\ell})|0\rangle = 0$. Since the quantisation procedure introduces ordering ambiguities, we decide to define the normal ordered form of a composite operator, in the magnetic sector, as the operator written with all $\phi^\dagger$'s to the left of all ϕ's. Therefore, the normal ordered "magnetic" Hamiltonian, associated to a "free" field,

$$\hat{H}_\Phi = \int \mathrm{d}^2x^i : \Big\{ -\frac{1}{2}\pi_\Phi(0,\vec{x})\big[\Delta^{-1}\pi_\Phi](0,\vec{x}) + \frac{1}{2}(\Delta\Phi)^2(0,\vec{x})\Big\} : \quad (4.42)$$

may be expanded in modes as follows:

$$\hat{H}_\Phi = \int \mathrm{d}^2k^i|\vec{k}|\phi^\dagger(\vec{k})\phi(\vec{k}). \qquad (4.43)$$

Treating $\hat{H}_\Phi$ as the free Hamiltonian and the other terms as interactions, considered in perturbation theory, we define the interaction picture field as

$$\Phi_I(t,\vec{x}) = e^{\mathrm{i}\hat{H}_\Phi t}\Phi(0,\vec{x})e^{-\mathrm{i}\hat{H}_\Phi t}. \qquad (4.44)$$

Using customary techniques, the free magnetic mode propagator, i.e. in absence of interaction, can be computed, producing the Feynman propagator

$$\langle 0|T\Phi_I(x^0,\vec{x})\Phi_I(0,\vec{0})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{-\mathrm{i}k^0x^0+\mathrm{i}\vec{k}.\vec{x}}}{|\vec{k}|^2}\frac{\mathrm{i}}{(k^0)^2-|\vec{k}|^2+\mathrm{i}\epsilon}. \quad (4.45)$$

The p-space propagator is illustrated by a curly line,

being a useful representation of the momentum space two-point function of the gauge invariant and physical magnetic mode. Incidentally, after the elimination of the longitudinal gauge mode, the spatial gauge potential is $A^i_T = \epsilon^{ij}\partial_j\Phi$. Using this last identity and translational invariance, we recover the transverse photon propagator $D^{ij}(x^0-y^0,\vec{x}-\vec{y}) = \langle 0|TA^i_T(x^0,\vec{x})A^j_T(y^0,\vec{y})|0\rangle$ with

$$D^{ij}(x^0,\vec{x}) = \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{-\mathrm{i}k^0x^0+\mathrm{i}\vec{k}.\vec{x}}\frac{\delta^{ij}-k^ik^j/\vec{k}^2}{(k^0)^2-|\vec{k}|^2+\mathrm{i}\epsilon} \quad (4.46)$$

as follows from the identity $\epsilon^{im}k^m\epsilon^{jn}k^n = \vec{k}^2\delta^{ij}-k^ik^j$ (for a reference concerning Coulomb gauge QED_{2+1}, see [40]).

4.3.2 Fermionic sector

In order to quantise the fermion sector, the classical spinor field is expanded in the basis of solutions of the free Dirac equation. The classical solutions to the Dirac equation in $2+1$ dimensions are constructed in terms of the spinors

$$u(k^\mu) = \begin{pmatrix} \frac{k^2+\mathrm{i}k^1}{\sqrt{k^0-m}} \\ \sqrt{k^0-m} \end{pmatrix}, v(k^\mu) = \begin{pmatrix} \frac{k^2+\mathrm{i}k^1}{\sqrt{k^0+m}} \\ \sqrt{k^0+m} \end{pmatrix}, \quad (4.47)$$

normalized as $u^\dagger(k^\mu)u(k^\mu) = v^\dagger(k^\mu)v(k^\mu) = 2k^0 > 0$ and where $k^\mu = (k^0, \vec{k})$. In the massless limit, the Dirac spinors $u(k^\mu) = v(k^\mu)$ are degenerate so that the mode expansions of the fields at $x^\mu = (x^0, \vec{x})$ become

$$\chi(x^\mu) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2k^0}} \left[b(\vec{k})e^{-\mathrm{i}k.x} + d^\dagger(\vec{k})e^{\mathrm{i}k.x}\right]u(\vec{k}), \tag{4.48}$$

$$\chi^\dagger(x^\mu) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2k^0}} \left[b^\dagger(\vec{k})e^{\mathrm{i}k.x} + d(\vec{k})e^{-\mathrm{i}k.x}\right]u^\dagger(\vec{k}), \tag{4.49}$$

where the last two expressions have to be evaluated at $k^0 = |\vec{k}|$, whereas $k.x = k^0x^0 - \vec{k}.\vec{x}$ stands for the Minkowski inner product. Quantisation is performed at the reference time $x^0 = 0$. Following from the algebra of classical Dirac brackets, in the quantised theory the fermionic creators-annihilators have to verify $\{b(\vec{p}), b^\dagger(\vec{q})\} = \delta^{(2)}(\vec{p} - \vec{q}) = \{d(\vec{p}), d^\dagger(\vec{q})\}$, while the fermionic Fock vacuum $|0\rangle$ is chosen to be annihilated by $b(\vec{p})$ and $d(\vec{p})$. Let us consider an operator AB, bilinear in b, d and their adjoints. Its contraction is defined to be,

$$\overset{\sqcap}{AB} = \langle 0|AB|0\rangle \tag{4.50}$$

while its normal ordered form, where the creators are positioned to the left of all annihilators, is given by

$$: AB := AB - \overset{\sqcap}{AB}. \tag{4.51}$$

With the help of these notations, the Hamiltonian operator is defined by a normal ordered form of the classical expression, where each charge density factor $\chi^\dagger\chi$ is also written in the normal order on its own:

$$\begin{aligned}\hat{H} = \; & \int \mathrm{d}^2x^i \{\frac{1}{2} : \bar{\chi}(t,\vec{x})\gamma^i(-\mathrm{i}\partial_i)\chi(t,\vec{x}) : + \frac{1}{2} : \mathrm{i}\partial_i\bar{\chi}(t,\vec{x})\gamma^i\chi(t,\vec{x}) : \} \\ & + \hat{H}_C \end{aligned} \tag{4.52}$$

where

$$\hat{H}_C = -\frac{e^2}{2}\int \mathrm{d}^2x^i \mathrm{d}^2y^i (: \chi^\dagger\chi :)(0,\vec{x})G(\vec{x},\vec{y})(: \chi^\dagger\chi :)(0,\vec{y}). \tag{4.53}$$

The Green function of the Laplacian $G(\vec{x}, \vec{y})$ is given by (4.33). Gauss' law constraint, which involves the charge operator

$$\hat{Q} = \int \mathrm{d}^2x^i : \chi^\dagger(0,\vec{x})\chi(0,\vec{x}) :, \tag{4.54}$$

annihilates the physical, i.e. gauge invariant quantum states, $\hat{Q}|\text{phys}\rangle = 0$, that is to say, the physical states should contain an equal number of fermions and anti-fermions, so that these states are electrically neutral. This constraint may be connected with the problem of the divergences at large distances which is a typical concern in $2+1$ dimensional gauge theories. Let us explain how with an elementary argument. It is noteworthy that the classical electrostatic energy of a single pointlike charge is infrared divergent due to the logarithmic behaviour of the Green function. However, the electrostatic potential of a system made of two opposite pointlike charges is well behaved at large distances, because it is proportional to

$$\ln \mu|\vec{x}-\vec{x}_1| - \ln \mu|\vec{x}-\vec{x}_2| = \ln \frac{|\vec{x}-\vec{x}_1|}{|\vec{x}-\vec{x}_2|}, \tag{4.55}$$

where $\vec{x}_1$ and $\vec{x}_2$ are the positions of the two opposite charges. This classical argument strongly suggests that gauge invariant states should not suffer difficulties in the infrared region. Accordingly, when $\hat{H}_F$ acts on a gauge invariant state, namely a state with a vanishing total charge, the result is not affected by the transformation $G(\vec{x},\vec{y}) \to G(\vec{x},\vec{y}) + \text{cst}$, given the specific ordering of the charge density operators in the Coulomb Hamiltonian.

Thus, when we consider states containing an equal number of particles and anti-particles, we may simply substitute the naive expression for the Green function

$$G(\vec{x},\vec{y}) = \int_{(\infty)} \frac{\mathrm{d}^2 p^i}{(2\pi)^2} \frac{-1}{|\vec{p}|^2} e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})}, \tag{4.56}$$

apparently infrared divergent, in the formula for the quantum Hamiltonian $\hat{H}_F$. We may expect that no gauge dependence will occur due to the specific ordering prescription, provided that $\hat{H}_F$ acts on physical states. However this will not be true in the case of a single charged particle or anti-particle, as will be seen in the next section.

4.4 Fermion condensate in massless QED_{2+1}

Because a non trivial vacuum structure is expected from the classical features of the theory, we would like to investigate the possibility of a pair condensation mechanism in the vacuum. The approach followed here puts forward an expression of a trial state which is likely to provide a satisfactory approximation of the exact vacuum state. The developements are somehow inspired by the microscopic theory of low temperature superconductivity. We will try to argue that the choice is sufficiently flexible to provide a consistent approximation of the non-perturbative nature of the vacuum state. The freedom introduced by the trial state is associated to a “wave function” which is to be determined through a procedure of minimization of the total energy, in the presence of the Coulomb interaction. Interestingly, a very similar variational procedure, non explicitely Lorentz covariant, was very recently undertaken by Reinhardt *et al.* in the case of Hamiltonian QCD_{3+1} in the Coulomb gauge [53, 54], opening the door to a novel approach. This “Hartree-Fock” procedure has the avantage to provide a consistent framework to the approximation.
By the way, a different strategy to probe the non-perturbative effects could rely on the functional formulation of quantum field theory. From this point of view, the problem would be to find a solution to the Schwinger-Dyson equations, with a specific truncation scheme and gauge fixing. Although these ideas might seem unrelated, we show that the problem to find a wave function minimizing the energy gives rise to an integral equation which can be formulated as Schwinger-Dyson equation for the fermion propagator.

Inspired by the techniques developped in [55–58], which resulted in a successful description of non-perturbative properties of the pion [59] and in a close analogy with the Bardeen-Cooper-Schrieffer ground state of a superconductor, we now introduce the coherent superposition

$$|\Psi\rangle = \frac{1}{N(\Psi)}\exp\,[-\int \mathrm{d}^2x^i \mathrm{d}^2y^i\ \tilde{\Psi}(|\vec{x}-\vec{y}|) : \bar{\chi}(\vec{x})\chi(\vec{y}) :]|0\rangle, \qquad (4.57)$$

where $\tilde{\Psi}(|\vec{x}|)$ is a function describing the distribution in space of condensate pairs. Because of its convenience, it is advantageous to write the previous definition in momentum space. To do so, we perform a Fourier transform and find the expression

$$|\Psi\rangle = \frac{1}{N(\Psi)} \exp \int \mathrm{d}^2 p^i \ \Psi(|\vec{p}|) b^\dagger(\vec{p}) d^\dagger(-\vec{p}) |0\rangle, \tag{4.58}$$

containing an arbitrary number of fermion/anti-fermion pairs of opposite momenta. Accordingly, it is guaranteed that the wave function is invariant under the spatial translations. The associated dimensionless wave function in momentum space $\Psi(p) = \Psi(|\vec{p}|)$ is chosen to be invariant under rotations in the plane. Because this function is complex valued, we can express it as the product of a modulus and a phase: $\Psi(p) = |\Psi(p)| \exp \, \mathrm{i}\phi(p)$ where $p = |\vec{p}|$. The purpose of our analysis is to determine if the dynamics triggers a pair condensate, whose profile is described by the wave function $\Psi(p)$ in p-space.

As a means to compute the normalization of the trial state, the integral over the momenta may be discretized, allowing to express the exponential as an infinite product. The normalisation of each of these factors may then be calculated individually and the continuum limit be taken subsequently. For the sake of completeness, the normalization of the coherent superposition of pairs

$$N(\Psi) = \prod_{p^i} \sqrt{1 + |\Psi(p)|^2}, \tag{4.59}$$

may be computed, the continuous product being approximated by a discretization of the momentum space into a lattice. For further use, let us define the functions of $p = |\vec{p}|$

$$\alpha(p) = \frac{1}{\sqrt{1 + |\Psi(p)|^2}}, \quad \beta(p) = \frac{\Psi(p)}{\sqrt{1 + |\Psi(p)|^2}}, \tag{4.60}$$

which can be associated to an angle $\Theta(p)$ defined by the relations $\cos\Theta(p) = \alpha(p)$ and $\sin\Theta(p) = |\beta(p)|$. Consequently, the trial state (4.58) may be formulated as a product of normalized factors

$$|\Psi\rangle = \prod_{p^i} \left[\alpha(p) + \beta(p) b^\dagger(\vec{p}) d^\dagger(-\vec{p}) \right] |0\rangle. \tag{4.61}$$

These definitions allow to better interpret the trial state as a fermionic "coherent state". In order to investigate its content in terms of fermionic components, we naturally remark now that the following identities:

$$b(\vec{p})|\Psi\rangle = \Psi(p)d^\dagger(-\vec{p})|\Psi\rangle, \quad d(-\vec{p})|\Psi\rangle = -\Psi(p)b^\dagger(\vec{p})|\Psi\rangle, \tag{4.62}$$

are somehow reminiscient of the property of the canonical coherent states, which are eigenstates of the annihilation operator. This property enjoins us to define a Bogoliubov transformation of the creators and annihilators

$$B(\vec{p}) = \alpha(p)b(\vec{p}) - \beta(p)d^\dagger(-\vec{p}), \tag{4.63}$$
$$B^\dagger(\vec{p}) = \alpha(p)b^\dagger(\vec{p}) - \beta^*(p)d(-\vec{p}), \tag{4.64}$$
$$D(-\vec{p}) = \alpha(p)d(-\vec{p}) + \beta(p)b^\dagger(\vec{p}), \tag{4.65}$$
$$D^\dagger(-\vec{p}) = \alpha(p)d^\dagger(-\vec{p}) + \beta^*(p)b(\vec{p}), \tag{4.66}$$

which verify $B(\vec{p})|\Psi\rangle = 0 = D(-\vec{p})|\Psi\rangle$ and satisfy the Fock algebra $\{B(\vec{p}), B^\dagger(\vec{q})\} = \delta^{(2)}(\vec{p}-\vec{q}) = \{D(-\vec{p}), D^\dagger(-\vec{q})\}$ while all other anti-commutators vanish. In a similar fashion, the inverse relations are provided by

$$b(\vec{p}) = \alpha(p)B(\vec{p}) + \beta(p)D^\dagger(-\vec{p}), \tag{4.67}$$
$$b^\dagger(\vec{p}) = \alpha(p)B^\dagger(\vec{p}) + \beta^*(p)D(-\vec{p}), \tag{4.68}$$
$$d(-\vec{p}) = \alpha(p)D(-\vec{p}) - \beta(p)B^\dagger(\vec{p}), \tag{4.69}$$
$$d^\dagger(-\vec{p}) = \alpha(p)D^\dagger(-\vec{p}) - \beta^*(p)B(\vec{p}). \tag{4.70}$$

Because the states created by $B^\dagger$ and $D^\dagger$ carry the same electric charge as the ones created by $b^\dagger$ and $d^\dagger$ and diagonalize the fermionic Hamiltonian H_F up to some residual Coulmb interactions, the former states can be regarded as physical fermionic particles excited over the condensate. Consequently it is useful to define a new ordering prescription associated to the condensate $|\Psi\rangle$ of any operator $\hat{O}$, to be denoted by $:\hat{O}:_\Psi$, such that all $B^\dagger$ and $D^\dagger$ operators are positioned to the left of all B and D operators. Technical tools developped in [56] can simplify the computations dramatically, as we shall outline briefly.

Considering a bilinear operator AB in these fermionic creation and annihilation operators, one may change the ordering prescription thanks to the formula

$$: AB : = : AB :_\Psi + \widehat{AB}, \quad \widehat{AB} = \langle\Psi|AB|\Psi\rangle - \langle 0|AB|0\rangle \qquad (4.71)$$

which will be used in the sequel in order to calculate the necessary matrix elements. Given the definition of the Bogoliubov operators, the mode expansions of the fermionic fields at $x^\mu = (0, \vec{x})$ are modified. Thus, a substitution gives readily the following expansions:

$$\chi(0,\vec{x}) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2k^0}}[B(\vec{k})N_1(k)u(\vec{k}) + D^\dagger(-\vec{k})N_2(k)u(-\vec{k})]e^{\mathrm{i}\vec{k}.\vec{x}},$$
$$\chi^\dagger(0,\vec{x}) = \int \frac{\mathrm{d}^2k^i}{2\pi\sqrt{2k^0}}[B^\dagger(\vec{k})u^\dagger(\vec{k})N_1^\dagger(k) + D(-\vec{k})u^\dagger(-\vec{k})N_2^\dagger(k)]e^{-\mathrm{i}\vec{k}.\vec{x}}.$$

For simplicity, the following matrices, whose definition are specific to the representation chosen for the Dirac matrices,

$$N_1(k) = \alpha(k) + \beta^*(k)\gamma^0, \qquad N_1^\dagger(k) = \alpha(k) + \beta(k)\gamma^0, \quad (4.72)$$
$$N_2(k) = \alpha(k) - \beta(k)\gamma^0, \qquad N_2^\dagger(k) = \alpha(k) - \beta^*(k)\gamma^0, \quad (4.73)$$

are introduced. Being equipped with suitable tools, we may now envisage to compute the average kinetic and interaction energy of the state $|\Psi\rangle$. Since we work in a space of infinite volume the most favourable state will be the one minimizing the energy per unit volume. More precisely, we would like to calculate the energy density of the coherent state (4.58), as given by

$$E = \frac{\langle\Psi|\hat{H}_F|\Psi\rangle}{(2\pi)^2\delta^{(2)}_{(p)}(0)}, \qquad (4.74)$$

where $(2\pi)^2\delta^{(2)}_{(p)}(0)$ is the spatial "volume" and $\delta^{(2)}_{(p)}(p)$ the Dirac delta function in momentum space, in order to find the best wave function $\Psi(p)$ minimizing this ratio. The computation of the energy density of the condensate requires the use of the Wick theorem to evaluate the product of normal ordered factors appearing in the Coulomb Hamiltonian

$$: \chi^\dagger_\alpha(\vec{x})\chi_\alpha(\vec{x}) : G(\vec{x},\vec{y}) : \chi^\dagger_\beta(\vec{y})\chi_\beta(\vec{y}) := \qquad (4.75)$$

$$: \chi^\dagger_\alpha(\vec{x})\chi_\alpha(\vec{x})G(\vec{x},\vec{y})\chi^\dagger_\beta(\vec{y})\chi_\beta(\vec{y}) : + : \chi^\dagger_\alpha(\vec{x})\chi_\alpha(\vec{x})G(\vec{x},\vec{y})\chi^\dagger_\beta(\vec{y})\chi_\beta(\vec{y}) :$$
$$+ : \chi_\alpha(\vec{x})\chi^\dagger_\alpha(\vec{x})G(\vec{x},\vec{y})\chi_\beta(\vec{y})\chi^\dagger_\beta(\vec{y}) : + \chi^\dagger_\alpha(\vec{x})\chi_\alpha(\vec{x})G(\vec{x},\vec{y})\chi^\dagger_\beta(\vec{y})\chi_\beta(\vec{y})$$

where the fields have been implicitly expressed at $x^0 = 0 = y^0$. It is necessary to compute the mean value of the last operator in the vacuum state $|\Psi\rangle$. To do so, following [55, 60], we may take advantage of the newly defined ordering prescription and express these same operators in the order $: :_\Psi$, so that the calculation of the matrix elements is made simpler. Making use of the relation (4.71), we find

$$\langle\Psi| : \chi^\dagger_\alpha(0,\vec{x})\chi_\beta(0,\vec{y}) : |\Psi\rangle = \chi^\dagger_\alpha\widehat{(0,\vec{x})\chi_\beta}(0,\vec{y}) \tag{4.76}$$

$$\langle\Psi| : \chi_\alpha(0,\vec{x})\chi^\dagger_\beta(0,\vec{y}) : |\Psi\rangle = \chi_\alpha\widehat{(0,\vec{x})\chi^\dagger_\beta}(0,\vec{y}) \tag{4.77}$$

and

$$\begin{aligned}&\langle\Psi| : \chi^\dagger_\alpha(0,\vec{x})\chi_\alpha(0,\vec{x})\chi^\dagger_\beta(0,\vec{y})\chi_\beta(0,\vec{y}) : |\Psi\rangle\\ &= \chi^\dagger_\alpha\widehat{(0,\vec{x})\chi_\beta}(0,\vec{y}) \quad \chi_\alpha\widehat{(0,\vec{x})\chi^\dagger_\beta}(0,\vec{y}),\end{aligned} \tag{4.78}$$

where α and β denote the spinor components. For conciseness, useful formulas to calculate the above expressions can be found in B.2. Theses results lead to the average energy

$$\langle\Psi|\hat{H}_F|\Psi\rangle = (2\pi)^2\delta^{(2)}_{(p)}(0)\int\frac{\mathrm{d}^2k^i}{(2\pi)^2}2|\vec{k}|\frac{|\Psi(k)|^2}{1+|\Psi(k)|^2} + \langle\Psi|\hat{H}_C|\Psi\rangle, \tag{4.79}$$

where the infrared finite mean interaction energy of the condensate is[7]

$$\begin{aligned}&\langle\Psi|\hat{H}_C|\Psi\rangle =\\ &-\tfrac{e^2}{2}\int \mathrm{d}^2x^i\mathrm{d}^2y^i\ \langle\Psi| : \chi^\dagger_\alpha(0,\vec{x})\chi_\alpha(0,\vec{x}) : G(\vec{x},\vec{y}) : \chi^\dagger_\beta(0,\vec{y})\chi_\beta(0,\vec{y}) : |\Psi\rangle\\ &= -\tfrac{e^2}{2}(2\pi)^2\delta^{(2)}_{(p)}(0)\int\tfrac{\mathrm{d}^2k^i\mathrm{d}^2\ell^i}{(2\pi)^4}\tfrac{-1}{(\vec{\ell}-\vec{k})^2}\Big\{\tfrac{1}{(1+|\Psi(k)|^2)(1+|\Psi(\ell)|^2)}\times\\ &\times\Big[-2|\Psi(k)||\Psi(\ell)|\cos\phi(\ell)\cos\phi(k)\\ &+\hat{\ell}.\hat{k}\Big(|\Psi(\ell)|^2+|\Psi(k)|^2-2|\Psi(\ell)||\Psi(k)|\sin\phi(\ell)\sin\phi(k)\Big)\Big]\\ &+\tfrac{1}{2}-\tfrac{1}{2}\hat{\ell}.\hat{k}\Big\}.\end{aligned} \tag{4.80}$$

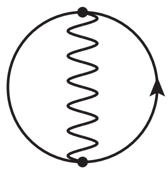
The expression of this mean interaction energy deserves some comments, because its finiteness is not self-evident. Indeed, the last line of (4.80),

[7] A term proportional to $\hat{\ell}\times\hat{k}$ was omitted in this expression. The reason is that it was shown to vanish after the integral over the relative angle between $\vec{k}$ and $\vec{l}$.

involving the factor of $\frac{1}{2} - \frac{1}{2}\hat{\ell}.\hat{k}$ is an infinite constant, corresponding to the term completely contracted in the last line of (4.75) and which may be understood as a quantum fluctuation of the vacuum energy. It is divergent in the ultraviolet but not in the infrared as one can see from the limit $\vec{k} \to \vec{\ell}$, so that we choose to regulate it by introducing a momentum cut-off $\Lambda > 0$,

$$-\frac{e^2}{2}(2\pi)^2\delta^{(2)}_{(p)}(0)\int_{|\vec{k}|<\Lambda}\frac{\mathrm{d}^2k^i}{(2\pi)^2}\int_{|\vec{\ell}|<\Lambda}\frac{\mathrm{d}^2\ell^i}{(2\pi)^2}\frac{-1}{(\vec{\ell}-\vec{k})^2}(\frac{1}{2}-\frac{1}{2}\hat{\ell}.\hat{k}), \quad (4.81)$$

In presence of the regulator and since this contribution is independent of the condensate wave function $\Psi(p)$, we can safely subtract (4.81) from the Hamiltonian. This contribution is proportional to the bubble diagram



where the exact meaning of this pictorial representation is given in terms of the Feynman rules listed in B.5.

Regarding the other terms in (4.80), the apparent singularity of the integral at $\vec{k} = \vec{\ell}$, where a denominator vanishes, is resolved because the denominator appropriately goes to zero at the same time. The infrared finiteness of the mean Coulomb energy and its independence of the parameter μ are specifically due to the choice of ordering prescription in the definition of the Coulomb interaction, which is crucial.

As it happens, the mean energy depends on both the modulus and the phase of the condensate wave function. However, simple considerations about the interaction energy can provide information about the influence of the phase of the wave function on the magnitude of the interaction. In order to minimize the energy density, we would like to make the interaction energy (4.80) as negative as possible. A possibility is to require, separately, a stationary variation with respect to the phase and to the modulus of the wave function. We may first consider to choose the optimal phase of the condensate $\phi(p)$ to minimize the Coulomb energy. Varying $\langle\Psi|\hat{H}_F|\Psi\rangle$ with respect to $\phi(p)$, requires to take simply $\sin\phi(p) = 0$ or $\cos\phi(p) = 0$ for any $p > 0$. Examining (4.80), we notice that, because $\hat{\ell}.\hat{k} \leq 1$, the best choice is to maximize $\cos\phi(p)$, so that we take $\phi(p) = 0$, leading to a real wave function for the fermion condensate. Consequently, we decide to write in the sequel $\Psi(p) = |\Psi(p)|$ to simplify the expressions.

4.4.1 Integral equation

Having formulated the expression of the expected energy density of the condensate, a necessary condition for finding an extremum of that quantity is given by the stationary variation of the energy density

$$\frac{\delta}{\delta\Psi(p)}\frac{\langle\Psi|\hat{H}_F|\Psi\rangle}{(2\pi)^2\delta^{(2)}_{(p)}(0)} = 0, \tag{4.82}$$

with respect to the wave function $\Psi(p)$. Dealing with the functional derivative in the case $p \neq 0$, the resulting nonlinear integral equation reads

$$p\Psi(p) = \tfrac{e^2}{8\pi^2}\int\tfrac{\mathrm{d}^2q^i}{(\vec{q}-\vec{p})^2}[(1-\Psi(p)^2)\tfrac{\Psi(q)}{1+\Psi(q)^2} + \hat{q}.\hat{p}\ \Psi(p)\tfrac{\Psi(q)^2-1}{\Psi(q)^2+1}]. \tag{4.83}$$

Owing to the invariance of the wave function under spatial rotations, the angular integral may be performed explicitly, with the help of formulas given in B.3, so that the integral equation simplifies to

$$\begin{aligned} p\Psi(p) = \ & \alpha\int_0^{+\infty}\mathrm{d}q\Big[q\frac{1-\Psi(p)^2}{|p^2-q^2|}\frac{\Psi(q)}{1+\Psi(q)^2} \\ & + \frac{\Psi(p)}{2p}(-1+\frac{p^2+q^2}{|p^2-q^2|})\frac{\Psi(q)^2-1}{\Psi(q)^2+1}\Big]. \end{aligned} \tag{4.84}$$

where $\alpha = e^2/4\pi$. The non-perturbative features of the modelled phenomenon are reflected by the nonlinearity of the integral equation.
Notably, the integration converges in a neighbourhood of $q = p$ thanks to a cancellation of the two terms in the *rhs* of (4.84). The reason for

the convergence at $q = p$ finds its origin in the choice of ordering prescription made for the Coulomb Hamiltonian. Although obtaining an explicit analytical solution of the equation may be arduous, a property of the solution can be found without effort. Actually, one may readily guess that, in order to ensure the convergence of the integral in the limit $p \to 0$, the wave function should verify $\Psi(0) = 1$. The solution of the linearized equation is expected to have a very different behaviour close to $p = 0$. The mathematical literature dealing with integral equations does not provide a suitable analytic method to find a solution to this kind of very non-linear equation with a singular kernel. As a consequence, we shall look for a numerical solution.

A possible concern about the integral equation could be the existence of solutions as the value of the coupling constant varies. To discuss the dependence on the parameter α, one can try to understand how the equation depends on the typical scale of the problem. In fact, it is possible to express the integral equation in terms of dimensionless variables, using $x = p/\alpha$ and $y = q/\alpha$,

$$\begin{aligned} x\psi(x) = &\int_0^{+\infty} \mathrm{d}y \Big[y \frac{1-\psi(x)^2}{|x^2-y^2|} \frac{\psi(y)}{1+\psi(y)^2} \\ &\qquad + \frac{\psi(x)}{2x} (-1 + \frac{x^2+y^2}{|x^2-y^2|}) \frac{\psi(y)^2-1}{\psi(y)^2+1} \Big], \end{aligned} \tag{4.85}$$

where, in terms of the wave function appearing in (4.83), $\psi(x) = \Psi(\alpha x)$. The conclusion is that, whatever the value of α, we have only one equation to solve, which does not depend on α. Actually, a solution to (4.85) is only a function of the argument $x = p/\alpha$. As a consequence, the required function $\Psi(p)$ solving (4.84) is then simply obtained by the formula $\Psi(p) = \psi(p/\alpha)$. Contrary to the case of QED$_{3+1}$, the rescaled solution $\Psi(\lambda p)$ with $\lambda > 0$ does not obey the same equation as $\Psi(p)$, i.e. equation (4.84). It is only a solution in a theory where e^2 is changed to e^2/λ. Therefore $\Psi(\lambda p)$ is not a stationary point of the energy (4.79).

In actual fact, nothing guarantees that the physical solution is $\Psi \neq 0$, rather than $\Psi = 0$. However, we could wonder if the condensate is energetically more favourable compared to empty Fock vacuum. If a non trivial solution to (4.84) exists, its energy density will be negative and hence lower than the energy density of the Fock vacuum $|0\rangle$, as we shall briefly show. The substitution of the integral equation (4.83) in the formula for the energy density $E = \langle\Psi|\hat{H}_F|\Psi\rangle/(2\pi)^2\delta^{(2)}_{(p)}(0)$ given by (4.79), where the infinite constant (4.81) has been subtracted out, gives the negative value

$$E = \frac{e^2}{2}\int \frac{\mathrm{d}^2k^i\mathrm{d}^2\ell^i}{(2\pi)^4}\frac{-1}{(\vec{\ell}-\vec{k})^2}\frac{\Psi(k)\Psi(\ell)}{(1+\Psi(k)^2)(1+\Psi(\ell)^2)}(\Psi(k)\hat{k}-\Psi(\ell)\hat{\ell})^2.$$

Since this energy density is less than the energy density of the Fock vacuum, we may expect that the Fock vacuum will be unstable to decay into the condensate state.

4.4.2 Numerical solution

A numerical iteration procedure can produce an approximate solution to the integral equation (4.85), written in the form

$$\psi(x) = O[\psi](x) \tag{4.86}$$

where O denotes the nonlinear integral operator which can be read from (4.85). The numerical recipy consists in finding the best trial function to solve the integral equation. An analytic formula for the wave function depending on a series of parameters was guessed and the values of the parameters were determined by an optimization procedure minimizing the squared difference between the trial function and the *rhs* of (4.85) evaluated on a lattice of points. The approximate solution is illustrated in Fig. 4.1.

4.4.3 Spontaneous parity violation

In the literature, reliable arguments support the absence of parity violation (or a parity anomaly) at the perturbative level [61,62], given massless fermions in the bare Lagrangian[8]. Nonetheless, it is not unexpected that

[8]Actually, the bilinear $\bar{\chi}(x)\chi(x)$ violates parity and time reversal, while it preserves charge conjugation.

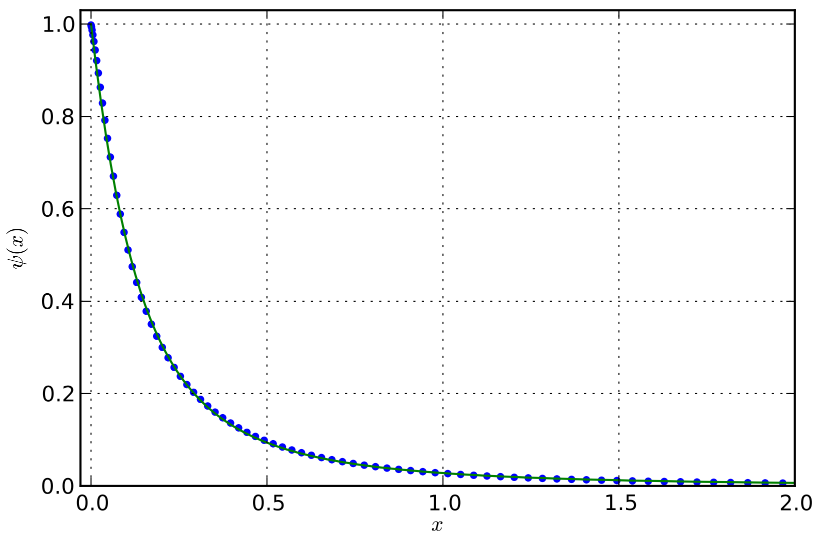


Figure 4.1: The figure compares the trial function (continuous line) with the value of the integral on the *rhs* of (4.85) (dots), as a function of $x = p/\alpha$.

non-perturbative effects may dynamically break this discrete symmetry, as claimed already in [45]. Incidentally, the question of spontaneous parity violation has also been studied in the context of multi-flavour QED$_3$ (see for example [47]).

The expectation value of the parity odd operator $:\bar{\chi}(x)\chi(x):$ is vanishing in the Fock vacuum $|0\rangle$. However, the same is not true for the pair condensate $|\Psi\rangle$. The expectation value in the condensate may be calculated with the help of (B.13) leading to

$$\langle\Psi| :\bar{\chi}(0,\vec{x})\chi(0,\vec{x}): |\Psi\rangle = -\int \frac{\mathrm{d}^2 p^i}{(2\pi)^2} \frac{2\Psi(p)}{1+\Psi(p)^2} \tag{4.87}$$

$$= -(\frac{e^2}{4\pi})^2 \int_0^{+\infty} \frac{\mathrm{d}y}{\pi} \frac{y\psi(y)}{1+\psi(y)^2}. \tag{4.88}$$

A quadrature using the numerical approximation for the condensate wave function gives the following result for the order parameter

$$\langle\Psi|\bar{\chi}\chi|\Psi\rangle \approx -3.2\cdot 10^{-2}\Big(\frac{e^2}{4\pi}\Big)^2. \tag{4.89}$$

Hence we conclude that the vacuum $|\Psi\rangle$, which is energetically more favoured, violates parity, as a straightforward consequence of the definition of the coherent state. Incidentally, the reader will notice that because $\langle 0| : \bar{\chi}(0,\vec{x})\chi(0,\vec{x}) : |0\rangle = 0$, we have

$$\langle\Psi|\bar{\chi}(0,\vec{x})\chi(0,\vec{x})|\Psi\rangle = \langle\Psi| : \bar{\chi}(0,\vec{x})\chi(0,\vec{x}) : |\Psi\rangle. \tag{4.90}$$

4.5 Definition of the Hamilton operator of the quasi-particles

The full quantum Hamiltonian is not yet thoroughly specified. Actually, it may be written completely in terms of the Bogoliubov operators, and should be defined so that its matrix elements are finite. Given that the wave function $\Psi(p)$ is real, one finds the exact result

$$\begin{aligned}\hat{H}_F = &\int \mathrm{d}^2p^i\ \omega(p)[B^\dagger(\vec{p})B(\vec{p}) + D^\dagger(-\vec{p})D(-\vec{p})] + \langle\Psi|\hat{H}_F|\Psi\rangle + \\ &+2\int \tfrac{\mathrm{d}^2p}{1+\Psi(p)^2}\Big\{p\Psi(p) - \tfrac{e^2}{8\pi^2}\int \tfrac{\mathrm{d}^2q^i}{(\vec{q}-\vec{p})^2}\Big[(1-\Psi(p)^2)\tfrac{\Psi(q)}{1+\Psi(q)^2} + \\ &\qquad +\hat{q}.\hat{p}\ \Psi(p)\tfrac{\Psi(q)^2-1}{\Psi(q)^2+1}\Big]\Big\}[B^\dagger(\vec{p})D^\dagger(-\vec{p}) + D(-\vec{p})B(\vec{p})] \\ &+ :\hat{H}_C:_\Psi,\end{aligned} \tag{4.91}$$

where the dispersion relation for the quasi-particles is given by the expression

$$\begin{aligned}\omega(p) = p\tfrac{1-\Psi(p)^2}{1+\Psi(p)^2} \\ +\tfrac{e^2}{2}\mathcal{P}\int \tfrac{\mathrm{d}^2q}{(2\pi)^2}\tfrac{4\Psi(p)\Psi(q)+\hat{p}.\hat{q}(1+\Psi(q)^2\Psi(p)^2-\Psi(p)^2-\Psi(q)^2)}{(\vec{p}-\vec{q})^2(1+\Psi(p)^2)(1+\Psi(q)^2)}\end{aligned}\ . \tag{4.92}$$

The Coulomb interaction Hamiltonian $:\hat{H}_C:_\Psi$ may not be put into a simple form. Examining the quantum Hamiltonian more closely, the new bilinear terms in the first line of (4.91) result from the reorganization of the whole Hamiltonian given in (4.52) and (4.53) as a sum of terms in the normal ordered form associated to the condensate, $: :_\Psi$. Hence, this reorganization generates diagonal terms multiplied by a new dispersion relation $\omega(|\vec{p}|)$ as well as off-diagonal terms. As a consequence of the integral equation (4.83), the off-diagonal terms in the expression for $\hat{H}_F$, which are of the type $B^\dagger(\vec{p})D^\dagger(-\vec{p})$ or $D(-\vec{p})B(\vec{p})$, vanish so that we can interpret the function $\omega(p)$ as the energy of an excitation of one "constituent" fermion or "quasi-particle", $B^\dagger(\vec{p})|\Psi\rangle$ or $D^\dagger(-\vec{p})|\Psi\rangle$. The energy dispersion relation of a quasi-particle may be rewritten as the sum of a finite and a gauge dependent contribution (the latter being potentially divergent),

$$\omega(p) = p\frac{1-\Psi(p)^2}{1+\Psi(p)^2} \tag{4.93}$$

$$+\frac{e^2}{2}\int\frac{\mathrm{d}^2q^i}{(2\pi)^2}\frac{4\Psi(p)\Psi(q)-2\hat{p}.\hat{q}(\Psi(p)^2+\Psi(q)^2)}{(\vec{p}-\vec{q})^2(1+\Psi(p)^2)(1+\Psi(q)^2)} \tag{4.94}$$

$$+\frac{e^2}{2}\mathcal{P}\int\frac{\mathrm{d}^2q^i}{(2\pi)^2}\frac{\hat{p}.\hat{q}}{(\vec{p}-\vec{q})^2} \tag{4.95}$$

where the contribution (4.93) is the "corrected" linear dispersion relation of a relativistic fermion with an asymptotic linear behaviour at large momenta, while the term (4.94) is a pure effect of the presence of the pair condensate. Actually, the integral (4.94) is convergent whenever $p > 0$, but diverges for $p = 0$.

In order to unravel the low momentum behaviour of the dispersion relation, a closer analysis of the behaviour of this integral at $p \to 0$ is required. We decide to perform the angular integration and to use a limited series expansion of the solution for the wave function

$$\Psi(k) = 1 + \Psi'(0)k + \dots \tag{4.96}$$

where $k = q$ or $k = p$ is in the interval $[0, \eta]$, while η is estimated by looking at the numerical solution. To be more specific, we find that the linear approximation is valid when $\eta \approx 0.1\alpha = 0.1(e^2/4\pi)$. In order to study the singular contribution as $p \to 0$, we limit the radial integral in (4.94) to the range $|\vec{q}| \in [0, \eta]$. The integration can then be performed and the result shows that the divergent contribution of (4.94) behaves like

$$\frac{e^2}{4\pi}\Big\{1-\ln 2+\ln(\frac{p+\eta}{2p})-\frac{\Psi'(0)^2}{4}\eta^2+\dots\Big\} \tag{4.97}$$

where the dots mean that we neglected terms vanishing in the limit $p\to 0$. The result of this approximation is that in the small p region the leading (divergent) behaviour of (4.94) is

$$\frac{e^2}{2}\int\frac{\mathrm{d}^2q^i}{(2\pi)^2}\frac{4\Psi(p)\Psi(q)-2\hat{p}.\hat{q}(\Psi(p)^2+\Psi(q)^2)}{(\vec{p}-\vec{q})^2(1+\Psi(p)^2)(1+\Psi(q)^2)}\sim_{p\ll\eta}\frac{e^2}{4\pi}\ln(\frac{\eta}{2p}). \tag{4.98}$$

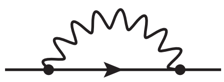
Therefore, the conclusion is that the influence of the condensate induces a divergent contribution to the energy dispersion relation in the infrared region.

In order to understand the origin of the term (4.95), it may be instructive to come back to the ordering prescription chosen for the definition of the Coulomb Hamiltonian of the form $-\frac{e^2}{2}\rho\Delta^{-1}\rho$ with $\rho=\chi^\dagger\chi$. In the quantum Hamiltonian, each charge density factor was ordered separately, i.e. we chose to define the Hamitonian as follows $:\rho:\Delta^{-1}:\rho:$ which had the advantage to remove the gauge dependence.

As may be observed from (4.75), the difference between this prescription and the choice to order the whole expression $:\rho\Delta^{-1}\rho:$ is the sum of a constant term (full contraction) and two bilinear terms. Considering only the two bilinear terms in (4.75), a straightforward calculation gives

$$\begin{aligned}
-\tfrac{e^2}{2}\textstyle\int \mathrm{d}^2x^i\mathrm{d}^2y^i \quad & \Big[:\chi^\dagger_\alpha(\vec{x})\chi_\alpha(\vec{x})G(\vec{x},\vec{y})\chi^\dagger_\beta(\vec{y})\chi_\beta(\vec{y}): \\
& +:\chi_\alpha(\vec{x})\chi^\dagger_\alpha(\vec{x})G(\vec{x},\vec{y})\chi_\beta(\vec{y})\chi^\dagger_\beta(\vec{y}):\Big] \\
=\tfrac{e^2}{2}\textstyle\int \mathrm{d}^2p^i\mathrm{d}^2q^i \quad & \frac{\hat{p}.\hat{q}}{(\vec{p}-\vec{q})^2}\Big[b^\dagger(\vec{p})b(\vec{p})+d^\dagger(-\vec{p})d(-\vec{p})\Big],
\end{aligned} \tag{4.99}$$

which is exactly the extra contribution in $b^\dagger(\vec{p})b(\vec{p})+d^\dagger(-\vec{p})d(-\vec{p})$ remaining when $\Psi(p)$ is sent to zero in the expression for (4.91). This means that the term in (4.95) is only a consequence of the choice of ordering in the definition of $\hat{H}_C$ in (4.53) and hence is not caused by the presence of the condensate. In fact, the operator (4.99) has to be understood as the "finite part" and is proportional to the diagram



where the wavy line is associated to the instantaneous "photon" propagator (similar to the Coulomb gauge photon) as explained in B.4. Incidentally, we may now notice that an infrared divergence appears if we made the choice of the naive Green function as in (4.56). However the Fourier transform of the Green function is actually given by the finite part

$$\frac{e^2}{2}\mathcal{P}\int\frac{\mathrm{d}^2q^i}{(2\pi)^2}\frac{\hat{p}.\hat{q}}{(\vec{p}-\vec{q})^2} \tag{4.100}$$

as explained before. We find

$$\begin{aligned}\tfrac{e^2}{2}\int_{|\vec{p}-\vec{q}|>\mu}\tfrac{\mathrm{d}^2q^i}{(2\pi)^2}\tfrac{\hat{p}.\hat{q}}{(\vec{p}-\vec{q})^2}+\tfrac{e^2}{2}\int_{|\vec{p}-\vec{q}|<\mu}\tfrac{\mathrm{d}^2q^i}{(2\pi)^2}\tfrac{\hat{p}.\hat{q}-1}{(\vec{p}-\vec{q})^2}\\ =\tfrac{e^2}{4\pi}\Big[\ln\tfrac{2p}{\mu}+\ln 2-1\Big]\end{aligned} \tag{4.101}$$

where $p=|\vec{p}|$ and $q=|\vec{q}|$. The details of the calculation leading to (4.101) are given in B.4. The scale μ is related to the scale present in the logarithm in the Coulomb Green function in x-space. The relation between the scales is given by (4.35).
Without further ado, we may now study the small p behaviour of the dispersion relation, by summing (4.97) and (4.101), to note that the divergent contributions coming from the logarithms cancel each other. This is confirmed by the numerical evaluation of the dispersion relation as plotted in Fig. 4.2.

To disentangle this situation, we may decide to separate the contribution coming from the condensate and the one originating from the self-energy as follows

$$\omega(|\vec{p}|)=\omega_0(|\vec{p}|)+\sigma(|\vec{p}|), \tag{4.102}$$

with

$$\sigma(|\vec{p}|)=\frac{e^2}{2}\mathcal{P}\int\frac{\mathrm{d}^2q^i}{(2\pi)^2}\frac{\hat{p}.\hat{q}}{(\vec{p}-\vec{q})^2}. \tag{4.103}$$

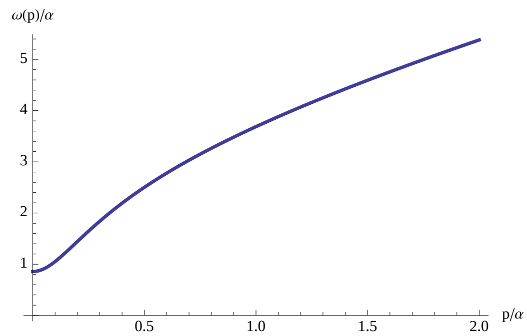


Figure 4.2: The "renormalized" dispersion relation in unit of $\alpha = e^2/4\pi$, where we chose $\mu = 0.1e^2/4\pi$.

The contribution from the condensate causes a low momentum divergence of the energy as we explained before. This behaviour is illustrated in Fig. 4.3, where the rise of the energy as $p \to 0$ is viewed as the signature of the confinement of charges. This will be made clear when we will study the energy of a state made of a pair of opposite charges.

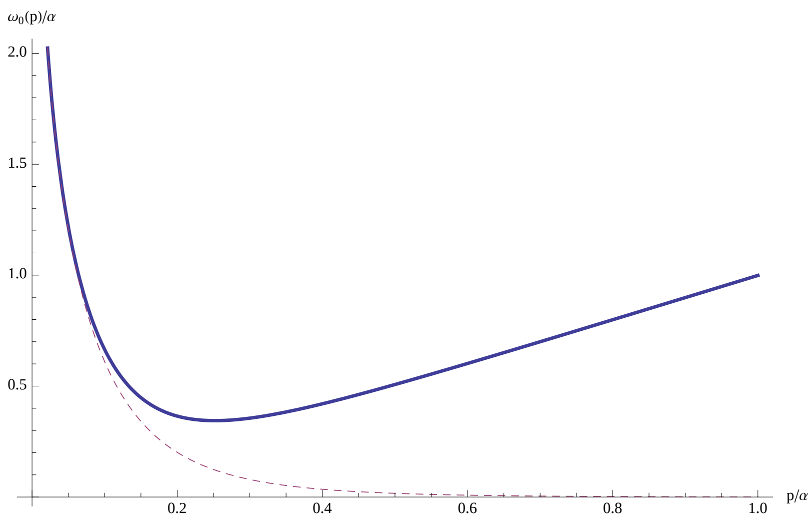


Figure 4.3: The dispersion relation ω_0 (thick line) in units of $\alpha = e^2/4\pi$. The dashed line represents the contribution of the term (4.94) to the dispersion relation.

In conclusion, we found that the contribution to the dispersion relation coming from the interaction with the condensate and the contribution coming from the self-energy had the exact opposite behaviour at small momentum. Hence a complete screening of the low momentum divergence is observed. The result is that $\omega(0)$ takes a finite value, which depends on the scale μ. This is not unexpected since the self-energy takes into account the interaction of the particle with its own Coulomb potential which is μ-dependent. By the way, a similar screening of divergencies is described in [63], based on a different treatment.
The dependence on μ is the fingerprint of the confining electrostatic potential, and is justified in the expression for the energy of a single charged particle because, by itself a state composed of a single charged particle is not gauge invariant. Nevertheless, in complete analogy with the classical situation, we will show hereafter that the mean energy of a particle/antiparticle pair is independent of μ and it is neither UV divergent, nor IR divergent. In section 4.7, we will show that we can understand $\omega_0(p)$ as the energy at a pole of the fermion propagator dressed by the Coulomb interaction.

4.6 Residual Coulomb interactions

As a matter of fact, the energy of a state composed of a single charged particle depends on the scale μ present in the Coulomb Green function. The reason for this observation is that such a state is not physical. On the contrary, a charge neutral state is physical and should have a gauge invariant energy. The goal of this section is to show that a bound state of the form

$$|f\rangle = \int \mathrm{d}^2 k^i f(|\vec{k}|) B^\dagger(\vec{k}) D^\dagger(-\vec{k})|\Psi\rangle, \tag{4.104}$$

has a finite Coulomb energy. The pair state can be interpreted as a positronium, at rest in the "center of mass" frame. As a perspective, once the value for the bound state energy is established, a "Schrödinger" equation can be derived from the variation

$$\frac{\delta}{\delta f^*(k)} \frac{\langle f|\hat{H}_0|f\rangle}{\langle f|f\rangle} = 0, \tag{4.105}$$

where the magnetic mode sector is ignored and considering a simplified Hamiltonian

$$\hat{H}_0 = \hat{H}_K + :\hat{H}_C:_\Psi, \tag{4.106}$$

with

$$\hat{H}_K = \int \mathrm{d}^2p^i \omega(|\vec{p}|)[B^\dagger(\vec{p})B(\vec{p}) + D^\dagger(-\vec{p})D(-\vec{p})]. \tag{4.107}$$

The solution of this integral equation would provide the wave function $f(|\vec{p}|)$ and the energy of the lowest excitation of the bound state. For instance, the numerical procedure could involve a Gauss-Laguerre quadrature method, which leads to a non-trivial problem, even in absence of a pair condensate. However we leave this possibility for future work.

In order to evaluate the energy of the bound state and before calculating the Coulomb interaction energy, a first trivial result is

$$\frac{\langle f|\hat{H}_K|f\rangle}{(2\pi)^2\delta^{(2)}_{(p)}(0)} = \int \frac{\mathrm{d}^2k^i}{(2\pi)^2} 2\omega(|\vec{k}|)|f(k)|^2 \tag{4.108}$$

where we decide to explicitly single out two terms in the dispersion relation

$$\omega(|\vec{k}|) = \omega_0(|\vec{k}|) + \frac{e^2}{2}\mathcal{P}\int \frac{\mathrm{d}^2q^i}{(2\pi)^2} \frac{\hat{k}.\hat{q}}{(\vec{k}-\vec{q})^2}, \tag{4.109}$$

which corresponds to a separation of the μ-dependent parts contributing to the dispersion relation.

4.6.1 Calculation of the residual Coulomb interactions

In order to compute the residual Coulomb interactions given by $:\hat{H}_C:_\Psi$, we will provide the details essential to obtain the necessary expressions.

A little algebra shows that

$$: \chi^\dagger(\vec{x})\chi(\vec{x}) := \int \frac{\mathrm{d}^2k^i\mathrm{d}^2\ell^i}{(2\pi)^2\sqrt{2|\vec{k}|2|\vec{\ell}|}} e^{\mathrm{i}(\vec{\ell}-\vec{k}).\vec{x}}$$
$$\Big\{M_1(\vec{k},\vec{\ell})B^\dagger(\vec{k})B(\vec{\ell}) - M_2(\vec{k},\vec{\ell})D^\dagger(-\vec{\ell})D(-\vec{k})$$
$$+M_3(\vec{k},\vec{\ell})B^\dagger(\vec{k})D^\dagger(-\vec{\ell}) + M_4(\vec{k},\vec{\ell})D(-\vec{k})B(\vec{\ell})\Big\} \quad (4.110)$$

where we have defined the following functions of the wave function of the condensate

$$\begin{aligned}
M_1(\vec{k},\vec{\ell}) = \;& u^\dagger(\vec{k})u(\vec{\ell})[\alpha(k)\alpha(\ell)+\beta(l)\beta(k)] \\
& -u^\dagger(\vec{k})u(-\vec{\ell})[\alpha(k)\beta(l)+\alpha(\ell)\beta(k)] \\
M_2(\vec{k},\vec{\ell}) = \;& u^\dagger(\vec{k})u(\vec{\ell})[\alpha(k)\alpha(\ell)+\beta(\ell)\beta(k)] \\
& +u^\dagger(\vec{k})u(-\vec{\ell})[\alpha(\ell)\beta(k)+\alpha(k)\beta(\ell)] \\
M_3(\vec{k},\vec{\ell}) = \;& u^\dagger(\vec{k})u(\vec{\ell})[\alpha(k)\beta(\ell)-\alpha(\ell)\beta(k)] \\
& +u^\dagger(\vec{k})u(-\vec{\ell})(\alpha(k)\alpha(\ell)-\beta(k)\beta(\ell)] \\
M_4(\vec{k},\vec{\ell}) = \;& u^\dagger(\vec{k})u(\vec{\ell})[\alpha(\ell)\beta(k)-\alpha(k)\beta(\ell)] \\
& +u^\dagger(\vec{k})u(-\vec{\ell})[\alpha(k)\alpha(\ell)-\beta(k)\beta(\ell)].
\end{aligned}$$

For the sake of completeness, we also give the following results

$$u^\dagger(\vec{k})u(\vec{\ell}) = \sqrt{|\vec{k}|.|\vec{\ell}|}(1+\hat{k}.\hat{\ell}+\mathrm{i}\hat{\ell}\times\hat{k}), \quad (4.111)$$

$$u^\dagger(\vec{k})u(-\vec{\ell}) = \sqrt{|\vec{k}|.|\vec{\ell}|}(1-\hat{k}.\hat{\ell}-\mathrm{i}\hat{\ell}\times\hat{k}). \quad (4.112)$$

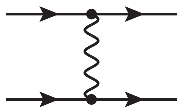
We shall now consider the interactions involving only one pair. Among all the possible Coulomb interactions, we find that the only terms contributing to (4.105) are

$$-\frac{e^2}{2}\int \mathrm{d}^2x^i\mathrm{d}^2y^i\Big[\, : (: \chi^\dagger(\vec{x})\chi(\vec{x}) : G(\vec{x},\vec{y}) : \chi^\dagger(\vec{y})\chi(\vec{y}) :) :_\Psi \Big]_{1P}$$
$$= -\frac{e^2}{2}\mathcal{P}\int \frac{\mathrm{d}^2\ell^i\mathrm{d}^2k^i\mathrm{d}^2p^i}{(2\pi)^2}\frac{-1}{|\vec{p}|^2}\frac{2}{\sqrt{2|\vec{\ell}|2|\vec{k}|2|\vec{\ell}+\vec{p}|2|\vec{k}-\vec{p}|}}$$
$$\Big\{-M_1(\vec{k},\vec{k}-\vec{p})M_2(\vec{\ell},\vec{\ell}+\vec{p})\times$$
$$\times B^\dagger(\vec{k})D^\dagger(-(\vec{\ell}+\vec{p}))D(-\vec{\ell})B(\vec{k}-\vec{p}) \quad (4.113)$$
$$+M_3(\vec{k},\vec{k}-\vec{p})M_4(\vec{\ell},\vec{\ell}+\vec{p})\times$$
$$\times B^\dagger(\vec{k})D^\dagger(-(\vec{k}-\vec{p}))D(-\vec{\ell})B(\vec{\ell}+\vec{p})\Big\}. \quad (4.114)$$

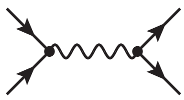
The following useful matrix element of the residual Coulomb Hamiltonian can be separated in two terms, corresponding to the first and second terms, respectively (4.113) and (4.114),

$$\frac{\langle f|(:\hat{H}_C:_\Psi)_{1P}|f\rangle}{(2\pi)^2\delta^{(2)}_{(p)}(0)} = T_1 + T_2, \tag{4.115}$$

where T_1 corresponds to the one Coulomb photon exchange inside the pair



and where T_2 is associated to the annihilation



of the pair into a Coulomb photon. We choose to study only the contribution of T_1, because T_2 is independent of the choice of zero of the potential μ. The inclusion of T_2 in the discussion is nonetheless straightforward. We find a result with a potential IR divergence at $\vec{k} = \vec{\ell}$, however the integration is considered as the "finite part",

$$\begin{aligned} T_1 &= \frac{e^2}{2}\mathcal{P}\int \frac{\mathrm{d}^2\ell^i \mathrm{d}^2k^i}{(2\pi)^4}\frac{-1}{(\vec{k}-\vec{\ell})^2} f^*(|\vec{k}|)f(|\vec{\ell}|)\times \\ &\times\Big[1+\hat{\ell}.\hat{k} - 2\frac{\psi(\ell)^2+\psi(k)^2-2\hat{\ell}.\hat{k}\psi(\ell)\psi(k)}{(1+\psi(\ell)^2)(1+\psi(k)^2)}\Big]. \end{aligned} \tag{4.116}$$

In the last equation (4.116), we have split the contribution coming from the pair condensate from the one already present in the Fock vacuum. The need for the "finite part" introduces a μ dependence in the expression. One may notice that the contribution coming from the condensate in (4.116) vanishes when $\vec{k} = \vec{\ell}$, while the term $1+\hat{\ell}.\hat{k}$ is divergent if

we set $\mu = 0$. The equation (4.116), when evaluated with $\Psi(p) = 0$, is completely analogous to the formula found in [55] which analysed a similar situation in 3+1 dimensions in the so-called Limited Fock Space Approximation. However, in $3 + 1$ dimensions, no infrared divergence is expected when $\vec{k} = \vec{\ell}$ because the double angular integration makes the singularity integrable. In this case the IR singularity behaves like $\ln|\frac{k+l}{k-l}|$.
Considering the sum of the kinetic and interaction mean energies, we find

$$\begin{aligned} E = \tfrac{\langle f|\hat{H}_K|f\rangle}{(2\pi)^2\delta^{(2)}_{(p)}(0)} + T_1 = \int \tfrac{\mathrm{d}^2k^i}{(2\pi)^2} 2\omega_0(|\vec{k}|)|f(k)|^2 + \\ + \tfrac{e^2}{2}\int \tfrac{\mathrm{d}^2\ell^i\mathrm{d}^2k^i}{(2\pi)^4}\tfrac{-1}{(\vec{k}-\vec{\ell})^2}\Big\{ f^*(|\vec{k}|)f(|\vec{\ell}|) \times \\ \times\Big[1 + \hat{\ell}.\hat{k} - 2\tfrac{\psi(\ell)^2+\psi(k)^2-2\hat{\ell}.\hat{k}\psi(\ell)\psi(k)}{(1+\psi(\ell)^2)(1+\psi(k)^2)}\Big] - 2\hat{\ell}.\hat{k}|f(|\vec{k}|)|^2\Big\}. \end{aligned} \tag{4.117}$$

where the potentially divergent terms in the second term of (4.109) and in (4.116) have cancelled each other. The result is that the finite part is not needed to render the value of the integral infrared finite. We emphasize once more that the mean energy is now independent of the scale μ. Remarquably, from the contribution of the dispersion relation, only the term $\omega_0(k)$, which is pictured in Fig. 4.3, remains. The divergence at $k \to 0$ of $\omega_0(k)$ is a signature of confinement, since it forces the wave function $f(k)$ to vanish at small momentum. We can symmetrize the last term in (4.117) and using the identity,

$$|f(k)|^2 + |f(\ell)|^2 = |f(k) - f(\ell)|^2 + f^*(l)f(k) + f^*(k)f(\ell) \tag{4.118}$$

we may reformulate the energy of the pair state as

$$\begin{aligned} E = \int \tfrac{\mathrm{d}^2k^i}{(2\pi)^2} 2\omega_0(|\vec{k}|)|f(k)|^2 + \tfrac{e^2}{2}\int \tfrac{\mathrm{d}^2\ell^i\mathrm{d}^2k^i}{(2\pi)^4}\tfrac{-1}{(\vec{k}-\vec{\ell})^2}\Big\{ f^*(k)f(\ell) \times \\ \times\Big[1 - \hat{\ell}.\hat{k} - 2\tfrac{\psi(\ell)^2+\psi(k)^2-2\hat{\ell}.\hat{k}\psi(\ell)\psi(k)}{(1+\psi(\ell)^2)(1+\psi(k)^2)}\Big] - \hat{\ell}.\hat{k}|f(k) - f(\ell)|^2\Big\}. \end{aligned} \tag{4.119}$$

This result allows us to conclude that the energy of a state made of a pair of opposite charge particles is indeed independent of the choice of zero of the potential. Therefore we confirm here that the energy of a gauge invariant state is perfectly infrared finite and gauge independent.

By the same token, the examination of the energy of a pair state (4.119) confirms the confinement scenario. Since the potential energy between the constituent fermions $2\omega_0(|\vec{p}|)$ is divergent at $\vec{p} = \vec{0}$, a likely and reasonable assumption is that the wave function of the pair, $f(|\vec{p}|)$, vanishes in order to solve the bound state equation

$$\begin{aligned}(2\omega_0(k) - E)f(k) + \tfrac{e^2}{2}\int \tfrac{\mathrm{d}^2\ell^i}{(2\pi)^2}\tfrac{-1}{(\vec{k}-\vec{\ell})^2} \times \\ \times\Big\{f(\ell)\Big[1 - \hat{\ell}.\hat{k} - 2\tfrac{\psi(\ell)^2+\psi(k)^2-2\hat{\ell}.\hat{k}\psi(l)\psi(k)}{(1+\psi(\ell)^2)(1+\psi(k)^2)}\Big] \\ -\hat{\ell}.\hat{k}[f(k) - f(\ell)]\Big\} = 0,\end{aligned} \tag{4.120}$$

obtained from (4.105). The study of the energy levels of this bound state necessitates the numerical solution of this eigenvalue integral equation. Beforehand, the interaction with the magnetic mode should be probably included in the variational principle in order to get a more consistent approximation. The numerical resolution of this equation is left open for future work.

4.7 Green function interpretation

4.7.1 Schwinger-Dyson equation

The Hamilton formalism has made clear that a variational procedure was an appropriate way to obtain the structure of the fermionic vacuum. As a complementary point of view on the condensation mechanism, we may understand the integral equation (4.83) as a (truncated) Schwinger-Dyson equation [57]. More precisely, the idea is to choose an ansatz for a p-space propagator, and to identify the relationship between a Schwinger-Dyson equation and the integral equation for the wave function. To do so, let us introduce the following formula for the fermion propagator in the condensate

$$S^{(3)}(p^0, \vec{p}) = \frac{\mathrm{i}}{\not{p} - \Sigma(p) + \mathrm{i}\epsilon} \tag{4.121}$$

with the parametrization $\Sigma(p) = |\vec{p}|A(p) + \vec{p}.\vec{\gamma}B(p)$. The functions $A(p) = A(|\vec{p}|)$ and $B(p) = B(|\vec{p}|)$ depend only on the modulus of $\vec{p}$.

When we substitute the parametrization in the Feynman propagator, we obtain

$$S^{(3)}(p^0,\vec{p}) = \mathrm{i}\frac{p^0\gamma^0 - \vec{p}.\vec{\gamma}(1+B(p)) + |\vec{p}|A(p)}{(p^0)^2 - |\vec{p}|^2(A^2(p)+(1+B(p))^2)+\mathrm{i}\epsilon}. \tag{4.122}$$

As a consequence, the calculation of the equal time propagator in the condensate

$$S(\vec{p}) = \int \frac{\mathrm{d}p^0}{2\pi} S^{(3)}(p^0,\vec{p}) \tag{4.123}$$

allows one to the obtain a relation between the wave function of the condensate and the functions $A(p)$ and $B(p)$. More precisely, we find the relation between the wave function of the condensate and the ansatz functions

$$\frac{1}{2}\frac{A(p)+\hat{p}.\vec{\gamma}(1+B(p))}{\sqrt{A^2(p)+(1+B(p))^2}} = \frac{1}{2}\Big[\frac{2\Psi(p)}{1+\Psi(p)^2} + \frac{1-\Psi(p)^2}{1+\Psi(p)^2}\hat{p}.\vec{\gamma}\Big], \tag{4.124}$$

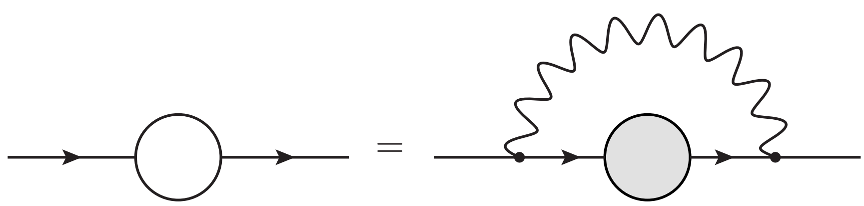
by integrating (4.123) using the parametrization (4.122) and identifying the result with the equal time propagator obtained from (B.14). This allows to identify

$$\frac{A(p)}{1+B(p)} = \frac{2\Psi(p)}{1-\Psi(p)^2}. \tag{4.125}$$

In order to fully understand $A(p)$ and $B(p)$ in terms $\Psi(p)$, we need to study the following Schwinger-Dyson equation

$$-\mathrm{i}\Sigma(\vec{p}) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{i}}{(\vec{p}-\vec{q})^2}(\mathrm{i}e\gamma^0)S^{(3)}(q)(\mathrm{i}e\gamma^0), \tag{4.126}$$

which can be pictorially represented as



where the Feynman rules for the associated diagrammatic formulation are listed in the B.5. The corresponding integral equation may be rewritten

$$pA(p) + \vec{p}.\vec{\gamma}B(p) = \frac{e^2}{8\pi^2}\int \frac{\mathrm{d}^2q^i}{(\vec{q}-\vec{p})^2}\Big[\frac{2\Psi(q)}{1+\Psi(q)^2} + \frac{1-\Psi(q)^2}{1+\Psi(q)^2}\hat{q}.\vec{\gamma}\Big], \tag{4.127}$$

with $p = |\vec{p}|$ and where the integral should be understood as the Hadamard finite part. If we multiply (4.127) by

$$N_1(\vec{p}) = \frac{(1+B(p)) + \hat{p}.\vec{\gamma}A(p)}{\sqrt{A^2(p)+(1+B(p))^2}} = \frac{1-\Psi(p)^2}{1+\Psi(p)^2} + \hat{p}.\vec{\gamma}\frac{2\Psi(p)}{1+\Psi(p)^2}, \tag{4.128}$$

and take the trace, we obtain the integral equation (4.83), as could have been anticipated,

$$4p\Psi(p) = \frac{e^2}{2\pi^2}\int \frac{\mathrm{d}^2q^i}{(\vec{q}-\vec{p})^2}[(1-\Psi(p)^2)\frac{\Psi(q)}{1+\Psi(q)^2} + \hat{q}.\hat{p}\ \Psi(p)\frac{\Psi(q)^2-1}{\Psi(q)^2+1}],$$

where we used in the calculation the relation

$$\frac{A(p)}{\sqrt{A^2(p)+(1+B(p))^2}} = \frac{2\Psi(p)}{1+\Psi(p)^2}. \tag{4.129}$$

Similarly, we can express the function $A(p)$ in terms of the wave function of the condensate. Taking the trace of (4.127) over the spinor indices, we find

$$|\vec{p}|A(p) = \frac{e^2}{(2\pi)^2}\mathcal{P}\int \mathrm{d}^2q^i\frac{1}{(\vec{p}-\vec{q})^2}\frac{\Psi(q)}{1+\Psi(q)^2}. \tag{4.130}$$

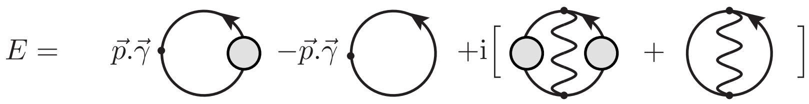
Finally, we notice that the pole structure of (4.122) provides the energy of the particle excitations: $|\vec{p}|\sqrt{A^2(p)+(1+B(p))^2}$. In order to obtain the formula for the dispersion relation, we can add $\vec{p}.\vec{\gamma}$ to (4.127) and multiply it by

$$N_2(\vec{p}) = \frac{A(p) + \hat{p}.\vec{\gamma}(1+B(p))}{\sqrt{A^2(p)+(1+B(p))^2}} = \frac{2\Psi(p)}{1+\Psi(p)^2} - \frac{1-\Psi(p)^2}{1+\Psi(p)^2}\hat{p}.\vec{\gamma}, \tag{4.131}$$

and finally take the trace. The result of this short manipulation gives

$$\begin{aligned} |\vec{p}|\sqrt{A^2(p)+(1+B(p))^2} &= p\tfrac{1-\Psi(p)^2}{1+\Psi(p)^2} \\ +\tfrac{e^2}{2}\mathcal{P}\int \tfrac{\mathrm{d}^2q^i}{(2\pi)^2}&\tfrac{4\Psi(p)\Psi(q)+\hat{p}.\hat{q}(1-\Psi(p)^2)(1-\Psi(q)^2)}{(\vec{p}-\vec{q})^2(1+\Psi(p)^2)(1+\Psi(q)^2)}, \end{aligned} \tag{4.132}$$

which is exactly the dispersion relation $\omega(p)$ found before in (4.92). In conclusion, we obtain that the energy of the quasi-particles created by $B^\dagger$ and $D^\dagger$ corresponds exactly to the energy of the physical pole of the propagator of the fermion field in the pair condensate. Hence this result supports the interpretation obtained before. In order to complete the analogy, we can reformulate the energy of the condensate in the diagramatic expression



which can be readily used to obtain the Schwinger-Dyson equation (4.126). The last term in the sum above is a constant divergent bubble diagram that was subtracted from the Hamiltonian when we discussed the value of the energy density of the condensate.

It should be emphasized that the approach developped here does not rely on the dimensional regularisation used more or less implicitly in the literature, but on the exact Fourier transform of the x-space Coulomb Green function.

4.7.2 Two-point function

In the previous section, an ansatz technique allowed to obtain a Schwinger-Dyson equation for the fermion propagator. However it is not clear why the propagator obtained in this manner has a gauge dependent pole. In order to explain this issue, it may be more instructive to understand the origin of the "constituent" fermion propagator in the condensate from the Fourier transform of a x-space correlation function. Indeed, the relationship between the propagator found above and the approximate (or perturbative) evaluation of a time ordered correlation function is not cristal clear. The Green function of the equation (4.122), obtained in the p-space in the last section, obviously exhibits a pole whose position is gauge dependent. The energy at the pole corresponds to the energy of a single excitation $B^\dagger(\vec{p})|\Psi\rangle$ or $D^\dagger(-\vec{p})|\Psi\rangle$, in expectation value. From this point of view, it is not a surprise since a charged state is not gauge invariant, however we may raise the question of the mass of these constituent fermions. This puzzle has its origin in the Coulomb interactions. From a perturbative perspective, this feature can be understood as follows. The total quantum Hamiltonian can be split in

$$\hat{H} = \hat{H}_0 + \hat{H}^I_C + \hat{H}_\Phi + \hat{H}^I_{\Phi\chi}, \tag{4.133}$$

where $\hat{H}_\Phi$ is given by (4.43), and where the Hamiltonian $\hat{H}^I_{\Phi\chi}$ is obtained thanks to the Hamiltonian density (4.31), while

$$\hat{H}_0 = \int \mathrm{d}^2p^i \omega_0(p)[B^\dagger(\vec{p})B(\vec{p}) + D^\dagger(\vec{p})D(\vec{p})], \tag{4.134}$$

$$\hat{H}^I_C = \int \mathrm{d}^2p^i \sigma(p)[B^\dagger(\vec{p})B(\vec{p}) + D^\dagger(\vec{p})D(\vec{p})] + :\hat{H}_C:_\Psi, \tag{4.135}$$

where the first term (4.134) is bilinear and gauge invariant, while the second term (4.135) is also separately gauge invariant and contains a bilinear and quadrilinear term. The reason for this separation is the ordering prescription taken for the Coulomb Hamiltonian, which insures that the sum of the two gauge dependent terms in (4.135) is in fact gauge invariant, when acting in the physical state space.

In a perturbative treatment, one should consider $\hat{H}_0$ as the "free" Hamiltonian, whereas $\hat{H}^I_C$ and $\hat{H}^I_{\Phi\chi}$ as the "interaction" Hamiltonians. In order to define a gauge invariant two-point function in the condensate, we decide to define the interaction picture field

$$\chi_I(t,\vec{x}) = e^{\mathrm{i}\hat{H}_0 t}\chi(0,\vec{x})e^{-\mathrm{i}\hat{H}_0 t}. \tag{4.136}$$

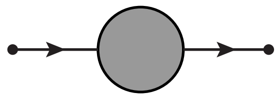
The time ordered and gauge invariant two-point function in the condensate can be calculated thanks to

$$\mathcal{S}(t,\vec{x}) = \langle\Psi|\chi_I(t,\vec{x})\bar{\chi}_I(0,\vec{0})|\Psi\rangle\Theta(t) - \langle\Psi|\bar{\chi}_I(0,\vec{0})\chi_I(t,\vec{x})|\Psi\rangle\Theta(-t),$$

where $\Theta(t)$ is the Heaviside step function. An explicit calculation allows to express the Fourier transform of the two-point function

$$\mathcal{S}(k^0,\vec{k}) = \int \mathrm{d}t\ \mathrm{d}^2x^i\ e^{\mathrm{i}k^0t-\mathrm{i}\vec{k}.\vec{x}}\mathcal{S}(t,\vec{x}) \tag{4.137}$$

which is represented by a fermion line with a dark blob



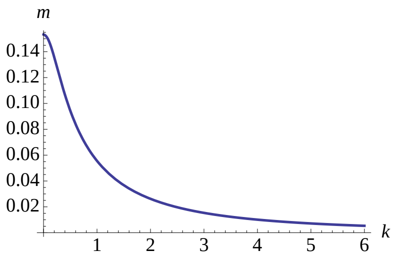
and given precisely by the expression

$$\mathcal{S}(k^0,\vec{k}) = \mathrm{i}\frac{k^0\gamma^0 - Z(k)[\vec{k}.\vec{\gamma} - m(k)]}{(k^0)^2 - \omega_0^2(k) + \mathrm{i}\epsilon}, \tag{4.138}$$

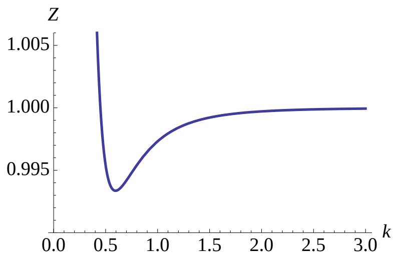
with $k = |\vec{k}|$ and

$$Z(k) = \frac{1-\psi^2(k)}{1+\psi^2(k)}\frac{\omega_0(k)}{k}, \quad m(k) = \frac{2k\psi(k)}{1-\psi^2(k)}. \tag{4.139}$$

The behaviour of the functions $m(k)$ and $Z(k)$ is illustrated in the figures (4.4a) and (4.4b). As expected, the dynamical mass tends to zero at large momentum, while the function $Z(k)$ goes to unity. Whereas the value $m(0)$ is finite, we observe that $Z(k)$ exhibits an integrable logarithmic divergence as $k \to 0$.



(a)



(b)

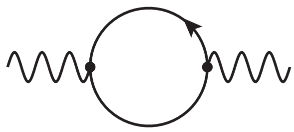
Figure 4.4: The functions $m(k)$, and $Z(k)$ in units of $e^2/4\pi$.

4.8 Correction to the magnetic mode propagator

While the previous sections treated the fermion sector in the sole presence of Coulomb interactions, the present section aims at examining the influence of the dynamics of the fermions on the propagation of the magnetic mode. The non-perturbative solution in the fermionic sector will serve the zeroth-order contribution in the perturbative expansion in the interactions with the magnetic sector.

In order to understand the effect of the fermion condensate and of parity violation on the magnetic mode sector, it is instructive to review the one-loop correction to the photon propagator in the absence of a condensate, with a massless fermion. Indeed, UV divergences in perturbation theory, due to the large momentum regime, will affect the magnetic mode progator, irrespective of the presence or not of the condensate.

In relativistic covariant perturbation theory, the leading order correction is the amputated diagram

$\mathrm{i}\Pi^{\mu\nu} =$ 

which reads

$$\mathrm{i}\Pi^{\mu\nu}(p) = -\int \frac{\mathrm{d}^3\ell}{(2\pi)^3} e^2 \mathrm{Tr}[\gamma^\mu \frac{\not{p} + \not{\ell}}{(p+\ell)^2 + \mathrm{i}\epsilon} \gamma^\nu \frac{\not{\ell}}{\ell^2 + \mathrm{i}\epsilon}]. \tag{4.140}$$

The integral is linearly divergent in power counting. While dimensional regularisation provides a finite result without a divergent contribution [30], we prefer here to use a cut-off regulator, because it is more instructive in this context, but at the expense of breaking gauge symmetry. After a Wick rotation $\ell^0 \to \mathrm{i}\ell^0_E$, and with the help of the Feynman parameter trick, we obtain the result

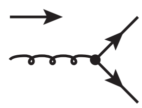
$$\mathrm{i}\Pi^{\mu\nu}(p) = -\frac{\mathrm{i}e^2}{3\pi^2}\Lambda\eta^{\mu\nu} - \frac{\mathrm{i}e^2}{16}(\eta^{\mu\nu}p^2 - p^\mu p^\nu)\frac{1}{\sqrt{-p^2 - \mathrm{i}\epsilon}}, \tag{4.141}$$

where the linearly divergent contribution in the first term is a gauge symmetry breaking term, whereas the second term is the finite result also given by the dimensional regularisation procedure[9]. The cut-off dependent term has to be subtracted exactly thanks to a covariant mass counter term in the Lagrangian, leaving no ambiguous finite term in order to preserve the Ward identity.

As a lesson from the form of the vacuum polarization contribution in the absence of the condensate, we expect also a linear divergence in the analogue diagram for the magnetic mode in the condensate. Namely, we are interested in the two-point function of the magnetic mode

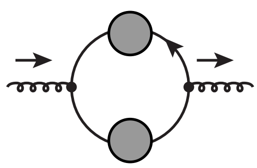
$$T\langle\Omega|\Phi(x^0,\vec{x})\Phi(y^0,\vec{y})|\Omega\rangle \tag{4.142}$$

with $|\Omega\rangle$ the full interacting vacuum. Working in p-space, we decide to perform a perturbative expansion, with the interaction Hamiltonians $\hat{H}^I_{\Phi\chi}$ and $\hat{H}^I_C$. Hence the Feynman rule for the vertex between the magnetic mode and the current



is given by: $e\epsilon^{ij}p^j\gamma^j$, when the magnetic mode momentum $(p^0,\vec{p})$ is incoming.

With the help of this Feynman rule, it is possible to formulate the first loop correction to the free propagator. In the presence of the condensate, the first contribution to vacuum polarization is

$-\mathrm{i}\pi(p^0,\vec{p}) =$ 

[9]Here, $p = (p^0,\vec{p})$ is the 3-vector associated to the momentum of the incoming photon.

where the fermion propagator in the condensate is the one given by (4.138). The polarization modified by the presence of the condensate can be written as the product

$$\pi(p^0,\vec{p}) = |\vec{p}|^2\pi_0(p^0,\vec{p}), \tag{4.143}$$

where the quantity of interest is the loop integral

$$\begin{aligned} -\mathrm{i}\pi_0(p^0,\vec{p}) = \\ = -\int \frac{\mathrm{d}^3\ell}{(2\pi)^3}2e^2 \times \left\{\frac{\ell^0(p^0+\ell^0)+Z(\ell)Z(|\vec{p}+\vec{\ell}|)[2\ell^2\sin^2\theta-\vec{\ell}.(\vec{\ell}+\vec{p})-m(\ell)m(|\vec{p}+\vec{\ell}|)]}{[(p^0+\ell^0)^2-\omega^2(|\vec{p}+\vec{\ell}|)+\mathrm{i}\epsilon][(\ell^0)^2-\omega^2(\ell)+\mathrm{i}\epsilon]}\right\} \end{aligned} \tag{4.144}$$

with $\ell = |\vec{\ell}|$ and where θ is the relative angle between the loop momentum $\vec{\ell}$ and the incoming spatial momentum $\vec{p}$. Denoting the free magnetic mode propagator by

$$D(p^0,\vec{p}) = \frac{1}{|\vec{p}|^2}\frac{\mathrm{i}}{(p^0)^2-|\vec{p}|^2+\mathrm{i}\epsilon}, \tag{4.145}$$

we can compute the full propagator as the sum of the one particle irreducible diagrams

$$\begin{aligned} D(p^0,\vec{p}) + (D(-\mathrm{i}\pi)D)(p^0,\vec{p}) + (D(-\mathrm{i}\pi)D(-\mathrm{i}\pi)D)(p^0,\vec{p}) + \ldots \\ = \mathrm{i}\Big\{|\vec{p}|^2\big[(p^0)^2-|\vec{p}|^2-\pi_0(p^0,\vec{p})+\mathrm{i}\epsilon\big]\Big\}^{-1}. \end{aligned} \tag{4.146}$$

Hence the investigation for a dynamical mass of the magnetic mode photon requires to solve the condition

$$(p^0)^2-|\vec{p}|^2-\pi_0(p^0,\vec{p}) = 0, \tag{4.147}$$

in order to find the position of a pole of order one in the resummed propagator. If we can find a solution to (4.147) in perturbation theory, we will be able to write a dispersion relation $p^0(|\vec{p}|)$, and will define a running mass squared as

$$M^2(|\vec{p}|) = \Big(p^0(|\vec{p}|)\Big)^2-|\vec{p}|^2. \tag{4.148}$$

As we will show, the solution verifies, to leading order in perturbation theory,

$$(p^0)^2 = |\vec{p}|^2+e^4\pi'(p^0,\vec{p}) \approx |\vec{p}|^2+e^4\pi'(|\vec{p}|,\vec{p}), \tag{4.149}$$

where we used $\pi_0(p^0,\vec{p}) = e^4\pi'(p^0,\vec{p})$, so that the running mass squared is approximately given by

$$M^2(|\vec{p}|) = \Big(p^0(|\vec{p}|)\Big)^2 - |\vec{p}|^2 \approx e^4\pi'(|\vec{p}|,\vec{p}). \tag{4.150}$$

The value that we will be interested in, is $M^2(0) \approx e^4\pi'(0,\vec{0})$. Hence, due to the technical difficulties, we shall only calculate the value of $\pi_0(0,\vec{0})$.

Because the computation of $\pi_0(p^0,\vec{p})$ involves a function known only numerically, we shall evaluate its first term in a power expansion in $|\vec{p}|$,

$$\pi(p^0,\vec{p}) = |\vec{p}|^2\Big[\pi_0(p^0,\vec{0}) + O(|\vec{p}|)\Big]. \tag{4.151}$$

The expression of $\pi_0(p^0,\vec{0})$ in (4.144) involves an integral over the temporal and spatial loop momentum of a non explicitly covariant function, so that Wick rotation does not seem to be appropriate. Nevertheless the Feynman parameter technique can be used and, afterwards, the expression can be simplified thanks to the shift $\ell^0 \to \ell^0 - xp^0$. The ℓ^0-integral is convergent and can be calculated by evaluating the residue of a double pole, leaving an integral over the spatial momentum $\vec{\ell}$. Performing the angular integral, the result is an integral over $\ell = |\vec{\ell}|$,

$$-\mathrm{i}\pi_0^{\Lambda}(p^0,\vec{0}) = \frac{-\mathrm{i}e^2}{2}\int_0^1 \mathrm{d}x\int_0^{\Lambda}\frac{\ell\mathrm{d}\ell}{2\pi}\frac{-Z^2(\ell)\ell^2 - 2Z^2(\ell)m^2(\ell)}{[\omega_0^2(\ell) - x(1-x)(p^0)^2]^{3/2}}, \tag{4.152}$$

whose linear divergence was regularised with a cut-off $|\vec{\ell}| < \Lambda$. The divergent behaviour lies of course in the ultraviolet regime and is exactly the same as in the absence of a condensate. Neglecting the condensate, that is to say putting $\Psi = 0$, we find the exact result

$$\begin{aligned}
-\mathrm{i}|\vec{p}|^2\pi_0^{\Lambda}(p^0,\vec{0}) &\overset{\Psi=0}{=} |\vec{p}|^2\frac{-\mathrm{i}e^2}{2}\int_0^1 \mathrm{d}x\int_0^{\Lambda}\frac{\ell\mathrm{d}\ell}{2\pi}\frac{-\ell^2}{[\ell^2 - x(1-x)(p^0)^2]^{3/2}},\\
&= |\vec{p}|^2\frac{-\mathrm{i}e^2\sqrt{-(p^0)^2}}{16} + |\vec{p}|^2\frac{\mathrm{i}e^2}{2}\frac{\Lambda}{2\pi}.
\end{aligned} \tag{4.153}$$

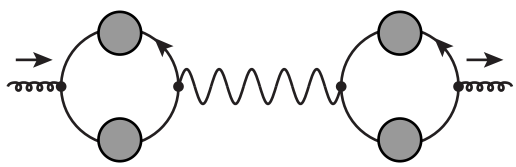
Because we expect that, in the large momentum limit, the theory with the condensate yields the same result as ordinary perturbative QED_{2+1}, the requirement of finiteness of this diagram gives us an unambiguous way to subtract the linear divergence of the same diagram in presence of the condensate. Hence, using (4.153), the renormalization of $\pi_0(p^0,\vec{0})$ gives a finite result

$$-\mathrm{i}\pi_0^{\mathrm{reg}}(p^0,\vec{0}) = \lim_{\Lambda\to+\infty}\Big\{-\mathrm{i}\pi_0^{\Lambda}(p^0,\vec{0}) - \frac{\mathrm{i}e^2}{2}\frac{\Lambda}{2\pi}\Big\}, \tag{4.154}$$

obtained thanks to the addition of a counter term proportional to $\Phi\Delta\Phi$ in the Lagrangian. Setting $p^0 = 0$ in order to evaluate the mass of the magnetic mode, a numerical integration yields the result

$$\begin{aligned}\pi_0^{\mathrm{reg}}(0,\vec{0}) &= -\frac{e^2}{4\pi}\int_0^{+\infty}\mathrm{d}\ell\Big\{\frac{\ell-\omega(\ell)}{\omega(\ell)} + \frac{\ell Z^2(\ell)m^2(\ell)}{\omega^3(\ell)}\Big\}\\ &\approx 0.14\Big(\frac{e^2}{4\pi}\Big)^2. \end{aligned}\tag{4.155}$$

The subtraction of the linear divergence from this one loop diagram leaves a finite contribution proportional to e^4. Other finite contributions proportional to e^4 will come from diagrams containing more loops. However, it is not excluded that two loop diagrams give rise to a divergent dynamical mass to the magnetic mode. Among them, the potentially problematic diagram denoted by $-\mathrm{i}\pi^{(1)}(p^0,\vec{p})$,

$-\mathrm{i}\pi^1(p^0,\vec{p}) =$ 

with an intermediate Coulomb propagator, could provide an additional contribution to the mass of the magnetic mode. Because of the intermediate Coulomb propagator $\mathrm{i}/|\vec{p}|^2$, we could expect that $-\mathrm{i}\pi^{(1)}(0,\vec{0}) = \mathrm{Cst} \neq 0$, so that it gives rise to a pole in the dynamical mass as $|\vec{p}| \to 0$. However, this is not the case. The diagram is of the form

$$-\mathrm{i}\pi^{(1)}(p^0,\vec{p}) = (-\mathrm{i}\kappa(p^0,\vec{p}))\frac{\mathrm{i}}{|\vec{p}|^2}(-\mathrm{i}\kappa(p^0,\vec{p})), \tag{4.156}$$

where the first order in the expansion in $|\vec{p}|$ and p^0 can be found thanks to

$$\kappa(p^0, \vec{p}) \approx |\vec{p}|^2 \frac{e^2}{4\pi} \kappa_0, \tag{4.157}$$

with the numerical coefficient given by the quadrature

$$\kappa_0 = \int_0^{+\infty} \ell \mathrm{d}\ell \; Z^2(\ell) \frac{m(\ell) - \ell m'(\ell)/2}{\omega_0^3(\ell)} \approx 0.58. \tag{4.158}$$

Defining $\pi^{(1)}(p^0, \vec{p}) = |\vec{p}|^2 \pi_0^{(1)}(p^0, \vec{p})$, we find the contribution to the mass of this diagram to be

$$\pi_0^{(1)}(0, \vec{0}) \approx 0.34 \Big(\frac{e^2}{4\pi}\Big)^2. \tag{4.159}$$

We may find the approximate value of the mass of the magnetic mode by summing the contributions coming from the two diagrams considered, i.e. $M^2(0) \approx 0.48(e^2/4\pi)^2$.

4.9 Conclusions

Thanks to the factorization of local gauge transformations and of gauge degrees of freedom, as well as the dressing of the fermion field, the dynamics of massless QED_{2+1} with one flavour of electrons could be reduced to the interaction of a dressed fermion field with a physical magnetic scalar mode. The decomposition of the gauge field and the factorization of the local gauge symmetry rendered manifest the relevance of the gauge invariant magnetic scalar, understood as the only propagating gauge invariant electromagnetic degree of freedom.
In the fermionic sector, a ground state of the BCS type was shown to be energetically more favourable than an "empty" Fock state. Furthermore, the wave function of the pair condensate was found by solving an integral equation, including non-perturbatively the effects of Coulomb interactions. As a result, the pseudo-particle excitations above the condensate, namely the constituent fermions, exhibit a peculiar dispersion relation, with a divergent behaviour at low momentum, being a signature

for the confinement of charged states. This interpretation was confirmed by the study of the energy of a bound state of two of these constituent fermions.
Due to pair condensation, parity symmetry is spontaneously broken. Hence, the propagation of the magnetic mode excitations is affected by the interactions with the pair condensate. Starting from the non-perturbative result for the ground state, we decided to expand in perturbation the effects of the residual Coulomb interactions and the interactions between the magnetic mode and the fermion current. Although the complete loop calculation seems to be too involved, the corrections to the magnetic mode propagator from the first relevant diagrams indicate the dynamical generation of a mass for the magnetic mode.

Among the drawbacks of the variational approach used here, the difficulty to evaluate the accuracy of the implied approximation is a disadvantage. In contradistinction to a perturbative treatment, no power expansion in a small parameter is performed to obtain the ground state. It is the form of the pair condensate state which dictates the form of the integral equation to be solved. Hence, in order to improve the reliability of the approximation, the flexibility of the ansatz wave function could be increased. As a perspective, it would be instructive to study the possibility of a condensation of magnetic modes, in interaction with condensed fermion pairs. This idea has been explored in a recent work in the case of QCD_{3+1}, in a "quenched" approximation of QCD_{3+1} [53]. Due to the factorization of the local gauge symmetry, the formulation used in this work has lost manifest Lorentz covariance, although it remains covariant under spatial translations and rotations. It is challenging to understand how the equations are changed under a Lorentz boost. We leave this analysis for a further work. Nevertheless, one conclusion seems to have been established definitely by the present work. The well-known exact solution to the Schwinger model, namely massless QED_{1+1}, shows that as soon as the gauge coupling constant is turned on however small its value, massless quantum electrodynamics in two spacetime dimensions is not a theory of interacting (and gauge non invariant) electrons and photons, but rather is a theory of a (gauge invariant) free massive pseudoscalar particle, namely essentially the electric field. Likewise massless quantum electrodynamics in three spacetime dimensions with a non vanishing gauge coupling constant however small its value, is not a theory of interacting (and gauge non invariant) electrons and photons, but rather is a theory of a (gauge invariant) massive magnetic mode scalar interacting with (gauge invariant) neutral paired electron-positron states. Furthermore, parity is spontaneously broken dynamically, while charged states cannot be separed at large distances and remain confined in the neutral paired electron-positron states.

CHAPTER 5

Fermion condensation in $2+1$ dimensions in a constant magnetic field

5.1 Introduction

The aim of this chapter is to describe the influence of an external homogeneous magnetic field on massless fermions in $2+1$ dimensions. As we have seen in the last chapter, because of the attractive interactions, a condensate of particle/anti-particle pairs is energetically more favourable than the perturbative vacuum in massless QED_{2+1} in the absence of a magnetic field. Furthermore, it is also expected from the literature that under the effect of a constant magnetic field, the ground state charge density is non vanishing. The pairing structure of the vacuum becomes therefore non-trivial (for a study with two fermion flavours see [64]). In the presence of one flavour of massive fermions, a vacuum charge density is induced, as is well-known in the context of fermion fractionization [65, 66], whereas similar conclusions were obtained by computing the effective action in QED_{2+1} [67–69]. However, the case of a massless fermion is intriguing. Indeed, if we compute the vacuum charge density for a massive fermion, the charge density will only be proportional to the sign of the fermion mass but otherwise independent of the value of that mass. Hence, the case of a massless fermion remains ambiguous. It is natural to wonder if there exists a quantisation where the ground state charge density is vanishing in some way. In order to clarify this situation, we shall study the features of the lowest energy state in the presence of one massless fermion flavour, in the Hamiltonian formalism.

The outline of the chapter is as follows. First, section 5.2 presents the classical solutions to the Dirac equation for one flavour of electrons in the external field. The relevant features of the spectrum, such as the asymmetry, will be emphasized. Next, the quantisation of the fermion field will be performed in section 5.3. The intuitive picture about the form of the Dirac spectrum in the absence of a magnetic field is shown to be modified by the "zero-energy" sector. Furthermore, this zero-energy level may contain excitations corresponding to the quantum analogue of electrons at rest. We will take this opportunity to detail their interpretation in terms of coherent states.
Subsequently, section 5.4 will address the issue of the gauge invariance of the vacuum state in the case of massless fermions, leading to a specific study of the vacuum structure in section 5.5. Finally, results will be summarised in section 5.6.

Before introducing the quantum model, we shall firstly provide a brief description of the dynamics of a classical spinless particle in a constant magnetic field. The understanding gained from the classical analysis will guide our intuition for the quantum dynamics.

5.1.1 Classical particle in an homogeneous magnetic field

The preliminary study of the classical dynamics of a non-relativistic pointlike charged particle will render manifest some of the features of the quantum version of the problem. Following references [70, 71], we consider the dynamics as given by the Lagrangian

$$L = \frac{1}{2} m \dot{\vec{x}}^2 + e \dot{\vec{x}}.\vec{A}(\vec{x}) \tag{5.1}$$

where the symmetric gauge, $A^i(\vec{x}) = -\frac{B}{2}\epsilon_{ij}x^j$ is chosen for the vector potential. Among the constants of motion, the conserved quantities, related to the space translation symmetry,

$$q_i = m\dot{x}^i - eB\epsilon_{ij}x^j = p_i - \frac{eB}{2}\epsilon_{ij}x^j \tag{5.2}$$

are the components of the analogue conserved momentum, while the canonical momenta conjugate to the x^i's are

$$p_i = m\dot{x}^i - \frac{eB}{2}\epsilon_{ij}x^j. \tag{5.3}$$

This observation will be crucial in the analysis of the quantum version of the model where the distinction between these two definitions of momenta is important in order to define properly the operators associated to spatial translations. The simple features of the classical translation group will be complicated in the quantum model, due to the definition of the canonical momenta p_i.
Concerning the classical equations of motion, their solutions are easily found to correspond to the cyclotron orbits of frequency $\omega_c = B/m$ given by

$$x^1(t) = \frac{q_2}{eB} + R\cos\omega_c(t - t_0), \tag{5.4}$$

$$x^2(t) = -\frac{q_1}{eB} - R\sin\omega_c(t - t_0), \tag{5.5}$$

where R and t_0 are integration constants. The conserved quantities q_i are indeed related to the spatial coordinates of the "guiding center" of the cyclotron orbits, which are obviously constants of motion. This is somehow surprising because there are built as conserved quantities associated to magnetic translations. The conserved energy[1] of the classical solution

$$E = \frac{m}{2}\dot{\vec{x}}^2 = \frac{m}{2}R^2\omega_c^2, \tag{5.6}$$

[1] It is well-known that the Lorenz force does not work.

is constant in time and depends only on the parameters describing the radius of the cyclotron motion. Trivially the state of minimal energy $E = 0$ is given by an electron at rest, placed arbitrarily on the plane. On the contrary, the minimal energy states in the relativistic version of the quantum model have an extremely rich structure.

The energy spectrum will be quantised in Landau levels in the quantum theory. As a matter of fact, the position of the center of the orbit does not change the value of the energy. Hence we may expect a degeneracy related to the position of the magnetic center in the spectrum of the quantum model, which is indeed a classical interpretation of the physical reason for the infinite degeneracy in each Landau level. In the circular gauge, a conserved quantity is the angular momentum, itself given by the sum of two terms of opposite signs

$$L_{\rm ang} = x^1 p_2 - x^2 p_1 = \frac{1}{2eB}(q_1^2 + q_2^2) - \frac{eB}{2} R^2, \tag{5.7}$$

where we can notice that the coordinates of the magnetic center position

$$r_1^{\rm MC} = q_2/eB, \quad r_2^{\rm MC} = -q_1/eB \tag{5.8}$$

explicitely contribute to the conserved angular momentum with a positive sign, while the radius of the cyclotron orbit contributes with a negative sign. Hence the conserved angular momentum may be expressed as a two-dimensional wedge product

$$L_{\rm ang} = \vec{r}^{\rm MC} \wedge \vec{q} - m\omega_c R^2, \tag{5.9}$$

where $\vec{q}$ plays the role of a momentum of the magnetic center in the first term and where the second term is the expression of the angular momentum of a particle on a circular orbit with angular frequency ω_c. This feature finds an analogue in the quantum version of the model, where these two terms will be separately quantised in integer units.

Finally whereas it is noticeable that the classical equations of motion do not change if we perform another gauge choice, the definition of the angular momentum is however not invariant under a gauge transformation. The next section will analyse the case of a massive spin 1/2 particle in an homogeneous magnetic field, in the framework of the Dirac equation, so that the model will also involve anti-particles.

5.2 Solutions to the Dirac equation in a magnetic field

After the review of the classical motions, the quantum dynamics of a single Dirac fermion can be analysed. In order to express the Dirac equation, a choice of gauge is necessary to account for the presence of a constant magnetic field. Because of the absence of an electric field, we have immediately $A^0 = 0$. The experience gained by solving the "ordinary" non-relativistic Landau problem [72] tells us to prefer the so-called symmetric gauge to the Landau gauge $A^1(\vec{x}) = -Bx^2$ and $A^2(\vec{x}) = 0$. The latter leads to a lack of normalisability and localizability of the states, which renders the physical interpretation difficult. On the contrary, the symmetric gauge $A^1(\vec{x}) = -\frac{B}{2}x^2$ and $A^2(\vec{x}) = \frac{B}{2}x^1$ with $B > 0$, leading to normalizable and localized wave functions, circumvents the difficulties of interpretation. The Dirac equation

$$[\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu) - m]\psi = 0 \tag{5.10}$$

is formulated with the following representation $\gamma^0 = \sigma^3$ and $\gamma^i = \mathrm{i}\sigma^i$ for $i = 1, 2$. In the sequel all indices are euclidian. We consider a static solution $\psi(t, \vec{x}) = \phi(\vec{x})\exp\,(-\mathrm{i}Et)$. Hence the equation takes the form

$$M\phi(\vec{x}) = 0$$

with

$$M = \begin{pmatrix} (E-m) & -\mathrm{i}(\hat{p}_1 + \frac{eB}{2}\hat{x}^2) - (\hat{p}_2 - \frac{eB}{2}\hat{x}^1) \\ -\mathrm{i}(\hat{p}_1 + \frac{eB}{2}\hat{x}^2) + (\hat{p}_2 - \frac{eB}{2}\hat{x}^1) & -(E+m) \end{pmatrix}$$

where we denoted for simplicity $\hat{p}_i = -\mathrm{i}\partial/\partial x^i$. Because we expect to obtain the eigenstates thanks to an algebraic procedure, we introduce the same definitions for the chiral Fock operators as in the ordinary Landau problem,

$$\hat{a}_i = \tfrac{1}{2}\sqrt{eB}(\hat{x}^i + \tfrac{2\mathrm{i}}{eB}\hat{p}_i), \quad \hat{a}_i^\dagger = \frac{1}{2}\sqrt{eB}(\hat{x}^i - \frac{2\mathrm{i}}{eB}\hat{p}_i) \tag{5.11}$$

$$\hat{a}_\pm = \tfrac{1}{\sqrt{2}}(\hat{a}_1 \mp \mathrm{i}\hat{a}_2), \qquad \hat{a}_\pm^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger \pm \mathrm{i}\hat{a}_2^\dagger) \tag{5.12}$$

verifying independent bosonic Fock algebras: $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ and $[\hat{a}_\pm, \hat{a}_\pm^\dagger] = 1$, while all the other commutators vanish[2]. The algebras can be represented thanks to the normalized Fock states $|n_+, n_-\rangle = |n_+\rangle \otimes |n_-\rangle$, where $|n_\pm\rangle = \frac{(\hat{a}_\pm^\dagger)^{n_\pm}}{\sqrt{n_\pm !}}|0\rangle$. The eigenvalue problem can be reformulated as

$$\begin{pmatrix} (E-m) & \sqrt{2Be}\ a_-^\dagger \\ -\sqrt{2Be}\ a_- & -(E+m) \end{pmatrix} \phi(\vec{x}) = 0. \tag{5.13}$$

Considering the case of positive mass $m > 0$ and magnetic field $B > 0$, we find that the normalizable eigenstates are of two types. The solutions in the symmetric spectrum are

$$\phi_{n_+,n_-}^{(m)}(\vec{x}) = N_E \begin{pmatrix} \frac{\sqrt{2Be(n_-+1)}}{-E+m} \langle x^1, x^2 | n_- + 1, n_+\rangle \\ \langle x^1, x^2 | n_-, n_+\rangle \end{pmatrix} \tag{5.14}$$

with the eigenvalues satisfying $E^2 - m^2 - 2Be(n_- + 1) = 0$, and where N_E is a normalisation factor. The single solution which is "unpaired" is

$$\phi_{n_+}^{(0)}(\vec{x}) = \begin{pmatrix} \langle x^1, x^2 | 0, n_+\rangle \\ 0 \end{pmatrix}, \tag{5.15}$$

and has positive eigenvalue $E = m$. Of course, if we change the sign of the mass and take $m < 0$, the unpaired solution has a negative energy $E = -|m|$. We observe a discrete infinite degeneracy in $n_+ = 0, 1, 2, \ldots$ in both the cases $E \neq 0$ and $E = 0$. These infinitely degenerate levels can be interpreted as "relativistic Landau levels". A crucial observation is that there is an asymmetry in the spectrum of the Dirac operator in the constant magnetic field. In the sequel, only the case $m = 0$ will be analysed.

Because of their infinite degeneracy in $n_+ = 0, 1, 2, \ldots$, each Landau level is itself a Hilbert space spanned by the orthonormal basis of (localized) solutions given above. Any state in a given Landau level may be expanded in series in the orthonormal basis. The states with $n_- = 1, 2, \ldots$ and $n_+ = 0$ can be interpreted as a "fuzzy picture" of a classical electron

[2]Despite the similarity of notation, the operators $\hat{a}_{1,2}$ are not related to the ones of Chapter 3.

(or positron) making a cyclotron motion of quantised radius around the origin. Although the basis presented here is very convenient because of its physical interpretation, one may alternatively choose to construct another basis in each Landau level.

5.2.1 Massless limit

Since we are interested in the massless limit, we define the following positive and negative energy solutions

$$\langle x^1, x^2|\phi^{(\pm)}_{n_+,n_-}\rangle = \begin{pmatrix} \mp\langle x^1, x^2|n_- + 1, n_+\rangle \\ \langle x^1, x^2|n_-, n_+\rangle \end{pmatrix} \tag{5.16}$$

with the corresponding energies $E^{\pm}_{n_-} = \pm\sqrt{2Be(n_- + 1)}$, where n_+ and n_- are positive integers. Incidentally, this typical spectrum was measured experimentally, see for example [73, 74].
The mutually orthogonal solutions of the Dirac equation with non-zero energy are therefore

$$\phi^{(\pm)}_{n_+,n_-}(t, \vec{x}) = e^{-\mathrm{i}E^{\pm}_{n_-}t}\phi^{(\pm)}_{n_+,n_-}(\vec{x}) \tag{5.17}$$

with $||\phi^{(\pm)}_{n_+,n_-}(t, \vec{x})||^2 = 2$. Moreover, these classical solutions are eigenstates of the angular momentum operator

$$\hat{L} = x^1\hat{p}_2 - x^2\hat{p}_1 + \frac{1}{2}\gamma_0 = a^\dagger_+ a_+ - a^\dagger_- a_- + \frac{1}{2}\gamma_0. \tag{5.18}$$

Hence, the eigenfunctions of the total angular momentum, given by the sum of the orbital and spin angular momentum, are specified by

$$\hat{L}\langle x^1, x^2|\phi^{(\pm)}_{n_+,n_-}, t\rangle = (n_+ - n_- - \frac{1}{2})\langle x^1, x^2|\phi^{(\pm)}_{n_+,n_-}, t\rangle. \tag{5.19}$$

To be more explicit, the wave functions [72] can be conveniently expressed with the help of

$$\langle x^1, x^2|n_+, n_-\rangle = \frac{(-1)^m}{\sqrt{2\pi}}\sqrt{\frac{eBm!}{(m+|l|)!}}u^{|l|/2}e^{\mathrm{i}l\theta}e^{-u/2}L^{|l|}_m(u), \tag{5.20}$$

which are orthonormal functions where $l = n_+ - n_-$, $m = \min(\mathrm{n}_-, \mathrm{n}_+)$, and $L_m^{|l|}(u)$ are the generalized Laguerre polynomials, while

$$u = \frac{eB}{2}\vec{x}^2, \qquad e^{\mathrm{i}\theta} = \frac{x^1 + \mathrm{i}x^2}{\sqrt{\vec{x}^2}}. \tag{5.21}$$

Normalizable zero-modes

It is noteworthy that normalizable zero energy eigenstates exist. More presicely, the infinitely degenerate set of orthonormal functions

$$\phi^{(0)}_{n_+}(\vec{x}) = \left(\begin{array}{c} \langle x^1, x^2 | 0, n_+ \rangle \\ 0 \end{array} \right), \tag{5.22}$$

for $n_+ = 0, 1, 2, \ldots$, spans the "lowest Landau level" (LLL). Their infinite degeneracy is related to the infinite magnetic flux through the plane. In fact, their wave function correspond to concentric orbits, mutually orthogonal. Nonetheless, the circular orbits are not the quantum mechanical equivalents of the classical cyclotron motion of an electron in a magnetic field.
Following the article [66], we remark that the matrix γ_0 commutes with the Hamiltonian in the absence of a mass term. Therefore it plays the role of a conjugation matrix. That is to say, the static non zero energy states are related by conjugation

$$\gamma_0 \phi^{(\pm)}_{n_+, n_-}(\vec{x}) = -\phi^{(\mp)}_{n_+, n_-}(\vec{x}), \tag{5.23}$$

emphasing that a state of energy E_{n_-} can be sent to a state the opposite energy $-E_{n_-}$. Contrary to the case of the non zero levels, the zero-modes are self-conjugate

$$\gamma_0 \phi^{(0)}_{n_+}(\vec{x}) = \phi^{(0)}_{n_+}(\vec{x}). \tag{5.24}$$

This peculiarity will have noticeable consequences in the (second) quantised theory.

5.2.2 Magnetic centers and magnetic translations

Because of the external magnetic field, and the gauge choice necessary to write the minimal coupling, spatial translation invariance takes a particular form. In order to keep the dynamics unchanged, a space translation should be accompanied by a gauge transformation. The generators corresponding to this simultaneous transformation are

$$\hat{T}_1 = \hat{p}_1 - \frac{eB}{2}x^2 = \mathrm{i}\sqrt{\frac{eB}{2}}(a_+^\dagger - a_+), \tag{5.25}$$

$$\hat{T}_2 = \hat{p}_2 + \frac{eB}{2}x^1 = \sqrt{\frac{eB}{2}}(a_+^\dagger + a_+), \tag{5.26}$$

and commute with the Hamiltonian: $[\hat{T}_i, \hat{H}] = 0$, for $i = 1, 2$. Their exponential realizes the "non-infinitesimal" magnetic translations on the wave functions. Indeed, the finite transformation, associated to the translation $x^i \to x^i + a^i$, for $i = 1, 2$ leaving the dynamics invariant is

$$\psi(x^1, x^2) \to e^{-\mathrm{i}a^1 x^2 eB/2} e^{\mathrm{i}a^2 x^1 eB/2} \psi(x^1 + a^1, x^2 + a^2). \tag{5.27}$$

Because of the combined translation and gauge transformation, magnetic translations do not commute with each other $[\hat{T}_1, \hat{T}_2] = \mathrm{i}eB$.

In order to understand the effects of the magnetic translators, it can be also useful to introduce the magnetic center coordinates (see for example [71]). To do so, we write

$$x^1 = \frac{1}{\sqrt{2eB}}(a_- + a_-^\dagger + a_+ + a_+^\dagger), \tag{5.28}$$

$$x^2 = \frac{\mathrm{i}}{\sqrt{2eB}}(a_-^\dagger - a_- + a_+ - a_+^\dagger). \tag{5.29}$$

Hence we decompose the positions as the sum of two contributions

$$x^1 = \hat{x}^1 + \hat{\eta}^1, \quad x^2 = \hat{x}^2 + \hat{\eta}^2, \tag{5.30}$$

where the magnetic center coordinates are

$$\hat{x}^1 = \frac{a_+ + a_+^\dagger}{\sqrt{2eB}}, \qquad \hat{x}^2 = \frac{\mathrm{i}}{\sqrt{2eB}}(a_+ - a_+^\dagger) \tag{5.31}$$

and the cyclotron coordinates which describe the relative motion around the guiding center are

$$\hat{\eta}^1 = \frac{a_- + a_-^\dagger}{\sqrt{2eB}}, \qquad \hat{\eta}^2 = \frac{\mathrm{i}}{\sqrt{2eB}}(a_-^\dagger - a_-). \tag{5.32}$$

They satisfy $[\hat{x}^1, \hat{x}^2] = -\mathrm{i}/eB$ and $[\hat{\eta}^1, \hat{\eta}^2] = \mathrm{i}/eB$, where $1/\sqrt{eB} = \ell_B$ can be interpreted as a magnetic length. As an immediate consequence, an uncertainty principle renders a simultaneous measurement of the coordinates of the magnetic center impossible, while the "typical" best possible accuracy is given by the magnetic length. More explicitely, one can translate the magnetic center using the magnetic translation operator, as we can guess from the commutation relations: $[\hat{x}^i, \hat{T}_j] = \mathrm{i}\delta_{ij}$.

By the way, we can consider an alternative to span the LLL, taking advantage of the structure of the translation group on the physical plane. Since the $\phi^{(0)}_{n_+}(\vec{x})$ are the eigenfunctions of the angular momentum in the LLL, other wave functions may be built in order to be eigenfunctions of the translation operators. Strikingly, this construction may only be achieved in a specific way that we detail here briefly.
The set of canonical coherent states

$$\langle \vec{x} | z \rangle = e^{-|z|^2/2} \sum_{n_+=0}^{\infty} \frac{z^{n_+}}{\sqrt{n_+!}} \phi^{(0)}_{n_+}(\vec{x}) \tag{5.33}$$

forms a complete set of wavefunctions, which are somehow the eigenstates of the "magnetic center position operator". By construction, they are the eigenstates of the operator $\hat{Z} = (\hat{x}^1 + \mathrm{i}\hat{x}^2)/\sqrt{2} = \ell_B a_+$,

$$\hat{Z}\langle \vec{x} | z \rangle = z \ell_B \langle \vec{x} | z \rangle. \tag{5.34}$$

The coherent states $\langle \vec{x} | z \rangle$ have a localized magnetic center, representing "fat" electrons, while their "thickness" is approximately given by $\ell_B = 1/\sqrt{eB}$. However they are not mutually orthogonal[3], so that they may not be used conveniently to perform a mode expansion of the "second quantised" fields. Being aware of this subtlety, we can proceed to the quantisation of the Dirac field.

[3] In a condensed matter context, reference [75] constructed a set of orthogonal and localized wave functions. Constructing the relationship between this formulation and the coherent states exposed here is an interesting and unresolved question.

5.3 Quantisation and mode expansion

Quantisation should realize the equal time anti-commutator algebra

$$\{\psi(t,\vec{x});\psi^\dagger(t,\vec{y})\} = \delta^{(2)}(\vec{x}-\vec{y}). \tag{5.35}$$

To do so we introduce naturally the expansion as the sum over the modes

$$\psi(\vec{x},t) = \sum_{n_+=0}^{\infty} c_{n_+}\phi^{(0)}_{n_+}(\vec{x})$$
$$+\sum_{n_+,n_-=0}^{\infty}\frac{1}{\sqrt{2}}[b_{n_+,n_-}e^{-\mathrm{i}E_{n_-}t}\phi^{(+)}_{n_+,n_-}(\vec{x}) + d^\dagger_{n_+,n_-}e^{\mathrm{i}E_{n_-}t}\phi^{(-)}_{n_+,n_-}(\vec{x})]$$

of the spinor field and its adjoint as given by hermitian conjugaison

$$\psi^\dagger(\vec{x},t) = \sum_{n_+=0}^{\infty} c^\dagger_{n_+}\phi^{(0)*}_{n_+}(\vec{x})$$
$$+\sum_{n_+,n_-=0}^{\infty}\frac{1}{\sqrt{2}}[b^\dagger_{n_+,n_-}e^{\mathrm{i}E_{n_-}t}\phi^{(+)*}_{n_+,n_-}(\vec{x}) + d_{n_+,n_-}e^{-\mathrm{i}E_{n_-}t}\phi^{(-)*}_{n_+,n_-}(\vec{x})],$$

where, due to the orthonormality and completeness properties of the classical solutions, the fermionic oscillators satisfy independent Fermionic Fock algebras

$$\{c_{n_+};c^\dagger_{n'_+}\} = \delta_{n_+,n'_+}, \tag{5.36}$$

$$\{b_{n_+,n_-};b^\dagger_{n'_+,n'_-}\} = \delta_{n_+,n'_+}\delta_{n_-,n'_-} = \{d_{n_+,n_-};d^\dagger_{n'_+,n'_-}\}, \tag{5.37}$$

while all the other anti-commutators vanish. Due to the conjugation properties of the non zero energy modes, we can very simply represent their algebra $b_{n_+,n_-}|0\rangle = d_{n_+,n_-}|0\rangle = 0$ and interpret the excitations created by $b^\dagger_{n_+,n_-}$ and $d^\dagger_{n_+,n_-}$ as particles and anti-particles respectively. The representation of the algebra for the zero-modes is not completely straightforward and will be the topic of the next section.

5.4 Gauge invariance and fractionization

5.4.1 Ordering prescription

In order to represent the algebra of the zero-modes, the two possible extremal states verify

$$c_{n_+}|\Omega_-\rangle = 0, \quad c^\dagger_{n_+}|\Omega_+\rangle = 0, \tag{5.38}$$

for $n_+ \in \mathbb{N}$. The generator of $U(1)$ global rotations is the total charge, ordered in the following way

$$\hat{Q} = \int \mathrm{d}^2x^i \; o[\chi^\dagger(x)\chi(x)] \tag{5.39}$$

$$= \sum_{n_+,n_-=0}^{\infty} (b^\dagger_{n_+,n_-} b_{n_+,n_-} - d^\dagger_{n_+,n_-} d_{n_+,n_-}) \tag{5.40}$$

$$+ \sum_{n_+=0}^{\infty} \frac{1}{2}(c^\dagger_{n_+} c_{n_+} - c_{n_+} c^\dagger_{n_+}) \tag{5.41}$$

which generates global $U(1)$ transformations

$$e^{-\mathrm{i}\alpha\hat{Q}}\psi(t,\vec{x})e^{\mathrm{i}\alpha\hat{Q}} = e^{\mathrm{i}\alpha}\psi(t,\vec{x}). \tag{5.42}$$

An argument supporting the above ordering prescription follows from the literature [65,66,76,77]. Going from the classical theory to the quantum theory, we have to choose an ordering prescription. The prescription denoted by $o[\chi^\dagger(x)\chi(x)]$ allows actually to write the operator without the need to subtract an infinite quantity. Hence, it is sufficient to remark that

$$o[\chi^\dagger(x)\chi(x)] = \frac{1}{2}\Big(\chi^\dagger(x)\chi(x) - \chi(x)\chi^\dagger(x)\Big) \tag{5.43}$$

implicitly eliminates the annoying infinite contribution, as was suggested by Jackiw [66,76], as well as Semenoff and Niemi [65,77]. Remarkably this definition has the advantage to order the operator in the zero-mode sector as well as in the non zero-mode sector. It is interesting to notice that charge conjugaison exchanges the particles and anti-particles, namely $b_{n_+,n_-} \to d_{n_+,n_-}$, $d_{n_+,n_-} \to b_{n_+,n_-}$ and $c_{n_+} \leftrightarrow c^\dagger_{n_+}$. As a consequence, the charge conjugate of $\hat{Q}$ is exactly given by $-\hat{Q}$, which is what is required. This is not true if we choose another ordering prescription than (5.43). In the non-zero mode sector, the subtraction realized by (5.43) could be interpreted, from a condensed matter point of view, as the subtraction of the contribution of the positive ions to the charge charge density so that the particle-hole symmetric point is "neutral".

Concentrating on the zero energy level, the fermion number operator is ordered in the sector n_+ as $N_{n_+} = \frac{1}{2}(c^\dagger_{n+}c_{n+} - c_{n+}c^\dagger_{n+})$, and we observe the well-known fermion number fractionization $N_{n_+}|\Omega_-\rangle = -\frac{1}{2}|\Omega_-\rangle$ and $N_{n_+}|\Omega_+\rangle = \frac{1}{2}|\Omega_+\rangle$. In the sequel we will make the abuse to write $|\Omega_-\rangle$ for the tensor product $|\Omega_-\rangle \otimes |0\rangle$, where $|0\rangle$ is chosen in order to obey $b_{n_+,n_-}|0\rangle = d_{n_+,n_-}|0\rangle = 0$.
As another consequence of the ordering prescription, the charge of the zero energy states is infinite

$$\hat{Q}|\Omega_-\rangle = -\infty|\Omega_-\rangle, \qquad \hat{Q}|\Omega_+\rangle = +\infty|\Omega_+\rangle \tag{5.44}$$

The result is that we cannot choose among the states of zero energy $|\Omega_-\rangle$ and $|\Omega_+\rangle$, in order to find a ground state. Thus, the vacuum charge induced by the magnetic field is infinite, because there is an infinite degeneracy in the lowest energy level. For instance, we can choose to compute the charge density in $|\Omega_-\rangle$ by the formula

$$\begin{aligned}\langle\rho(\vec{x})\rangle_{\Omega_-} &= \langle\Omega_-|o[\chi^\dagger(x)\chi(x)]|\Omega_-\rangle \\ &= \frac{-1}{2}\sum_{n_+=0}^{\infty}\frac{eB}{2\pi n_+!}[\frac{eB}{2}\vec{x}^2]^{n_+}\exp\,[-\frac{eB}{2}\vec{x}^2] \\ &= -\frac{eB}{4\pi}\end{aligned}$$

while the induced charge in $|\Omega_+\rangle$ is $\langle\rho(\vec{x})\rangle_{\Omega_+} = eB/4\pi$. The upshot is that we recover the famous result [65, 66]

$$\langle j^0(0,\vec{x})\rangle = \pm\frac{1}{2}\frac{eB}{2\pi}. \tag{5.45}$$

Due to the selfconjugation property of the zero-modes

$$\gamma_0\phi^{(0)}_{n_+}(\vec{x}) = \phi^{(0)}_{n_+}(\vec{x}) \tag{5.46}$$

the order parameter $\bar{\psi}\psi$ takes a non vanishing value, that is to say

$$\langle\bar{\psi}(\vec{x})\psi(\vec{x})\rangle_{\Omega_+} = \langle\psi^\dagger(\vec{x})\psi(\vec{x})\rangle_{\Omega_+} = eB/4\pi. \tag{5.47}$$

The reason is that only the zero-modes contribute to the expectation value.

5.4.2 Interpretation of the induced charge density

The above results were obtained in the absence of a mass term. Let us study the case $m \neq 0$. In the presence of a mass term, the unpaired Landau level has a non vanishing energy, $E = m$. If we consider the case $m > 0$, the unpaired level belongs to the set of "positive energy states". Following the usual prescription of the "Dirac sea", the negative energy states should be filled and the positive energy states should stay empty. The consequence is that we have to choose $|\Omega_-\rangle$ as the vacuum state with charge density $-eB/4\pi$. In the opposite situation $m < 0$, the unpaired level has a negative energy and should therefore be filled by the Dirac sea. Hence the vacuum state should be $|\Omega_+\rangle$, with a charge density $eB/4\pi$. In the limit $m \to 0$, the ambiguity for the choice of the vacuum state arises due to the sign ambiguity of m.

In the case $m \neq 0$, the results of the literature suggest to add to the action an abelian Chern-Simons term $\mathcal{L} \ni \frac{\kappa}{2}\frac{e^2}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$. Choosing the Coulomb gauge, we can define $A_i = \epsilon_{ij}\partial_j\Phi$. As a result, the Gauss law is modified

$$\kappa\frac{e^2}{4\pi}\Delta\Phi(x) + e\chi^\dagger(x)\chi(x) = 0 \tag{5.48}$$

where $\Delta\Phi = B$. According to the sign of m, one can choose the parameter $\kappa = \pm 1$ so that the ground state can obey the Gauss law.

5.4.3 Induced angular momentum

A straightforward corollary of the induced charge density concerns the angular momentum. If the lowest energy state is for instance chosen to be $|\Omega_-\rangle$, there is an induced angular momentum density [78, 79] in this lowest energy level. Because of the infinite degeneracy, the total angular momentum of $|\Omega_-\rangle$ is infinite. The Noether theorem gives the conserved angular momentum density

$$\ell^0(\vec{x}) = \frac{\mathrm{i}}{2}\epsilon^{ij}x^j(\chi^\dagger\partial_i\chi - \partial_i\chi^\dagger\chi) + \frac{1}{2}\chi^\dagger\gamma_0\chi \tag{5.49}$$

In the $E = 0$ sector, the spin angular momentum is 1/2, so that it is convenient to define $L = a_+^\dagger a_+ - a_-^\dagger a_- + 1/2$. The quantum operator associated to the angular momentum density is

$$\hat{\ell}^0(\vec{x})|\Omega_-\rangle = -\frac{1}{2}\frac{1}{2\pi\ell_B^2}\Big\{\frac{1}{2} + \frac{\vec{x}^2}{2\ell_B^2}\Big\}|\Omega_-\rangle. \tag{5.50}$$

These comments concludes the disgression.

5.5 The level $E = 0$ in the massless case

This section is devoted to the study of the structure of the zero energy level, in the case of massless fermions. The discussion begins with some preliminary remarks.

In this section, the wave functions which are the solutions to the Dirac equation are always formulated in position space. Because it is not convenient for our purposes we do not go to momentum space. Although canonical coherent states play an ubiquitous role, we do not use them to provide a representation of the functional space constituting the $E = 0$ level. Nonetheless we will define states in the quantum Hilbert space of states which are analogues of the coherent states. To avoid possible confusions and ambiguities, the states in the quantum Hilbert space will be denoted by “kets” $|\ \rangle$, while it can be sometimes more appropriate to denote a function in the space of zero energy solutions of a Dirac equation by $(\vec{x}|f)$. We also recall the definition of the magnetic length $\ell_B = \sqrt{1/eB}$ in natural units.

5.5.1 Mode expansion in the orthonormal basis

Because we restrict ourselves to the $E = 0$ level, we introduce more convenient notations in order to reduce the complexity of the expressions. As we know, the spinor and its adjoint, projected in the $E = 0$ level, can be expanded in modes

$$\psi_0(\vec{x}) = \sum_{n=0}^{\infty} c_m\phi_m(\vec{x}), \quad \psi_0^\dagger(\vec{x}) = \sum_{n=0}^{\infty} c_n^\dagger\phi_n^*(\vec{x}) \tag{5.51}$$

where $\phi_n(\vec{x}) = (\vec{x}|n)$ is the Fock space basis of solutions in the $E = 0$ energy level, and where $\{c_n; c_m^\dagger\} = \delta_{n,m}$. The orthonormal basis of "ring shaped" x-space functions in the $E = 0$ level, corresponding to the upper component of (5.22), are given as

$$\phi_n(\vec{x}) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{eB}{n!}}[\sqrt{\frac{eB}{2}}(x^1 + \mathrm{i}x^2)]^n \exp\,[-\frac{1}{4}eB\vec{x}^2], \tag{5.52}$$

where $\vec{x}$ is associated to the physical spatial coordinate of the point where the wave function is evaluated. Because the set of functions chosen for the mode expansion is an orthonormal *basis*,

$$\int \mathrm{d}^2x^i \phi_n^*(\vec{x})\phi_m(\vec{x}) = \delta_{n,m}, \tag{5.53}$$

the adjoint of the fermionic field coincides with its conjugate.

As a result of the projection in the level $E = 0$, the anti-commutation relations of the fields are

$$\{\psi_0(\vec{x}); \psi_0^\dagger(\vec{y})\} = \frac{1}{2\pi\ell_B^2}e^{-\frac{1}{2}|z_{\vec{x}}-z_{\vec{y}}|^2}e^{\frac{1}{2}(\bar{z}_{\vec{x}}z_{\vec{y}}-\bar{z}_{\vec{y}}z_{\vec{x}})} \neq \delta^{(2)}(\vec{x}-\vec{y}) \tag{5.54}$$

where $z_{\vec{x}} = \frac{x^1-\mathrm{i}x^2}{\sqrt{2}\ell_B}$ and $z_{\vec{y}} = \frac{y_1-\mathrm{i}y_2}{\sqrt{2}\ell_B}$. We consider the charge density:

$$\begin{aligned}\rho(\vec{x}) &= (\psi_0^\dagger(\vec{x})\psi_0(\vec{x}) - \psi_0(\vec{x})\psi_0^\dagger(\vec{x}))/2 & (5.55)\\ &= \sum_{n,m=0}^{\infty}[c_m^\dagger c_n - \frac{1}{2}\delta_{m,n}]\phi_m^*(\vec{x})\phi_n(\vec{x}), & (5.56)\\ &= \psi_0^\dagger(\vec{x})\psi_0(\vec{x}) - \frac{1}{2}\frac{1}{2\pi\ell_B^2}. & (5.57)\end{aligned}$$

As a matter of fact, the term $-1/4\pi\ell_B^2 = -eB/4\pi$ is the average charge density of the vacuum state $|\Omega_-\rangle$, while the expected charge density of $|\Omega_+\rangle$ is merely the opposite. Hence we can easily evaluate, for example, the average charge density of $c_k^\dagger|\Omega_-\rangle$, thanks to

$$\langle\Omega_-|c_k\psi_0^\dagger(\vec{x})\psi_0(\vec{x})c_k^\dagger|\Omega_-\rangle = |\phi_k(\vec{x})|^2, \tag{5.58}$$

which is the probability density, centered at $\vec{x} = 0$, associated to the presence of an electron zero-mode. The property $\{\psi_0(\vec{x}), c_k^\dagger\} = \phi_k(\vec{x})$, which is straightforward to establish, gives the "wave function" associated to the one-particle state

$$\langle \Omega | \psi_0(\vec{x}) c_k^\dagger | \Omega \rangle = \phi_k(\vec{x}). \tag{5.59}$$

We should emphasize that these wave functions are orthogonal and are localized in space, namely $\phi_0(\vec{x})$ has a gaussian profile whereas for $n > 0$ they are ring-shaped.

5.5.2 Peculiarities of the state space in the $E = 0$ level

In the mode expansion introduced above, the point of coordinate $\vec{x} = 0$ seems to play a specific role. The natural question of translation invariance could be raised. We could want to introduce functions which are the (magnetic) translations in space of the wave function $(\vec{x}|0)$. To do so, we recall that a coherent state in the space of solutions $E = 0$ is given by

$$\begin{aligned}
(\vec{x}|z) &= e^{-|z|^2/2} \sum_{k=0}^{+\infty} \frac{z^k}{\sqrt{k!}} \phi_k(\vec{x}) && (5.60)\\
&= \frac{\sqrt{eB}}{2\pi} e^{-\frac{1}{2}|z|^2 + \sqrt{\frac{eB}{2}}(x^1 + \mathrm{i}x^2) z} e^{-\frac{1}{4} eB \vec{x}^2} && (5.61)\\
&= \frac{1}{2\pi \ell_B} e^{-\frac{1}{2}|\bar{z} - \frac{x^1 + \mathrm{i}x^2}{\sqrt{2}\ell_B}|^2} e^{\frac{1}{2}(z \frac{x^1 + \mathrm{i}x^2}{\sqrt{2}\ell_B} - \bar{z} \frac{x^1 - \mathrm{i}x^2}{\sqrt{2}\ell_B})}, && (5.62)
\end{aligned}$$

where the complex variable $z = \frac{y_1 - \mathrm{i}y_2}{\sqrt{2}\ell_B}$ is related to the guiding center coordinate $\vec{y}$. We can also notice that the wave function $\phi_n(\vec{x})$ is the following overlap

$$\phi_n(\vec{x}) = \frac{1}{\sqrt{2\pi}\ell_B} (z_{\vec{x}}|n) \tag{5.63}$$

where $z_{\vec{x}} = \frac{x^1 - \mathrm{i}x^2}{\sqrt{2}\ell_B}$. The set of canonical coherent states is overcomplete. They are properly normalized to satisfy a resolution of the identity in the $E = 0$ level

$$\int \frac{\mathrm{d}^2 z}{\pi} |z)(z| = 1. \tag{5.64}$$

If we wish to obtain a state $|Z\rangle$ whose charge density is the probability density associated to the coherent state $(\vec{x}|z)$, we can define the state

$$|Z\rangle = c^\dagger(z)|\Omega\rangle = e^{-|z|^2/2}\sum_{k=0}^{+\infty}\frac{z^k}{\sqrt{k!}}c_k^\dagger|\Omega\rangle. \tag{5.65}$$

It should be noted that the wave function associated to the quantum state $|Z\rangle$ is the x-space representation of the coherent state $|z)$, i.e.

$$\langle\Omega|\psi_0(\vec{x})|Z\rangle = (\vec{x}|z), \tag{5.66}$$

which is interpreted as the wave function, evaluated at a point of coordinate $\vec{x}$, of a localized electron zero-mode centered at the "guiding center" in the plane associated to the complex number z in magnetic length units. Hence the probability density of such a state is

$$\langle Z|\psi_0^\dagger(\vec{x})\psi_0(\vec{x})|Z\rangle = |(\vec{x}|z)|^2, \tag{5.67}$$

with a mean charge density $\langle\rho(\vec{x})\rangle_Z = |(\vec{x}|z)|^2 - eB/4\pi$.

In order to realize the translation in the plane of the localized electron states, it is useful to introduce the "creator of an electron at the guiding center coordinate z" ,

$$c^\dagger(z) = e^{-|z|^2/2}\sum_{k=0}^{+\infty}\frac{z^k}{\sqrt{k!}}c_k^\dagger. \tag{5.68}$$

It should be noted that $c^\dagger(z)$ is not an holomorphic function of z. The creator and the associated annihilator verify the anti-commutation relation $\{c(z), c^\dagger(z')\} = (z|z') = \exp\,(-|z|^2/2 - |z'|^2/2 + \bar{z}z')$. This means that the states created by $c^\dagger(z)$ and $c^\dagger(z')$ for $z \neq z'$ are *correlated*. The correlation decays on a typical length given by the magnetic length.
In order to relate the operators $\psi_0(\vec{x})$ and $c(z)$, we can formulate a series of remarks. Firstly, these operators verify $\{\psi_0(\vec{x}), c^\dagger(z)\} = (\vec{x}|z)$, where the complex number $z\ell_B = (y_1 - \mathrm{i}y_2)/\sqrt{2}$ can be associated to the coordinate of the guiding center $\vec{y}$ in the physical plane. Secondly, as an interesting consequence of the previous definitions, we may rewrite the fields as expansions in the set of coherent states

$$\psi_0(\vec{x}) = \int\frac{\mathrm{d}^2z}{\pi}c(z)(\vec{x}|z),\ \psi_0^\dagger(\vec{x}) = \int\frac{\mathrm{d}^2z}{\pi}c^\dagger(z)(z|\vec{x}). \tag{5.69}$$

The conclusion is that the field $\psi_0^\dagger(\vec{x})$ is exactly associated to the creator of a coherent state

$$\sqrt{2\pi}\ell_B \psi_0^\dagger(\vec{x}) = c^\dagger(z_{\vec{x}}). \tag{5.70}$$

This explains the anti-commutation relations between $\psi_0(\vec{x})$ and its adjoint (5.54).

Since the states in the $E = 0$ level seem to be localized in space, the way the translation symmetry is realized in this level is not self-evident. In analogy with the definition of the magnetic translation operators in the previous section, the following annihilation and creation operator are introduced

$$a = \sum_{k=1}^{+\infty} \sqrt{k} c_{k-1}^\dagger c_k, \quad a^\dagger = \sum_{k=1}^{+\infty} \sqrt{k} c_k^\dagger c_{k-1}, \tag{5.71}$$

verifying $[a, a^\dagger] = \hat{N}$ where the number operator $\hat{N} = \sum_{n=0}^{\infty} c_n^\dagger c_n$ commutes with a and its adjoint[4]. As a consequence of our definitions we find the expected property

$$c^\dagger(z) = e^{-\bar{z}a + za^\dagger} c_0^\dagger \, e^{\bar{z}a - za^\dagger}. \tag{5.72}$$

The composition of two magnetic translation operators is

$$e^{-\bar{w}a + wa^\dagger} e^{-\bar{z}a + za^\dagger} = e^{-\frac{1}{2}(\bar{w}z - \bar{z}w)\hat{N}} e^{-(\bar{z}+\bar{w})a + (z+w)a^\dagger}, \tag{5.73}$$

where the magnetic flux through the surface of the parallelogram drawn in the plane by z and w is responsible for the "cocycle" factor in the composition law.
Because of the non-commutative nature of spatial translations, the issue of translational invariance is subtle. If the translation vectors are chosen among a well designed lattice, they may commute with themselves. This question is closely related to the issue of choosing a "minimal" (complete) set of coherent states among the continous set of canonical coherent states. We can define a lattice made of the combinaisons $z_{nm} = n\omega_1 +$

[4]The operator a defined here is obviously different from the operator of the same name in Chapter 1.

$m\omega_2$ where the surface of the parallelogram drawn in the complex plane is $S = \mathrm{Im}\bar{\omega}_1\omega_2 = \pi$. The spatial lattice is associated to the so-called von Neumann lattice $|z_{n,m})$ of canonical coherent states [80]. The physical surface of the fundamental cell of the lattice is $2\pi\ell_B^2$, because the complex variable associated to the position of the guiding center of coordinate $\vec{y}$ is $z_{\vec{y}} = (y_1 - \mathrm{i}y_2)/(\sqrt{2}\ell_B)$.

We can reformulate the translation operators in order to "hide" the projective structure (of the representation) of the magnetic group. To do so, we notice that $\mathrm{Im}(\bar{z}_{n_1 n_2} z_{m_1 m_2}) = \pi(n_1 m_2 + n_2 m_1)$. The "improved" translation operators of a lattice vector can be defined, following [81], by

$$D(n,m) = (-1)^{(n+m+mn)\hat{N}} e^{-\bar{z}_{nm}a + z_{nm}a^\dagger}, \tag{5.74}$$

where the additional phase factor is included to appropriately account for the Aharomov-Bohm effect due to the magnetic field flux through the elementary surface, so that the composition law is naturally realized

$$D(n,m)D(k,\ell) = D(n+k, m+\ell). \tag{5.75}$$

The operator $\hat{N} = \sum_{n=0}^{+\infty} c_n^\dagger c_n$ is the number operator. Because the classical continuous group of spatial translations is reduced (or "broken") to a discrete group, we may expect momentum space to have a periodic structure. The eigenstates of the translation operators in the one particle state sector can be built easily. Let us introduce the "Fourier transformation"

$$c^\dagger(\theta^1,\theta^2) = \sum_{n_1,n_2} e^{\mathrm{i}(n_1\theta^1 + n_2\theta^2)} D(n_1,n_2) c_0^\dagger D^\dagger(n_1,n_2) \tag{5.76}$$

with the property of transforming covariantly under the lattice translations

$$D(k_1,k_2)c^\dagger(\theta^1,\theta^2)D^\dagger(k_1,k_2) = e^{-\mathrm{i}(k_1\theta^1 + k_2\theta^2)} c^\dagger(\theta^1,\theta^2). \tag{5.77}$$

The one particle state

$$|\theta^1,\theta^2\rangle = c^\dagger(\theta^1,\theta^2)|\Omega_-\rangle \tag{5.78}$$

is an eigenstate of the translation operator, with $(\theta^1,\theta^2) \in [0,2\pi[\times[0,2\pi[$, which plays the role of a Brillouin zone. This is true because the extremal state $|\Omega_-\rangle$ is invariant under the action of the translation operator, that is to say, $D(k_1, k_2)|\Omega_-\rangle = |\Omega_-\rangle$. The result is that the state space becomes similar to the one of a solid state problem.

The following natural question may be raised: Is there a (one particle) state that is invariant under the discrete lattice translations, when the lattice is the minimal lattice? The expected candidate is $|\theta^1 = 0, \theta^2 = 0\rangle$. Actually this candidate state vanishes identically due to the relation

$$\sum_{n,m} (-1)^{n+m+nm} |z_{n,m}) = 0 \tag{5.79}$$

where the coherent states $|z_{n,m})$ form a von Neumann lattice [81, 82]. This "no-go" property is a non trivial consequence of the analyticity properties of the coherent states. As a conclusion, we may state that there is no "one particle translationally invariant" state in the $E = 0$ level, when the lattice of translation vectors has a cell area of $2\pi\ell_B^2$.

5.5.3 Gauge invariant vacua

Since the states $|\Omega_-\rangle$ and $|\Omega_+\rangle$ have a non vanishing expected charge density, the states created over these extremal states will not be gauge invariant, because they are charged. We shall analyse the possibility to build a set of gauge invariant states. In order to define the physical state space[5], the required condition is that any physical states $|\phi\rangle$ and $|\psi\rangle$ should verify

$$\langle\phi|\hat{Q}|\psi\rangle = 0. \tag{5.80}$$

The first step is to define a "vacuum state". Undoubtebly, the most obvious states with an homogeneous charge density, in expectation value, are $|\Omega_-\rangle$ and $|\Omega_+\rangle$, where

$$c_n^\dagger|\Omega_+\rangle = 0 = c_n|\Omega_-\rangle \tag{5.81}$$

[5] It is somehow in that sense that gauge invariant states are constructed in bosonic string theory

for any $n \in \mathbb{N}$. In particular, they are also the only eigenstates of the charge density. However it should not be forgotten that a linear combination of these two states has also an homogeneous charge density (in expectation value). This clue suggests to define the following normalized combinations:

$$|\pm\rangle = \Big(|\Omega_-\rangle \pm |\Omega_+\rangle\Big)/\sqrt{2}, \tag{5.82}$$

satisfying $\langle +|-\rangle = 0$. Because of the property $\rho(x)|\Omega_\pm\rangle = \pm\rho_0|\Omega_\pm\rangle$, with $\rho_0 = 1/4\pi\ell_B^2$, we have

$$\rho(x)|\pm\rangle = -\rho_0|\mp\rangle. \tag{5.83}$$

As a consequence of the last definitions, the property

$$\langle\pm|\rho(x)|\pm\rangle = 0, \qquad \langle -|\rho(x)|+\rangle = -\rho_0 \neq 0, \tag{5.84}$$

implies that we need to choose among the two orthonormal vacua in order to build a physical state space, because $|+\rangle$ and $|-\rangle$ cannot belong together to the set of the physical states.

5.5.4 Construction of the physical state space

The purpose is to build states verifying

$$\langle\phi|\rho(x)|\psi\rangle = 0, \tag{5.85}$$

or simply $\langle\phi|\hat{Q}|\psi\rangle = 0$. Therefore, we expect to create neutral excitations on the states $|\pm\rangle$. To follow this intuition, the following hermitian operators are defined

$$\gamma_k = c_k + c_k^\dagger, \quad \tilde{\gamma}_k = \frac{c_k - c_k^\dagger}{\mathrm{i}}, \tag{5.86}$$

satisfying $\{\gamma_k, \gamma_l\} = 2\delta_{k,l} = \{\tilde{\gamma}_k, \tilde{\gamma}_l\}$ and $\{\gamma_k, \tilde{\gamma}_l\} = 0$. For simplicity, the excitations on the vacuum $|+\rangle$ will be first considered. In order to get an indication of the significance of the γ operators, we compute

$$\gamma_k|+\rangle = \Big(c_k|\Omega_+\rangle + c_k^\dagger|\Omega_-\rangle\Big)/\sqrt{2}, \tag{5.87}$$

so that we observe that the state obtained involves a symmetric mixture of $|\Omega_+\rangle$ and $|\Omega_-\rangle$.

From the symmetry point of vue, charge conjugation is represented by a unitary operator $\mathcal{C}$ which performs the transformation $b \to d$ and $c \leftrightarrow c^\dagger$, and flips the sign of the magnetic field at the same time $B \to -B$. We deduce that the action of the charge conjugaison on the γ_k and $\tilde{\gamma}_k$ is

$$\mathcal{C}\tilde{\gamma}_k\mathcal{C}^\dagger = -\tilde{\gamma}_k, \quad \mathcal{C}\gamma_k\mathcal{C}^\dagger = \gamma_k, \tag{5.88}$$

while we have $\mathcal{C}|\pm\rangle = \pm|\pm\rangle$.

Acting with an arbitrary sequence of γ operators, we expect that the states obtained this way

$$|+\rangle, \quad \gamma_k|+\rangle, \quad \gamma_k\gamma_l|+\rangle, \quad \ldots \tag{5.89}$$

and their linear combinations, are physical.

Firstly, we notice the following matrix elements of the charge density vanish

$$\langle +|\rho(x)\gamma_k|+\rangle = 0 = \langle +|\gamma_k\rho(x)\gamma_k|+\rangle, \tag{5.90}$$

$$\langle +|\gamma_k\rho(x)\gamma_k\gamma_l|+\rangle = 0 = \langle +|\gamma_l\gamma_k\rho(x)\gamma_k\gamma_l|+\rangle, \tag{5.91}$$

for $k \neq l$, and so on. As it can be checked thanks to a straightforward calculation, this is true because, by construction, the diagonal matrix elements of the charge density have to vanish, while the matrix elements involving states with a different number of γ's give merely zero. However the matrix elements of the charge density operators in the states (5.89) do not always vanish. Indeed we can also compute[6]

$$\langle +|\gamma_k\rho(x)\gamma_l|+\rangle = \frac{1}{2}\Big(\phi_k^*(x)\phi_l(x) - \phi_l^*(x)\phi_k(x)\Big), \tag{5.92}$$

so that, due to the orthogonality of the functions $\phi_n(x)$, the expectation value of the charge operator vanishes

$$\langle +|\gamma_k\hat{Q}\gamma_l|+\rangle = \int \mathrm{d}^2x\langle +|\gamma_k\rho(x)\gamma_l|+\rangle = 0. \tag{5.93}$$

[6]To do so, the identity $[\rho(x); c_k^\dagger] = \phi_k(x)\sum_{n=0}^{+\infty}\phi_n^*(x)c_n^\dagger$ can be useful.

Similarly, we find

$$\begin{aligned}&\langle +|\gamma_m\gamma_n\rho(x)\gamma_k\gamma_l|+\rangle\\ &= \delta_{n,k}[\phi_l(x)\phi_m^*(x) - \phi_l^*(x)\phi_m(x)] - \delta_{m,k}[\phi_l(x)\phi_n^*(x) - \phi_l^*(x)\phi_n(x)]\\ &+\delta_{m,l}[\phi_k(x)\phi_n^*(x) - \phi_k^*(x)\phi_n(x)] - \delta_{n,l}[\phi_k(x)\phi_m^*(x) - \phi_k^*(x)\phi_m(x)],\end{aligned}$$

for $n \neq m$ and $k \neq l$, so that

$$\int \mathrm{d}^2x \langle +|\gamma_m\gamma_n\rho(x)\gamma_k\gamma_l|+\rangle = 0. \tag{5.94}$$

Because we want to prove this feature more generally, we use the following result: in fact, the (off-diagonal) matrix element of an arbitrary sequence of γ's vanishes in the states $|+\rangle$ and $|-\rangle$, i. e.

$$\langle +|\gamma_{n_1}\dots\gamma_{n_N}|-\rangle = 0, \tag{5.95}$$

as a consequence of the charge conjugation properties of $|+\rangle$ and $|-\rangle$. Namely, it is directly shown making use of $[\hat{Q},\gamma_k] = -\mathrm{i}\tilde{\gamma}_k$, and thanks to the properties $\rho(x)|+\rangle = -\rho_0|-\rangle$ and $\tilde{\gamma}_k|+\rangle = \mathrm{i}\gamma_k|-\rangle$.

This means that acting with the an arbitrary sequence γ's on the state $|+\rangle$ and taking all the linear combinations, we obtain the physical states. On the other hand, we may wonder if this set of states

$$|+\rangle, \quad \gamma_k|+\rangle, \quad \gamma_k\gamma_l|+\rangle, \quad \dots \tag{5.96}$$

which are mutually physical, is a maximal set. We should notice that, acting with one $\tilde{\gamma}$ on $|+\rangle$, we do not get a physical state, because

$$\langle +|\gamma_k\rho(x)\tilde{\gamma}_k|+\rangle = \mathrm{i}|\phi_k(x)|^2. \tag{5.97}$$

On the other hand, it is direct to show that a state with an even number of $\tilde{\gamma}$'s does not leave the physical state space. This is a simple consequence of

$$\tilde{\gamma}_k\tilde{\gamma}_l|+\rangle = -\gamma_k\gamma_l|+\rangle + 2\delta_{k,l}|+\rangle, \tag{5.98}$$

which is straightforwardly generalized to an arbitrary even number of $\tilde{\gamma}$'s. So it is also possible to show that acting with an odd number of $\tilde{\gamma}$'s on $|+\rangle$, we never obtain a physical state. The reason is the relation

$$\tilde{\gamma}_k|+\rangle = \mathrm{i}\gamma_k|-\rangle, \quad \tilde{\gamma}_k|-\rangle = \mathrm{i}\gamma_k|+\rangle. \tag{5.99}$$

This means that a state with an arbitrary number of γ's and one $\tilde{\gamma}$ acting on $|+\rangle$, can be written as a sequence of γ's acting on $|-\rangle$.

As a conclusion, we may classify the mutually gauge invariant states according to their parity under $\mathcal{C}$.

Let us focus first on the parity even case. Using the relations (5.99), we can show that the space spanned by the states of the same parity of same number of γ (or $\tilde{\gamma}$) excitations are equal, so that the state space is classified in the following sectors

$$|+\rangle \tag{5.100}$$
$$\text{span}\{\gamma_i|+\rangle\} = \text{span}\{\tilde{\gamma}_i|-\rangle\}, \tag{5.101}$$
$$\text{span}\{\gamma_{i_1}\gamma_{i_2}|+\rangle\} = \text{span}\{\tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}|+\rangle\}, \tag{5.102}$$
$$\text{span}\{\gamma_{i_1}\gamma_{i_2}\gamma_{i_3}|+\rangle\} = \text{span}\{\tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}\tilde{\gamma}_{i_3}|-\rangle\}, \tag{5.103}$$
$$\cdots$$

where the indices $i_1, i_2, \dots$ are mutually different. The space of states generated by the linear combinations of states belonging to the subspaces with a fixed number of γ (or $\tilde{\gamma}$) excitations is gauge invariant.

Similarly, in the parity odd sector, we can show that the state space is divided in

$$|-\rangle \tag{5.104}$$
$$\text{span}\{\gamma_i|-\rangle\} = \text{span}\{\tilde{\gamma}_i|+\rangle\}, \tag{5.105}$$
$$\text{span}\{\gamma_{i_1}\gamma_{i_2}|-\rangle\} = \text{span}\{\tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}|-\rangle\}, \tag{5.106}$$
$$\text{span}\{\gamma_{i_1}\gamma_{i_2}\gamma_{i_3}|-\rangle\} = \text{span}\{\tilde{\gamma}_{i_1}\tilde{\gamma}_{i_2}\tilde{\gamma}_{i_3}|+\rangle\}, \tag{5.107}$$
$$\cdots$$

where the indices $i_1, i_2, \dots$ are mutually different.

It is likely that the inclusion of the effects of the Coulomb interaction will select among the possible states $|+\rangle$ and $|-\rangle$. Indeed, the other excitations have a higher interaction energy.

5.6 Conclusions

In the case of many fermion flavours in $2+1$ dimensions, it has been argued that a constant magnetic field induces a vacuum pairing structure of the Nambu–Jona-Lasinio type models. This phenomenon of magnetic pairing causes a spontaneous chiral symmetry breaking, as explained in [83, 84].

The study presented here investigated the case of only one fermion flavour. It has been shown that the structure of the lowest energy level of fermions in a constant magnetic field is rich. Spatial translations are realized in the quantum theory projectively, so that the formulation of a trial state analogous to a BCS pair condensate is non trivial. Therefore, an analysis of the vacuum state in the presence of a constant magnetic field, parallel to the study of Chapter 4, may not be straightforwardly performed.

When considering a massive fermion field, as it is well-known, a non-vanishing charge density is induced in the ground state. This conclusion may be reached by the computation of the effective action in the functional formalism [67–69]. We observed that, in order to reconcile the theory with the gauge symmetry, a Chern-Simons term may be added to the Lagrangian. In the functional formalism, the non-vanishing vacuum charge density also induces the presence of a Chern-Simons term in the effective action.
In this chapter, the analysis was restricted to the lowest Landau level in Hamiltonian formalism. In the case of a massless fermion, it was shown that a state with vanishing charge density, in expectation value, can be designed. Mutually gauge invariant states were constructed, relying on the definition of hermitian fermionic operators. The operators built here from a superposition of a creator and annihilator are similar to the Majorana fermion operators [85], which are of interest in the context of superconductivity and in close relation with the representations of the braid group. In a certain class of superconductors, the Majorana zero-modes (or particles, for a review see [86]) are predicted to have interesting consequences. Considering a finite set of hermitian fermionic

operators $\{\gamma_i\}_{i=0,\dots,N-1}$, verifying $\{\gamma_i;\gamma_j\} = 2\delta_{ij}$, operators realizing the braid group are defined as

$$U_{j,j+1} = (1+\gamma_j\gamma_{j+1})/\sqrt{2}. \tag{5.108}$$

For instance, one may verify, $U_{0,1}\gamma_0 U_{0,1}^\dagger = -\gamma_1$ and $U_{0,1}\gamma_1 U_{0,1}^\dagger = \gamma_0$. The relation between the braid group and the study presented in this chapter is intriguing, tough speculative.

An interesting perspective of research is the inclusion of the Coulomb interactions which may lift the degeneracy in the zero-mode sector. Moreover, a pairing of excitations belonging to higher Landau level is also expected. Because the magnetic field controls the distance between the Landau levels, a pair condensate due to the Coulomb interaction will not be energetically favourable for a large magnetic field. On the other hand, the energy density of a condensate of particle/anti-particles, pairing among the non zero Landau levels, may be negative at low magnetic field, and therefore be the most favourable state. This question was already addressed by P. Cea in [87, 88], showing that the perturbative vacuum is unstable yielding the formation of a uniform condensate. As mentioned in [88], a vacuum state partially occupied by the zero-modes in the lowest Landau level may be designed. In the context of the fractional quantum Hall effect, there is an obvious interest for a state with an intermediate occupation between the empty vacuum state and the filled state. This observation raises the question of the Coulomb interactions between the zero-modes.
To be more precise, an interesting prospect consists in the study of a trial states of the type

$$|\theta,\phi\rangle = \prod_{n=0}^{+\infty}(\cos\theta_n + \sin\theta_n e^{\mathrm{i}\phi_n} c_n^\dagger)|\Omega_-\rangle, \tag{5.109}$$

with $0 \leq \theta_n \leq \pi$ and $0 \leq \phi_n < 2\pi$, where each factor of the product is a superposition of the empty and filled state. Because of the property

$$\langle\theta,\phi|\frac{1}{2}(c_k^\dagger c_k - c_k c_k^\dagger)|\theta,\phi\rangle = -\frac{1}{2}\cos 2\theta_k, \tag{5.110}$$

the states of the type $|\theta, \phi\rangle$ correspond to partially filled zero energy levels. The dynamics in the vacuum sector should determine the total vacuum charge density. This perspective deserves a more detailed study.

CHAPTER 6

Conclusions and perspectives

This thesis intended to explore with non-perturbative techniques the dynamics of models of low dimensional quantum electrodynamics.
After the general introduction, chapter 2 laid the foundations for the developements exposed about QED_{1+1}. In the context of the Schwinger model, chapter 3 highlighted the role of the topological sector of this gauge theory, and especially in relationship with the axial anomaly. The non-perturbative character of the large gauge transformations and their role for the construction of the full solution were made clear. The quantisation of the model followed a factorisation procedure, providing an approach devoid of gauge fixing. To say the least, the large gauge transformations have shown to be of crucial importance in order to establish the bosonisation. Let us emphasize again that the solution – a free massive pseudo-scalar boson – has its origin in the interactions between the fermions and the gauge field, so that the physical spectrum can be interpreted as the dynamics of a bound state of the fermions and the electric field in interaction. Undeniably, the massive boson does certainly not emanate only from the bosonization of the massless fermions. This is in

contrast to the well-known equivalence between the Sine-Gordon model and the $1+1$-dimensional Thirring model, where the "meson" of the first model is the bosonized fermion of the second, while the fermion can be understood as a coherent state of bosons. On the contrary, in the Schwinger model, the interactions responsible for the existence of the bound state are more complex and, somehow, they defy description.
As an afterthought, we should mention that an extension of the technique to $2+1$ dimensions was also considered during the thesis. Such a procedure was applied formally to QED_{2+1} on a spatial torus with periodic boundary conditions, and due to the technical difficulties, no significant results could be obtained. Furthermore, the decompactification limit of the theory defined on a compact space remains a momentous issue, mainly because the discrete momentum space has to become a continous space, this last procedure being ill defined from the mathematical viewpoint.
Although it seems to be a non trivial issue, a similar analysis could be undertaken in QCD_{3+1} on a manifold $\mathbb{R}\times\Sigma$, where Σ is a three-dimensional compact manifold.

As for chapter 4, the variational analysis of the ground state of massless QED_{2+1} was however successful, leading to a solution in the form of a pair condensate. The parity symmetry of the classical theory is dynamically broken due to the vacuum structure.
As a consequence of condensation in massless QED_{2+1}, constituent fermions could be identified with a dispersion relation modified at low momentum. The energy of a gauge invariant constituent fermion/antifermions pair was studied, leading to an interpretation in favour of their confinement. It stands to reason that the confinement property is mainly due to the collective fermion behaviour in the vacuum, namely a nonperturbative dynamical effect. Indeed, the account provided here goes beyond the classical observation of the confining nature of the logarithmic Coulomb potential, and gave a dynamical description of the classical intuition. Contrary to the approach of Polyakov, the research presented here focused on the dynamics in the matter sector, in the non-compact version of QED_{2+1}, and therefore provides complementary insights.

Furthermore, concerning the gauge sector, it appears here that the propagation of the transverse electromagnetic degree of freedom, namely the so-called magnetic mode, is affected by the condensate in such a way that a finite dynamical mass is generated. Incidentally, a major asset of QED_{2+1} is it excellent UV behaviour. As a result, no counterterm had to supplement the action in order to subtract the divergences. In a sense, it is certainly possible to envisage a similar study in a higher dimension theory but the effects of renormalisation should be included carefully.

Regarding the quality of the approximation, the investigation followed a variational procedure whose accuracy may not be in any simple way estimated, whereas the flexibility of the ansatz is the main constraint of the analysis. Among the questions requiring further research, the definition of the ansatz for the pair condensate may be, in all likelihood, improved. The inclusion of a magnetic mode condensate could refine the analysis. As for Lorentz covariance of the ansatz, it seems that a Lorentz boots acts on this state in a non trivial way. At the least, the solution found here provides a first approximation of a very rich vacuum structure.
Let us disgress a little about the perspectives of this work in condensed matter physics. From this viewpoint, the downside of the approach is that the Coulomb potential between the particles was considered as purely two-dimensional, namely corresponding to an interaction confined to the plane. In contrast, the Coulomb potential in a two dimensional material is not logarithmic but behaves rather like $1/|\vec{p}|$. On the contrary, the version of QED_{2+1} exposed here is closer to an effective theory of the high temperature superconductivity.

Finally, in chapter 5, the case of massless fermions confined in a plane under the influence of a constant magnetic field has been analysed. As mentioned before, it is an example of fractionization of the fermion number. Thanks to the work by Jackiw, Semenoff and Niemi, the induced charge density in the vacuum of QED_{2+1} with massive fermions is an established effect.
Dealing with the subtle case of massless fermions, in this work, the rich-

ness of the zero-energy state structure was unravelled. Before performing canonical quantisation, the classical field had to be expanded in an orthonormal basis of functions. When restricting the analysis to the zero-energy level, the choice of basis for the expansion raised the question of the realization of translational symmetry, since it appears that finding a basis “equivalent” to the plane wave basis is not straightforward. Hence, the choice was made to expand the fermion field in a set of orthonormal functions which are localized in x-space. The fermion field projected onto the zero-energy level was understood as a coherent state operator, interpreted as a maximally localized electron translated in the plane. Furthermore, at the quantum level, the fate of the classical translation group was to be “broken” to a discrete abelian translation group, while the full translation group is realized projectively.
In the investigation for a vacuum state preserved by the gauge symmetry, we proposed a set of mutually gauge invariant states, realized as entangled superpositions of “pure” states. Seemingly, due to the Coulomb interaction, only two such states are selected as minimizing the interaction energy, and are distinguished by their parity under chage conjugaison. Because of their construction, the charge density, in expectation value, of these two states vanishes.
As a matter of fact, the question of finding the lowest energy state in presence of the Coulomb interaction is very close to the issues arising in the descriptions of the various forms of the Quantum Hall Effect. However, a significant difference between the present work and the condensed matter procedures is that the solid state problem is always concerned with a finite number of pseudo-particles, in comparison with the infinite number of potential excitations of a quantum field theory. As a perspective, the developpement of an approach to the Quantum Hall Effect, similar to the quantum field theory formulation of the article [89], would be an appealing prospect.

APPENDIX A

Regularisation of divergent series

In order to extract finite contributions out of otherwise divergent quantities, some regularisation procedure is required, for which either a gaussian or a zeta function regularisation has been considered. The details of either regularisation leading to the results quoted in the main text are discussed in this Appendix[1]. For simplicity calculations are developed hereafter when the real variable a is non integer. Extending results to the case when $a \in \mathbb{Z}$ is discussed in the main text where appropriate.

[1]After completion of this work we realized that certain series obtained in this Appendix were already available in [90]. We thank Andreas Wipf for calling this reference to our attention.

A.1 Divergences in the charge operators

Gaussian regularisation

The Poisson resummation formula may be used to establish the relation,

$$\sum_{m=-\infty}^{+\infty} \Theta(m+a)e^{-\alpha(m+a)^2} \stackrel{\alpha\to 0}{=} \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} + \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n a}}{2\mathrm{i}\pi n}, \tag{A.1}$$

so that the subtraction of the short distance divergence consists in removing the term in $(1/2)\sqrt{\pi/\alpha}$.

To prove this result, one applies the Poisson resummation formula to the expression on the *lhs* of this relation, in terms of the function $f(x) = \Theta(x+a)\exp\left[-\alpha(x+a)^2\right]$ of which the Fourier transform is, where $k \in \mathbb{R}$,

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-\mathrm{i}kx}\Theta(x+a)e^{-\alpha(x+a)^2} = e^{\mathrm{i}ka}\, I^0_\alpha(k),$$

with the definition

$$I^0_\alpha(k) = \int_0^{+\infty} dx\ e^{-\mathrm{i}kx-\alpha x^2},$$

so that,

$$\sum_{m=-\infty}^{+\infty} \Theta(m+a)e^{-\alpha(m+a)^2} = \sum_{n=-\infty}^{+\infty} \tilde{f}(2\pi n). \tag{A.2}$$

Quite obviously $I^0_\alpha(0) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$, while for $k \neq 0$ the integral $I^0_\alpha(k)$ is expressed in terms of the parabolic cylinder function $D_{-1}(z)$ [91],

$$I^0_\alpha(k) = \frac{1}{\sqrt{2\alpha}} \exp\left(-\frac{k^2}{8\alpha}\right) D_{-1}\left(\frac{ik}{\sqrt{2\alpha}}\right). \tag{A.3}$$

Since the asymptotic behaviour of $D_{-1}(z)$ is known as $|z| \to +\infty$ [91], in the small α limit one finds that $I^0_\alpha(k)$ behaves such that for $n \neq 0$,

$$I^0_\alpha(2\pi n) \stackrel{\alpha\to 0}{=} \frac{1}{2\mathrm{i}\pi n}\left(1 + \mathcal{O}(\alpha)\right). \tag{A.4}$$

Consequently, one has established relation (A.1), with the further observation that the infinite series contribution on the *rhs* is the Fourier series of a simple function of a, when a is non integer,

$$\sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n a}}{2\mathrm{i}\pi n} = \sum_{n=1}^{+\infty} \frac{\sin(2\pi n a)}{\pi n} = \frac{1}{2} - (a - \lfloor a \rfloor). \tag{A.5}$$

Zeta function regularisation

A regularisation of the ζ function type[2] of the same infinite series takes the following form, with $\alpha > 0$ and in the limit $\alpha \to 0$,

$$\begin{aligned}\sum_{m=-\infty}^{+\infty} \Theta(m+a) e^{-\alpha(m+a)} &= e^{-\alpha(a+\lfloor -a \rfloor)} \left(\frac{1}{1-e^{-\alpha}} - 1 \right) \\ &= \frac{1}{\alpha} - (a - \lfloor a \rfloor) + \frac{1}{2} + \mathcal{O}(\alpha),\end{aligned}$$

when using $\lfloor -a \rfloor = -\lfloor a \rfloor - 1$ (which applies when a is non integer). Similarly,

$$\sum_{m=-\infty}^{+\infty} \Theta(-m-a) e^{\alpha(m+a)} = \frac{1}{\alpha} + (a - \lfloor a \rfloor) - \frac{1}{2} + \mathcal{O}(\alpha).$$

Hence either regularisation prescription produces the same finite contribution as a function of a from the divergent series $\sum_{m=-\infty}^{+\infty} \Theta(m+a)$.

[2]This is also the regularisation used in [10, 21].

A.2 Divergences in the bilinear fermion Hamiltonian

Gaussian regularisation

We need also to show that

$$\begin{aligned}
&\sum_{m=-\infty}^{+\infty} |m+a| \\
&\overset{\alpha\to 0}{=} \quad \sum_{m=-\infty}^{+\infty} (m+a)\big(\Theta(m+a)-\Theta(-m-a)\big)e^{-\alpha(m+a)^2} \\
&= \quad \frac{1}{\alpha} - 2\sum_{n=-\infty, n\neq 0}^{+\infty} \frac{e^{2\mathrm{i}\pi n a}}{(2\pi n)^2} + \mathcal{O}(\alpha).
\end{aligned} \tag{A.6}$$

To make use of the Poisson resummation formula consider the function

$$g(x) = \big(\Theta(x+a)-\Theta(-x-a)\big)(x+a)\exp\,[-\alpha(x+a)^2],$$

of which the Fourier transform is, with $k \in \mathbb{R}$,

$$\tilde{g}(k) = \exp\,(\mathrm{i}ka)(I_\alpha(k)+I_\alpha(-k)),$$

where

$$I_\alpha(k) = \int_0^{+\infty} dx\ x\ \exp\,(-\mathrm{i}kx)\exp\,(-\alpha x^2),$$

whose value is expressed in terms of yet another parabolic cylinder function [91],

$$I_\alpha(k) = \frac{1}{2\alpha}\Gamma(2)\exp\,(-\frac{k^2}{8\alpha})D_{-2}(\frac{\mathrm{i}k}{\sqrt{2\alpha}}).$$

Given the asymptotic behaviour of $D_{-2}(z)$ [91], for $n \neq 0$ one finds in the limit $\alpha \to 0$,

$$I_\alpha(n) \overset{\alpha\to 0}{=} -\frac{1}{n^2},$$

while for $n = 0$, $I_\alpha(0) = \frac{1}{2\alpha}$. In conclusion, one has established that

$$\sum_{m=-\infty}^{+\infty} g(m) = \sum_{n=-\infty}^{+\infty} \tilde{g}(2\pi n)$$
$$\overset{\alpha \to 0}{=} \frac{1}{\alpha} - 2\sum_{n=-\infty, n\neq 0}^{+\infty} \frac{\exp\ (2i\pi n a)}{(2\pi n)^2} = \frac{1}{\alpha} - (a - \lfloor a \rfloor - \tfrac{1}{2})^2 + \tfrac{1}{12}, \quad \text{(A.7)}$$

which is the relation in (A.6).

Zeta function regularisation

Using a ζ function regularisation leads to the same result, namely,

$$\sum_{m=-\infty}^{+\infty} |m + a|$$
$$\overset{\alpha \to 0}{=} \sum_{m=-\infty}^{+\infty} (m + a)(\Theta(m + a)e^{-\alpha(m+a)} - \Theta(-m - a)e^{\alpha(m+a)})$$
$$\overset{\alpha \to 0}{=} \frac{2}{\alpha^2} - (a - \lfloor a \rfloor - \frac{1}{2})^2 + \frac{1}{12}. \quad \text{(A.8)}$$

By defining

$$S_+ = \sum_{m=-\infty}^{+\infty} (m + a)\Theta(m + a)e^{-\alpha(m+a)},$$

one observes that,

$$S_+ = -\frac{\partial}{\partial \alpha}\left(e^{-\alpha(a+\lfloor -a \rfloor)}(\frac{1}{1 - e^{-\alpha}} - 1)\right),$$

of which a Laurent series expansion in α produces,

$$S_+ = \frac{1}{\alpha^2} - \frac{1}{2}(a - \lfloor a \rfloor - \frac{1}{2})^2 + \frac{1}{24}.$$

Similarly given

$$S_- = \sum_{m=-\infty}^{+\infty} (m + a)\Theta(-m - a)e^{\alpha(m+a)},$$

this quantity takes the form

$$S_- = -\frac{1}{\alpha^2} + \frac{1}{2}(a - \lfloor a \rfloor - \frac{1}{2})^2 - \frac{1}{24} = -S_+.$$

Hence indeed the relation (A.8) has been established.

APPENDIX B

Technical results in QED_{2+1}

B.1 The Hadamard finite part and the photon mass term

The Fourier transform of the x-space Green function is not a function but a distribution. It may be more convincing to obtain the Hadamard finite part in terms of a limiting case of a more intuitive situation. The naive $\frac{-1}{|\vec{p}|^2}$ infrared divergent p-space Green function can be regularised using a mass regulator. If one adds a mass term in the Green function in p-space, one finds the following x-space Green function

$$G_\mu(x,y) = \int \frac{\mathrm{d}^2 p^i}{(2\pi)^2} \frac{-1}{|\vec{p}|^2 + \mu^2} e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})} = -\frac{1}{2\pi} K_0(\mu|\vec{x}-\vec{y}|), \qquad \text{(B.1)}$$

where $K_0(\mu|\vec{x}-\vec{y}|)$ is a modified Bessel function of the second kind. The IR behaviour of $G_\mu(x,y)$ completely changes however small the value for μ is, as illustrated in Fig. B.1. Even for a very small μ, the “potential” $G_\mu(x,y)$ is no longer confining!

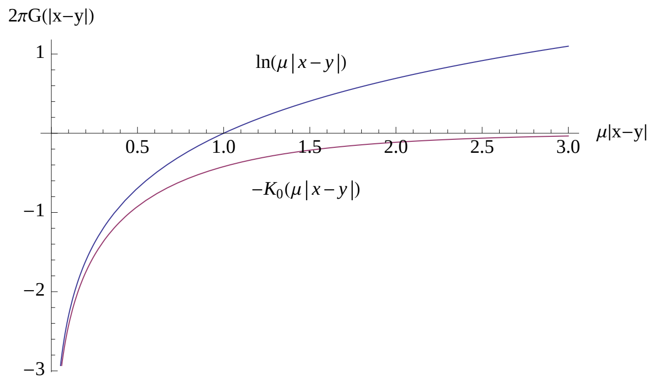


Figure B.1: The figure compares the behaviour of the x-space Green function in presence and absence of a mass term for the photon. The large distance behaviours are very different.

A brutal substitution $\mu = 0$ in the last Fourier transform gives us the naive Fourier transform of the Green function. However we know that the limit $\mu \to 0$ should be taken with care. Setting $\mu = 0$ barely makes sense. The reason for this is that when μ goes to zero, the integration in (B.1) still involves values of $\vec{p}$ with $|\vec{p}| < \mu$. In order to identify the divergence resulting from the limit $\mu \to 0$, one may clearly separate the safe regions of integration from the potentially divergent regions. To do so, one introduces $\epsilon > \mu$, which will be kept constant in the limit $\mu \to 0$. Hence we can rewrite

$$G_\mu(x, y) = I_1^\epsilon + I_2^\epsilon, \tag{B.2}$$

with

$$I_1^\epsilon = \int_0^\epsilon \frac{p\mathrm{d}p}{2\pi} \frac{-1}{p^2+\mu^2} J_0(p|\vec{x} - \vec{y}|), \tag{B.3}$$

$$I_2^\epsilon = \int_\epsilon^\infty \frac{p\mathrm{d}p}{2\pi} \frac{-1}{p^2+\mu^2} J_0(p|\vec{x} - \vec{y}|). \tag{B.4}$$

It is now straightforward to take the limit of the second term

$$\lim_{\mu\to 0} I_2^\epsilon = \int_\epsilon^\infty \frac{\mathrm{d}p}{2\pi} \frac{-1}{p} J_0(p|\vec{x} - \vec{y}|), \tag{B.5}$$

where J_0 is a Bessel function of the first kind. One may also consider the first term and extract its divergent contribution when $\mu \to 0$. Integrating it by parts one finds

$$I_1^\epsilon = \frac{1}{2\pi}(-\frac{1}{2}\ln\frac{\epsilon^2+\mu^2}{\epsilon^2}J_0(\epsilon|\vec{x}-\vec{y}|)+\ln\frac{\mu}{\epsilon})+ \tag{B.6}$$

$$+\int_0^\epsilon \frac{\mathrm{d}p}{2\pi}\frac{1}{2}\ln(\frac{p^2+\mu^2}{\epsilon^2})|\vec{x}-\vec{y}|J_1(p|\vec{x}-\vec{y}|), \tag{B.7}$$

where, as before J_1 denotes a Bessel function of the first kind. The second term in the last equation is perfectly convergent when $\mu \to 0$. We have succeeded in pinpointing the divergent contribution occuring when the mass goes to zero. It is now completely obvious that the behaviour of I_1^ϵ in the limit is

$$\lim_{\mu\to 0} I_1^\epsilon = \lim_{\mu\to 0}\frac{1}{2\pi}\ln\frac{\mu}{\epsilon}+\int_0^\epsilon \frac{\mathrm{d}p}{2\pi}\frac{1}{2}\ln(\frac{p^2}{\epsilon^2})|\vec{x}-\vec{y}|J_1(p|\vec{x}-\vec{y}|). \tag{B.8}$$

The only source of divergence is the term $\frac{1}{2\pi}\ln\frac{\mu}{\epsilon}$ that needs to be subtracted from I_1^ϵ to make sense of the limit. One notices also that the quantity that has to be added to I_1^ϵ to ensure the subtraction is

$$-\frac{1}{2\pi}\ln\frac{\mu}{\epsilon} = \int_\mu^\epsilon \frac{\mathrm{d}p}{2\pi}\frac{1}{p} = \int_0^\epsilon \frac{\mathrm{d}p}{2\pi}\frac{1}{p}\theta(p-\mu). \tag{B.9}$$

Adding this term to (B.3), and taking the limit, one finds

$$\begin{aligned}\lim_{\mu\to 0} I_1^\epsilon - \tfrac{1}{2\pi}\ln\tfrac{\mu}{\epsilon} &= \lim_{\mu\to 0}\int_0^\epsilon \left\{\frac{\mathrm{d}p}{2\pi}\frac{-p}{p^2+\mu^2}J_0(p|\vec{x}-\vec{y}|)+\frac{1}{p}\theta(p-\mu)\right\}\\ &= \int_0^\epsilon \frac{\mathrm{d}p}{2\pi}[\frac{-1}{p}J_0(p|\vec{x}-\vec{y}|)+\frac{1}{p}]\end{aligned} \tag{B.10}$$

Restoring now the angular integral by replacing the Bessel function by its integral representation, the final result of this procedure is

$$\lim_{\mu\to 0} G_\mu(x,y) - \frac{1}{2\pi}\ln\frac{\mu}{\epsilon} \tag{B.11}$$

$$= \int_{|\vec{p}|<\epsilon}\frac{\mathrm{d}^2p^i}{(2\pi)^2}\frac{-1}{|\vec{p}|^2}(e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})}-1)+\int_{|\vec{p}|>\epsilon}\frac{\mathrm{d}^2p^i}{(2\pi)^2}\frac{-1}{|\vec{p}|^2}e^{\mathrm{i}\vec{p}.(\vec{x}-\vec{y})}. \tag{B.12}$$

Hence in conclusion, the Hadamard finite part can indeed be interpreted as the limit of the Green function regularised with a mass term for the photon. The presence of the scale ϵ is unavoidable because it is essential to help us to make sense of the limit $\mu \to 0$ which is a limit of a dimensionful quantity. The scale ϵ is somehow a remnant of the mass term.

B.2 Matrix elements and contractions

Some useful matrix elements are

$$\langle\Psi|\chi^\dagger_\alpha(0,\vec{x})\chi_\beta(0,\vec{y})|\Psi\rangle = \int \frac{\mathrm{d}^2p^i}{2p^0}\Big[p^0 - (1-2|\beta(p)|^2)\gamma^0\vec{\gamma}.\vec{p} \tag{B.13}$$
$$-p^0\gamma^0\alpha(p)[\beta(p)+\beta^*(p)] + \vec{p}.\vec{\gamma}\alpha(p)[\beta(p)-\beta^*(p)]\Big]_{\beta\alpha}\frac{e^{-\mathrm{i}\vec{p}(\vec{x}-\vec{y})}}{(2\pi)^2},$$
$$\langle\Psi|\chi_\alpha(0,\vec{x})\chi^\dagger_\beta(0,\vec{y})|\Psi\rangle = \int \frac{\mathrm{d}^2p^i}{2p^0}\Big[p^0 + (1-2|\beta(p)|^2)\gamma^0\vec{\gamma}.\vec{p}$$
$$+p^0\gamma^0\alpha(p)[\beta(p)+\beta^*(p)] - \vec{p}.\vec{\gamma}\alpha(p)[\beta(p)-\beta^*(p)]\Big]_{\alpha\beta}\frac{e^{\mathrm{i}\vec{p}(\vec{x}-\vec{y})}}{(2\pi)^2}. \tag{B.14}$$

The contractions needed to compute the matrix elements of the normal ordered operators are

$$\chi^\dagger_\alpha(0,\widehat{\vec{x})\chi_\beta}(0,\vec{y}) = \int \frac{\mathrm{d}^2p^i}{2p^0}\Big[2|\beta(p)|^2\gamma^0\vec{\gamma}.\vec{p} \tag{B.15}$$
$$-p^0\gamma^0\alpha(p)[\beta(p)+\beta^*(p)] + \vec{p}.\vec{\gamma}\alpha(p)[\beta(p)-\beta^*(p)]\Big]_{\beta\alpha}\frac{e^{-\mathrm{i}\vec{p}(\vec{x}-\vec{y})}}{(2\pi)^2},$$
$$\chi_\alpha(0,\widehat{\vec{x})\chi^\dagger_\beta}(0,\vec{y}) = \int \frac{\mathrm{d}^2p^i}{2p^0}\Big[-2|\beta(p)|^2\gamma^0\vec{\gamma}.\vec{p} \tag{B.16}$$
$$+p^0\gamma^0\alpha(p)[\beta(p)+\beta^*(p)] - \vec{p}.\vec{\gamma}\alpha(p)[\beta(p)-\beta^*(p)]\Big]_{\alpha\beta}\frac{e^{\mathrm{i}\vec{p}(\vec{x}-\vec{y})}}{(2\pi)^2}.$$

B.3 Useful integrals

The following integrals have to be computed with great care:

$$\int \frac{\mathrm{d}\theta}{p^2+q^2-2pq\cos\theta} = \frac{2}{|p^2-q^2|}\mathrm{Atan}\Big\{\frac{\mathrm{p+q}}{|\mathrm{p-q}|}\tan\theta/2\Big\}, \tag{B.17}$$
$$\int \frac{\cos\theta\,\mathrm{d}\theta}{p^2+q^2-2pq\cos\theta} = \frac{1}{2pq}\Big\{-\theta + 2\frac{p^2+q^2}{|p^2-q^2|}\mathrm{Atan}[\frac{\mathrm{p+q}}{|\mathrm{p-q}|}\tan\frac{\theta}{2}]\Big\}, \tag{B.18}$$
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{p^2+q^2-2pq\cos\theta} = \frac{2\pi}{|p^2-q^2|}, \tag{B.19}$$
$$\int_0^{2\pi} \frac{\cos\theta\,\mathrm{d}\theta}{p^2+q^2-2pq\cos\theta} = \frac{2\pi}{2pq}\Big\{-1 + \frac{p^2+q^2}{|p^2-q^2|}\Big\}, \tag{B.20}$$

where the evaluation of the definite integrals takes into account the presence of a discontinuity in the corresponding primitives.

B.4 The self-energy contribution to the dispersion relation

At equation (4.101) we found an interesting result and provide here some details for its derivation. We had to evaluate the finite part of the problematic integral

$$\sigma(p) = \frac{e^2}{2}\mathcal{P}\int \frac{\mathrm{d}^2 q^i}{(2\pi)^2}\frac{\vec{p}.\vec{q}}{|\vec{p}||\vec{q}|}\frac{1}{(\vec{p}-\vec{q})^2}$$
$$= \tfrac{e^2}{2(2\pi)^2}\int_0^{+\infty}\mathrm{d}q\tfrac{1}{q}\left[\int_0^{2\pi}\mathrm{d}\theta\tfrac{p+q\cos\theta}{\sqrt{p^2+q^2+2pq\cos\theta}} - 2\pi H(\mu-q)\right]$$

where $H(x)$ is the Heaviside step function. Using

$$\frac{\partial}{\partial q}\left(\frac{p+q\cos\theta}{\sqrt{p^2+q^2+2pq\cos\theta}}\right) = \frac{-pq\sin^2\theta}{(p^2+q^2+2pq\cos\theta)^{3/2}} \tag{B.21}$$

and an integration by parts (with vanishing boundary terms), we find

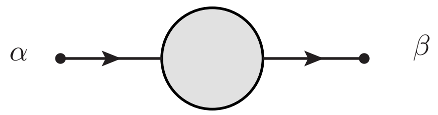
$$\sigma(p) = \frac{e^2}{4\pi}\ln\frac{c}{\mu} + \frac{e^2}{8\pi^2}\int_0^{+\infty}\mathrm{d}q\ \ln\frac{q}{c}\int_0^{2\pi}\mathrm{d}\theta\frac{pq\sin^2\theta}{(p^2+q^2+2pq\cos\theta)^{3/2}}$$

where c is an integration constant. One can first perform a change of variables $q = ps$ and then calculate the s-integral. The final result is a function of θ, which can be integrated from 0 to 2π. The integration constant simplifies, and the result is

$$\sigma(p) = \frac{e^2}{4\pi}[\ln(\frac{2p}{\mu}) + \ln 2 - 1].$$

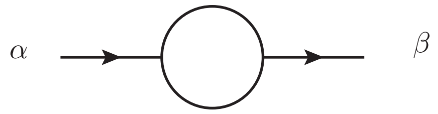
B.5 Feynman rules

The Feynman rules associated to the Schwinger-Dyson equations of Section 4.7.1 are:



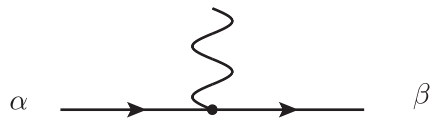
$$= S^{(3)}_{\alpha\beta}(p)$$

$$= S^{(3)}_0(p)_{\alpha\beta}$$



$$= -\mathrm{i}\Sigma_{\alpha\beta}(\vec{p})$$

$$= \mathrm{i}/|\vec{q}|^2$$



$$= \mathrm{i}e(\gamma^0)_{\alpha\beta}$$

where

$$S^{(3)}(p^0,\vec{p}) = \frac{\mathrm{i}}{\not{p} - \Sigma(p^0,\vec{p}) + \mathrm{i}\epsilon}, \tag{B.22}$$

$$S_0^{(3)}(p^0,\vec{p}) = \frac{\mathrm{i}}{\not{p} + \mathrm{i}\epsilon}, \tag{B.23}$$

$$\Sigma(p^0,\vec{p}) = |\vec{p}|A(|\vec{p}|) + \vec{p}.\vec{\gamma}B(|\vec{p}|). \tag{B.24}$$

B.6 Clifford-Dirac algebra

Up to unitary transformations, there exist two inequivalent irreducible representation of the Dirac algebra for 2-spinors. One is given by

$$\gamma^0 = \sigma_3,\ \gamma^1 = \mathrm{i}\sigma_1,\ \gamma^2 = \mathrm{i}\sigma_2 \tag{B.25}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.26}$$

and for the "mostly minus" signature of the Lorentzian metric. Obviously no chirality matrix exist for this representation. The other non-equivalent representation of the Clifford algebra can be obtained if one multiplies the above matrices γ^μ by an overall minus sign. The representation of the Lorentz group is nevertheless the same even if from the Clifford point of view these two representations are not unitarily equivalent.

A possible way to overcome these aspects is to define a *reducible* 4 by 4 representation combining the above two representations as

$$\Gamma^0 = \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}, \ \Gamma^1 = \begin{pmatrix} \gamma^1 & 0 \\ 0 & -\gamma^1 \end{pmatrix}, \ \Gamma^2 = \begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix} \tag{B.27}$$

which obey the Clifford algebra $\left\{\Gamma^\mu, \Gamma^\nu\right\} = 2\eta^{\mu\nu}$. Using these matrices one can build a hermitian matrix commuting with all the Dirac matrices [92]

$$\tau_3 = \mathrm{i}\Gamma^0\Gamma^1\Gamma^2 \tag{B.28}$$

which takes the form

$$\tau_3 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} \tag{B.29}$$

and which can be used to project onto each of the two irreducible subspaces.

Additional Features There exists *two* hermitian matrices which anticommute with all the Γ's:

$$\Gamma_3 = \begin{pmatrix} 0 & -\mathrm{i}1_{2\times 2} \\ \mathrm{i}1_{2\times 2} & 0 \end{pmatrix}; \ \Gamma_5 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \tag{B.30}$$

such that $\left\{\Gamma_{3,5}, \Gamma^\mu\right\} = 0$ and $\Gamma_{3,5}^2 = 1_{4\times4}$. When considered together $\Sigma_1 = \Gamma_5$, $\Sigma_2 = \Gamma_3$ and $\Sigma_3 = \tau_3$ form a representation of $su(2)$:

$$\left[\Sigma^i, \Sigma^j\right] = 2\mathrm{i}\epsilon^{ijk}\Sigma^k \tag{B.31}$$

and they square to the unit operator:

$$\left\{\Sigma^i, \Sigma^j\right\} = 2\delta_{ij}. \tag{B.32}$$

Note This $SU(2)$ acts by rotating the two "species" of spinors into each other.

B.7 Discrete symmetries

In $2+1$ dimensions discrete symmetries act in a peculiar way on the fields. Let us briefly review these transformations.

Time Reversal

$$\begin{array}{rcl} A^0(x^i,t) & \to & A^0(x^i,t) \\ A^i(x^i,t) & \to & -A^i(x^i,t) \\ \psi(x^i,t) & \to & \gamma^2\psi(x^i,-t) \end{array}$$

Parity

$$\begin{array}{rcl} A^0(x,y,t) & \to & A^0(-x,y,t) \\ A^1(x,y,t) & \to & -A^1(-x,y,t) \\ A^2(x,y,t) & \to & A^2(-x,y,t) \\ \psi(x,y,t) & \to & \gamma^1\psi(-x,y,t) \end{array}$$

Charge Conjugation

$$\begin{aligned} A^\mu(x,y,t) &\rightarrow -A^\mu(x,y,t) \\ \psi(x,y,t) &\rightarrow \psi_c(x,y,t) = \gamma^2\gamma^0\psi^*(x,y,t) \end{aligned}$$

Mass terms Any fermionic mass term of the form $\psi^\dagger\gamma_0\psi$ breaks parity. However combining two spinors ψ_1 and ψ_2 in the reducible representation, each in a different irreducible representation, allows to write down a parity conserving mass term

$$m\overline{\Psi}\Psi = m\Psi^\dagger\Gamma^0\Psi = m(\psi_1^\dagger\gamma^0\psi_1 - \psi_2^\dagger\gamma^0\psi_2) \tag{B.33}$$

with

$$\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}. \tag{B.34}$$

Indeed, parity acts as

$$\psi_1(x,y,t) \rightarrow -\mathrm{i}\gamma^1\psi_2(-x,y,t), \tag{B.35}$$

$$\psi_2(x,y,t) \rightarrow -\mathrm{i}\gamma^1\psi_1(-x,y,t). \tag{B.36}$$

Then $m\overline{\Psi}\Psi$ is not parity violating.

B.8 Pseudo-chiral symmetries

A fermionic kinetic term $\overline{\Psi}\mathrm{i}\Gamma^\mu D_\mu\Psi$ is invariant against the two global "pseudo-chiral" symmetries generated by Γ_3 and Γ_5 :

$$\begin{aligned} \Psi &\rightarrow e^{\mathrm{i}\alpha\Gamma_{3,5}}\Psi \\ \Psi^\dagger &\rightarrow \Psi^\dagger e^{-\mathrm{i}\alpha\Gamma_{3,5}}. \end{aligned}$$

A mass term $m\overline{\Psi}\Psi$ breaks the pseudo-chiral symmetries generated by Γ_3 and Γ_5 but not by τ_3. On the other hand $m\overline{\Psi}\tau_3\Psi$ breaks parity but not the symmetries generated by Γ_3 and Γ_5.

APPENDIX C

Additional research

C.1 Affine quantisation and the initial cosmological singularity

The initial cosmological singularity is a question which may be addressed in many possible theories of quantum gravity. While String Theory and Loop Quantum Gravity are competitive frameworks to address this issue, we suggested in [93] another approach based on an alternative quantisation procedure. We applied the "Enhanced Affine Quantisation" program, suggested by John Klauder [94], to a toy model of Friedman-Lemaître-Robertson-Walker cosmology. A major feature of the work is that it followed the proposal to quantise the affine algebra of a metric-like variable rather than the Heisenberg algebra. Associated affine coherent states are constructed and used in order to build a classical action containing quantum corrections. The corrections to the classical dynamics were shown to provide a potential barrier term responsible for bouncing solutions.

C.2 The $\mathcal{N} = 1$ supersymmetric Wong equations and the non-abelian Landau problem

The motion of a non-abelian charged particle in a classical non-abelian gauge field is described by the Wong equations. A recent study of the non-abelian Landau problem, i.e. a quantum particle confined to a plane and subjected to a static and homogeneous perpendicular magnetic field, has shown that the effects of specific choices of non-abelian gauge potentials corresponding to homogeneous coloured magnetic fields could account for the presence of spin-orbit interactions. The consequences of having in addition a supersymmetric invariant realization of the quantised system corresponding to the motion of a coloured particle in a classical external static non-abelian gauge field are discussed in [95]. We consider the case of a particle with arbitrary spin in a unitary (irreducible) representation of a compact gauge group. Furthermore, a canonical quantisation of the classical formulation is constructed. Subsequently, as a particular illustration, the spectrum of the $\mathcal{N} = 1$ supersymmetric non-abelian Landau problem is obtained in the specific case of a spin 1/2 particle in a non trivial static non-abelian background magnetic field. Finally, the inclusion of an electric potential term is discussed.

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